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Some results of $n \times n$ block matrices of linear relations in Banach spaces

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Abstract. In this paper, we consider in the product of Banach spaces $X_1 \otimes X_2 \otimes \ldots \otimes X_n$, the $n \times n$-block matrices of linear relations in the form,

$$\mathcal{M} := \begin{pmatrix} A_{1,1} & \ldots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \ldots & A_{n,n} \end{pmatrix},$$

where the entries of the matrix are in general unbounded linear relations and satisfy the following conditions:

$$A_{i,j} : X_j \to X_i, \ \ \forall i, j \in \{1, ..., n\}.$$

Studying the spectral properties of $\mathcal{M}$, it is natural to take stability of closedness for this matrix. So, we have to study this problem in the present paper. In addition, we show under some suitable conditions that $\mathcal{M}$ is a Fredholm linear relation.

1. Introduction

The theory of multivalued linear operator (or linear relation) arises frequently in the analysis of single valued linear operators motivated by the need to consider the adjoint of non densely defined operators, the closure, the inverse and the completion of linear operators. One of the works ashieved on the multivalued linear operators is the study of some Cauchy problems associated with parabolic type equations in Banach spaces (see [12]).Let X and Y be two Banach spaces. A linear relation $T : X \to Y$ is a mapping from a subspace the domain of T, into the collection of nonempty subsets of Y such that
$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$, for all nonzero scalars α_1, α_2 and $x_1, x_2 \in D(T)$. For $x \in X$, $x \notin D(T)$ we define $Tx = \emptyset$. With this convention, we have

$$D(T) := \Big\{x \in X : Tx \neq \emptyset\Big\}.$$

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The class of all linear relations from X to Y is indicated by $\mathcal{LR}(X,Y)$. The set of all closed linear relations from X to Y is indicated by $\mathcal{CR}(X,Y)$. If $T \in \mathcal{LR}(X,Y)$, then the graph of T is the subset $G(T)$ of $X \times Y$ defined by

$$G(T) := \Big\{(x,y) \in X \times Y : x \in D(T), y \in Tx\Big\}.$$

The inverse of T is the relation T^{-1} given by

$$G(T^{-1}) := \Big\{(y,x) \in Y \times X : (x,y) \in G(T)\Big\}.$$

If T maps the points in its domain to singletons, then T is said to be a single valued or an operator. Let $M \subset X$ be a subset, we write

$$T(M) := \cup\Big\{T(m) : m \in M \cap \mathcal{D}(T)\Big\}$$

called the image of M, with $\mathcal{R}(T) := T(X) = T(\mathcal{D}(T))$ called the range of T. If $\mathcal{R}(T) = Y$, then T is called surjective. Moreover, from [10, Proposition 2.3.], we have $y \in Tx \Longleftrightarrow Tx = y + T(0)$, where $x \in D(T)$. Thus, we say that $T \in \mathcal{LR}(X,Y)$ is single valued or operator if and only if $T(0) = \{0\}$, if and only if T^{-1} is injective. Now, let $T \in \mathcal{LR}(X,Y)$, we define a selection A of T by

$$T = A + T - T \text{ and } D(T) = D(A).$$

If A is a selection of T, then we have,

$$\|Tx\| \leq \|Ax\|, \ \ \forall x \in D(T).$$

Several problems in mathematical physics are defined by the system of partial or ordinary differential equations or linearizations thereof. In applications, the time evolution of a physical system is governed by block operator matrices. Hence, the spectral theory of these matrices plays a very important role. In the last decades, F. V. Atkinson, H. Langer, R. Mennicken, and A. A. Shkalikov (see [15]) studied the Wolf essential spectrum of a block operator matrix. An account of the research and a large panorama of methods to investigate the spectrum of block operator matrices were presented by in C. Tretter in [16] and A. Jeribi [13].

T. Álvarez, A. Ammar and A. Jeribi in their work [3] thought to expand these results for block operator matrix to block matrices of multivalued operators. In [6] A. Ammar, S. Fakhfakh and A. Jeribi expand firstly the main results of Tretter in [16] to linear relations and gave a necessary and sufficient condition for a 2×2 block matrices of linear relations $\mathcal{L}$ to become closed and closable. Secondly, they studied the stability of the essential spectrum of this matrix linear relation. In [5] A. Ammar, T. Diagana, and A. Jeribi, have studied the spectral properties of a 3×3 block matrices of linear relations $\mathcal{A}$.

In this work, we consider in the product of Banach spaces $X_1 \otimes X_2 \otimes \ldots \otimes X_n$, the $n \times n$ block matrices of linear relations defined by

$$\mathcal{M} := \begin{pmatrix} A_{1,1} & \ldots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \ldots & A_{n,n} \end{pmatrix}, \tag{1}$$

where the entries of the matrix are in general unbounded linear relations and satisfy the following conditions:

$$A_{i,j} : X_j \to X_i, \ \ \forall i,j \in \{1,...,n\}.$$

$\mathcal{M}$ is defined by its graph as follows:

$$\left\{\begin{array}{l} G(\mathcal{M}) = \left\{ \left(\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \right) \in (X_1 \otimes X_2 \otimes \ldots \otimes X_n)^2 \ : \ \begin{array}{c} v_1 \in \sum\limits_{i=1}^{n} A_{1,i} u_i \\ \vdots \\ v_n \in \sum\limits_{i=1}^{n} A_{n,i} u_i \end{array} \right\}. \\ \mathcal{D}(\mathcal{M}) = \Big(\bigcap\limits_{i=1}^{n} \mathcal{D}(A_{i,1}) \Big) \times \Big(\bigcap\limits_{i=1}^{n} \mathcal{D}(A_{i,2}) \Big) \times \ldots \times \Big(\bigcap\limits_{i=1}^{n} \mathcal{D}(A_{i,n}) \Big). \end{array}\right.$$

