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Porosity and pointwise product in X^p spaces

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Abstract. In this paper, we study some important porous subsets of the product space $X_1^{p_1} \times \ldots \times X_n^{p_n}$, where $p_1, \ldots, p_n \geq 1$ and $X_1, \ldots, X_n$ are Banach function spaces. The obtained results give interesting information regarding the closedness of X^p spaces with respect to poitwise product. Also, some applications for weighted Lebesgue and Orlicz spaces are given. The conclusions are generalizations of similar facts regarding Lebesgue and Orlicz spaces.

1. Introduction and preliminaries

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $X^0(\Omega)$ be the set of all μ-equivalence classes of measurable functions on Ω. A subset X of $X^0(\Omega)$ is called a *Banach function space* on Ω, if there is a norm $\|\cdot\|_X$ on X such that $(X, \|\cdot\|_X)$ is Banach space. In this case, X is called *solid* if for each $f \in X$ and $g \in X^0(\Omega)$ satisfying $|g| \leq |f|$ we have $g \in X$ and $\|g\|_X \leq \|f\|_X$. For each $p \in [1, \infty)$, the Lebesgue space

$$L^p(\Omega) = \left\{ f \in X^0(\Omega) : \|f\|_p = \left(\int_\Omega |f|^p \right)^{\frac{1}{p}} d\mu < \infty \right\}$$

is the important Banach function space on Ω. Also if by a weight on Ω we mean a $\mathcal{A}$-measurable function $w : \Omega \to (0, \infty)$, then the weighted Lebesgue space $L^p(\Omega, w) = \{f \in X^0(\Omega) : \|f\|_{p,w} = \left(\int_\Omega |fw|^p \right)^{\frac{1}{p}} d\mu < \infty\}$ is a Banach space.
For any Banach space $(X, \|\cdot\|_X)$ and $p \in [1, \infty)$, a generalization of $L^p(\Omega)$, the Banach function space $X^p(\Omega)$ is defined by

$$X^p(\Omega) = \{f \in X^0(\Omega) : |f|^p \in X\},$$

with the norm $\|f\|_{X^p} = (\||f|^p\|_X)^{\frac{1}{p}}$. In particular, $\|\chi_E\|_{X^p}^p = \|\chi_E\|_X$. Also Orlicz spaces are genuine generalizations of the usual L^p-spaces. A brief definition of Orlicz spaces is provided in the second section.

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Now, we recall the definition of a porous set. Let X be a metric space. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. For a given number $0 < c \leq 1$, a subset M of X is called c-lower porous if

$$\liminf_{R\to 0+} \frac{\gamma(x, M, R)}{R} \geq \frac{c}{2}$$

for all $x \in M$, where

$$\gamma(x, M, R) = \sup\{r > 0 : \exists z \in X, B(z, r) \subseteq B(x, R)\backslash M\}.$$

It is clear that M is c-lower porous if and only if

$$\forall x \in M, \forall \alpha \in (0, c/2), \exists r_0 > 0, \forall r \in (0, r_0), \exists z \in X, B(z, \alpha r) \subseteq B(x, r)\backslash M.$$

A set is called σ-c-lower porous if it is a countable union of c-lower porous sets with the same constant $c > 0$. See [3] for more details and information.

If $p_1, \ldots, p_n$ and r are positive real numbers, then by [2, Theorem 9] if $\inf\{\mu(A) : \mu(A) > 0\} = 0$ and $\frac{1}{p_1} + \ldots + \frac{1}{p_n} > \frac{1}{r}$, then the set

$$E = \{(f_1, f_2, \ldots, f_n) \in L^{p_1} \times \cdots \times L^{p_n} : f_1 \cdots f_n \in L^r\}$$

is a σ-porous subset of $L^{p_1} \times \cdots \times L^{p_n}$. Similary by [2, Theorem 10] if $\inf\{\mu(A) : \mu(A) < \infty\} = \infty$ and $\frac{1}{p_1} + \ldots + \frac{1}{p_n} < \frac{1}{r}$, then E is a σ-porous subset of $L^{p_1} \times \cdots \times L^{p_n}$. In the cases $\frac{1}{p_1} + \ldots + \frac{1}{p_n} = \frac{1}{r}$ or $\inf\{\mu(A) : \mu(A) > 0\} > 0$ and $\frac{1}{p_1} + \ldots + \frac{1}{p_n} > \frac{1}{r}$, or $\inf\{\mu(A) : \mu(A) < \infty\} < \infty$ and $\frac{1}{p_1} + \ldots + \frac{1}{p_n} < \frac{1}{r}$, by [2, proposition 2 and Theorem 8] we have $E = L^{p_1} \times \cdots \times L^{p_n}$.
In section 2, generalizations of these results in X^p spaces are expressed. Also, results about the spaceability of the complement of the desired subsets of the product of X^p spaces have been proved in section 3.

2. On σ-c-lower porousness

In this section some generalization of porosity theorems regarding Lebesgue or Orlicz spaces is presented for X^p spaces. We have the following useful lemma regarding X^p spaces whose proof is straightforward.

Lemma 2.1. *Let X be a Banach function space. Then*

1. *X is solid if and only if X^p is solid for all $p \geq 1$.*
2. *$\|\cdot\|_X$ has the absolute continuous property if and only if $\|\cdot\|_{X^p}$ has the absolute continuous property.*

As a first main result we express the following theorem, which was proved for Lebesgue spaces in [2, Theorem 6].

