



On the structure of Gelfand transform and BED property for abstract Segal algebras

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Abstract. Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. In this paper, we first show that $\mathcal{A}$ is Tauberian if and only if $\mathcal{B}$ is Tauberian. Then we prove that $\widehat{\mathcal{A}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{A}))$ if and only if $\widehat{\mathcal{B}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{B}))$. Afterwards, we conclude that whenever $\mathcal{A}$ is a BED algebra, then $\mathcal{B}$ is a BED algebra if and only if $C_{\text{BSE}}^0(\Delta(\mathcal{B})) \subseteq \widehat{\mathcal{B}}$.

1. Introduction and Preliminaries

Takahasi and Hatori introduced and investigated the notion of BSE algebras [17]. Then in 2007, Inoue and Takahasi introduced and studied the concept of BED algebras [10]. Subsequently, several authors investigated these concepts for various classes of commutative and semisimple Banach algebras; see for example [8] and [9], as some valuable survey works, as well as some recent works of the author with others, such as [1], [2], [3], [4] and [5].

To provide the basic definitions of BSE and BED algebras, we present some preliminaries and frameworks, as the following; see [13] and [14] for more information.

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative Banach algebra with the dual space $\mathcal{A}^*$. Denote by $\Delta(\mathcal{A})$ the Gelfand (character) space of $\mathcal{A}$, consisting of all nonzero multiplicative elements in $\mathcal{A}^*$. Also $\Delta(\mathcal{A})$ is considered with the weak* topology, induced by $\mathcal{A}^*$ and is always a locally compact Hausdorff space [13, Theorem 2.2.3]. Also the linear span of $\Delta(\mathcal{A})$ is indicated by $\text{span}(\Delta(\mathcal{A}))$. It should be noted that every element p of $\text{span}(\Delta(\mathcal{A}))$ is written as $p = \sum_{\varphi} \widehat{p}(\varphi)\varphi$, such that $\widehat{p}: \Delta(\mathcal{A}) \rightarrow \mathbb{C}$ is a function with finite support. Moreover, the Banach algebra consisting of all continuous and bounded complex valued functions on $\Delta(\mathcal{A})$, is denoted by $C_b(\Delta(\mathcal{A}))$, which is equipped with the pointwise product. Furthermore, the Gelfand mapping of $\mathcal{A}$ is defined as

$$\mathcal{A} \rightarrow C_b(\Delta(\mathcal{A})) \quad a \mapsto \widehat{a},$$

where $\widehat{a}(\varphi) = \varphi(a)$ ($\varphi \in \Delta(\mathcal{A})$). Set $\widehat{\mathcal{A}} = \{\widehat{a} : a \in \mathcal{A}\}$. Then $\mathcal{A}$ is called semisimple if its Gelfand mapping is injective. Moreover, $\mathcal{A}$ is called Tauberian if $\mathcal{A}_0$, which is the set of all $x \in \mathcal{A}$ such that $\widehat{x}$ has compact support, is dense in $\mathcal{A}$. Throughout the paper, we assume that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a commutative and semisimple

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Banach algebra. The function $\sigma \in C_b(\Delta(\mathcal{A}))$ is called a BSE function if there exists a constant $K > 0$ such that the inequality

$$\left| \sum_{j=1}^n d_j \sigma(\psi_j) \right| \leq K \left\| \sum_{j=1}^n d_j \psi_j \right\|_{\mathcal{A}^*} \quad (1)$$

holds, for every finite number of $\psi_1, \dots, \psi_n$ in $\Delta(\mathcal{A})$ and also the same number of $d_1, \dots, d_n$ in $\mathbb{C}$ [17]. Then the BSE norm of σ ($\|\sigma\|_{\text{BSE}, \mathcal{A}}$), is the infimum of all such M , satisfying (1). The set of all BSE-functions is indicated by $C_{\text{BSE}}(\Delta(\mathcal{A}))$. In [17, Lemma 1], Takahasi and Hatori showed that $(C_{\text{BSE}}(\Delta(\mathcal{A})), \|\cdot\|_{\text{BSE}, \mathcal{A}})$, is a commutative and semisimple Banach algebra with pointwise product. We always have $\widehat{\mathcal{A}} \subseteq C_{\text{BSE}}(\Delta(\mathcal{A}))$ and also

$$\|\widehat{x}\|_{\infty} \leq \|\widehat{x}\|_{\text{BSE}, \mathcal{A}} \leq \|x\| \quad (x \in \mathcal{A}).$$