We decompose $\mathcal{M}$ as follows:

$$\mathcal{M} = \mathcal{D} + \sum_{i=1}^{n-1} \mathcal{B}_i, \tag{2}$$

where

$$\mathcal{D} = \begin{pmatrix} A_{1,1} & 0 & \ldots & 0 & 0 \\ 0 & A_{2,2} & 0 & \vdots & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \ddots & 0 & A_{n-1,n-1} & 0 \\ 0 & \ldots & 0 & 0 & A_{n,n} \end{pmatrix}$$

and, $\forall i \in \{1, ..., n-1\}$:

$$\mathcal{B}_i = \begin{pmatrix} 0 & \ldots & \ldots & 0 & A_{1,n-(i-1)} & 0 & \ldots & 0 \\ \vdots & 0 & \ldots & 0 & 0 & A_{2,n-(i-2)} & \ddots & \vdots \\ \vdots & \vdots & 0 & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \vdots & \vdots & 0 & 0 & 0 & 0 & A_{i,n} \\ A_{i+1,1} & 0 & 0 & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & A_{i+2,2} & 0 & \vdots & 0 & 0 & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 & \ldots & \ldots & 0 & \vdots \\ 0 & \ldots & 0 & A_{n,n-i} & 0 & \ldots & \ldots & 0 \end{pmatrix}.$$

Our main objective is to use the decomposition of $\mathcal{M}$ with two different situations (see the hypothesis (H) and the hypotheses (H_1) and (H_2)). First we consider the situation where each $\mathcal{B}_i$ is $\mathcal{D}$-bounded. Next we consider the situation where $\mathcal{B}_1$ is $\mathcal{D}$-bounded and $\forall i \in \{1, ..., n-1\}$, $\mathcal{B}_{i+1}$ is $\mathcal{B}_i$-bounded. In both situations, we study two different problems. The first problem is to study the closure of $\mathcal{M}$. The second problem is to study the Fredholm properties of a perturbed linear relation. Moreover with the help of the previous decomposition, we study the Fredholm properties of $\mathcal{M}$.

Our paper is organized as follows:

In Section 2, we give some definitions and auxiliary results, sometimes purely algebraic, which are used to prove the main results. In Section 3, we study the perturbation of linear relation. More precisely, we prove under some hypotheses that a perturbed linear relation is a Fredholm linear relation. Moreover, we study the closure of $\mathcal{M}$. More precisely, with the help of the decomposition (2), we prove that $\mathcal{M}$ is closed if and only if $\mathcal{D}$ is so (see Theorem 3.9). Moreover, we prove in Theorem 3.11, that $\mathcal{M}$ is closed if and only if $A_{1,1}$, ..., $A_{n,n}$ are closed. Finally, we show under the hypothesis (H) (respectively the hypotheses (H_1) and (H_2)), that $\mathcal{M}$ is a Fredholm relation.

2. Preliminaries and Auxiliary Results

The main aim of this section is to supply some auxiliary results which will be needed in the sequel.

Definition 2.1. *Let $T \in \mathcal{LR}(X,Y)$ and $S \in \mathcal{LR}(Y,Z)$ where $\mathcal{R}(T) \cap D(S) \neq \emptyset$. The composition (or product) ST in $\mathcal{LR}(X,Z)$ is defined as follows:*

$$(ST)(x) := \{S(Tx), \ \ (x \in X)\}$$

where $D(ST) = \{x \in X : Tx \cap D(S) \neq \emptyset\}$.
Hence,

$$D(ST) = T^{-1}(D(S)).$$

From the definition of ST it is easily seen that

$$G(ST) = \Big\{(x,z) \in X \times Z : (x,y) \in G(T) \text{ and } (y,z) \in G(S), \text{ for some } y \in Y\Big\}.$$

Definition 2.2. *Let $T \in \mathcal{LR}(X,Y)$ be a linear relation, we denote by Q_T is the natural quotient map of Y onto $Y/\overline{T(0)}$. It is clear that Q_T is single valued.*
We define

$$\|Tx\| = \|Q_T Tx\|, \text{ for all } x \in \mathcal{D}(T) \text{ and } \|T\| = \|Q_T T\|$$

called the norm of Tx and T respectively.

Proposition 2.3. *[11, Prposition II.1.7] Let S, $T \in \mathcal{LR}(X,Y)$ we have*

$$\|S + T\| \leq \|S\| + \|T\|$$

and

$$\|\alpha T\| = |\alpha| \|T\|, \quad \text{where } \ \alpha \in \mathbb{K}.$$

Definition 2.4. *The minimum modulus of $T \in \mathcal{LR}(X,Y)$ is the quantity*

$$\gamma(T) := \sup\{\lambda : \|Tx\| \geq \lambda d(x, \mathcal{N}(T)) \text{ for } x \in D(T)\}.$$

We have the formula $\gamma(T) = \|T^{-1}\|^{-1}$.

Definition 2.5. *Given $T \in \mathcal{LR}(X,Y)$, let X_T denote the vector space $D(T)$ normed by*

$$\|x\|_T := \|x\| + \|Tx\|, \ \ \forall x \in D(T).$$

Let $G_T \in \mathcal{LR}(X_T, X)$ be the identity injection of X_T into X, ie.

$$D(G_T) = X_T, \quad G_T x = x \ \ \forall x \in X_T.$$

G_T is called the graph operator of T.

Remark 2.6. *If T is a closed linear relation on a Banach space X. It is clear by the closedness of T that $(D(T), \|.\|_T)$ is a Banach space.*

Now, we give the definition of Fredholm relation. Several authors studied this theory, see for instance [2, 4, 5, 17].

Definition 2.7. *Let $T \in \mathcal{LR}(X,Y)$ be a closed linear relation where X and Y are Banach spaces, then the classes of Fredholm, upper semi-Fredholm and lower semi-Fredholm linear relations are defined, respectively, by:*

$$\Phi(X,Y) = \{T \in \mathcal{CR}(X,Y) : \ \mathcal{R}(T) \text{ is closed}, \ \alpha(T) < \infty \text{ and } \beta(T) < \infty\},$$

$$\Phi_+(X,Y) = \{T \in \mathcal{CR}(X,Y) : \ \mathcal{R}(T) \text{ is closed and } \alpha(T) < \infty\}$$

and

$$\Phi_-(X,Y) = \{T \in \mathcal{CR}(X,Y) : \ \mathcal{R}(T) \text{ is closed and } \beta(T) < \infty\}.$$

Moreover, $\alpha(T) := \dim T^{-1}(0) := \dim \mathcal{N}(T)$ and $\beta(T) := \dim Y/\mathcal{R}(T)$, called the nullity and deficiency of T, respectively. The index $ind(T)$ of T is defined as $ind(T) := \alpha(T) - \beta(T)$.