Theorem 2.2. *Let $1 \leq p_1, \ldots p_n, r < \infty$ and $X_1, \ldots, X_n, \mathcal{Z}$ be solid function spaces. Assume that there exists a sequence $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$ with $\mu(A_k) > 0$ for all k such that for some $a_i, b_i > 0$, $(i = 1, \ldots, n)$, and for every $B_k \subseteq A_k$, $(k \in \mathbb{N})$ we have*

$$a_i \|\chi_{B_k}\|_{\mathcal{Z}} \leq \|\chi_{B_k}\|_{X_i} \leq b_i \|\chi_{B_k}\|_{\mathcal{Z}}. \tag{1}$$

Let one of the conditions (i) *or* (ii) *be valid:*

(i) $\sum_{i=1}^{n} \frac{1}{p_i} > \frac{1}{r}$ *and*

$$\lim_{k\to\infty} \left(\|\chi_{A_k}\|_{\mathcal{Z}} + \sum_{i=1}^{n} \left\|\chi_{A_k} f_i\right\|_{X_i} \right) = 0$$

for all $(f_1, \ldots, f_n) \in \prod_{i=1}^{n} X_i$,

(ii) $\sum_{i=1}^{n} \frac{1}{p_i} < \frac{1}{r}$ *and*

$$\lim_{k\to\infty}\left(\frac{1}{\|\chi_{A_k}\|_{\mathcal{Z}}} + \sum_{i=1}^{n} \left\|\chi_{A_k} f_i\right\|_{X_i}\right) = 0$$

for all $(f_1,\ldots,f_n) \in \prod_{i=1}^{n} X_i$.

Then, there is $c > 0$ *such that*

$$E := \{(f_1,\ldots,f_n) \in X_1^{p_1} \times \ldots \times X_n^{p_n} : f_1 \ldots f_n \in \mathcal{Z}^r\}$$

is σ*-c-lower porous in* $X_1^{p_1} \times \ldots \times X_n^{p_n}$.

Proof. For each $m \in \mathbb{N}$ we denote

$$E_m := \left\{(f_1,\ldots,f_n) \in X_1^{p_1} \times \ldots \times X_n^{p_n} : f_1 \ldots f_n \in \mathcal{Z}^r \text{ and } \|f_1 \ldots f_n\|_{\mathcal{Z}^r} \le m\right\}.$$

Trivially, $E = \bigcup_{m=1}^{\infty} E_m$. Fix an $m \in \mathbb{N}$, and let $R > 0$. There is some $c > 0$ such that for every $0 < \delta < \frac{c}{2}$,

$$1 - \delta > (1+n)\frac{b_i}{a_i}\delta$$

for all $i \in \{1,2,\ldots,n\}$. So, there exists some η with

$$\max\left\{(1+n)\frac{b_i}{a_i}\delta : i \in \{1,2,\ldots,n\}\right\} < \eta < 1 - \delta.$$

Therefore, $\left(\frac{\delta}{\eta}\right)^{p_i} < \frac{a_i}{b_i(n+1)}$ for all $i = 1 \ldots n$. So, for some $0 < d_1, d_2 < 1$,

$$\left(\frac{\delta}{\eta}\right)^{p_i} < \frac{a_i(1-d_2)^{p_i}}{b_i(n+d_1+1)} \quad (i = 1,\ldots,n). \tag{2}$$

In the case (i), one can pick some $T > 0$ such that

$$m < \frac{(d_1+1)^{\frac{1}{r}}\,(d_2\eta R)^n}{(b_1^{\frac{1}{p_1}} \ldots b_n^{\frac{1}{p_n}})(n+d_1+1)^{\frac{1}{r}}\,t^{\left(\frac{1}{p_1}+\ldots+\frac{1}{p_n}-\frac{1}{r}\right)}}$$

for all $0 < t \le T$. Also, in the case (ii), there exists some $T > 0$ such that

$$m < \frac{(d_1+1)^{\frac{1}{r}}(d_2\eta R)^n t^{\left(\frac{1}{r}-\frac{1}{p_1}-\ldots-\frac{1}{p_n}\right)}}{(b_1^{\frac{1}{p_1}} \ldots b_n^{\frac{1}{p_n}})(n+d_1+1)^{\frac{1}{r}}}$$

for all $t \ge \frac{1}{T}$.

Let $(f_1,\ldots,f_n) \in X_1^{p_1} \times \ldots \times X_n^{p_n}$. If (i) or (ii) holds, then there is some $k_0 \in \mathbb{N}$ such that setting $A := A_{k_0}$ we have

$$\left\|\chi_A f_i^{p_i}\right\|_{X_i} < \min\{T, ((1-\delta-\eta)R)^{p_i}\}, \qquad (i = 1,\ldots,n) \tag{3}$$

and also in the case (*i*), $\|\chi_A\|_{\mathcal{Z}} < T$ and in the case (*ii*), $\frac{1}{\|\chi_A\|_{\mathcal{Z}}} < T$. Denote $M_i := \|\chi_A\|_{X_i}^{\frac{-1}{p_i}} \eta R$ for each $i = 1,\ldots,n$. Then, setting

$$\tilde{f_i} := M_i\chi_A + f_i\chi_{A^c}, \qquad (i = 1,\ldots,n),$$

we have

$$\begin{aligned}\|f_i - \tilde{f}_i\|_{X_i^{p_i}} &= \|(f_i - M_i)\chi_A\|_{X_i^{p_i}}\\ &\le \|f_i\chi_A\|_{X_i^{p_i}} + M_i\,\|\chi_A\|_{X_i^{p_i}}\\ &\le \|f_i^{p_i}\chi_A\|_{X_i}^{\frac{1}{p_i}} + M_i\|\chi_A\|_{X_i}^{\frac{1}{p_i}}\\ &< (1-\delta-\eta)R + \eta R = R - \delta R.\end{aligned}$$

