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Characterizations of M-fuzzifying topological operators and its applications

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Abstract. This paper mainly discuss eight kinds of equivalent characterizations of an M-fuzzifying topological closure operator $cl : 2^X \to M^X$, an M-fuzzifying topological interior operator $int : 2^X \to M^X$, and an M-fuzzifying topological derived operator $der : 2^X \to M^X$, in which M is a completely distributive De Morgan algebra. As its applications, we give the fuzzy constructions of M-fuzzifying topological operators cl^d, int^d, der^d induced by an M-fuzzifying pseudo metric d in the sense of Morsi's fuzzy metric and prove that the continuity between M-fuzzifying topological operator spaces and M-fuzzifying pseudo metric spaces is maintained. Finally, we show that the M-fuzzifying topological $\mathcal{T}$ induced by d is exactly the M-fuzzifying topological induced by cl^d, int^d and der^d. Namely, $\mathcal{T}^d = \mathcal{T}^{cl^d} = \mathcal{T}^{int^d} = \mathcal{T}^{der^d}$.

1. Introduction

In 1968, C.L. Chang [2] innovate fuzzy theory into topology. The open sets are fuzzy in a Chang's topology, however the topology composed by those open sets is a crisp subset of the I-powerset I^X, where I is the unit interval $[0, 1]$. Later, J. A. Goguen [7] replaced I with an arbitrary complete infinitely distributive lattice L (now called a quantale with unit) in 1973. Then he obtained the concept of L-fuzzy topology. However, in a totally different research direction, Höhle [10] introduced the concept of a fuzzy topology which is treated as an L-subset of a powerset 2^X in 1982. In 1991, Ying [34, 35] studied Höhle's topology in a logical sense, while naming it a fuzzifying topology.

With the great development of fuzzy sets[36] in recent years, various kinds of fuzzy structures have been researched, such as fuzzy convergence structures [18, 19], fuzzy topological structures [11, 17, 20–22, 39, 40, 43, 45], fuzzy convex structures [13, 28, 31, 33, 37, 42, 44], fuzzy matroid [12, 29, 41] and so on. Closure operators and interior operators play a crucial part in many mathematical branches, involving the realm of algebra [1], topology [3], lattices and order [5], matroid theory [16], convex structure [26], etc..

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Fuzzy closure operators and fuzzy interior operators are widely studied in many fuzzy structures from different angles. In 2001, Xu [32] discussed eight kinds of characterizations of a fuzzifying topological interior which was regarded as a set $P = \{p_x \mid x \in X\}$ of maps $p_x : 2^X \longrightarrow [0, 1]$. In 2009, Shi [21] proposed the notions of L-fuzzy closures and L-fuzzy interiors, and studied the relations among them and an L-fuzzy topology. In 2010, Fang and Yue [6] introduced the notions of a strong L-fuzzy closure operator and a strong L-fuzzy closure system, and show that a strong L-fuzzy closure system is precisely the fuzzy system in opposition to the crisp system. In 2013, Shi and Pang [23] investigated the isomorphic categories of the category of L-fuzzy closure system spaces. In 2017, Guo et al. [8] investigated algebraic fuzzy closure operators and algebraic fuzzy closure L-systems on a fuzzy complete lattice. And they have shown that there is a categorical isomorphism between algebraic fuzzy closure operators and fuzzy convex structures. In 2021, using fuzzy closure operators as tools, Han and Wang [9] demonstrated that the category CFPos of complete fuzzy posets and their fuzzy-join preserving maps is a reflective full subcategory of FPosu which denotes the category of fuzzy posets and their fuzzy-existing-join preserving maps. In 2022, Ojeda-Hernández et al. [15] discussed different alternatives to define the desired fuzzy closure systems and proved that it exist a one-to-one relation between fuzzy closure operators and fuzzy closure systems. Afterwards, Zhao and Pang explored the relationships among L-closure (interior) operators, L-closure (interior) systems and L-enclosed (internal) relations in [38].

Now, we consider whether we can characterize fuzzy topological closure operators, fuzzy topological interior operators and fuzzy topological derived operators in the M-fuzzifying case of mappings $cl/int/der : 2^X \to M^X$, and whether we can give the M-fuzzifying constructions of cl^d, int^d, der^d from the perspective of an M-fuzzifying peseudo metric d in the sense of Morsi's fuzzy metric, where M is a completely distributive De Morgan algebra These are the problems that need to be solved in the field of fuzzy topological structures, which are the main research purposes of this paper.

This paper consists of the following sections. In Section 2, M-fuzzy non-negative real number, M-fuzzifying pseudo metrics and M-fuzzifying topological operators are reviewed. In Section 3, we discuss eight kinds of equivalent characterizations of an M-fuzzifying topological closure operator $cl : 2^X \to M^X$, an M-fuzzifying topological interior operator $int : 2^X \to M^X$, and an M-fuzzifying topological derived operator $der : 2^X \to M^X$. In Section 4, we construct M-fuzzifying topological operators cl^d, int^d, der^d induced by an M-fuzzifying pseudo metric d and study its related properties. In Section 5, we show $\mathcal{T}^d = \mathcal{T}^{cl^d} = \mathcal{T}^{int^d} = \mathcal{T}^{der^d}$.

2. Preliminaries

Throughout this paper, M denote a completely distributive De Morgan algebra, i.e., a completely distributive lattice with an sequential involution $'$. Use $\perp_M$ and $\top_M$ to represent the smallest as well as the largest element in M, respectively.

For $a, b \in M$, we use $a \prec b$ to indicate that a way-below b, that is, if for each subset $D \subseteq M$, $b \leq \bigvee D$ suggest $d \geq a$ when $d \in D$. The set $\{a \in M : a \prec b\}$ is a greatest minimal family of b in [27], denoted by $\beta(b)$. In a completely distributive lattice, $b = \bigvee \beta(b) = \bigvee\{a \in M : a \prec b\}$ for each $b \in M$. Also, the wedge below relation has the interpolation property, that is, $a \prec b$ implies $a \prec c \prec b$ when exists $c \in M$. In addition, we know $a \prec \bigvee_{j \in J} b_j$ implies there exists some b_j such that $a \prec b_j$. [25].

An element a is called co-prime, if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The symbol $J(M)$ is used to represent the set of non-zero co-prime elements. An element a is called prime, if $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$. The symbol $P(M)$ is used to represent the set of non-unit prime elements.

For $A \in M^X$ and $\alpha \in M$, the following notations are referenced:

$$A_{[\alpha]} = \{x \in X \mid A(x) \geq \alpha\}, \quad A^{(\alpha)} = \{x \in X \mid A(x) \nleq \alpha\}.$$

As we know, there exists a bijection between non-negative real number $\bar{a}$ and the interval $[0, a]$. The non-negative real number $\bar{a}$ can be viewed as the mapping $\bar{a} : [0, +\infty) \to [0, 1]$, $r \mapsto \bar{a}(r)$ defined by $\bar{a}(r) = \begin{cases} 1, & 0 \leq r \leq a\,; \\ 0, & r > a\,. \end{cases}$. It is easily to get the mapping $\bar{a}$ is decreasing and $\bigvee\limits_{r \in [0,+\infty)} \bar{a}(r) = \top_M$, $\bigwedge\limits_{r \in [0,+\infty)} \lambda(r) = \perp_M$.

Based on these, we extend it to M-fuzzy cases in the following definition.

Definition 2.1. ([22]) An equivalency class $[\lambda]$ of reverse-order maps $\lambda : [0,+\infty) \to M$ is called an *M-fuzzy non-negative real number* if it satisfies

$$\bigvee_{r\in[0,+\infty)} \lambda(r) = \top_M, \qquad \bigwedge_{r\in[0,+\infty)} \lambda(r) = \bot_M,$$

.

We denote $[0,\infty)(M)$ as the set of all M-fuzzy real numbers.

In 1988, Morsi gave the definition of a fuzzy pseudo metric $d : X\times X \to [0,\infty)([0,1])$ by using non-negative $[0,1]$-fuzzy real number [14]. If Morsi's fuzzy pseudo metrics degenerates to the classical case, then a mapping $\chi_d : X\times X \to [0,\infty)\,([0,1])$ is defined by

$$\chi_d(x,y)(r) = \begin{cases} 1, & d(x,y) \geq r\,; \\ 0, & d(x,y) < r\,. \end{cases}.$$

In 2018, Wang and Shi proposed the concept of an M-fuzzifying pseudo metric [24], in which they generalize the value $[0,1]$ of Morsi's fuzzy pseudo metric to a completely distributive lattice M as follows.