Lemma 2.8. *Let $G, S \in \mathcal{LR}(X, Y)$ be two linear relations.*
(i) *[11, Exercise I.2.14(b)] If $D(G) = D(S)$ and $G(0) = S(0)$, then $G = S$ or the graphs of G and S are incomparable.*
(ii) *[17, Proposition 2.7] If $G \in \Phi(X, Y)$, $D(S) \supset D(G)$, $S(0) \subset \overline{G(0)}$ and $\|S\| < \gamma(G)$, then $ind(S + G) = ind(G)$.*
(iii) *[11, Proposition II.5.3] G is closed if, and only if, $Q_G G$ is closed closed and $G(0)$ is closed.*
(iv) *[5, Lemma 2.3] $G - S + S = G$ if, and only if, $Q_G(S)$ is a single valued operator and $\|Q_G(S)\| \leq \|Q_S(S)\|$.*

Proposition 2.9. *Let S and T be two $n \times n$ block matrices of linear relations defined by*

$$S = \begin{pmatrix} S_{1,1} & \dots & S_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} & \dots & S_{n,n} \end{pmatrix} \text{ and } T = \begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{n,1} & \dots & T_{n,n} \end{pmatrix},$$

where the entries of the two matrices are in general unbounded linear relations and satisfies the following conditions:

$$S_{i,j} : X_j \to X_i \text{ and } T_{i,j} : X_j \to X_i, \ \ \forall i, j \in \{1, ..., n\}.$$

Then,

(i) $S + T = \begin{pmatrix} S_{1,1} + T_{1,1} & \dots & S_{1,n} + T_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} + T_{n,1} & \dots & S_{n,n} + T_{n,n} \end{pmatrix}.$

(ii) $\begin{pmatrix} S_{1,1} & \dots & S_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} & \dots & S_{n,n} \end{pmatrix}\begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{n,1} & \dots & T_{n,n} \end{pmatrix} \subseteq \begin{pmatrix} \sum_{i=1}^{n} S_{1,i}T_{i,1} & \dots & \sum_{i=1}^{n} S_{1,i}T_{i,n} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} S_{n,i}T_{i,1} & \dots & \sum_{i=1}^{n} S_{n,i}T_{i,n} \end{pmatrix}.$

Proof. (i) We have

$$\begin{aligned}
(S+T)\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} &= \left(\begin{pmatrix} S_{1,1} & \dots & S_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} & \dots & S_{n,n} \end{pmatrix} + \begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{n,1} & \dots & T_{n,n} \end{pmatrix} \right)\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} S_{1,1} & \dots & S_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} & \dots & S_{n,n} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{n,1} & \dots & T_{n,n} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} S_{1,1}(0) + T_{1,1}(0) & \dots & S_{1,n}(0) + T_{1,n}(0) \\ \vdots & \ddots & \vdots \\ S_{n,1}(0) + T_{n,1}(0) & \dots & S_{n,n}(0) + T_{n,n}(0) \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} (S_{1,1}+T_{1,1})(0) & \dots & (S_{1,n}+T_{1,n})(0) \\ \vdots & \ddots & \vdots \\ (S_{n,1}+T_{n,1})(0) & \dots & (S_{n,n}+T_{n,n})(0) \end{pmatrix}$$

Hence,

$$(S+T)\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} S_{1,1}+T_{1,1} & \dots & S_{1,n}+T_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1}+T_{n,1} & \dots & S_{n,n}+T_{n,n} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Moreover, we have

$$\begin{aligned} D(S+T) &= D(S)\bigcap D(T) \\ &= \Big(\bigcap_{i=1}^{n} D(S_{i,1})\Big)\times\Big(\bigcap_{i=1}^{n} D(S_{i,2})\Big)\times\ldots\times\Big(\bigcap_{i=1}^{n} D(S_{i,n})\Big) \\ &\quad \bigcap\Big(\bigcap_{i=1}^{n} D(T_{i,1})\Big)\times\Big(\bigcap_{i=1}^{n} D(T_{i,2})\Big)\times\ldots\times\Big(\bigcap_{i=1}^{n} D(T_{i,n})\Big) \\ &= \Big(\bigcap_{i=1}^{n} D(S_{i,1})\bigcap\bigcap_{i=1}^{n} D(T_{i,1})\Big)\times\Big(\bigcap_{i=1}^{n} D(S_{i,2})\bigcap\bigcap_{i=1}^{n} D(T_{i,2})\Big) \\ &\quad \times\ldots\times\Big(\bigcap_{i=1}^{n} D(S_{i,n})\bigcap\bigcap_{i=1}^{n} D(T_{i,n})\Big) \\ &= \Big(\bigcap_{i=1}^{n} D(S_{i,1}+T_{i,1})\Big)\times\Big(\bigcap_{i=1}^{n} D(S_{i,2}+T_{i,2})\Big)\times\ldots\times\Big(\bigcap_{i=1}^{n} D(S_{i,n}+T_{i,n})\Big) \\ &= D\left(\begin{pmatrix} S_{1,1}+T_{1,1} & \dots & S_{1,n}+T_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1}+T_{n,1} & \dots & S_{n,n}+T_{n,n} \end{pmatrix}\right). \end{aligned}$$

On the other hand, let

$$\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}\right) \in G(S+T).$$

Then there exist $\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix}$ and $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}$, such that $\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}$, with

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} \in \begin{pmatrix} S_{1,1} & \dots & S_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} & \dots & S_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

and

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} \in \begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{n,1} & \dots & T_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_{n-1} + v_{n-1} \\ u_n + v_n \end{pmatrix} \in \begin{pmatrix} (S_{1,1} + T_{1,1})x_1 + (S_{1,2} + T_{1,2})x_2 + \dots + (S_{1,n} + T_{1,n})x_n \\ (S_{2,1} + T_{2,1})x_1 + (S_{2,2} + T_{2,2})x_2 + \dots + (S_{2,n} + T_{2,n})x_n \\ \vdots \\ (S_{n-1,1} + T_{n-1,1})x_1 + \dots + (S_{n-1,n} + T_{n-1,n})x_n \\ (S_{n,1} + T_{n,1})x_1 + (S_{n,2} + T_{n,2})x_2 + \dots + (S_{n,n} + T_{n,n})x_n \end{pmatrix}.$$