This implies that $B((\tilde{f}_1,\dots,\tilde{f}_n);\delta R) \subseteq B((f_1,\dots,f_n);R)$. Now, we assume that $(u_1,\dots,u_n) \in B((\tilde{f}_1,\dots,\tilde{f}_n);\delta R)$. Then,

$$\begin{aligned}\delta R &> \|u_i - \tilde{f}_i\|_{X_i^{p_i}}\\ &\ge \|(u_i - \tilde{f}_i)\chi_A\|_{X_i^{p_i}}\\ &= \|(u_i - M_i)\chi_A\|_{X_i^{p_i}},\end{aligned}$$

for each $i = 1,\dots,n$. By (2) this implies that

$$\left\|\left(\frac{u_i}{M_i} - 1\right)\chi_A\right\|_{X_i^{p_i}} \le \frac{\delta R}{M_i} = \frac{\delta}{\eta}\,\|\chi_A\|_{X_i}^{\frac{1}{p_i}} \le \left(\frac{a_i\,\|\chi_A\|_{X_i}}{b_i(n+d_1+1)}\right)^{\frac{1}{p_i}}\ (1-d_2),\ \ (i=1,\dots,n).$$

For every $i = 1,\dots,n$ put

$$B_i := \{x \in A : \frac{u_i(x)}{M_i} < d_2\}.$$

Then

$$\begin{aligned}\frac{1}{(n+d_1+1)}\,\|\chi_A\|_{X_i} &\ge \left\|\left(\frac{u_i}{M_i} - 1\right)\chi_A\right\|_{X_i^{p_i}}^{p_i}\ (1-d_2)^{-p_i}\,\frac{b_i}{a_i}\\ &\ge \left\|\left(\frac{u_i}{M_i} - 1\right)\chi_{B_i}\right\|_{X_i^{p_i}}^{p_i}\ (1-d_2)^{-p_i}\,\frac{b_i}{a_i}\\ &\ge \left\|(1-d_2)\,\chi_{B_i}\right\|_{X_i^{p_i}}^{p_i}\ (1-d_2)^{-p_i}\,\frac{b_i}{a_i}\\ &= (1-d_2)^{p_i}\,\|\chi_{B_i}\|_{X_i}\ (1-d_2)^{-p_i}\,\frac{b_i}{a_i}\\ &= \|\chi_{B_i}\|_{X_i}\,\frac{b_i}{a_i}\end{aligned}$$

Hence by (1) for every $i = 1,\dots,n$ we have

$$\begin{aligned}a_i\|\chi_{B_i}\|_{\mathcal{Z}} &\le \|\chi_{B_i}\|_{X_i}\\ &\le \frac{\|\chi_A\|_{X_i}}{(n+d_1+1)}\,\frac{a_i}{b_i}\\ &\le \frac{b_i\|\chi_A\|_{\mathcal{Z}}}{(n+d_1+1)}\,\frac{a_i}{b_i}\\ &= \frac{a_i\|\chi_A\|_{\mathcal{Z}}}{(n+d_1+1)},\end{aligned}$$

so

$$\|\chi_{B_i}\|_{\mathcal{Z}} \le \frac{\|\chi_A\|_{\mathcal{Z}}}{(n+d_1+1)}.$$

Then,

$$\begin{aligned}
\|u_1\dots u_n\|^r_{\mathcal{Z}^r} &\ge M_1\dots M_n \left\| \left| \frac{u_1}{M_1}\cdots\frac{u_n}{M_n}\chi_A\right|^r \right\|_{\mathcal{Z}} \\
&\ge M_1^r\dots M_n^r \left\| \left| \frac{u_1}{M_1}\cdots\frac{u_n}{M_n}\right|^r \chi_{A-\bigcup_{i=1}^n B_i} \right\|_{\mathcal{Z}} \\
&\ge M_1^r\dots M_n^r d_2^{rn}\|\chi_{A-\bigcup_{i=1}^n B_i}\|_{\mathcal{Z}} \\
&\ge M_1^r\dots M_n^r d_2^{rn}\left(\|\chi_A\|_{\mathcal{Z}} - \sum_{i=1}^n \|\chi_{B_i}\|_{\mathcal{Z}}\right) \\
&\ge M_1^r\dots M_n^r d_2^{rn}\|\chi_A\|_{\mathcal{Z}}\left(1 - \sum_{i=1}^n \frac{1}{n+d_1+1}\right) \\
&= M_1^r\dots M_n^r d_2^{rn}\|\chi_A\|_{\mathcal{Z}}\left(1 - \frac{n}{n+d_1+1}\right) \\
&= M_1^r\dots M_n^r d_2^{rn}\|\chi_A\|_{\mathcal{Z}}\left(\frac{d_1+1}{n+d_1+1}\right) \\
&\ge \frac{1}{b_1^{\frac{r}{p_1}}\dots b_n^{\frac{r}{p_n}}}\left(\frac{d_1+1}{n+d_1+1}\right)(d_2\eta R)^{rn}\,\|\chi_A\|_{\mathcal{Z}}^{1-\frac{r}{p_1}-\dots\frac{r}{p_n}} \\
&\ge m^r.
\end{aligned}$$