Definition 2.2. ([22, 24]) A mapping $d : X\times X \longrightarrow [0,\infty)(M)$ is called *an M-fuzzifying pseudo metric* if it satisfies: $\forall x,y,z\in X$ and $\forall r,s>0$,
(MF1) $d(x,x)(0+) = \bigvee_{r>0} d(x,x)(r) = \bot_M$, *i.e.*, $\forall r>0, d(x,x)(r) = \bot_M$;
(MF2) $d(x,y)(r) = d(y,x)(r)$;
(MF3) $d(x,y)(r+s) \leq d(x,z)(r) \vee d(z,y)(s)$.
Then (X,d) is an M-fuzzifying pseudo metric space, and $d(x,y)(r)$ represents the degree to which the distance between x and y is greater than or equal to r.

Example 2.3. Let $M = \{\top_M, a, \bot_M\}$. Define d: $X\times X \longrightarrow [0,\infty)(M)$ by

$$d(x,y)(t) = \begin{cases} \top_M, & t = 0; \\ a, & 0 < t \leq \mid x-y \mid; \\ \bot_M, & t > \mid x-y \mid. \end{cases}$$

Then d is an M-fuzzifying pseudo metric.

Proof. Firstly, we need to check that $d(x,y)(-) : [0,+\infty) \to M$ is well defined, which means $d(x,y)(-)$ is an M-fuzzy non-negative real number. From its construction, it is obvious that $d(x,y)(-)$ satisfies the conditions in Definition 2.1. Secondly, we need to prove that the mapping $d : X\times X \longrightarrow [0,\infty)(M)$ is an M-fuzzifying pseudo metric, which means d should satisfy the conditions (MF1)-(MF3) in Definition 2.2. (MF1) and (MF2) are easily to be verified. We only need to check (MF3). If neither of $d(x,z)(r)$ and $d(z,y)(s)$ equal $\bot_M$, then $d(x,y)(r+s) \leq d(x,z)(r) \vee d(z,y)(s)$ holds. If $d(x,z)(r) = \bot_M$ and $d(z,y)(s) = \bot_M$, which means $r > \mid x-z\mid$ and $s > \mid z-y\mid$, then $r+s > \mid x-z\mid + \mid z-y\mid \geq \mid x-y\mid$. This implies $d(x,y)(r+s) = \bot_M$. Hence $d(x,y)(r+s) \leq d(x,z)(r) \vee d(z,y)(s)$. □

Remark 2.4. Actually, (MF3) is a generalization of the classical triangle inequality: $d(x,y) \leq d(x,z) + d(z,y)$. In fact, by Theorem 1.2 in [14], we know

$$(d(x,z)\oplus d(z,y))(t) = \bigwedge_{r+s=t} (d(x,z)(r) \vee d(z,y)(s)) = \bigvee_{r+s=t} (d(x,z)(r) \wedge d(z,y)(s)).$$

Based on that, we get $\forall t>0,\ d(x,y)(t) \leq (d(x,z)\oplus d(z,y))(t) \Leftrightarrow \forall r,s>0,\ d(x,y)(r+s) \leq d(x,z)(r) \vee d(z,y)(s)$.

Let (X, d_X) and (X, d_Y) be two M-fuzzifying pseudo metric spaces. If a mapping $f : (X, d_X) \to (X, d_Y)$ satisfies the following formula: for any $r > 0$ and for all $x_1, x_2 \in X$, we have $d_X(x_1, x_2)(r) \geq d_Y(f(x_1), f(x_2))(r)$, then f is called *continuous*.

Next, we shall recall some concepts of M-fuzzifying topological spaces.

Definition 2.5. ([34]) A mapping $\mathcal{T} : 2^X \to M$ is called *an M-fuzzifying topology* if it satisfies:
(MFT1) $\mathcal{T}(\emptyset) = \mathcal{T}(X) = \top_M$;
(MFT2) $\forall A_1, A_2 \in 2^X, \mathcal{T}(A_1) \bigwedge \mathcal{T}(A_2) \leq \mathcal{T}(A_1 \bigcap A_2)$;
(MFT3) $\forall \{A_j : j \in J\} \subseteq 2^X$, $\bigwedge_{j\in J} \mathcal{T}(A_j) \leq \mathcal{T}(\bigcup_{j\in J} A_j)$.

In the theory of topology, we know that the topological closure and interior operator are defined from the perspective of sets. To be specific, A crisp topological closure operator is a mapping $cl : 2^X \to 2^X$ defined by $cl(A) = \bigcap\{B \mid cl(B) \subseteq B, A \subseteq B\}$. A crisp topological interior operator is a mapping $int : 2^X \to 2^X$ defined by $int(A) = \bigcup\{B \mid B \subseteq int(B), B \subseteq A\}$.

In 2001, Xu [32] gave the definition of a fuzzy topological closure and interior operator which were regarded as a set $P = \{p_x \mid x \in X\}$ of maps $p_x : 2^X \longrightarrow [0, 1]$. Compared with crisp topological closure and interior operators, Xu's fuzzy topological operators may not the most appropriate one.

In 2009, Shi [21] proposed the concepts of an M-fuzzifying closure operator $cl : 2^X \to M^X$ and an M-fuzzifying interior operator $int : 2^X \to M^X$ which are compatible with the crisp cases.

Definition 2.6. ([21]) A mapping $cl : 2^X \to M^X$ is called *an M-fuzzifying topological closure operator* if it satisfies:
(MFCL1) $cl(\emptyset)(x) = \bot_M$;
(MFCL2) $\forall x \in A, cl(A)(x) = \top_M$;
(MFCL3) $cl(B_1 \bigcup B_2) = cl(B_1) \bigvee cl(B_2)$;
(MFCL4) $cl(A)(x) = \bigwedge_{x\notin B\supseteq A} \bigvee_{y\notin B} cl(B)(y)$.

Let (X, cl_X) and (Y, cl_Y) be two M-fuzzifying topological closure spaces. If a map $f : X \to Y$ satisfies $cl_X(A)(x) \leq cl_Y(f^{\to}(A))(f(x))$ for any $x \in X$ and $A \in 2^X$, where $f^{\to}(A) = \{f(x) \mid x \in A\}$, then f is called *continuous*.

Similarly, we present the definition of an M-fuzzifying interior operator.

Definition 2.7. ([21]) A mapping $int : 2^X \to M^X$ is called *an M-fuzzifying topological interior operator* if it satisfies:
(MFIN1) $int(X)(x) = \top_M$;
(MFIN2) $\forall x \notin A, int(A)(x) = \bot_M$;
(MFIN3) $int(A_1 \bigcap A_2) = int(A_1) \bigwedge int(A_2)$;
(MFIN4) $int(A)(x) = \bigvee_{x\in B\subseteq A} \bigwedge_{y\in B} int(B)(y)$.

Let (X, int_X) and (Y, int_Y) be two M-fuzzifying topological interior spaces. If a map $f : X \to Y$ satisfies $int_Y(B)(f(x)) \leq int_X(f^{\leftarrow}(B))(x)$ for any $x \in X$ and $B \in 2^Y$, where $f^{\leftarrow}(B) = \{x \in X \mid f(x) \in B\}$, then f is called *continuous*.

In addition to closure operators and interior operators, derived operators also have a one-to-one correspondence with topologies. A topological derived operator is a mapping $der : 2^X \to 2^X$ satisfying the conditions: (DER1) $der(\emptyset) = \emptyset$; (DER2) $x \notin der(\{x\})$; (DER3) $\forall A_1, A_2 \subseteq X$, $der(A_1 \bigcup A_2) = der(A_1) \bigcup der(A_2)$; (DER4) $der(der(A)) \subseteq A \bigcup der(A)$.

In 2019, F.H. Chen, Y. Zhong and F.G. Shi [4] generalized the crisp topological derived operators to the M-fuzzifying case in the following.

Definition 2.8. ([4]) A mapping $der : 2^X \to M^X$ is called *an M-fuzzifying topological derived operator* if it satisfies:
(MFDER1) $\forall x \in X, der(\emptyset)(x) = \bot_M$;
(MFDER2) $\forall x \in X, der(\{x\})(x) = \bot_M$;
(MFDER3) $der(A_1 \bigcup A_2) = der(A_1) \bigvee der(A_2)$;
(MFDER4) $der(A)(x) = \bigwedge_{x\notin B\supseteq A-\{x\}} \bigvee_{y\notin B} der(B)(y)$.