We obtain,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_{n-1} + v_{n-1} \\ u_n + v_n \end{pmatrix} \in \begin{pmatrix} S_{1,1} + T_{1,1} & \dots & S_{1,n} + T_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} + T_{n,1} & \dots & S_{n,n} + T_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}.$$

Consequently,

$$G(S + T) \subseteq G\left(\begin{pmatrix} S_{1,1} + T_{1,1} & \dots & S_{1,n} + T_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} + T_{n,1} & \dots & S_{n,n} + T_{n,n} \end{pmatrix} \right).$$

In the end, the result follows from Lemma 2.8 (i).
(ii) Let

$$\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} \right) \in G\left(\begin{pmatrix} S_{1,1} & \dots & S_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} & \dots & S_{n,n} \end{pmatrix} \begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{n,1} & \dots & T_{n,n} \end{pmatrix} \right).$$

Then there exists $\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} \in X_1 \times X_2 \times ... \times X_n$, such that

$$\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} \right) \in G\left(\begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{n,1} & \dots & T_{n,n} \end{pmatrix} \right)$$

and

$$\left(\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} \right) \in G\left(\begin{pmatrix} S_{1,1} & \dots & S_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} & \dots & S_{n,n} \end{pmatrix} \right).$$

We get

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} \in \begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{n,1} & \dots & T_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

and

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} \in \begin{pmatrix} S_{1,1} & \dots & S_{1,n} \\ \vdots & \ddots & \vdots \\ S_{n,1} & \dots & S_{n,n} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix}.$$

Hence, $\forall k \in \{1, ..., n\}$

$$z_k \in \sum_{j=1}^{n} T_{k,j} x_j \text{ and } y_k \in \sum_{i=1}^{n} S_{k,i} z_i.$$

Thus, $\forall k \in \{1, ..., n\}$

$$y_k \in \sum_{j=1}^{n} \sum_{i=1}^{n} S_{k,i} T_{i,j} x_j.$$

Consequently,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} \in \begin{pmatrix} \sum_{i=1}^{n} S_{1,i} T_{i,1} & \dots & \sum_{i=1}^{n} S_{1,i} T_{i,n} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} S_{n,i} T_{i,1} & \dots & \sum_{i=1}^{n} S_{n,i} T_{i,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}.$$

$\square$

3. Main results

The aim of this section is to study the perturbation of linear relations

Definition 3.1. *[11, Definition VII.2.1] Let G, S be two linear relations on some Banach space. We say that S is relatively bounded with respect to G, or simply G-bounded, if $D(G) \subset D(S)$ and there exist $a \geq 0$ and $b \geq 0$ such that*

$$\|S\varphi\| \leq a\|\varphi\| + b\|G\varphi\|, \quad \forall \varphi \in D(G). \tag{3}$$

In this case the minimal constant b such that (3) holds, is called the G-bound of S.

Remark 3.2. *The inequality (3) is equivalent to,*

$$\|S\varphi\|^2 \leq a_1^2\|\varphi\|^2 + b_1^2\|G\varphi\|^2, \quad \forall \varphi \in D(G),$$

where $a_1 = \sqrt{a^2 + ab}$ and $b_1 = \sqrt{b^2 + ab}$.

Proposition 3.3. *Let A_1, ...,A_m and B be $m + 1$ linear relations defined on a Banach space X. Then, the following assertions holds true:*

(i) If, $\forall i \in \{1, ..., m\}$, A_i is B-bounded with relative bound $b_i < \frac{1}{m}$, then $\sum_{i=1}^{m} A_i$ is B-bounded with relative bound < 1.

(ii) If, $\forall i \in \{1, ..., m-1\}$, A_{i+1} is A_i-bounded and A_1 is B-bounded, then $\sum_{i=1}^{m} A_i$ is B-bounded with relative bound < 1.

Proof. (i) Since $\forall i \in \{1, ..., m\}$, A_i is B-bounded, then we have

$$\|A_i\varphi\| \leq a_i\|\varphi\| + b_i\|B\varphi\|, \quad \forall \varphi \in D(B).$$

Now, using triangle inequality, we obtain: $\forall \varphi \in D(B)$,

$$\|\sum_{i=1}^{m} A_i\varphi\| \leq \alpha\|\varphi\| + \beta\|B\varphi\|,$$

where, $\alpha = \sum_{i=1}^{m} a_i$ and $\beta = \sum_{i=1}^{m} b_i$.

Moreover, since $b_i < \frac{1}{m}$, then $\beta = \sum_{i=1}^{m} b_i < 1$. Thus, $\sum_{i=1}^{m} A_i$ is B-bounded with relative bound $\beta < 1$.

(ii) By hypothesis, $\forall i \in \{1, ..., m-1\}$, we have

$$\|A_{i+1}\varphi\| \leq a_{i+1}\|\varphi\| + b_{i+1}\|A_i\varphi\|, \quad \forall \varphi \in D(A_i),$$

and

$$\|A_1\varphi\| \leq a_1\|\varphi\| + b_1\|B\varphi\|, \quad \forall \varphi \in D(B).$$

Hence, it is clear that, $\forall i \in \{1, ..., m\}$, $\forall \varphi \in D(B)$,

$$\|A_i\varphi\| \leq \Big(\sum_{j=1}^{i} \Big(a_j \prod_{k=j+1}^{i} b_k\Big)\Big)\|\varphi\| + \prod_{j=1}^{i} b_j\|B\varphi\|.$$

Thus,

$$\|\sum_{i=1}^{m} A_i\varphi\| \leq \alpha\|\varphi\| + \beta\|B\varphi\|, \quad \forall \varphi \in D(B)$$

where $\alpha = \sum_{i=1}^{m} \sum_{j=1}^{i} a_j \prod_{k=j+1}^{i} b_k$ and $\beta = \sum_{i=1}^{m} \prod_{j=1}^{i} b_j$.

On the other hand, It is not difficult to see that $\beta = \sum_{i=1}^{m} \prod_{j=1}^{i} b_j < 1$ □

Now, we study the perturbation of Fredholm linear relation

Theorem 3.4. *Let $S, G \in \mathcal{LR}(X, Y)$ such that $S(0) \subset \overline{G(0)}$ and let $\widehat{G}$ be the bijection associated with it G. Assume that there exists a constant $c < \gamma(\widehat{G})$ such that*

$$\|S\varphi\| \leq c(\|\varphi\| + \|G\varphi\|), \quad \forall \varphi \in D(G).$$

If $G \in \Phi(X, Y)$, then $S + G \in \Phi(X, Y)$. Moreover, $ind(S + G) = ind(G)$, $\alpha(S + G) \leq \alpha(G)$ and $\beta(T + S) \leq \beta(T)$.