This implies that $(u_1,\dots,u_n)\notin E_m$. Hence,

$$B\Big((\tilde{f_1},\dots,\tilde{f_n}),\delta R) \subseteq B\Big((f_1,\dots,f_n),R\Big) - E_m,$$

and the proof is complete. □

Setting $X_1 = X_2 = \dots = X_n = \mathcal{Z}$ in the previous theorem, we obtain the following result. Just note that if $c := \frac{2}{n+1}$, then for every $0 < \delta < \frac{c}{2}$ we have $1-\delta > (1+n)\delta$.

Theorem 2.3. *Let $1 \le p_1,\dots p_n, r < \infty$ and $\mathcal{Z}$ be a solid Banach space. Assume that there exists a sequence $\{A_k\}_{k=1}^\infty \subseteq \mathcal{A}$ with $\mu(A_k) > 0$ for all k such that one of the followings holds:*

(i) $\sum_{i=1}^n \frac{1}{p_i} > \frac{1}{r}$ *and* $\left(\|\chi_{A_k}\|_{\mathcal{Z}} + \left\|\chi_{A_k} f\right\|_{\mathcal{Z}}\right) \to 0$, *for all* $f \in \mathcal{Z}$,

(ii) $\sum_{i=1}^n \frac{1}{p_i} < \frac{1}{r}$ *and* $\left(\frac{1}{\|\chi_{A_k}\|_{\mathcal{Z}}} + \left\|\chi_{A_k} f\right\|_{\mathcal{Z}}\right) \to 0$, *for all* $f \in \mathcal{Z}$.

Then,

$$E := \{(f_1,\dots,f_n) \in \mathcal{Z}^{p_1}\times\dots\times\mathcal{Z}^{p_n} : f_1\dots f_n \in \mathcal{Z}^r\}$$

is σ-$\frac{2}{n+2}$-lower porous in $\mathcal{Z}^{p_1}\times\dots\times\mathcal{Z}^{p_n}$.

Remark 2.4. Recall that a Banach function space $(X, \|\cdot\|_X)$ has the absolute continuous norm whenever $\lim_{\mu(A)\to 0}\|f\chi_A\|_X = 0$ for all $f \in X$. So if we assume that $X_1,\dots,X_n$ has absolute continuous norm, then conditions (i) and (ii) in Theorem 2.2 can be expressed as follows:

(i) if $\frac{1}{p_1}+\dots+\frac{1}{p_n} > \frac{1}{r}$, then $\lim_{k\to\infty}(\|\chi_{A_k}\|_{\mathcal{Z}} + \mu(A_k)) = 0$,

(ii) if $\frac{1}{p_1} + \ldots + \frac{1}{p_n} < \frac{1}{r}$, then $\lim_{k\to\infty}\left(\frac{1}{\|\chi_{A_k}\|_Z} + \mu(A_k)\right) = 0$.

Let us recall some notation and terminology concerning Orlicz spaces. A function $\Phi : \mathbb{R} \to [0, \infty)$ is called a *Young function* if Φ is convex, even and

$$\Phi(0) = \lim_{x\to 0} \Phi(x), \qquad \lim_{x\to\infty} \Phi(x) = \infty.$$

We write $\Phi \in \Delta_2$ whenever there are $M > 0$ and $x_0 \geq 0$ such that for any $x \geq x_0$,

$$\Phi(2x) \leq M\,\Phi(x).$$

Let Φ be a Young function. For each $f \in X^0(\Omega)$ we define

$$\rho_\Phi(f) := \int_\Omega \Phi(|f(x)|)d\mu(x).$$

The Orlicz space $L^\Phi(\Omega)$ is defined by

$$L^\Phi(\Omega) := \left\{f \in X^0(\Omega) : \rho_\Phi(af) < \infty, \text{ for some } a > 0\right\}.$$

Then $L^\Phi(\Omega)$ is a Banach space under the norm

$$N_\Phi(f) = \inf\{k > 0 : \rho_\Phi(f/k) \leq 1\},$$

where $f \in L^\Phi(\Omega)$. By [5, Corollary 7, pqge 78] if Φ is a coninuous Young function with $\Phi(x) = 0$ if and only if $x = 0$, then for each $A \in \mathcal{A}$ with $\mu(A) < \infty$ we have

$$N_\Phi(\chi_A) = \left[\Phi^{-1}\left(\frac{1}{\mu(A)}\right)\right]^{-1}. \tag{4}$$

Note that classical Lebesgue spaces $L^p(\Omega), 1 \leq p \leq \infty$, are elementary examples of Orlicz spaces $L^p(\Omega)$ with $\Phi = |\cdot|^p$.