Let (X, der_X) and (Y, der_Y) be two M-fuzzifying topological derived spaces. If a map $f : X \to Y$ satisfies $der_X(A)(x) \le f^{\to}(A)(f(x)) \vee der_Y(f^{\to}(A))(f(x))$, where

$$f^{\to}(A)(f(x)) \vee der_Y(f^{\to}(A))(f(x)) = \begin{cases} \top, & f(x) \in f^{\to}(A)\,; \\ der_Y(f^{\to}(A))(f(x)), & f(x) \notin f^{\to}(A)\,. \end{cases}$$

then f is called continuous.

Remark 2.9. ([4]) If a mapping $der : 2^X \to M^X$ satisfies (MFDER1)-(MFDER3), then $der(A)(x) = der(A-\{x\})(x)$ for all $x \in X$ and $A \in 2^X$.

3. Equivalent characterizations of M-fuzzifying topological operators

3.1. Equivalent characterizations of an M-fuzzifying topological closure operator

Before discussing equivalent characterizations of an M-fuzzifying topological closure operator, We need the following lemma which describes the equivalent characterizations of a crisp topological closure operator.

Lemma 3.1. *Let a mapping $cl : 2^X \to 2^X$ be a crisp topological closure operator defined by $cl(A) = \bigcap\{B \subseteq X \mid A \subseteq B, cl(B) \subseteq B\}$. Then*

$$x \in cl(A) \Leftrightarrow \forall B \subseteq X, A \subseteq B, cl(B) \subseteq B, s.t.\, x \in B.$$

$$\begin{aligned} i.e.,\ x \notin cl(A) \quad &\Leftrightarrow \quad (1)\ \exists B \subseteq X, A \subseteq B, cl(B) \subseteq B, such that.t.\, x \notin B \\ &\Leftrightarrow \quad (2)\ \exists B \subseteq X, A \subseteq B, cl(B) \subseteq B, s.t.\, x \notin B, x \notin cl(B) \\ &\Leftrightarrow \quad (3)\ \exists B \subseteq X, A \subseteq B, cl(A) \subseteq B, s.t.\, x \notin B, x \notin cl(B) \\ &\Leftrightarrow \quad (4)\ \exists B \subseteq X, A \subseteq B, cl(B) \subseteq B, s.t.\, x \notin B, x \notin cl(A) \end{aligned}$$

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) and (2) $\Rightarrow$ (3) are obvious. We only show (3) $\Rightarrow$ (2).

Since $x \notin cl(B)$, it follows from (3) that there exists some B_x with $A \subseteq B_x$, $cl(A) \subseteq B_x$ such that $x \notin B_x$ and $x \notin cl(B_x)$. Let

$$W = \bigcap\{B_\alpha \subseteq X \mid A \subseteq B_\alpha, cl(A) \subseteq B_\alpha\}.$$

Then $B_x \supseteq W \supseteq A$. Further $x \notin W \supseteq A$.

Next, we verify $cl(W) \subseteq W$. For any $y \notin W$, there exists some B_y with $A \subseteq B_y$ and $cl(A) \subseteq B_y$ such that $y \notin B_y$. So $y \notin cl(A)$. By (3), we know there exists $\widetilde{B}_y$ with $A \subseteq \widetilde{B}_y$ and $cl(A) \subseteq \widetilde{B}_y$ such that $y \notin \widetilde{B}_y$ and $y \notin cl(\widetilde{B}_y)$. Then $\widetilde{B}_y \supseteq W$, which implies $cl(\widetilde{B}_y) \supseteq cl(W)$. Hence $y \notin cl(W)$. By the arbitrariness of y, we get $cl(W) \subseteq W$. Therefore, there exists $W \subseteq X$ with $A \subseteq X$, $cl(W) \subseteq W$ such that $x \notin W$ and $x \notin cl(W)$, which means (2) holds. □

Next, we will generalize the conclusions of a topological closure operator in Lemma 3.1 to the M-fuzzifying cases.

Theorem 3.2. *Let a mapping $cl : 2^X \to M^X$ satisfying the conditions (MFYC1)-(MFYC3). Then the followings are equivalent.*

(MFYC4-1) $cl(A)(x) = \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} cl(B)(y)$;
(MFYC-2) $cl(A)(x) = \bigwedge_{x \notin B \supseteq A} \left(cl(B)(x) \vee \bigvee_{y \notin B} cl(B)(y)\right)$;
(MFYC4-3) $cl(A)(x) = \bigwedge_{x \notin B \supseteq A} \left(cl(B)(x) \vee \bigvee_{y \notin B} cl(A)(y)\right)$;
(MFYC4-4) $cl(A)(x) = \bigwedge_{x \notin B \supseteq A} \left(cl(A)(x) \vee \bigvee_{y \notin B} cl(B)(y)\right)$;
(MFYC4-5) $cl(A)(x) = \bigwedge_{B \supseteq A} \left(cl(B)(x) \vee \bigvee_{y \notin B} cl(B)(y)\right)$;
(MFYC4-6) $cl(A)(x) = \bigwedge_{B \supseteq A} \left(cl(B)(x) \vee \bigvee_{y \notin B} cl(A)(y)\right)$;
(MFYC4-7) $cl(A)(x) = \bigwedge_{B \supseteq A} \left(cl(A)(x) \vee \bigvee_{y \notin B} cl(B)(y)\right)$;
(MFYC4-8) $\forall \alpha \in J(M)$, $cl(cl(A)_{[\alpha]})_{[\alpha]} \subseteq cl(A)_{[\alpha]}$, *where* $cl(A)_{[\alpha]} = \{x \in A \mid cl(A)(x) \ge \alpha\}$.

Proof. (1) Firstly, we prove that (MFYC4-1) $\Leftrightarrow$ (MFYC4-3) $\Leftrightarrow$ (MFYC4-8). Since Shi and Pang have proved (MFYC4-1) $\Leftrightarrow$ (MFYC4-8) in [23], we need to show (MFYC4-1) $\Leftrightarrow$ (MFYC4-3).
i) By Definition 2.6 , we know the mapping cl is order-preserving. Then $\bigwedge_{x\notin B\supseteq A}\left(cl(B)(x)\vee\bigvee_{y\notin B}cl(A)(y)\right)\geq cl(A)(x)$. On the other hand, it follows from (MFYC4-1) that

$$cl(A)(x)=\bigwedge_{x\notin B\supseteq A}\bigvee_{y\notin B}cl(B)(y)\geq\bigwedge_{x\notin B\supseteq A}\left(cl(B)(x)\vee\bigvee_{y\notin B}cl(A)(y)\right).$$

Therefore $cl(A)(x)=\bigwedge_{x\notin B\supseteq A}\left(cl(B)(x)\vee\bigvee_{y\notin B}cl(A)(y)\right)$. This shows (MFYC4-1) $\Rightarrow$ (MFYC4-3).
ii) For any $x\notin B\supseteq A$, we know $\bigvee_{y\notin B}cl(B)(y)\geq cl(B)(x)\geq cl(A)(x)$. Then $cl(A)(x)\leq\bigwedge_{x\notin B\supseteq A}\bigvee_{y\notin B}cl(B)(y)$. In order to prove (MFYC4-1), it suffices to verify $cl(A)(x)\geq\bigwedge_{x\notin B\supseteq A}\bigvee_{y\notin B}cl(B)(y)$, i.e.,

$$cl(A)(x)'\leq\bigvee_{x\notin B\supseteq A}\bigwedge_{y\notin B}cl(B)(y)'.$$

Assume that $\alpha\in J(M)$ with $\alpha\prec cl(A)(x)'$. By (MFYC4-3), we get $cl(A)(x)'=\bigvee_{x\notin B\supseteq A}\left(cl(B)(x)'\wedge\bigwedge_{y\notin B}cl(A)(y)'\right)$. On the other hand, for any $y\notin B_x$, there exists some B_x with $x\notin B_x\supseteq A$ such that $cl(A)(y)'\succ\alpha$ and $cl(B_x)(x)'\succ\alpha$. Let

$$U=\bigcap\{B_u\subseteq X\mid u\notin B_u\supseteq A,\alpha\prec cl(B_u)(u)',\forall y\notin B_u,\alpha\prec cl(A)(y)'\}$$

Then $B_x\supseteq U$. Further $x\notin U\supseteq A$.