Proof. According to [5, Theorem 3.1], it is clear that $S + G$ is a closed linear relation. Let G_1, S_1 be the restrictions of the relations G, S to X_G. Evidently, G is a Fredholm linear relation and S_1 is a bounded linear relation. Furthermore, it is clear to prove that (see, [11, Theorem III.5.3]),

$$\|S_1\| \leq \gamma(\widehat{G}) = \gamma(\widehat{G'}).$$

Thus, by [11, Theorem V.5.12] and [11, Theorem V.3.2], we get $S_1 + G_1 \in \Phi(X, Y)$ and hence, $S + G \in \Phi(X, Y)$.

Now, the use of Lemma 2.8 (ii) and [11, Theorem III.7.4] leads to $\text{ind}(S + G) = \text{ind}(G)$, $\alpha(S + G) \leq \alpha(G)$ and $\beta(S + G) \leq \beta(G)$ and the Theorem is proved. □

Now, we use the decomposition (2) to study some spectral properties of $\mathcal{M}$ in two different situations. Firstly, we assume the following hypothesis:

(H) For all $1 \leq i \neq j \leq n$: $A_{i,j}$ is $A_{j,j}$-bounded with relative bound $< \frac{1}{n}$ and $A_{i,j}(0) \subset \overline{A_{j,j}(0)}$.

Proposition 3.5. *Under the hypothesis (H), $\forall k \in \{1, ..., n-1\}$, $\mathcal{B}_k$ is $\mathcal{D}$-bounded with relative bound $< \frac{1}{n}$.*

Proof. According to hypothesis (H) and by taking into account the Remark 3.2, we obtain, $\forall U = (u_1, u_2, ..., u_n) \in D(A_{1,1}) \times D(A_{2,2}) \times ... \times D(A_{n,n})$,

$$\begin{array}{rcl}
\|A_{k+1,1}u_1\|^2 & \leq & a_1^2\|u_1\|^2 + b_1^2\|A_{1,1}u_1\|^2 \\
\vdots & & \vdots \\
\|A_{n,n-k}u_{n-k}\|^2 & \leq & a_{n-k}^2\|u_{n-k}\|^2 + b_{n-k}^2\|A_{n-k,n-k}u_{n-k}\|^2 \\
\|A_{1,n-(k-1)}u_{n-(k-1)}\|^2 & \leq & a_{n-(k-1)}^2\|u_{n-(k-1)}\|^2 + b_{n-(k-1)}^2\|A_{n-(k-1),n-(k-1)}u_{n-(k-1)}\|^2 \\
\vdots & & \vdots \\
\|A_{k,n}u_n\|^2 & \leq & a_n^2\|u_n\|^2 + b_n^2\|A_{n,n}u_n\|^2.
\end{array}$$

Hence, $\forall U \in D(A_{1,1}) \times D(A_{2,2}) \times ... \times D(A_{n,n})$,

$$\begin{array}{rcl}
\|\mathcal{B}_kU\|^2 & = & \|A_{k+1,1}u_1\|^2 + ... + \|A_{n,n-k}u_{n-k}\|^2 + \|A_{1,n-(k-1)}u_{n-(k-1)}\|^2 + ... + \|A_{k,n}u_n\|^2 \\
& \leq & \sum_{i=1}^{n} a_i^2\|u_i\|^2 + b_i^2\|A_{i,i}u_i\|^2 \\
& \leq & \alpha_1\|U\|^2 + \beta_1\|\mathcal{D}U\|^2,
\end{array}$$

where, $\alpha_1 = \max\limits_{1\le i\le n} a_i^2$ and $\beta_1 = \max\limits_{1\le i\le n} b_i^2$. Thus, $\mathcal{B}_k$ is $\mathcal{D}$-bounded with relative bound $< \frac{1}{n}$ and the proposition is proved. □

Now, we give another hypotheses. For this we suppose that the entries of $\mathcal{M}$ satisfy the following conditions:
(H_1) $\forall i \in \{1, ..., n-1\}$ and $\forall j \in \{1, ..., n\}$ where $i+1 \neq j$, $A_{i+1,j}$ is $A_{i,j}$-bounded and $A_{i+1,j}(0) \subset \overline{A_{i,j}(0)}$.
(H_2) $\forall j \in \{1, ..., n\}$, $A_{1,j}$ is $A_{n,j}$-bounded and $A_{1,j}(0) \subset \overline{A_{n,j}(0)}$.

Proposition 3.6. *Under the hypotheses (H_1) and (H_2), we have*
(i) $\mathcal{B}_1$ is $\mathcal{D}$-bounded.
(ii) $\forall i \in \{1, ..., n-1\}$, $\mathcal{B}_{i+1}$ is $\mathcal{B}_i$-bounded.

Proof. (i) According to hypotheses (H_1) and (H_2) and by taking into account the Remark 3.2, we obtain $\forall U = (u_1, u_2, ..., u_n) \in D(A_{1,1}) \times D(A_{2,2}) \times ... \times D(A_{n,n})$:

$$\begin{array}{rcl} \|A_{2,1}u_1\|^2 & \le & a_1^2\|u_1\|^2 + b_1^2\|A_{1,1}u_1\|^2 \\ \|A_{3,2}u_2\|^2 & \le & a_2^2\|u_2\|^2 + b_2^2\|A_{2,2}u_2\|^2 \\ \vdots & & \vdots \\ \|A_{n,n-1}u_{n-1}\|^2 & \le & a_{n-1}^2\|u_{n-1}\|^2 + b_{n-1}^2\|A_{n-1,n-1}u_{n-1}\|^2 \\ \|A_{1,n}u_n\|^2 & \le & a_n^2\|u_n\|^2 + b_n^2\|A_{n,n}u_n\|^2. \end{array}$$

Hence, $\forall U \in D(A_{1,1}) \times D(A_{2,2}) \times ... \times D(A_{n,n})$

$$\begin{array}{rcl} \|\mathcal{B}_1 U\|^2 & = & \|A_{2,1}u_1\|^2 + \|A_{3,2}u_2\|^2 + ... + \|A_{n,n-1}u_{n-1}\|^2 + \|A_{1,n}u_n\|^2 \\ & \le & \displaystyle\sum_{i=1}^{n} a_i^2\|u_i\|^2 + b_i^2\|A_{i,i}u_i\|^2 \\ & \le & \alpha\|U\|^2 + \beta\|\mathcal{D}U\|^2, \end{array}$$

where, $\alpha = \max\limits_{1\le i\le n} a_i^2$ and $\beta = \max\limits_{1\le i\le n} b_i^2$. Thus, $\mathcal{B}_1$ is $\mathcal{D}$-bounded.