Moreover, $\mathcal{A}$ is called a BSE norm algebra (or has a BSE norm) if there exists $M > 0$ such that $\|x\| \leq M \|\widehat{x}\|_{\text{BSE}, \mathcal{A}}$ ($x \in \mathcal{A}$) [9, 18]. Let

$$\mathcal{M}(\mathcal{A}) = \{\sigma \in C_b(\Delta(\mathcal{A})) : \sigma \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}\};$$

see [14], for more information. Now suppose that $\mathcal{A}$ is a commutative and semisimple Banach algebra. Following [17], $\mathcal{A}$ is called a BSE algebra if

$$C_{\text{BSE}}(\Delta(\mathcal{A})) = \mathcal{M}(\mathcal{A}).$$

A bounded net $\{x_{\alpha}\}_{\alpha \in \Lambda}$ in $\mathcal{A}$ is called a bounded Δ -weak approximate identity for $\mathcal{A}$, if $\lim_{\alpha} \varphi(x x_{\alpha}) = \varphi(x)$ ($x \in \mathcal{A}, \varphi \in \Delta(\mathcal{A})$) [12]. In [17, Corollary 5], the authors proved that $\mathcal{A}$ has a bounded Δ -weak approximate identity if and only if

$$\mathcal{M}(\mathcal{A}) \subseteq C_{\text{BSE}}(\Delta(\mathcal{A})).$$

Finally, we introduce the concept of BED algebras. Note that the basic definition of BED algebras, presented in [10], is based on the notion of quasi-topologies of $\text{span}(\Delta(\mathcal{A}))$. But another introduction is as follows; see [10]. Let $\mathcal{K}$ be the set of all compact subsets of $\Delta(\mathcal{A})$ and for each $f \in C_{\text{BSE}}(\Delta(\mathcal{A}))$ and any $K \in \mathcal{K}$ let

$$\|f\|_{\text{BSE}, \mathcal{K}} = \sup \left\{ \left| \sum_{\varphi} \widehat{p}(\varphi) f(\varphi) \right| : p \in \text{span}(\Delta(\mathcal{A})), \|p\|_{\mathcal{A}^*} \leq 1, \widehat{p}(\varphi) = 0 \ (\varphi \in K) \right\}.$$

Moreover, let $\|f\|_{\text{BSE}, \infty} = \inf_{K \in \mathcal{K}} \|f\|_{\text{BSE}, K}$ and

$$C_{\text{BSE}}^0(\Delta(\mathcal{A})) = \{f \in C_{\text{BSE}}(\Delta(\mathcal{A})) : \|f\|_{\text{BSE}, \infty} = 0\},$$

which is a closed ideal in $C_{\text{BSE}}(\Delta(\mathcal{A}))$ [10, Corollary 3.9]. Then $\mathcal{A}$ is called a BED algebra if $\widehat{\mathcal{A}} = C_{\text{BSE}}^0(\Delta(\mathcal{A}))$. It is noticeable that since $C_{\text{BSE}}^0(\Delta(\mathcal{A}))$ is closed under the norm $\|\cdot\|_{\text{BSE}, \mathcal{A}}$, all BED algebras have BSE norm.