Next, we shall show $\alpha\leq\bigwedge_{z\notin U}(cl(U)(z))'$. For any $z\notin U$, there exists some B_u with $u\notin B_u\supseteq A$, $\alpha\prec cl(B_u)(u)'$, $\forall y\notin B_u,\alpha\prec cl(A)(y)'$ such that $z\notin B_u$. It follows that $\alpha\prec cl(A)(z)'$. By (MFYC4-3), there exists some B_z and $z\notin B_z\supseteq A$ such that $\alpha\prec cl(B_z)(z)'$ and $\alpha\prec cl(A)(w)'$ for any $w\notin B_z$. By the construction of U, we get $B_z\supseteq U$. Then $cl(B_z)(z)\geq cl(U)(z)$, i.e., $cl(U)(z)'\geq cl(B_z)(z)'\geq\alpha$. Further

$$\alpha\leq\bigwedge_{z\notin U}cl(U)(z)'\text{ and }\alpha\leq\bigvee_{x\notin B\supseteq A}\bigwedge_{y\notin B}cl(B)(y)'.$$

Due to the arbitrariness of α, we obtain

$$(cl(A)(x))'\leq\bigvee_{x\notin B\supseteq A}\bigwedge_{y\notin B}cl(B)(y)'.$$

Hence

$$(cl(A)(x))'=\bigvee_{x\notin B\supseteq A}\bigwedge_{y\notin B}cl(B)(y)'.$$

Therefore $cl(A)(x)=\bigwedge_{x\notin B\supseteq A}\bigvee_{y\notin B}cl(B)(y)$. This shows (MFYC4-3) $\Rightarrow$ (MFYC4-1).

(2) Since $cl(B)(x)=\top_M$ for all $x\in B$, we get

$$\begin{aligned}cl(A)(x)=&\bigwedge_{B\supseteq A}\left(cl(B)(x)\vee\bigvee_{y\notin B}cl(B)(y)\right)\\=&\left(\bigwedge_{x\in B\supseteq A}\left(cl(B)(x)\vee\bigvee_{y\notin B}cl(B)(y)\right)\right)\\&\wedge\left(\bigwedge_{x\notin B\supseteq A}\left(cl(B)(x)\vee\bigvee_{y\notin B}cl(B)(y)\right)\right)\\=&\top_M\wedge\left(\bigwedge_{x\notin B\supseteq A}\left(cl(B)(x)\vee\bigvee_{y\notin B}cl(B)(y)\right)\right)\\=&\bigwedge_{x\notin B\supseteq A}\left(cl(B)(x)\vee\bigvee_{y\notin B}cl(B)(y)\right)\end{aligned}$$

This shows (MFYC4-2) $\Leftrightarrow$ (MFYC4-5).

(3) The proofs of (MFYC4-3) $\Leftrightarrow$ (MFYC4-6), (MFYC4-4) $\Leftrightarrow$ (MFYC4-7) are similar to that of (MFYC4-2) $\Leftrightarrow$ (MFYC4-5), Here we omit. □

3.2. Equivalent characterizations of an M-fuzzifying topological interior operator

Before discussing equivalent characterizations of an M-fuzzifying topological interior operator, we need the following lemma which describes the equivalent characterizations of a crisp topological interior operator.

Lemma 3.3. *Let a mapping* $int : 2^X \to 2^X$ *be a crisp topological interior operator defined by* $int(A) = \bigcup\{B \subseteq X \mid B \subseteq int(B), B \subseteq A\}$. *Then*

$$\begin{aligned} x \in int(A) &\Leftrightarrow (1)\ \exists B \subseteq X, B \subseteq A, B \subseteq int(B), s.t.\, x \in B \\ &\Leftrightarrow (2)\ \exists B \subseteq X, B \subseteq A, B \subseteq int(B), s.t.\, x \in B, x \in int(B) \\ &\Leftrightarrow (3)\ \exists B \subseteq X, B \subseteq A, B \subseteq int(A), s.t.\, x \in B, x \in int(B) \\ &\Leftrightarrow (4)\ \exists B \subseteq X, B \subseteq A, B \subseteq int(B), s.t.\, x \in B, x \in int(A) \end{aligned}$$

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) and (2) $\Rightarrow$ (3) are obvious. We only show (3) $\Rightarrow$ (2). Since $B \subseteq int(A)$ and $x \in B$, it follows that $x \in int(A)$. By (3), we know there exists some B_x with $B_x \subseteq A, B_x \subseteq int(A)$ such that $x \in B_x, x \in int(B_x)$. Let

$$U = \bigcup\{B_\alpha \mid B_\alpha \subseteq A, B_\alpha \subseteq int(A)\}.$$

Then $B_x \subseteq U \subseteq A$ and $x \in U$.

Next, we verify $U \subseteq int(U)$. For any $y \in U$, there exists some B_y with $B_y \subseteq A, B_y \subseteq int(A)$ such that $y \in B_y$. So $y \in int(A)$. By (3), we know there exist some $\widetilde{B_y}$ with $\widetilde{B_y} \subseteq A, \widetilde{B_y} \subseteq int(A)$ such that $y \in \widetilde{B_y}$, $y \in int(\widetilde{B_y})$. Then $\widetilde{B_y} \subseteq U$. Further $y \in int(\widetilde{B_y}) \subseteq int(U)$. Hence $y \in int(U)$. Because of the arbitrariness of y, we get $U \subseteq int(U)$. Therefore there exists $U \subseteq X$ with $U \subseteq A$, $U \subseteq int(U)$ such that $x \in U$ and $x \in int(U)$, which means (2) holds. □

In what follows, eight equivalent characterizations of an M-fuzzifying topological interior operator will be presented.

Theorem 3.4. Let a mapping $int : 2^X \to M^X$ satisfying conditions (MFIN1)-(MFIN3). Then the followings are equivalent.

(MFIN4-1) $int(A)(x) = \bigvee_{x\in B\subseteq A} \bigwedge_{y\in B} int(B)(y)$;

(MFIN4-2) $int(A)(x) = \bigvee_{x\in B\subseteq A} \left(int(B)(x) \wedge \bigwedge_{y\in B} int(B)(y)\right)$;

(MFIN4-3) $int(A)(x) = \bigvee_{x\in B\subseteq A} \left(int(B)(x) \wedge \bigwedge_{y\in B} int(A)(y)\right)$;

(MFIN4-4) $int(A)(x) = \bigvee_{x\in B\subseteq A} \left(int(A)(x) \wedge \bigwedge_{y\in B} int(B)(y)\right)$;

(MFIN4-5) $int(A)(x) = \bigvee_{B\subseteq A} \left(int(B)(x) \wedge \bigwedge_{y\in B} int(B)(y)\right)$;

(MFIN4-6) $int(A)(x) = \bigvee_{B\subseteq A} \left(int(B)(x) \wedge \bigwedge_{y\in B} int(A)(y)\right)$;

(MFIN4-7) $int(A)(x) = \bigvee_{B\subseteq A} \left(int(A)(x) \wedge \bigwedge_{y\in B} int(B)(y)\right)$;

(MFIN4-8) $\forall \alpha \in P(M)$, $int(A)^{(\alpha)} \subseteq int(int(A)^{(\alpha)})^{(\alpha)}$, *where* $int(A)^{(\alpha)} = \{x \mid int(A)(x) \not\leq \alpha\}$.

Proof. (1) Firstly, we want to prove that (MFIN4-1) $\Leftrightarrow$ (MFIN4-3) $\Leftrightarrow$ (MFIN4-8). Since Shi and Pang have proved (MFIN4-1) $\Leftrightarrow$ (8) in [23], we only need to show (MFIN4-1) $\Leftrightarrow$ (MFIN4-3).

i) According to the order-preserving properties of int, we know that $int(A)(x) \geq \bigvee_{x\in B\subseteq A} \left(int(B)(x) \wedge \bigwedge_{y\in B} int(B)(y)\right)$. By (MFIN4-1), we get

$$int(A)(x) = \bigvee_{x\in B\subseteq A} \bigwedge_{y\in B} int(B)(y) \leq \bigvee_{x\in B\subseteq A} \left(int(B)(x) \wedge \bigwedge_{y\in B} int(A)(y)\right).$$

Therefore $int(A)(x) = \bigvee_{x\in B\subseteq A} \left(int(B)(x) \wedge \bigwedge_{y\in B} int(B)(y)\right)$. This shows (MFIN4-1)$\Rightarrow$ (MFIN4-3).

ii) To show (MFIN4-1), we only prove $\bigvee_{x\in B\subseteq A} \bigwedge_{y\in B} int(B)(y) \geq int(A)(x)$.