It was proved in [5] that the Orlicz space L^Φ has absolute continuous norm whenever $\Phi \in \Delta_2$ and $\Phi(x) = 0$ if and only if $x = 0$. Note that if Φ is a Young function and $p > 1$, then $(L^\Phi)^p = L^{\Phi_p}$, where $\Phi_p(x) := \Phi(x^p)$. Then, thanks to the relation (4) we have the following fact.

Theorem 2.5. *Let $1 \leq p_1, \ldots p_n, r < \infty$, and $\Phi_1, \ldots, \Phi_n, \Phi$ be Young functions such that for each $i \in \{1, 2, \ldots, n\}$, $\Phi_i \in \Delta_2$ and $\Phi_i(x) > 0$ for any $x > 0$. Assume that there is $\{A_k\}_{k=1}^\infty \subseteq \mathcal{A}$ with $0 < \mu(A_k) < \infty$ for all k such that (i) or (ii) holds:*

(i) *if $\frac{1}{p_1} + \ldots + \frac{1}{p_n} > \frac{1}{r}$, then $\lim_{k\to\infty} \mu(A_k) = 0$,*

(ii) *if $\frac{1}{p_1} + \ldots + \frac{1}{p_n} < \frac{1}{r}$, then $\lim_{k\to\infty} \mu(A_k) = \infty$.*

Then,

$$E := \{(f_1, \ldots, f_n) \in L^{\Phi_{p_1}} \times \ldots \times L^{\Phi_{p_n}} : f_1 \ldots f_n \in L^{\Phi_r}\}$$

is σ-$\frac{2}{n+2}$-lower porous in $L^{p_1} \times \ldots \times L^{p_n}$.

Thanks to Remark 2.4, we conclude the following two Corollaries from Theorem 2.3.

Corollary 2.6. *Let X be a solid Banach function space such that $X \subseteq L^1(\Omega)$ and $\inf\{\|\chi_A\|_X : A \in \mathcal{A}, \mu(A) > 0\} = 0$. Then, for every $1 \leq p_1, \ldots p_n, r < \infty$ with $\frac{1}{p_1} + \ldots + \frac{1}{p_n} > \frac{1}{r}$, we have*

$$E := \{(f_1, \ldots, f_n) \in X^{p_1} \times \ldots \times X^{p_n} : f_1 \ldots f_n \in X^r\}$$

is σ-$\frac{2}{n+2}$-lower porous in $X^{p_1} \times \ldots \times X^{p_n}$.

Corollary 2.7. *Let X be a solid Banach function space such that $L^1(\Omega) \subseteq X$ and $\sup\{\|\chi_A\|_X : A \in \mathcal{A}\} = \infty$. Then, for every $1 \le p_1, \ldots p_n, r < \infty$ with $\frac{1}{p_1} + \ldots + \frac{1}{p_n} < \frac{1}{r}$, we have*

$$E := \{(f_1, \ldots, f_n) \in X^{p_1} \times \ldots \times X^{p_n} : f_1 \ldots f_n \in X^r\}$$

is σ-$\frac{2}{n+2}$-lower porous in $X^{p_1} \times \ldots \times X^{p_n}$.

Also, if we put $\mathcal{Z} = L^1(\Omega, w)$ in Theorem 2.3, where w is a weight on Ω, then we know that $L^1(\Omega, w)^{p_i} = L^{p_i}(\Omega, w^{\frac{1}{p_i}})$ for each $i = 1, \ldots, n$. So we have the following Corollary from Theorem 2.2 for weighted Lebesgue spaces.

Corollary 2.8. *Let $1 \le p_1, \ldots p_n, r < \infty$, w be a weight on Ω, and there exists a sequence $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$ such that $\lim_{k\to\infty} \mu(A_k) = 0$ and one of the following are hold:*

(i) *if $\frac{1}{p_1} + \ldots + \frac{1}{p_n} > \frac{1}{r}$, then $\lim_{k\to\infty} \int_{A_k} w d\mu = 0$,*

(ii) *if $\frac{1}{p_1} + \ldots + \frac{1}{p_n} < \frac{1}{r}$, then $\lim_{k\to\infty} \frac{1}{\int_{A_k} w d\mu} = 0$.*

Then,

$$E := \{(f_1, \ldots, f_n) \in L^{p_1}(\Omega, w^{\frac{1}{p_1}}) \times \ldots \times L^{p_n}(\Omega, w^{\frac{1}{p_n}}) : f_1 \ldots f_n \in L^r(\Omega, w^{\frac{1}{r}})\}$$

is σ-$\frac{2}{n+2}$-lower porous in $L^{p_1}(\Omega, w^{\frac{1}{p_1}}) \times \ldots \times L^{p_n}(\Omega, w^{\frac{1}{p_n}})$.

Also, setting $p_1 = \ldots = p_n = 1$ we have the following Corollary.

Corollary 2.9. *Assume that Φ is a Δ_2-regular Young function and there is $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$ with $\lim_{k\to\infty} \mu(A_k) = 0$ such that (i) or (ii) holds:*

(i) *if $nr > 1$, then $\lim_{k\to\infty} \|w\chi_{A_k}\|_\Phi = 0$,*

(ii) *if $nr < 1$, then $\lim_{k\to\infty} \frac{1}{\|w\chi_{A_k}\|_\Phi} = 0$.*

Then,

$$E := \{(f_1, \ldots, f_n) \in L^\Phi(\Omega) \times \ldots \times L^\Phi(\Omega) : (f_1 \ldots f_n)^r \in L^\Phi(\Omega)\}$$

is σ-$\frac{2}{n+2}$-lower porous in $L^\Phi(\Omega) \times \ldots \times L^\Phi(\Omega)$.