Our aim of this paper is investigating the BED property for the abstract Segal algebras of any BED algebra $\mathcal{A}$. We first show that $\mathcal{A}$ is Tauberian if and only if $\mathcal{B}$ is Tauberian. Then we prove that $\widehat{\mathcal{A}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{A}))$ if and only if $\widehat{\mathcal{B}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{B}))$. Moreover, we provide an example to show that $C_{\text{BSE}}^0(\Delta(\mathcal{A})) \subseteq \widehat{\mathcal{A}}$ does not necessarily imply the inclusion $C_{\text{BSE}}^0(\Delta(\mathcal{B})) \subseteq \widehat{\mathcal{B}}$. Thus we conclude that $\mathcal{B}$ is a BED algebra if and only if $C_{\text{BSE}}^0(\Delta(\mathcal{B})) \subseteq \widehat{\mathcal{B}}$. Finally, we provide some examples for clarification.

2. main results

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra. Then the Banach algebra $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is called an abstract Segal algebra with respect to $\mathcal{A}$ if the following conditions are satisfied.

(i) $\mathcal{B}$ is a dense ideal in $\mathcal{A}$, which is essential; i.e.

$$\mathcal{B} = \mathcal{A} \cdot \mathcal{B} = \{ab : a \in \mathcal{A}, b \in \mathcal{B}\}.$$

(ii) For any $b \in \mathcal{B}$, $\|b\|_{\mathcal{A}} \leq \|b\|_{\mathcal{B}}$.

(iii) For all $a, b \in \mathcal{B}$, $\|ab\|_{\mathcal{B}} \leq \|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$.

Since in the definition of abstract Segal algebras, the essentiality of $\mathcal{B}$ is assumed, it follows from [7, Theorem 2.1] that $\mathcal{B}$ is semisimple. Moreover, $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$ are homeomorphic; see [7, Theorem 2.1]. Indeed, by [6, Lemma 2.2] the function $\widetilde{\varphi} = \varphi|_{\mathcal{B}}$ belongs to $\Delta(\mathcal{B})$, for each $\varphi \in \Delta(\mathcal{A})$ and

$$\Delta(\mathcal{B}) = \{\widetilde{\varphi} : \varphi \in \Delta(\mathcal{A})\};$$

Note that abstract Segal algebras were presented as a generalization of Segal algebras, introduced in [16]. In fact, all Segal algebras are abstract Segal algebras with respect to $L^1(G)$, where G is an abelian locally compact Hausdorff group.

We commence our results with the following proposition, which is interesting in its own right.

Proposition 2.1. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then $\mathcal{A}$ is Tauberian if and only if $\mathcal{B}$ is Tauberian.*

Proof. First let $\mathcal{B}$ be Tauberian and take $a \in \mathcal{A}$. For any $\varepsilon > 0$, there exists $b \in \mathcal{B}$ such that $\|a - b\|_{\mathcal{A}} < \varepsilon/2$. By the hypothesis, there exists $b_0 \in \mathcal{B}_0$ such that $\|b - b_0\|_{\mathcal{B}} < \varepsilon/2$. Note that $\text{supp } \widehat{b_0}$ is compact in $\Delta(\mathcal{B})$ and so in $\Delta(\mathcal{A})$. Consequently, $b_0 \in \mathcal{A}_0$ and so

$$\|a - b_0\|_{\mathcal{A}} \leq \|a - b\|_{\mathcal{A}} + \|b - b_0\|_{\mathcal{A}} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

It follows that $\mathcal{A}_0$ is dense in $\mathcal{A}$ and thus $\mathcal{A}$ is Tauberian.

Conversely, suppose that $\mathcal{A}$ be Tauberian and take $b \in \mathcal{B}$ to be arbitrary. By the essentiality of $\mathcal{B}$, there are $a \in \mathcal{A}$ and $c \in \mathcal{B}$ such that $b = ac$. By the hypothesis, there exists $a_0 \in \mathcal{A}_0$ such that

$$\|a - a_0\|_{\mathcal{A}} < \frac{\varepsilon}{\|c\|_{\mathcal{B}}}.$$

Now we have $a_0c \in \mathcal{B}$ and since

$$\text{supp } \widehat{a_0c} \subseteq \text{supp } \widehat{a_0},$$

thus $\text{supp } \widehat{a_0c}$ is compact in $\Delta(\mathcal{B})$. Moreover,

$$\|b - a_0c\|_{\mathcal{B}} = \|ac - a_0c\|_{\mathcal{B}} \leq \|a - a_0\|_{\mathcal{A}}\|c\|_{\mathcal{B}} \leq \frac{\varepsilon}{\|c\|_{\mathcal{B}}}\|c\|_{\mathcal{B}} = \varepsilon.$$