On one hand, $\bigvee_{x\in B\subseteq A}\bigwedge_{y\in B} int(B)(y) \le int(A)(x)$ is trivial. On the other hand, assume that $\alpha \prec int(A)(x)$ and $\alpha \in J(M)$. By (MFIN4-3), $int(A)(x) = \bigvee_{x\in B\subseteq A}\left(int(B)(x) \wedge \bigwedge_{y\in B} int(A)(y)\right)$. For any $y \in B_x$, $\alpha \prec int(A)(y)$ and $\alpha \prec int(B_x)(x)$, there exists some B_x with $x \in B_x \subseteq A$. Let

$$V = \bigcup\{B_u \mid u \in B_u \subseteq A, \alpha \prec int(B_u)(u), \forall y \in B_u, \alpha \prec int(A)(y)\}$$

Then $B_x \subseteq V$. Further $x \in V \subseteq A$.

Next, we shall show $\alpha \le \bigwedge_{z\in V} int(V)(z)$. For any $z \in V$, there exists some B_u with $u \in B_u \subseteq A$, $\alpha \prec int(B_u)(u)$, $\forall y \in B_u, \alpha \prec int(A)(y)$ such that $z \in B_u$. It follows that $\alpha \prec int(A)(z)$. By(MFIN4-3), there exists some B_z with $z \in B_z \subseteq A$ such that $\alpha \prec int(B_z)(z)$ and $\alpha \prec int(A)(w)$ for any $w \in B_z$. By the construction of V, we get $B_z \subseteq V$. Then $\alpha \le int(B_z)(z) \le int(V)(z)$. Further,

$$\alpha \le \bigwedge_{z\in V} int(V)(z) \text{ and } \alpha \le \bigvee_{x\in B\subseteq A}\bigwedge_{y\in B} int(B)(y).$$

Because of the arbitrariness of α, we have

$$int(A)(x) \le \bigvee_{x\in B\subseteq A}\bigwedge_{y\in B} int(B)(y).$$

Therefore

$$int(A)(x) = \bigvee_{x\in B\subseteq A}\bigwedge_{y\in B} int(B)(y).$$

This shows (MFIN4-3) $\Rightarrow$ (MFIN4-1).

(2) Due to $int(A)(x) = \perp_M$ for any $x \notin A$, we get

$$\begin{aligned} int(A)(x) &= \textstyle\bigvee_{B\subseteq A}\left(int(B)(x) \wedge \bigwedge_{y\in B} int(B)(y)\right) \\ &= \left(\textstyle\bigvee_{x\notin B\subseteq A}\left(int(B)(x) \wedge \bigwedge_{y\in B} int(B)(y)\right)\right) \\ &\quad \vee \left(\textstyle\bigvee_{x\in B\subseteq A}\left(int(B)(x) \wedge \bigwedge_{y\in B} int(B)(y)\right)\right) \\ &= \perp_M \vee \left(\textstyle\bigvee_{x\in B\subseteq A}\left(int(B)(x) \wedge \bigwedge_{y\in B} int(B)(y)\right)\right) \\ &= \textstyle\bigvee_{x\in B\subseteq A}\left(int(B)(x) \wedge \bigwedge_{y\in B} int(B)(y)\right) \end{aligned}$$

This shows (MFIN4-2) $\Leftrightarrow$ (MFIN4-5).

(3) The proofs of (MFIN4-3) $\Leftrightarrow$ (MFIN4-6) and (MFIN4-4) $\Leftrightarrow$ (MFIN4-7) are similar to that of (MFIN4-2) $\Leftrightarrow$ (MFIN4-5), Here we omit. □

3.3. *Equivalent characterization of an M-fuzzifying topological derived operator*

In [4], it has shown that if cl is an M-fuzzifying topological closure operator, then $der^{cl}: 2^X \to M^X$ defined by $der^{cl}(A)(x) = cl(A - \{x\})(x)$ is an M-fuzzifying topological derived operator. On the contrary, if der is an M-fuzzifying topological derived operator, then $cl^{der}: 2^X \to M^X$ defined by $cl^{der}(A) = \begin{cases} \top_M, & x \in A; \\ der(A)(x), & x \notin A. \end{cases}$ is an M-fuzzifying topological closure operator. Besides, $der^{cl^{der}} = der$ and $cl^{der^{cl}} = cl$.

Namely, there exists one-to-one correspondence between M-fuzzifying topological derived operators and M-fuzzifying topological closure operator. Due to Theorem 3.2 and the above conclusions, it is easy to get the following eight characterizations of an M-fuzzifying topological derived operator.

Theorem 3.5. Let a map $der : 2^X \to M^X$ satisfying the conditions (MFDER1)-(MFDER3). Then the followings are equivalent.

(MFDER4-1) $der(A)(x) = \bigwedge_{x\notin B\supseteq A-\{x\}}\bigvee_{y\notin B} der(B)(y)$;

(MFDER4-2) $der(A)(x) = \bigwedge_{x\notin B\supseteq A-\{x\}}\left(der(B)(x) \vee \bigvee_{y\notin B} der(B)(y)\right)$;

(MFDER4-3) $der(A)(x) = \bigwedge_{x\notin B\supseteq A-\{x\}}\left(der(B)(x) \vee \bigvee_{y\notin B} der(A-\{x\})(y)\right)$;

(MFDER4-4) $der(A)(x) = \bigwedge_{x\notin B\supseteq A-\{x\}} \left(der(A-\{x\})(x) \vee \bigvee_{y\notin B} der(B)(y)\right)$;
(MFDER4-5) $der(A)(x) = \bigwedge_{B\supseteq A-\{x\}} \left(der(B)(x) \vee \bigvee_{y\notin B} der(B)(y)\right)$;
(MFDER4-6) $der(A)(x) = \bigwedge_{B\supseteq A-\{x\}} \left(der(B)(x) \vee \bigvee_{y\notin B} der(A-\{x\})(y)\right)$;
(MFDER4-7) $der(A)(x) = \bigwedge_{B\supseteq A-\{x\}} \left(der(A-\{x\})(x) \vee \bigvee_{y\notin B} der(B)(y)\right)$;
(MFDER4-8) $\forall \alpha \in J(M)$, $der(der(A)_{[\alpha]})_{[\alpha]} \subseteq A \bigcup der(A)_{[\alpha]}$, *where* $der(A)_{[\alpha]} = \{x \in A \mid der(A)(x) \geq \alpha\}$.

4. M-fuzzifying topological operators cl^d, int^d, der^d induced by an M-fuzzifying pseudo metric d

In this section, we shall show the fuzzy construction of an M-fuzzifying topological closure operator $cl^d : 2^X \to M^X$, an M-fuzzifying topological interior operator $int^d : 2^X \to M^X$, and an M-fuzzifying topological derived operator $der^d : 2^X \to M^X$ induced by an M-fuzzifying pseudo metric d respectively. Also, we will show that the continuity between M-fuzzifying topological operator spaces and M-fuzzifying pseudo metric spaces is maintained.

Firstly, let us recall the construction of the closure of a subset $A \subseteq X$ induced by a metric d. In classical case, let (X, d) be a metric space. Define $d(x, A) = inf\{d(x, y) \mid y \in A\}$ be the distance from the point x to the subset A. Then

$$cl^d(A) = \{x \in X \mid d(x, A) = 0\} = \{x \in X \mid \forall r > 0, \exists y \in A, d(x, y) < r\}.$$

As it was mentioned in preliminaries, a crisp metric d can be equivalently regarded as a mapping $\chi_d : X \times X \to [0, \infty)([0, 1])$ defined by $\chi_d(x, y)(r) = \begin{cases} 1, & d(x, y) \geq r\,; \\ 0, & d(x, y) < r\,. \end{cases}$

Based on that, we extend cl^d to an M-fuzzifying case.

Theorem 4.1. *Let (X, d) be an M-fuzzifying pseudo metric space. Define a map $cl^d : 2^X \to M^X$ by $\forall A \in 2^X$, $x \in X$*

$$cl^d(A)(x) = \bigwedge_{r>0} \bigvee_{y\in A} d(x, y)(r)'.$$

Then cl^d is an M-fuzzifying closure operator.