(ii) Using the same reasoning as (i), we obtain (ii) and the proposition is proved. □

Remark 3.7. *From the Propositions 3.3 and 3.5 (respectively Propositions 3.3 and 3.6), we have $\sum\limits_{i=1}^{n-1} \mathcal{B}_i$ is $\mathcal{D}$-bounded with relative bound < 1. Hence there exist a and b such that*

$$\|\sum_{i=1}^{n-1} \mathcal{B}_i\varphi\| \le a\|\varphi\| + b\|\mathcal{D}\varphi\|, \quad \forall \varphi \in D(\mathcal{D}).$$

Example 3.8. *Let us consider in $L_2(\mathbb{R}^3) \otimes L_2(\mathbb{R}^3) \otimes \ldots \otimes L_2(\mathbb{R}^3)$, the $n \times n$ block matrices of Schrödinger relations defined as follows:*

$$\mathcal{M} = \begin{pmatrix} \Delta & \mathcal{A}_{1,2} & \ldots & \mathcal{A}_{1,n-1} & \mathcal{A}_{1,n} \\ \mathcal{A}_{2,1} & \Delta & \mathcal{A}_{2,3} & \vdots & \mathcal{A}_{2,n} \\ \mathcal{A}_{3,1} & V_{3,2} & \ddots & \mathcal{A}_{n-2,n-1} & \vdots \\ \vdots & \ddots & V_{n-1,n-2} & \Delta & \mathcal{A}_{n-1,n} \\ \mathcal{A}_{n,1} & \ldots & \mathcal{A}_{n,n-2} & \Delta_{n,n-2} & \Delta \end{pmatrix},$$

where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ *is the Laplacian and* $\forall i, j \in \{1,...,n\}$, $\mathcal{A}_{i,j} \in L_{2,loc}(\mathbb{R}^3)$ *are linear relations defined by* $\forall i, j \in \{1,...,n\}$,

$$\mathcal{A}_{i,j} := V_{i,j} + \mathcal{A}_{i,j} - \mathcal{A}_{i,j},$$

with $\forall i, j \in \{1,...,n\}$, $V_{i,j} \in L_{2,loc}(\mathbb{R}^3)$ *is a positive potential given by* $V_{i,j}(x) = -\frac{\alpha_{i,j}}{|x|}$ *for some constant* $\alpha_{i,j} > 0$ *(hydrogen atom with Coulomb interaction).*
From [14, Theorem X.15 p 165] we have $\forall i, j \in \{1,...,n\}$, $V_{i,j}$ *is* Δ*-bounded.*
Since $\forall i, j \in \{1,...,n\}$, $V_{i,j}$ *is a selection of* $\mathcal{A}_{i,j}$, *then* $\forall i, j \in \{1,...,n\}$, $\mathcal{A}_{i,j}$ *is* Δ*-bounded.*
Now, it is clear that $\mathcal{A}_{i,j}$ *satisfied the condition of* (H), *hence, using Proposition 3.5, we obtain* $\forall k \in \{1,...,n-1\}$, $\mathcal{B}_k$ *is* $\mathcal{D}$*-bounded. Moreover, from Remark 3.7, we get* $\sum_{i=1}^{n-1} \mathcal{B}_i$ *is* $\mathcal{D}$*-bounded.*

The main results of this subsection are given as follows

Theorem 3.9. *Under the hypothesis* (H) *(respectively the hypotheses* (H_1) *and* (H_2)*), suppose, moreover, that the relative bounds are sufficiently small. Then*

$$\mathcal{M} \text{ is closed if and only if } \mathcal{D} \text{ is so.}$$

Proof. It follows from Remark 3.7, that $\sum_{i=1}^{n-1} \mathcal{B}_i$ is $\mathcal{D}$-bounded with relative bound $b < 1$.
On the other hand, Since, $\forall 1 \le i \ne j \le n$, $A_{i,j}(0) \subset \overline{A_{j,j}(0)}$, it is clear, $\forall U \in D(\mathcal{D})$,

$$\begin{aligned} \|\mathcal{D}U\| &\le \|\mathcal{M}U\| + \|(\sum_{i=1}^{n-1} \mathcal{B}_i)U\| \\ &\le \|\mathcal{M}U\| + a\|U\| + b\|\mathcal{D}U\| \end{aligned}$$

Since, $b < 1$, then $1 - b > 0$ and hence, we obtain:

$$\|\mathcal{D}U\| \le \frac{1}{1-b}\|\mathcal{M}U\| + \frac{a}{1-b}\|U\|. \tag{4}$$

Moreover, we have

$$\|\mathcal{M}U\| \le a\|U\| + (b+1)\|\mathcal{D}U\|. \tag{5}$$

Now, we suppose that $\mathcal{D}$ is closed. Let us consider two cases:

Case 1: For all $i, j \in \{1,...,n\}$, $A_{i,j}$ is single valued.
Let $(U_n)_n$ be a sequence in $D(\mathcal{D})$ such that $U_n \longrightarrow U$ and $\mathcal{M}U_n \longrightarrow V$ as $n \longrightarrow \infty$. The use of Eq. (4), leads to:

$$\|\mathcal{D}(U_n - U_m)\| \le \frac{1}{1-b}\|\mathcal{M}(U_n - U_m)\| + \frac{a}{1-b}\|U_n - U_m\|.$$

Thus, the sequence $(\mathcal{D}U_n)_n$ is a Cauchy sequence in the product of Banach spaces and so, is convergent. Since, $\mathcal{D}$ is closed, then $U \in D(\mathcal{D})$ and $\mathcal{D}U_n \longrightarrow \mathcal{D}U$ as $n \longrightarrow \infty$. Hence, the Eq. (5) enables us to conclude that, $\mathcal{M}U_n \longrightarrow \mathcal{M}U$ as $n \longrightarrow \infty$ and $V = \mathcal{M}U$. Consequently, $\mathcal{M}$ is closed.