Therefore $\mathcal{B}_0$ is dense in $\mathcal{B}$ and so $\mathcal{B}$ is Tauberian. $\square$

To present the main results, we state some basic propositions. Note that for any $p \in \text{span}(\Delta(\mathcal{A}))$ we have $\|p\|_{\mathcal{B}^*} \leq \|p\|_{\mathcal{A}^*}$, immediately.

Proposition 2.2. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$ and $\sigma \in C_{\text{BSE}}^0(\Delta(\mathcal{B}))$. Then $\bar{\sigma}$, defined as $\bar{\sigma}(\varphi) = \sigma(\widetilde{\varphi})$, ($\varphi \in \Delta(\mathcal{A})$) belongs to $C_{\text{BSE}}^0(\Delta(\mathcal{A}))$.*

Proof. It is obvious that $\bar{\sigma} \in C_b(\Delta(\mathcal{A}))$ and $\|\bar{\sigma}\|_{\infty} = \|\sigma\|_{\infty}$. Moreover, by [17, Theorem 4] there exists the bounded net $\{b_{\lambda}\}_{\lambda}$ in $\mathcal{B}$ such that $\widehat{b_{\lambda}}(\widetilde{\varphi}) \rightarrow \sigma(\widetilde{\varphi})$ ($\widetilde{\varphi} \in \Delta(\mathcal{B})$). Thus $\{b_{\lambda}\}_{\lambda}$ is bounded in $\mathcal{A}$ and $\widehat{b_{\lambda}}(\varphi) \rightarrow \bar{\sigma}(\varphi)$ ($\varphi \in \Delta(\mathcal{A})$). It follows that $\bar{\sigma} \in C_{\text{BSE}}(\Delta(\mathcal{A}))$. Furthermore, for any $\varepsilon > 0$ there exists the compact subset K of $\Delta(\mathcal{B})$ such that for all $p \in \text{span}(\Delta(\mathcal{B}) \setminus K)$ as $p = \sum_{\widetilde{\varphi} \notin K} \widetilde{p}(\widetilde{\varphi})\widetilde{\varphi}$ with $\|p\|_{\mathcal{B}^*} \leq 1$ we have

$$\left| \sum_{\widetilde{\varphi} \notin K} \widetilde{p}(\widetilde{\varphi})\sigma(\widetilde{\varphi}) \right| < \varepsilon. \quad (2)$$

Let

$$F = \{\varphi : \widetilde{\varphi} \in K\},$$

which is compact in $\Delta(\mathcal{A})$. Then for any $p \in \text{span}(\Delta(\mathcal{A}) \setminus F)$ as $p = \sum_{\varphi \notin F} \widehat{p}(\varphi) \varphi$ with $\|p\|_{\mathcal{A}^*} \leq 1$ we have

$$\|p\|_{\mathcal{B}^*} \leq \|p\|_{\mathcal{A}^*} \leq 1$$

and by (2) we obtain

$$\left| \sum_{\varphi \notin F} \widehat{p}(\varphi) \overline{\sigma}(\varphi) \right| = \left| \sum_{\widetilde{\varphi} \notin K} \widehat{p}(\varphi) \sigma(\widetilde{\varphi}) \right| < \varepsilon.$$

Therefore $p \in C_{\text{BSE}}^0(\Delta(\mathcal{A}))$ and the proof is completed. $\square$

One can also indicate the symbol $\overline{\sigma}$ with σ , if there is no ambiguity.