Proof. We need to check that cl meet with (MFYC1)-(MFYC4).
(MFYC1) and (MFYC2) are trivial.
(MFYC3) On one side, $cl^d(A_1) \leq cl^d(A_2)$ is trivial for any $A_1 \subseteq A_2$. Then $cl^d(A_1) \vee cl^d(A_2) \leq cl^d(A_1 \cup A_2)$. On the other side, we have

$$\begin{aligned} cl^d(A_1)(x) \vee cl^d(A_2)(x) &= \left(\bigwedge_{s>0} \bigvee_{y_1\in A_1} d(x, y_1)(s)'\right) \vee \left(\bigwedge_{t>0} \bigvee_{y_2\in A_2} d(x, y_2)(t)'\right) \\ &= \bigwedge_{s>0,t>0} \bigvee_{y_1\in A_1, y_2\in A_2} (d(x, y_1)(s)' \vee d(x, y_2)(t)') \geq \bigwedge_{r>0} \bigvee_{y\in A_1\cup A_2} d(x, y)(r)' = cl^d(A_1 \cup A_2)(x). \end{aligned}$$

Next, we shall prove the inequality in the above.

Take any $a \in M \setminus \{\perp\}$ with $a \nleq \bigwedge_{s>0,t>0} \bigvee_{y_1\in A_1, y_2\in A_2} (d(x, y_1)(s)' \vee d(x, y_2)(t)')$, then there exist some $s > 0$ and $t > 0$ such that $a \nleq d(x, y_1)(s)'$ for all $y_1 \in A_1$ and $a \nleq d(x, y_2)(t)'$ for all $y_2 \in A_2$.

Let $0 < r < s \wedge t$. For any $y \in A_1 \cup A_2$ (that is, $y \in A_1$ or $y \in A_2$), since $d(x, y)(-)$ is non-increasing, it follows that $d(x, y)(r) \geq d(x, y)(s)$, that is, $d(x, y)(r)' \leq d(x, y)(s)'$. If $y \in A_1$, then $a \nleq d(x, y)(s)'$. So $a \nleq d(x, y)(r)'$. If $y \in A_2$, we get $a \nleq d(x, y)(r)'$ similarly. Hence $a \nleq \bigwedge_{r>0} \bigvee_{y\in A_1\cup A_2} d(x, y)(r)'$. By the arbitrariness a, we have

$$\bigwedge_{s>0,t>0} \bigvee_{y_1\in A_1, y_2\in A_2} (d(x, y_1)(s)' \vee d(x, y_2)(t)') \geq \bigwedge_{r>0} \bigvee_{y\in A_1\cup A_2} d(x, y)(r)'.$$

(MFCL4) By theorem 3.2, it suffices to prove (MFCL4-2), that is,

$cl^d(A)(x) = \bigwedge_{x\notin B\supseteq A}\left(cl^d(B)(x) \vee \bigvee_{y\notin B} cl^d(A)(y)\right).$

Since a mapping cl is order-preserving, we have

$$cl^d(A)(x) \leq \bigwedge_{x\notin B\supseteq A}\left(cl^d(B)(x) \vee \bigvee_{y\notin B} cl^d(A)(y)\right).$$

What remains is to prove

$$cl^d(A)(x) \geq \bigwedge_{x\notin B\supseteq A}\left(cl^d(B)(x) \vee \bigvee_{y\notin B} cl^d(A)(y)\right),$$

$$\text{i.e.,}\quad cl^d(A)(x)' \leq \bigvee_{x\notin B\supseteq A}\left(cl^d(B)(x)' \wedge \bigwedge_{y\notin B} cl^d(A)(y)'\right).$$

In fact, take any $a \in J(M)$ with $a \prec cl^d(A)(x)' = \bigvee_{r>0}\bigwedge_{y\in A} d(x,y)(r)$, there exists $r_0 > 0$ such that $a \leq d(x,y)(r_0)$ for all $y \in A$. Let

$$B = \{y \in X \mid d(x,y)\left(\frac{r_0}{2}\right) \geq a\}.$$

Then $x \notin B \supseteq A$ (Actually, we have $d(x,y)(\frac{r_0}{2}) \ngeq a$ for any $y \notin B$, and $d(x,y)(\frac{r_0}{2}) \geq d(x,y)(r_0)$. Then $d(x,y)(r_0) \ngeq a$, which implies $y \notin A$. So $B \supseteq A$. Besides $d(x,x)(\frac{r_0}{2}) \ngeq a$, it follows that $x \notin B \supseteq A$).

Note that

$$cl^d(B)(x)' = \bigvee_{s>0}\bigwedge_{y\in B} d(x,y)(s) \geq \bigwedge_{y\in B} d(x,y)\left(\frac{r_0}{2}\right) \geq a,$$

and

$$\bigwedge_{y\notin B} cl^d(A)(y)' = \bigwedge_{y\notin B}\bigvee_{t>0}\bigwedge_{z\in A} d(y,z)(t) \geq \bigwedge_{y\notin B}\bigwedge_{z\in A} d(y,z)\left(\frac{r_0}{2}\right).$$

Then we shall verify $\bigwedge_{y\notin B} cl^d(A)(y)' \geq a$. For all $y \notin B$, $d(x,y)(\frac{r_0}{2}) \ngeq a$ and for all $z \in A$, $d(x,z)(r_0) \geq a$. By $a \in J(M)$ and $d(x,z)(r_0) \leq d(x,y)\left(\frac{r_0}{2}\right) \vee d(y,z)\left(\frac{r_0}{2}\right)$, we get $d(y,z)(\frac{r_0}{2}) \geq a$. This implies $\bigwedge_{y\notin B} cl^d(A)(y)' \geq a$ and $a \leq \bigvee_{x\notin B\supseteq A}\left(cl^d(B)(x)' \wedge \bigwedge_{y\notin B} cl^d(A)(y)'\right)$. Because of the arbitrariness of a, we obtain

$$cl(A)^d(x)' \leq \bigvee_{x\notin B\supseteq A}\left(cl^d(B)(x)' \wedge \bigwedge_{y\notin B} cl^d(A)(y)'\right).$$

Therefore (MFCL4-2) holds, which is equivalent to condition (MFCL4) holds. □

In what follows, we shall show that the continuity of f between M-fuzzifying pseudo metric spaces and their induced M-fuzzifying topological closure spaces remains consistent.

Theorem 4.2. *Let (X, d_X) and (Y, d_Y) be two M-fuzzifying pseudo metric spaces. If $f : (X, d_X) \to (Y, d_Y)$ is continuous, then $f : (X, cl^{d_X}) \to (Y, cl^{d_Y})$ is also continuous.*

Proof. It required to show that $cl^{d_X}(A)(x) \leq cl^{d_Y}(f^{\to}(A))(f(x))$ for any $x \in X$ and for all $A \in 2^X$. Since $f : (X, d_X) \to (X, d_Y)$ is continuous, it follows that $(d_Y(f(x_1), f(x_2))(r))' \geq (d_X(x_1,x_2)(r))'$ for any $r > 0$ and for all $x_1, x_2 \in X$. Then

$$cl^{d_Y}(f^{\to}(A))(f(x)) = \bigwedge_{r>0}\bigvee_{z\in f^{\to}(A)} (d_Y(f(x),z)(r))' \geq \bigwedge_{r>0}\bigvee_{y\in A}(d_Y(f(x),f(y))(r))' \geq \bigwedge_{r>0}\bigvee_{y\in A}(d_X(x,y)(r))' = cl^{d_X}(A)(x).$$

□

Let (X, d) be a crisp metric space and let $A \subseteq X$. Define the open ball $B(x, r) = \{y \in X \mid d(x, y) < r\}$. Then

$$int^d(A) = \{x \in X \mid \exists r > 0, B(x, r) \subseteq A\} = \{x \in X \mid \exists r > 0, \forall y \notin A, d(x, y) \geq r\}.$$

As it is mentioned in preliminaries, a crisp metric d can be equivalently regarded as a mapping χ_d : $X \times X \to [0, \infty)([0, 1])$ defined by $\chi_d(x, y)(r) = \begin{cases} 1, & d(x, y) \geq r\,; \\ 0, & d(x, y) < r\,. \end{cases}$ According to that, int^d could be extended to the M-fuzzifying case in the following theorem.