3. On spaceability

In [1, Theorem 5] for each solid Banach function space and $1 \le p, q < r$ it is proved that the set $\{(f, g) \in X^p \times X^q : fg \notin X^r\}$ is spaceable. In this section, a generalization of this result is stated.

Recall that a subset S of a Banach space $\mathcal{E}$ is called *spaceable* in $\mathcal{E}$ if $S \cup \{0\}$ contains a closed infinite-dimensional subspace of $\mathcal{E}$. For each function f in $X^0(\Omega)$ we denote $E_f := \{x \in \Omega : f(x) \neq 0\}$, the set-theoretical support of f.

Definition 3.1. Let $\mathcal{E}$ be a topological vector space. We say that a relation $\sim$ on $\mathcal{E}$ has property (D) if the following conditions hold:

1. If (x_n) is a sequence in $\mathcal{E}$ such that $x_n \sim x_m$ for all distinct index m, n, then for each disjoint finite subsets A, B of $\mathbb{N}$ we have
$$\sum_{n\in A} \alpha_n x_n \sim \sum_{m\in B} \beta_m x_m,$$
where α_n and β_m's are arbitrary scalars.

2. If a sequence (x_n) converges to x in $\mathcal{E}$ and for some $y \in \mathcal{E}$, $x_n \sim y$ for all $n \in \mathbb{N}$, then $x \sim y$.

We use the following theorem applicable to prove spaceability.

Theorem 3.2. ([1, Theorem 4]) *Let $(\mathcal{E}, \|\cdot\|)$ be a Banach space, $\sim$ be a relation on $\mathcal{E}$ with property (D), and E be a nonempty subset of $\mathcal{E}$. Assume that:*

1. *there is a constant $k > 0$ such that $\|x + y\| \geq k\|x\|$ for all $x, y \in \mathcal{E}$ with $x \sim y$;*
2. *E is a cone;*
3. *if $x, y \in \mathcal{E}$ such that $x + y \in E$ and $x \sim y$ then $x, y \in K$;*
4. *there is an infinite sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{E} \setminus E$ such that for each distinct $m, n \in \mathbb{N}$, $x_m \sim x_n$.*

Then, $\mathcal{E} \setminus E$ is spaceable in $\mathcal{E}$.

Now we are ready to state the main results of this section.

Theorem 3.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, $1 \leq p_1, \ldots p_n, r < \infty$ and $X_1, \ldots, X_n, \mathcal{Z}$ be solid function spaces. Assume that $\sum_{i=1}^{n} \frac{1}{p_i} > \frac{1}{r}$ and there exists $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$ with $\mu(A_k) > 0$ for all k, such that $\lim_{n\to\infty} \|\chi_{A_n}\|_{\mathcal{Z}} = 0$. Assume that there are $b_i > 0$ $(i = 1, \ldots, n)$ such that for any $B_k \subseteq A_k$ $(k \in \mathbb{N})$ we have*

$$\|\chi_{B_k}\|_{X_i} \leq b_i \|\chi_{B_k}\|_{\mathcal{Z}}. \tag{5}$$

Then, $\{(f_1, \ldots, f_n) \in X_1^{p_1} \times \ldots \times X_n^{p_n} : f_1 \ldots f_n \notin \mathcal{Z}^r\}$ is spaceable in $X_1^{p_1} \times \ldots \times X_n^{p_n}$.

Proof. By [1], we can assume that if $i \neq j$ then $A_i \cap A_j = \emptyset$, and $0 < \|\chi_{A_i}\|_{\mathcal{Z}} < \frac{1}{2^i}$ for all $i \in \mathbb{N}$. Since $\epsilon_0 = \frac{1}{r} - \frac{1}{p_1} - \ldots - \frac{1}{p_n} < 0$, for each $i = 1, \ldots, n$ we can choose $\alpha_i > \frac{-1}{p_i}$ such that $\frac{1}{r} + \alpha_1 + \ldots + \alpha_n < 0$ (for example put $\alpha_i = \frac{-1}{p_i} + \frac{-\epsilon_0}{2n}$). By (5), for each $i = 1, \ldots, n$,

$$\begin{aligned}\sum_{k=1}^{\infty} \|\chi_{A_k}\|_{\mathcal{Z}}^{\alpha_i p_i} \|\chi_{A_k}\|_{X_i} &\leq b_i \sum_{k=1}^{\infty} \|\chi_{A_k}\|_{\mathcal{Z}}^{\alpha_i p_i + 1} \\ &\leq b_i \sum_{k=1}^{\infty} \left(\frac{1}{2^{\alpha_i p_i + 1}}\right)^n < \infty.\end{aligned}$$