Proposition 2.3. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$ and $\sigma \in C_{\text{BSE}}^0(\Delta(\mathcal{A}))$. Suppose that $\overline{\sigma}$ is a complex valued function on $\Delta(\mathcal{B})$, defined as $\overline{\sigma}(\widetilde{\varphi}) = \sigma(\varphi)$ ($\widetilde{\varphi} \in \Delta(\mathcal{B})$). Then $\overline{\sigma}b \in C_{\text{BSE}}^0(\Delta(\mathcal{B}))$, for all $b \in \mathcal{B}$.*

Proof. It is obvious that $\overline{\sigma} \in C_b(\Delta(\mathcal{B}))$ and $\|\overline{\sigma}\|_{\infty} = \|\sigma\|_{\infty}$. It is sufficient to prove the statement for the members in the unit ball of $\mathcal{B}$. Thus take $b \in \mathcal{B}$ with $\|b\|_{\mathcal{B}} \leq 1$. Since $\sigma \in C_{\text{BSE}}(\Delta(\mathcal{A}))$, [17, Theorem 4] implies that there exists the bounded net $\{a_{\lambda}\}_{\lambda}$ in $\mathcal{A}$ such that $\widehat{a_{\lambda}}(\varphi) \rightarrow \sigma(\varphi)$ ($\varphi \in \Delta(\mathcal{A})$). Thus $\{a_{\lambda}b\}_{\lambda}$ is bounded in $\mathcal{B}$ and $\widehat{a_{\lambda}b}(\widetilde{\varphi}) \rightarrow \overline{\sigma}b(\widetilde{\varphi})$ ($\widetilde{\varphi} \in \Delta(\mathcal{B})$). It follows that $\overline{\sigma}b \in C_{\text{BSE}}(\Delta(\mathcal{B}))$. Moreover, for any $\varepsilon > 0$, there exists the compact subset K of $\Delta(\mathcal{A})$ such that for all $p \in \text{span}(\Delta(\mathcal{A}) \setminus K)$ as $p = \sum_{\varphi \notin K} \widehat{p}(\varphi) \varphi$ with $\|p\|_{\mathcal{A}^*} \leq 1$ we have

$$\left| \sum_{\varphi \notin K} \widehat{p}(\varphi) \sigma(\varphi) \right| < \varepsilon. \quad (3)$$

Now let

$$\widetilde{K} = \{\widetilde{\varphi} : \varphi \in K\},$$

which is compact in $\Delta(\mathcal{B})$. Now for all $p \in \text{span}(\Delta(\mathcal{A}) \setminus \widetilde{K})$ as $p = \sum_{\widetilde{\varphi} \notin \widetilde{K}} \widehat{p}(\widetilde{\varphi}) \widetilde{\varphi}$ with $\|p\|_{\mathcal{B}^*} \leq 1$, suppose that

$$\bar{p} = \sum_{\varphi \notin K} \widehat{p}(\widetilde{\varphi}) \varphi(b) \varphi.$$

Thus

$$\begin{aligned} \|\bar{p}\|_{\mathcal{A}^*} &= \sup_{\|a\|_{\mathcal{A}} \leq 1} \left| \sum_{\varphi \notin K} \widehat{p}(\widetilde{\varphi}) \varphi(b) \varphi(a) \right| \\ &= \sup_{\|a\|_{\mathcal{A}} \leq 1} \left| \sum_{\varphi \notin K} \widehat{p}(\widetilde{\varphi}) \varphi(ab) \right| \\ &\leq \sup_{\|c\|_{\mathcal{B}} \leq 1} \left| \sum_{\varphi \notin K} \widehat{p}(\widetilde{\varphi}) \varphi(c) \right| \\ &= \|p\|_{\mathcal{B}^*} \\ &\leq 1. \end{aligned}$$

By (3) we obtain

$$\left| \sum_{\varphi \notin K} \widehat{p}(\widetilde{\varphi}) \varphi(b) \sigma(\varphi) \right| < \varepsilon.$$

It follows that

$$\left| \sum_{\widetilde{\varphi} \notin K} \widehat{p}(\widetilde{\varphi}) \widetilde{\sigma} \widehat{b}(\widetilde{\varphi}) \right| < \varepsilon.$$

Therefore $\widehat{\sigma} \widehat{b} \in C_{\text{BSE}}^0(\Delta(\mathcal{B}))$, as claimed. $\square$

One can also denote the symbol $\widetilde{\sigma}$ with σ , if there is no ambiguity.