Theorem 4.3. *Let (X, d) be an M-fuzzifying pseudo metric space. Define a map $int^d : 2^X \to M^X$ by $\forall A \in 2^X$, $\forall x \in X$*

$$int^d(A)(x) = \bigvee_{r>0} \bigwedge_{y \notin A} d(x, y)(r).$$

Then int^d is an M-fuzzifying interior operator.

Proof. We need to prove (MFIN1)-(MFIN4).
(MFIN1) and (MFIN2) are trivial.
(MFIN3) On one hand, it is obvious that $int^d(A_1) \leq int^d(A_2)$ for any $A_1 \subseteq A_2$. Then $int^d(A_1 \cap A_2) \leq int^d(A_1) \wedge int^d(A_2)$. On the other side,

$$\begin{aligned} int^d(A_1) \wedge int^d(A_2) &= \left(\bigvee_{s>0} \bigwedge_{y_1 \notin A_1} d(x, y_1)(s) \right) \wedge \left(\bigvee_{t>0} \bigwedge_{y_2 \notin A_2} d(x, y_2)(t) \right) \\ &= \bigvee_{s>0, t>0} \bigwedge_{y_1 \notin A_1, y_2 \notin A_2} d(x, y_1)(s) \wedge d(x, y_1)(t) \leq \bigvee_{r>0} \bigwedge_{y \notin A_1 \cap A_2} d(x, y)(r) = int^d(A_1 \cap A_2) \end{aligned}$$

Next, we shall prove the inequality in the above.

Assume $a \in M \setminus \{\perp\}$ with $a \prec \bigvee_{s>0, t>0} \bigwedge_{y_1 \notin A_1, y_2 \notin A_2} d(x, y_1)(s) \wedge d(x, y_2)(t)$. there exist $s > 0$ and $t > 0$ such that $a \prec d(x, y_1)(s) \wedge d(x, y_2)(t)$ for all $y_1 \notin A_1$ and $y_2 \notin A_2$. This implies $a \leq d(x, y_1)(s)$ for any $y_1 \notin A_1$ and $a \leq d(x, y_2)(t)$ for any $y_2 \notin A_2$. Let $0 < r < s \wedge t$. Take any $y \notin A_1 \cap A_2$ (that is, $y \notin A_1$ or $y \notin A_2$). If $y \notin A_1$, then $d(x, y)(s) \geq a$ and $d(x, y)(r) \geq d(x, y)(s) \geq a$. If $y \notin A_2$, we get $d(x, y)(r) \geq a$ similarly. So $d(x, y)(r) \geq a$. Further $a \leq \bigvee_{r>0} \bigwedge_{y \notin A_1 \cap A_2} d(x, y)(r)$. Because of the arbitrariness of a, we get

$$\bigvee_{s>0, t>0} \bigwedge_{y_1 \notin A_1, y_2 \notin A_2} d(x, y_1)(s) \wedge d(x, y_2)(t) \leq \bigvee_{r>0} \bigwedge_{y \notin A_1 \cap A_2} d(x, y)(r).$$

Hence $int^d(A_1 \cap A_2) = int^d(A_1) \wedge int^d(A_2)$.

(MFIN4) By Theorem 3.4, we only need to check (MFIN4-2), that is,

$$int^d(A)(x) = \bigvee_{x \in B \subseteq A} \left(int^d(B)(x) \wedge \bigwedge_{y \in B} int^d(A)(y) \right).$$

It follows from (MFIN3) that the map $int^d(\cdot)$ is order-preserving, it could draw the conclusion that

$$int^d(A)(x) \geq \bigvee_{x \in B \subseteq A} \left(int^d(B)(x) \wedge \bigwedge_{y \in B} int^d(A)(y) \right).$$

It suffices to show

$$int^d(A)(x) \leq \bigvee_{x \in B \subseteq A} \left(int^d(B)(x) \wedge \bigwedge_{y \in B} int^d(A)(y) \right).$$

Take any $a \in P(M)$ with $a \prec int^d(A) = \bigvee_{r>0} \bigwedge_{y \notin A} d(x,y)(r)$, there exists $r_0 > 0$ such that $a \leq d(x,y)(r_0)$ for all $y \notin A$. Let

$$B = \{y \in X \mid d(x,y)(\frac{r_0}{2}) \ngeq a\}.$$

Then $x \in B \subseteq A$. (In fact, since $d(x,y)(r_0) \geq a$ for all $y \notin A$, and $d(x,y)(-)$ is non-increasing, it follows that $d(x,y)(\frac{r_0}{2}) \geq d(x,y)(r_0)$. This shows $d(x,y)(\frac{r_0}{2}) \geq a$ which means $y \notin B$. So $B \subseteq A$. Besides $d(x,x)(r_0) = \perp_M \leq a$, we get $x \in B \subseteq A$).

Note that

$$int^d(B)(x) = \bigvee_{r>0} \bigwedge_{y \notin B} (d(x,y)(r)) \geq \bigwedge_{y \notin B} d(x,y)(\frac{r_0}{2}) \geq a,$$

and

$$\bigwedge_{y \in B} int^d(A)(y) = \bigwedge_{y \in B} \bigvee_{r>0} \bigwedge_{z \notin A} d(y,z)(r) \geq \bigwedge_{y \in B} \bigwedge_{z \notin A} d(y,z)(\frac{r_0}{2}).$$

Due to $a \in P(M)$ and $d(x,z)(r_0) \leq d(x,y)(\frac{r_0}{2}) \vee d(y,z)(\frac{r_0}{2})$. So $d(y,z)(\frac{r_0}{2}) \geq a$, which shows $\bigwedge_{y \in B} int^d(A)(y) \geq a$. Further $a \leq \left(int^d(B)(x) \wedge \bigwedge_{y \in B} int^d(A)(y)\right)$. Because of the arbitrariness of a, we obtain

$$int^d(A)(x) \leq \bigvee_{x \in B \subseteq A} \left(int^d(B)(x) \wedge \bigwedge_{y \in B} int^d(A)(y) \right).$$

Therefore (MFIL4-2) holds, which is equivalent to condition (MFIL4) holds. □

In the following, it shall shows that the continuity of f between M-fuzzifying pseudo-metric spaces and their induced M-fuzzifying interior spaces is consistent.

Theorem 4.4. *Let (X, d_X) and (Y, d_Y) be two M-fuzzifying pseudo metric spaces. If $f : (X, d_X) \to (Y, d_Y)$ is continuous, then $f : (X, int^{d_X}) \to (Y, int^{d_Y})$ is also continuous.*

Proof. It suffices to prove that $int^{d_Y}(B)(f(x)) \leq int^{d_X}(f^{\leftarrow}(B))(x)$ for any $x \in X$ and $B \in 2^Y$. Due to $f : (X, d_X) \to (X, d_Y)$ is continuous, we know $d_X(x_1, x_2)(r) \geq d_Y(f(x_1), f(x_2))(r))$ for any $r > 0$ and for all $x_1, x_2 \in X$. Then

$$\begin{aligned} int^{d_Y}(B)(f(x)) &= \bigvee_{r>0} \bigwedge_{z \notin B} (d_Y(f(x), z)(r)) \leq \bigvee_{r>0} \bigwedge_{y \notin f^{\leftarrow}(B)} (d_Y(f(x), f(y))(r)) \\ &\leq \bigvee_{r>0} \bigwedge_{y \notin f^{\leftarrow}(B)} (d_X(x,y)(r)) = int^{d_X}(f^{\leftarrow}(B))(x). \end{aligned}$$

□

In [4], it has shown that M-fuzzifying topological derived operators and M-fuzzifying topological closure operators exists a one-to-one correspondence. By $der^{cl}(A)(x) = cl(A - \{x\})(x)$ and Theorem 4.1, the following theorem could be obtained.

Theorem 4.5. *Let (X, d) be an M-fuzzifying pseudo metric space. For all $A \in 2^X$ and $x \in X$. Defined a mapping $der^d : 2^X \to M^X$ by $\forall A \in 2^X, \forall x \in X$,*

$$der^d(A)(x) = \bigwedge_{r>0} \bigvee_{y \in A - \{x\}} d(x,y)(r)'.$$

Therefore der^d is an M-fuzzifying derived operator.

Finally, we shall show that the continuity of f between M-fuzzifying pseudo metric and their induced M-fuzzifying topological derived spaces remains consistent.