So, the series $\sum_{k=1}^{\infty} \|\chi_{A_k}\|_{\mathcal{Z}}^{\alpha_i p_i} \chi_{A_k}$ is absolutely convergent in X_i. Since X_i is complete, this series is convergent in X_i. By the identity

$$\left|\sum_{k=1}^{\infty} \|\chi_{A_k}\|_{\mathcal{Z}}^{\alpha_i} \chi_{A_k}\right|^{p_i} = \sum_{k=1}^{\infty} \|\chi_{A_k}\|_{\mathcal{Z}}^{\alpha_i p_i} \chi_{A_k}$$

we conclude that

$$f_i := \sum_{k=1}^{\infty} \|\chi_{A_k}\|_{\mathcal{Z}}^{\alpha_i} \chi_{A_k} \in X_i^{p_i}.$$

for all $i = 1, \ldots, n$. On the other hand, in contrast, assume that $f_1 \ldots f_n \in \mathcal{Z}^r$. Then,

$$\begin{aligned}\|f_1 \ldots f_n\|_{\mathcal{Z}^r} &= \left\|\sum_{i=1}^{\infty} \|\chi_{A_i}\|_{\mathcal{Z}}^{\alpha_1 + \ldots + \alpha_n} \chi_{A_i}\right\|_{\mathcal{Z}^r} \\ &\geq \left\|\|\chi_{A_m}\|_{\mathcal{Z}}^{\alpha_1 + \ldots + \alpha_n} \chi_{A_m}\right\|_{\mathcal{Z}^r} \\ &= \left(\|\chi_{A_m}\|_{\mathcal{Z}}^{r(\alpha_1 + \ldots + \alpha_n) + 1}\right)^{\frac{1}{r}} \\ &= \|\chi_{A_m}\|_{\mathcal{Z}}^{\frac{1}{r} + \alpha_1 + \ldots + \alpha_n} \\ &\geq 2^{-m(\frac{1}{r} + \alpha_1 + \ldots + \alpha_n)},\end{aligned}$$

for every $m \in \mathbb{N}$. But $-(\frac{1}{r} + \alpha_1 + \ldots + \alpha_n) > 0$, so $\lim_{m\to\infty} 2^{m(-\frac{1}{r}-\alpha_1-\ldots-\alpha_n)} = \infty$, a contradiction. Therefore, $f_1 \ldots f_n \notin \mathcal{Z}^r$. Now, we define the relation $\sim$ on

$$X_1^{p_1} \times \ldots \times X_n^{p_n}$$

by

$$(f_1, \ldots, f_n) \sim (g_1, \ldots, g_n) \text{ if and only if } E_{f_i} \cap E_{g_i} = \emptyset, \text{ for all } i = 1, \ldots, n. \tag{6}$$

So, the relation $\sim$ satisfies the condition (D), and if we put

$$E := \{(f_1, \ldots, f_n) \in X_1^{p_1} \times \ldots \times X_n^{p_n} : f_1 \ldots f_n \in \mathcal{Z}^r\},$$

then the conditions $1-3$ in Theorem 3.2 are valid. Assume that $\{N_1, N_2, \ldots\}$ is a partition of $\mathbb{N}$ such that N_i is infinite for all i. Now, setting

$$K_j := \bigcup_{i \in N_j} A_i, \qquad (j = 1, 2, \ldots),$$

$\{K_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ is a sequence with pairwise disjoint terms and $f_1 \ldots f_n \chi_{K_i} \notin \mathcal{Z}^r$ for each $i \in \mathbb{N}$. Then,

$$\{(f_1\chi_{K_i}, \ldots, f_n\chi_{K_i})\}_{i=1}^{\infty} \subseteq X_1^{p_1} \times \ldots \times X_n^{p_n} \setminus E,$$

and for each distinct $i, j \in \mathbb{N}$ we have $(f_1\chi_{K_i}, \ldots, f_n\chi_{K_i}) \sim (f_1\chi_{K_j}, \ldots, f_n\chi_{K_j})$. So by Theorem 3.2 the proof is complete. □

By Theorems 2.2 and 3.3 we have the following result.

Theorem 3.4. *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, $1 \le p_1, \ldots p_n, r < \infty$ and $X_1, \ldots, X_n$ and X be solid Banach function spaces. Assume that there exists a sequence $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$ such that for some $a_i, b_i > 0$, $(i = 1, \ldots, n)$, and for every $B_k \subseteq A_k$, $(k \in \mathbb{N})$ we have*

$$a_i \|\chi_{B_k}\|_X \le \|\chi_{B_k}\|_{X_i} \le b_i \|\chi_{B_k}\|_X.$$

If

$$\sum_{i=1}^{n} \frac{1}{p_i} > \frac{1}{r} \text{ and } \left(\|\chi_{A_k}\|_X + \sum_{i=1}^{n} \left\|\chi_{A_k} f\right\|_{X_i}\right) \to 0, \text{ for all } f \in X,$$

then for the set

$$E := \{(f_1, \ldots, f_n) \in X_1^{p_1} \times \ldots \times X_n^{p_n} : f_1 \ldots f_n \in X^r\}$$

we have

1. *E is σ-c-lower porous in $X_1^{p_1} \times \ldots \times X_n^{p_n}$ for some $c > 0$;*
2. *$X_1^{p_1} \times \ldots \times X_n^{p_n} \setminus E$ is spaceable in $X_1^{p_1} \times \ldots \times X_n^{p_n}$.*

Simillar to the proof of Theorem 3.3 we can state the following Theorem.