Theorem 2.4. Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then $\widehat{\mathcal{B}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{B}))$ if and only if $\widehat{\mathcal{A}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{A}))$.

Proof. First, suppose that $\widehat{\mathcal{B}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{B}))$. Proposition 2.2 implies that $\widehat{\mathcal{B}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{A}))$. Now take $a \in \mathcal{A}$. For any $\varepsilon > 0$ there exists $b_0 \in \mathcal{B}$ such that $\|a - b_0\|_{\mathcal{A}} < \varepsilon/2$. Since $\widehat{b}_0 \in C_{\text{BSE}}^0(\Delta(\mathcal{B}))$, there exists the compact subset K of $\Delta(\mathcal{B})$ such that for all $p \in \text{span}(\Delta(\mathcal{B}) \setminus K)$, as $p = \sum_{\widetilde{\varphi} \notin K} \widehat{p}(\widetilde{\varphi}) \widetilde{\varphi}$ with $\|p\|_{\mathcal{B}^*} \leq 1$ we have

$$\left| \sum_{\widetilde{\varphi} \notin K} \widehat{p}(\widetilde{\varphi}) \widetilde{\varphi}(b_0) \right| < \varepsilon/2. \quad (4)$$

Suppose that

$$F = \{\varphi : \widetilde{\varphi} \in K\},$$

which is a compact subset of $\Delta(\mathcal{A})$. Thus for all $p \in \text{span}(\Delta(\mathcal{A}) \setminus F)$, as $p = \sum_{\varphi \notin F} \widehat{p}(\varphi) \varphi$ with $\|p\|_{\mathcal{A}^*} \leq 1$ we have $\|p\|_{\mathcal{B}^*} \leq 1$ and so by (4) we obtain

$$\left| \sum_{\varphi \notin F} \widehat{p}(\varphi) \varphi(b_0) \right| < \varepsilon/2.$$

Consequently, by the inequality (4) we have

$$\begin{aligned} \left| \sum_{\varphi \notin F} \widehat{p}(\varphi) \varphi(a) \right| &\leq \left| \sum_{\varphi \notin F} \widehat{p}(\varphi) \varphi(a - b_0) \right| + \left| \sum_{\varphi \notin F} \widehat{p}(\varphi) \varphi(b_0) \right| \\ &\leq \left\| \sum_{\varphi \notin F} \widehat{p}(\varphi) \varphi \right\|_{\mathcal{A}^*} \|a - b_0\|_{\mathcal{A}} + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

It follows that $\widehat{a} \in C_{\text{BSE}}^0(\Delta(\mathcal{A}))$ and so $\widehat{\mathcal{A}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{A}))$.

Conversely, suppose that $\widehat{\mathcal{A}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{A}))$. We show that $\widehat{\mathcal{B}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{B}))$. To this end, take $b \in \mathcal{B}$. There exist $a \in \mathcal{A}$ and $c \in \mathcal{B}$ such that $b = ac$. By the assumption $\widehat{a} \in C_{\text{BSE}}^0(\Delta(\mathcal{A}))$. Also Proposition 2.3 implies that $\widehat{b} = \widehat{a} \widehat{c}$ belongs to $C_{\text{BSE}}^0(\Delta(\mathcal{B}))$. Therefore $\widehat{\mathcal{B}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{B}))$, as claimed. $\square$