Case 2: For all $i, j \in \{1,...,n\}$, $A_{i,j}$ is linear relation.

Since, $\forall 1 \le i \ne j \le n$, $A_{i,j}(0) \subset \overline{A_{j,j}(0)} = A_{j,j}(0)$, then $\sum_{i=1}^{n-1} \mathcal{B}_i(0) \subset \mathcal{D}(0)$. Hence, it is clear that $Q_{\mathcal{D}} = Q_{\sum_{i=1}^{n-1} \mathcal{B}_i + \mathcal{D}} =$

$Q_{\mathcal{M}}$. Then $Q_{\mathcal{M}}(\mathcal{M}) = Q_{\mathcal{D}}(\mathcal{D}) + Q_{\mathcal{D}}(\sum_{i=1}^{n-1} \mathcal{B}_i)$. Moreover, by Lemma 2.8 (iv), we have $Q_{\mathcal{D}}(\sum_{i=1}^{n-1} \mathcal{B}_i)$ is single valued and $\forall \varphi \in D(\mathcal{D})$,

$$\begin{aligned} \|Q_{\mathcal{D}}(\sum_{i=1}^{n-1} \mathcal{B}_i)\varphi\| &\leq \|Q_{\sum_{i=1}^{n-1} \mathcal{B}_i}(\sum_{i=1}^{n-1} \mathcal{B}_i)\varphi\| = \|\sum_{i=1}^{n-1} \mathcal{B}_i\varphi\| \\ &\leq a\|\varphi\| + b\|\mathcal{D}\varphi\|. \end{aligned}$$

Thus,

$$\|Q_{\mathcal{D}}(\sum_{i=1}^{n-1} \mathcal{B}_i)\varphi\| \leq a\|\varphi\| + b\|Q_{\mathcal{D}}(\mathcal{D})\varphi\|, \quad \forall \varphi \in D(\mathcal{D}).$$

On the other hand, $Q_{\mathcal{D}}(\mathcal{D})$ is closed, then $Q_{\mathcal{D}}(\mathcal{D}) + Q_{\mathcal{D}}(\sum_{i=1}^{n-1} \mathcal{B}_i)$ is closed single valued. This means, since $(\sum_{i=1}^{n-1} \mathcal{B}_i + \mathcal{D})(0) = \mathcal{D}(0)$ is closed, that $\mathcal{M} = \sum_{i=1}^{n-1} \mathcal{B}_i + \mathcal{D}$ is closed. Conversely, assume that $\mathcal{M}$ is closed. Since, $\sum_{i=1}^{n-1} \mathcal{B}_i$ is $\mathcal{D}$-bounded with relative bound $b < 1$. It is clear that $\forall \varphi \in D(\mathcal{D})$:

$$\begin{aligned} \|\sum_{i=1}^{n-1} \mathcal{B}_i\varphi\| &\leq a\|\varphi\| + b\|\mathcal{D}\varphi\| \\ &\leq a\|\varphi\| + b\|(\mathcal{D} + \sum_{i=1}^{n-1} \mathcal{B}_i - \sum_{i=1}^{n-1} \mathcal{B}_i)\varphi\| \\ &\leq a\|\varphi\| + b\|(\mathcal{D} + \sum_{i=1}^{n-1} \mathcal{B}_i)\varphi\| + b\|\sum_{i=1}^{n-1} \mathcal{B}_i\varphi\|. \end{aligned}$$

Hence, since $b < 1$, we have $\forall \varphi \in D(\mathcal{D})$:

$$\| - \sum_{i=1}^{n-1} \mathcal{B}_i\varphi\| = \|\sum_{i=1}^{n-1} \mathcal{B}_i\varphi\| \leq \frac{a}{1-b}\|\varphi\| + \frac{b}{1-b}\|(\mathcal{D} + \sum_{i=1}^{n-1} \mathcal{B}_i)\varphi\|.$$

In the light of the above, $\mathcal{D} + \sum_{i=1}^{n-1} \mathcal{B}_i - \sum_{i=1}^{n-1} \mathcal{B}_i$ is closed. Now, according to Lemma 2.8(iv), we obtain $\mathcal{D}$ is closed. $\square$

Now, we give a proposition that we will need in the sequel.

Proposition 3.10. *We have,*

$$Q_{\mathcal{D}}\mathcal{D} = \begin{pmatrix} Q_{A_{1,1}}(A_{1,1}) & 0 & \dots & 0 & 0 \\ 0 & Q_{A_{2,2}}(A_{2,2}) & 0 & \vdots & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \ddots & 0 & Q_{A_{n-1,n-1}}(A_{n-1,n-1}) & 0 \\ 0 & \dots & 0 & 0 & Q_{A_{n,n}}(A_{n,n}) \end{pmatrix}$$

and, $\forall i \in \{1, ..., n-1\}$*:*

$$Q_{\mathcal{B}_i}\mathcal{B}_i = \begin{pmatrix} 0 & \cdots & 0 & Q_{A_{1,n-(i-1)}}(A_{1,n-(i-1)}) & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & 0 & \ddots & & 0 \\ \vdots & \vdots & & \vdots & 0 & 0 & Q_{A_{i,n}}(A_{i,n}) \\ Q_{A_{i+1,1}}(A_{i+1,1}) & 0 & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & & \vdots & 0 & 0 & \vdots & \vdots \\ \vdots & \ddots & 0 & \cdots & \cdots & 0 & \vdots \\ 0 & 0 & Q_{A_{n,n-i}}(A_{n,n-i}) & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Proof. Let $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in D(A_{1,1}) \times ... \times D(A_{n,n})$ and $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathcal{D}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then $y_1 \in A_{1,1}x_1, ..., y_n \in A_{n,n}x_n$.