Theorem 3.5. *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, $1 \le p_1, \ldots p_n, r < \infty$ and $X_1, \ldots, X_n$ and X be solid Banach function spaces. Assume that $\sum_{i=1}^{n} \frac{1}{p_i} < \frac{1}{r}$ and there exists a sequence $\{B_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$ such that $\|\chi_{B_k}\|_X \ge 2^k$, and for some $b_i > 0$, $(i = 1, \ldots, n)$, we have*

$$\|\chi_{B_k}\|_{X_i} \le b_i \|\chi_{B_k}\|_X, \tag{7}$$

for all $k \in \mathbb{N}$. Then $\{(f_1, \ldots, f_n) \in X_1^{p_1}, \ldots, X_n^{p_n} : f_1 \ldots f_n) \notin X^r\}$ is spaceable in $X_1^{p_1}, \ldots, X_n^{p_n}$.

Proof. Let $\alpha_i < \frac{-1}{p_i}$ such that $\frac{1}{r} > -(\alpha_1 + \ldots + \alpha_n)$. (for example we can put $\alpha_i = \frac{-1}{p_i} + \sum_{i=1}^{n} \frac{1}{2np_i} - \frac{1}{2nr}$). Now let $f_i = \sum_{k=1}^{\infty} \|\chi_{B_k}\|_X^{\alpha_i} \chi_{B_k}$, $(i = 1, \ldots, n)$. Then, since $\alpha_i p_i + 1 < 0$ by (7) we have $f_i \in X_i^{p_i}$ for all $i = 1, \ldots, n$, and $f_1 \ldots f_n \notin X^r$. Now we define $\sim$ on $X_1^{p_1} \times \ldots \times X_n^{p_n}$ as (6) then the relation $\sim$ satisfies the condition (D), and we can complete the proof similar to the proof of Theorem 3.3. □

Also we can state the following result from Theorems 3.3 and 3.5 in the case $X_1 = \ldots = X_n = X$.

Theorem 3.6. *Let X be a solid Banach function space with an absolute continuous norm and $1 \le p_1, \ldots p_n, r < \infty$. Assume that one of the followings holds:*

(i) $\frac{1}{p_1} + \ldots + \frac{1}{p_n} > \frac{1}{r}$ *and there is a mutually disjoint sequence $\{A_i\}_{i=1}^{\infty}$ in $\mathcal{A}$ such that $0 < \|\chi_{A_i}\|_X \le \frac{1}{2^i}$ for all $i \in \mathbb{N}$,*

(ii) $\frac{1}{p_1} + \ldots + \frac{1}{p_n} < \frac{1}{r}$ *and there is a mutually disjoint sequence $\{B_i\}_{i=1}^{\infty}$ in $\mathcal{A}$ such that $2^i < \|\chi_{B_i}\|_X$ for all $i \in \mathbb{N}$.*

Then

1. $\{(f_1, \ldots, f_n) \in X^{p_1} \times \ldots \times X^{p_n} : f_1 \ldots f_n \in X^r\}$ *is σ-c-lower porous in $X^{p_1} \times \ldots \times X^{p_n}$ for some $c \le \frac{2}{n+2}$;*
2. $\{(f_1, \ldots, f_n) \in X^{p_1} \times \ldots \times X^{p_n} : f_1 \ldots f_n \notin X^r\}$ *is spaceable in $X^{p_1} \times \ldots \times X^{p_n}$.*

Remark 3.7. (i) Note that, by the Hölder inequality [4, Theorem 2.1], in the case $\frac{1}{p_1} + \ldots + \frac{1}{p_n} = \frac{1}{r}$, we have

$$\{(f_1, \ldots, f_n) \in X^{p_1} \times \ldots \times X^{p_n} : f_1 \ldots f_n \in X^r\} = X^{p_1} \times \ldots \times X^{p_n}.$$

(ii) If $\sup\{\|\chi_A\|_X : A \in \mathcal{A} \text{ and } \chi_A \in X\} < \infty$ and $\frac{1}{p_1} + \ldots + \frac{1}{p_n} < \frac{1}{r}$, then

$$\{(f_1, \ldots, f_n) \in X^{p_1} \times \ldots \times X^{p_n} : f_1 \ldots f_n \in X^r\} = X^{p_1} \times \ldots \times X^{p_n}.$$

Indeed, for each $(f_1, \ldots, f_n) \in X^{p_1} \times \ldots \times X^{p_n}$ setting

$$h := \max\{|f_1|^{p_1}, \ldots, |f_n|^{p_n}\},$$

we have $h \in X$. By the assumption and thanks to [6, Theorem 2.4], $\rho := r(\frac{1}{p_1} + \ldots + \frac{1}{p_n}) < 1$ implies that $X \subseteq X^{\rho}$. So, $|h|^{\rho} \in X$. On the other hand,

$$|f_1, \ldots, f_n|^r \le |h|^{r(\frac{1}{p_1} + \ldots + \frac{1}{p_n})} = |h|^{\rho}.$$

So, $f_1, \ldots, f_n \in X^r$ by solidity of X.

(iii) Similar to the previous item and applying [6, Theorem 2.1], if

$$\inf\{\|\chi_A\|_X : A \in \mathcal{A}, \mu(A) > 0 \text{ and } \chi_A \in X\} > 0$$

and $\frac{1}{p_1} + \ldots + \frac{1}{p_n} > \frac{1}{r}$, then

$$\{(f_1, \ldots, f_n) \in X^{p_1} \times \ldots \times X^{p_n} : f_1 \ldots f_n \in X^r\} = X^{p_1} \times \ldots \times X^{p_n}.$$

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