Remark 2.5. The obvious question that is raised, is the relationship between the inclusions $C_{\text{BSE}}^0(\Delta(\mathcal{A})) \subseteq \widehat{\mathcal{A}}$ and $C_{\text{BSE}}^0(\Delta(\mathcal{B})) \subseteq \widehat{\mathcal{B}}$. We provide an example to show that the inclusion $C_{\text{BSE}}^0(\Delta(\mathcal{A})) \subseteq \widehat{\mathcal{A}}$ does not necessarily imply the inclusion $C_{\text{BSE}}^0(\Delta(\mathcal{B})) \subseteq \widehat{\mathcal{B}}$. Let G be an abelian locally compact Hausdorff group, $\mathcal{A}$ be the group algebra $L^1(G)$ and $1 < p < \infty$. Moreover, suppose that $\mathcal{B}$ is $S_p(G)$, defined as

$$S_p(G) = L^1(G) \cap L^p(G)$$

and for any $f \in \mathcal{B}$ let

$$\|f\|_{S_p(G)} = \max\{\|f\|_1, \|f\|_p\}.$$

Thus $\mathcal{B}$ is an abstract Segal algebra with respect to $\mathcal{A}$ [16]. By [10, Remark 4.14], $\mathcal{A}$ is a BED algebra. Furthermore, by [10, Theorem 6.3] $\mathcal{B}$ is not a BED algebra. Theorem 2.4 implies that $C_{\text{BSE}}^0(\Delta(\mathcal{B})) \not\subseteq \widehat{\mathcal{B}}$. However, $C_{\text{BSE}}^0(\Delta(\mathcal{A})) \subseteq \widehat{\mathcal{A}}$.

According to Remark 2.5, in the following we present some results that are slightly different from what we expected.

First we have the following lemma, which is immediately obtained.

Lemma 2.6. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then $\mathcal{A}$ has BSE norm if and only if $\widehat{\mathcal{B}}$ is dense in $(\widehat{\mathcal{A}}, \|\cdot\|_{\text{BSE}, \mathcal{A}})$.*

Proposition 2.7. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. If $\mathcal{A}$ is a BSE norm algebra then $\mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{B})$.*

Proof. Note that we always have $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{B})$. In fact, take $\sigma \in \mathcal{M}(\mathcal{A})$ and $b \in \mathcal{B}$. By the essentiality of $\mathcal{B}$ we have $b = ac$, for some $a \in \mathcal{A}$ and $c \in \mathcal{B}$. According to the choice of σ , there exists $d \in \mathcal{A}$ such that $\sigma\widehat{a} = \widehat{d}$. Now we have $dc \in \mathcal{B}$ and

$$\widehat{\sigma b} = \sigma\widehat{ac} = \widehat{dc} = \widehat{dc},$$

which implies that $\widehat{\sigma b} \in \widehat{\mathcal{B}}$. Thus $\sigma\widehat{\mathcal{B}} \subseteq \widehat{\mathcal{B}}$ and so $\sigma \in \mathcal{M}(\mathcal{B})$. Consequently, $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{B})$.

Now we prove the reverse of the inclusion, by using the BSE norm property of $\mathcal{A}$. Take $\sigma \in \mathcal{M}(\mathcal{B})$ and $a \in \mathcal{A}$. There exists the sequence $\{b_n\}_n$ in $\mathcal{B}$ such that

$$\lim_{n \rightarrow \infty} \|a - b_n\|_{\mathcal{A}} = 0.$$

Thus

$$\lim_{n \rightarrow \infty} \|\widehat{a} - \widehat{b_n}\|_{\text{BSE}, \mathcal{A}} = 0$$

and so

$$\lim_{n \rightarrow \infty} \|\sigma\widehat{a} - \sigma\widehat{b_n}\|_{\text{BSE}, \mathcal{A}} = 0.$$

By the fact that $\sigma \in \mathcal{M}(\mathcal{B})$, we have $\sigma\widehat{b_n} \in \widehat{\mathcal{B}}$ ($n \in \mathbb{N}$). Since $\mathcal{A}$ has BSE norm, Lemma 2.6 implies that $\sigma\widehat{a} \in \widehat{\mathcal{A}}$. Therefor $\sigma\widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$ and so $\sigma \in \mathcal{M}(\mathcal{A})$. $\square$