This yields,

$$Q_{\mathcal{D}}\mathcal{D}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = Q_{\mathcal{D}}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Now, let us find the expression of $Q_{\mathcal{D}}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

Notice that

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in Q_{\mathcal{D}}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

if, and only if,

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} - \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \begin{pmatrix} \overline{A_{1,1}(0)} \\ \vdots \\ \overline{A_{n,n}(0)} \end{pmatrix}.$$

Then,

$$\begin{cases} z_1 - y_1 \in \overline{A_{1,1}(0)}, \\ \quad \vdots \\ z_n - y_n \in \overline{A_{n,n}(0)}. \end{cases}$$

This is equivalent to

$$\begin{cases} z_1 \in Q_{A_{1,1}}(y_1), \\ \quad \vdots \\ z_n \in Q_{A_{n,n}}(y_n). \end{cases}$$

This shows that

$$Q_{\mathcal{D}}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} Q_{A_{1,1}}(y_1) \\ \vdots \\ Q_{A_{n,n}}(y_n) \end{pmatrix} = \begin{pmatrix} Q_{A_{1,1}}(A_{1,1}x_1) \\ \vdots \\ Q_{A_{n,n}}(A_{n,n}x_n) \end{pmatrix}.$$

Thus, $\forall \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in D(A_{1,1}) \times \ldots \times D(A_{n,n})$

$$Q_{\mathcal{D}}\mathcal{D}\begin{pmatrix} x_1 \\ \\ \vdots \\ \\ x_n \end{pmatrix} = \begin{pmatrix} Q_{A_{1,1}}(A_{1,1}) & 0 & \cdots & 0 & 0 \\ 0 & Q_{A_{2,2}}(A_{2,2}) & 0 & \vdots & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \ddots & 0 & Q_{A_{n-1,n-1}}(A_{n-1,n-1}) & 0 \\ 0 & \cdots & 0 & 0 & Q_{A_{n,n}}(A_{n,n}) \end{pmatrix}\begin{pmatrix} x_1 \\ \\ \vdots \\ \\ x_n \end{pmatrix}.$$

By the same reasoning as before, we get the result to $Q_{\mathcal{B}_i}\mathcal{B}_i$, with $i \in \{1, ..., n-1\}$. □

Theorem 3.11. *Under the hypothesis* (H) *(respectively the hypotheses* (H_1) *and* (H_2)*), suppose, moreover, that the relative bounds are sufficiently small. Then*

$$\mathcal{M} \text{ is closed if and only if } A_{1,1}, ..., A_{n,n} \text{ are closed.}$$

Proof. Suppose that $\mathcal{M}$ is closed. Then by Theorem 3.9, $\mathcal{D}$ is closed. According to Lemma 2.8 (iii) and Proposition 3.10, it follows that

$$\begin{pmatrix} Q_{A_{1,1}}(A_{1,1}) & 0 & \cdots & 0 & 0 \\ 0 & Q_{A_{2,2}}(A_{2,2}) & 0 & \vdots & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \ddots & 0 & Q_{A_{n-1,n-1}}(A_{n-1,n-1}) & 0 \\ 0 & \cdots & 0 & 0 & Q_{A_{n,n}}(A_{n,n}) \end{pmatrix}$$

is a closed linear operator and that $\begin{pmatrix} \overline{A_{1,1}(0)} \\ \vdots \\ \overline{A_{n,n}(0)} \end{pmatrix} = \begin{pmatrix} A_{1,1}(0) \\ \vdots \\ A_{n,n}(0) \end{pmatrix}$. Hence, $Q_{A_{1,1}}(A_{1,1})$, $Q_{A_{2,2}}(A_{2,2})$, ..., $Q_{A_{n,n}}(A_{n,n})$ are closed and $\overline{A_{1,1}(0)} = A_{1,1}(0)$, $\overline{A_{2,2}(0)} = A_{2,2}(0)$, ..., $\overline{A_{n,n}(0)} = A_{n,n}(0)$. Consequently, $A_{1,1}, A_{2,2}, ..., A_{n,n}$ are closed linear relations. Conversely, if we assume that $A_{1,1}, A_{2,2}, ..., A_{n,n}$ are closed linear relations, then by Lemma 2.8 (iii), we have $Q_{A_{1,1}}(A_{1,1})$, $Q_{A_{2,2}}(A_{2,2})$, ..., $Q_{A_{n,n}}(A_{n,n})$ are closed and $\overline{A_{1,1}(0)} = A_{1,1}(0)$, $\overline{A_{2,2}(0)} = A_{2,2}(0)$, ..., $\overline{A_{n,n}(0)} = A_{n,n}(0)$. Hence $Q_{\mathcal{D}}\mathcal{D}$ is a closed linear operator and $\mathcal{D}(0)$ is closed. Thus applying Lemma 2.8 (iii) we deduce that $\mathcal{D}$ is closed. According to Theorem 3.9, we obtain that $\mathcal{M}$ is closed. □

Remark 3.12. *By the same reasoning as before, we can study the closure of* $\mathcal{M}$ *for generalized subordinate perturbations (see [1, Definition 2.1]).*

Now, we show under the hypothesis (H) (respectively the hypotheses (H_1) and (H_2)), that $\mathcal{M}$ is a Fredholm relation.

Theorem 3.13. *Let* $\widehat{\mathcal{D}}$ *be the bijection associated with* $\mathcal{D}$*. Under the hypothesis* (H) *(respectively the hypotheses* (H_1) *and* (H_2)*), suppose moreover that the relative bounds is sufficiently small and* $\max(a, b) < \gamma(\widehat{\mathcal{D}})$*. Then, if* $\forall i \in \{1, ..., n\}$*,* $A_{i,i}$ *is Fredholm linear relation then* $\mathcal{M}$ *is Fredholm linear relation. Moreover,* $ind(\mathcal{M}) = ind(\mathcal{D})$*,* $\alpha(\mathcal{M}) \leq \alpha(\mathcal{D})$ *and* $\beta(\mathcal{M}) \leq \beta(\mathcal{D})$*.*

Proof. Since, $\forall i \in \{1, ..., n\}$, $A_{i,i}$ is Fredholm linear relation then, it is clear by Cross that $\forall i \in \{1, ..., n\}$, $Q_{A_{i,i}}(A_{i,i})$ is Fredholm linear operator. Hence $Q_{\mathcal{D}}\mathcal{D}$ is a Fredholm operator. Thus $\mathcal{D}$ is a Fredholm linear relation.

Finally the results follow from Theorem 3.4. □

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