Proposition 2.8. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Moreover, suppose that $\mathcal{A}$ has BSE norm. If $C_{\text{BSE}}^0(\Delta(\mathcal{B})) \subseteq \widehat{\mathcal{B}}$ then $C_{\text{BSE}}^0(\Delta(\mathcal{A})) \subseteq \mathcal{M}(\mathcal{A})$.*

Proof. Take $\sigma \in C_{\text{BSE}}^0(\Delta(\mathcal{A}))$. Since $\mathcal{A}$ has BSE norm, it is sufficient by Proposition 2.7 to show that $\sigma \in \mathcal{M}(\mathcal{B})$. To this end, take $b \in \mathcal{B}$ to be arbitrary. By the assumption and also Proposition 2.3, $\sigma\widehat{b} \in \widehat{\mathcal{B}}$. It follows that $\sigma \in \mathcal{M}(\mathcal{B})$. Thus the proof is completed. $\square$

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a BSE algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then by [11, Theorem 9.10] and also [15], $\mathcal{B}$ is a BSE algebra if and only if $\mathcal{B}$ admits a bounded Δ -weak approximate identity; equivalently $\mathcal{M}(\mathcal{B}) \subseteq C_{\text{BSE}}(\Delta(\mathcal{B}))$ [17, Corollary 5]. Analogously, we have the main result of the present work, as the following.

Theorem 2.9. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a BED algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then $\mathcal{B}$ is a BED algebra if and only if $C_{\text{BSE}}^0(\Delta(\mathcal{B})) \subseteq \widehat{\mathcal{B}}$.*

Proof. First, let $C_{\text{BSE}}^0(\Delta(\mathcal{B})) \subseteq \widehat{\mathcal{B}}$. Since $\mathcal{A}$ is a BED algebra, thus

$$\widehat{\mathcal{A}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{A})).$$

By Theorem 2.4 we have $\widehat{\mathcal{B}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{B}))$. It follows that $\mathcal{B}$ is BED. The converse is obvious. $\square$

We conclude the paper with the following results, which are obtained from Proposition 2.1 and Theorem 2.4, immediately.

Corollary 2.10. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Tauberian commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then $\mathcal{B}$ is a BED algebra if and only if $C_{\text{BSE}}^0(\Delta(\mathcal{B})) \subseteq \widehat{\mathcal{B}}$.*

Corollary 2.11. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Moreover, suppose that $\widehat{\mathcal{A}} \subseteq C_{\text{BSE}}^0(\Delta(\mathcal{A}))$. Then $\mathcal{B}$ is a BED algebra if and only if $C_{\text{BSE}}^0(\Delta(\mathcal{B})) \subseteq \widehat{\mathcal{B}}$.*

Examples 2.12. We provide some examples, for clarification.

- (1). Let G be an infinite abelian compact group and $1 < p < \infty$. Then $L^p(G)$ is a proper Segal algebra with respect to $L^1(G)$; see [16] for more details. Since $L^1(G)$ is Tauberian, by Proposition 2.1 $L^p(G)$ is also Tauberian. Moreover, by [18, Theorem 4]

$$\widehat{L^p(G)} = C_{\text{BSE}}(\Delta(L^p(G))),$$

which implies that

$$\widehat{L^p(G)} = C_{\text{BSE}}^0(\Delta(L^p(G))) = C_{\text{BSE}}(\Delta(L^p(G))).$$

Thus $L^p(G)$ is a BED algebra.

- (2). Let G be an abelian locally compact group and $1 < p < \infty$. Then $S_p(G)$, defined in Remark 2.5, is an abstract Segal algebra with respect to $L^1(G)$ which is not BED.
- (3). Let X be a nonempty set. It is known that $\ell^1(X)$ is a Banach algebra under pointwise product, which is an abstract Segal algebra with respect to $c_0(X)$. By [10, Theorem 5.10], $c_0(X)$ is a BED algebra. Furthermore, by [10, corollary 4.2] and also [4, Theorem 3.2] $\ell^1(X)$ is a BED algebra.

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