Theorem 4.6. *Let* (X, d_X) *and* (Y, d_Y) *be two M-fuzzifying pseudo metric spaces. If* $f : (X, d_X) \to (Y, d_Y)$ *is continuous, then* $f : (X, der^{d_X}) \to (Y, der^{d_Y})$ *is also continuous.*

Proof. We need to show $\forall x \in X$, $\forall A \in 2^X$, $der^{d_X}(A)(x) \leq f^{\to}(A)(f(x)) \vee der^{d_Y}(f^{\to}(A))(f(x))$. Since $f : (X, d_X) \to (X, d_Y)$ is continuous, we have $(d_Y(f(x_1), f(x_2))(r))' \geq (d_X(x_1, x_2)(r))'$ for any $r > 0$ and for all $x_1, x_2 \in X$.
(1) If $f(x) \in f^{\to}(A)$, then

$$f^{\to}(A)(f(x)) \vee der^{d_Y}(f^{\to}(A))(f(x)) = \top_M \geq der^{d_X}(A)(x).$$

(2) If $f(x) \notin f^{\to}(A)$, then

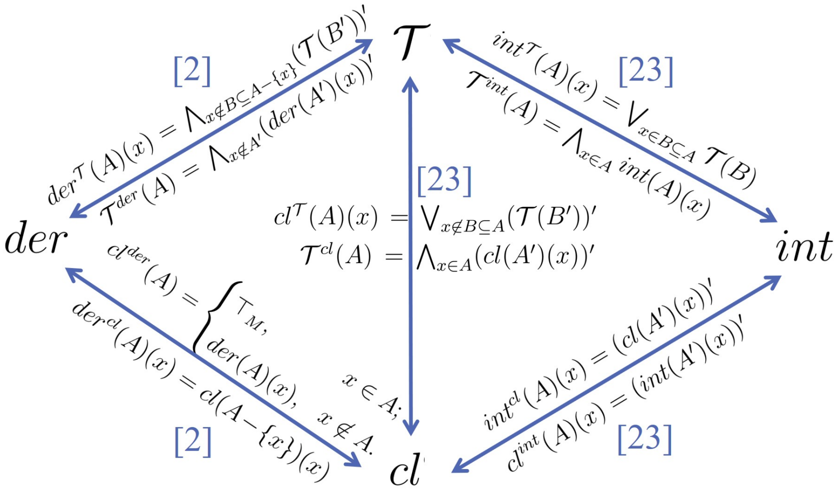
$$der^{d_Y}(f^{\to}(A))(f(x)) = \bigwedge_{r>0} \bigvee_{z \in f^{\to}(A) - f(x)} (d_Y(f(x), z)(r))'$$
$$= \bigwedge_{r>0} \bigvee_{z \in f^{\to}(A)} (d_Y(f(x), z)(r))' \geq \bigwedge_{r>0} \bigvee_{y \in A} (d_Y(f(x), f(y))(r))'$$
$$\geq \bigwedge_{r>0} \bigvee_{y \in A} (d_X(x, y)(r))' \geq \bigwedge_{r>0} \bigvee_{y \in A - \{x\}} (d_X(x, y)(r))' = der^{d_X}(A)(x).$$

□

5. M-fuzzifying topology $\mathcal{T}^d = \mathcal{T}^{cl^d} = \mathcal{T}^{int^d} = \mathcal{T}^{der^d}$

In the classical case, there is one-to-one correspondences among closure operators, interior operators, derived operators and topologies. Besides, it has shown that the topology $\mathcal{T}^d$ induced by a pseudo d is exactly the topology induced by cl^d, int^d, der^d. Namely, $\mathcal{T}^d = \mathcal{T}^{cl^d} = \mathcal{T}^{cl^d} = \mathcal{T}^{int^d} = \mathcal{T}^{der^d}$. Naturally, we would like to know whether the above conclusions hold or not, when it is extended to M-fuzzifying cases.

Many researchers [4, 6, 21, 32, 34] have shown the relations among M-fuzzifying topological closure operators, M-fuzzifying topological interior operators, M-fuzzifying topological derived operators, M-fuzzifying neighborhood system, and M-fuzzifying topologies. Now, we use the following chart to recall some important relationships.



In [22], it has proved that if d is an M-fuzzifying pseudo metric, then the M-fuzzifying topology $\mathcal{T}^d : 2^X \to M$ induced by d is defined by $\forall A \in 2^X$,

$$\mathcal{T}^d(A) = \bigwedge_{x \in A} \bigvee_{r>0} \bigwedge_{y \notin A} d(x, y)(r).$$

Next, we shall show the most meaningful conclusions of this section.

Theorem 5.1. $\mathcal{T}^d = \mathcal{T}^{cl^d} = \mathcal{T}^{int^d} = \mathcal{T}^{der^d}$.

Proof. For any $A \in 2^X$,

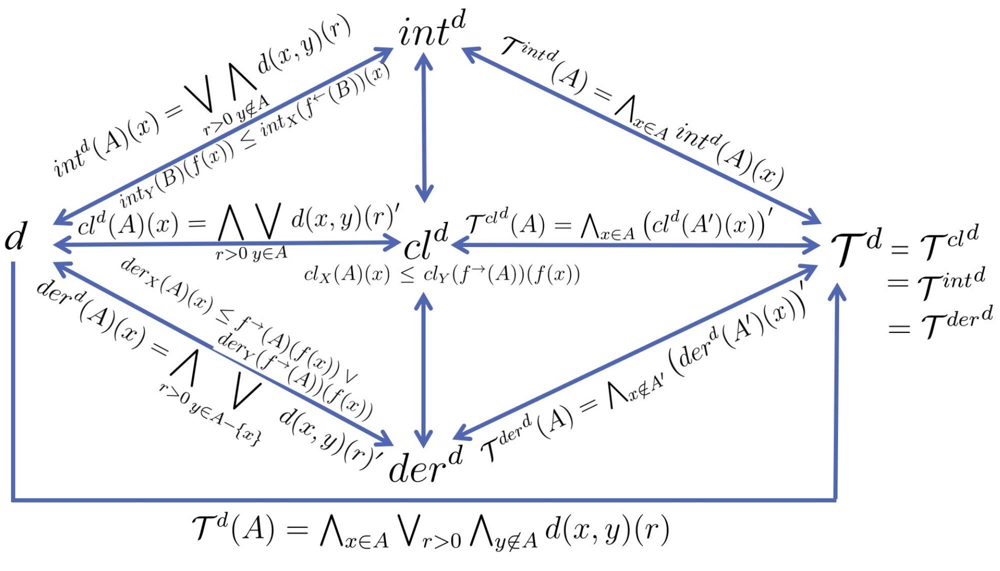
$$\mathcal{T}^{cl^d}(A) = \bigwedge_{x\in A}\left(cl^d(A')(x)\right)' = \bigwedge_{x\in A}\left(\bigwedge_{r>0}\bigvee_{y\in A'}(d(x,y)(r))'\right)' = \bigwedge_{x\in A}\bigvee_{r>0}\bigwedge_{y\notin A} d(x,y)(r) = \mathcal{T}^d(A).$$

$$\mathcal{T}^{int^d}(A) = \bigwedge_{x\in A} int^d(A)(x) = \bigwedge_{x\in A}\bigvee_{r>0}\bigwedge_{y\notin A} d(x,y)(r) = \mathcal{T}^d(A).$$

$$\begin{aligned}\mathcal{T}^{der^d}(A) &= \bigwedge_{x\notin A'}\left(der^d(A')(x)\right)' = \bigwedge_{x\in A}\left(\bigwedge_{r>0}\bigvee_{y\in A'-\{x\}} d(x,y)(r)'\right)' \\ &= \bigwedge_{x\in A}\bigvee_{r>0}\bigwedge_{y\in A'-\{x\}} d(x,y)(r) = \bigwedge_{x\in A}\bigvee_{r>0}\bigwedge_{y\in A'} d(x,y)(r) \\ &= \bigwedge_{x\in A}\bigvee_{r>0}\bigwedge_{y\notin A} d(x,y)(r) = \mathcal{T}^d(A).\end{aligned}$$

□

In the end, we shall show a summary diagram that illustrates the research content of this paper.



6. Conclusions

In this paper, we firstly mainly discussed eight equivalent characterizations of an M-fuzzifying topological closure operators, an M-fuzzifying topological interior operators, an M-fuzzifying topological derived operators. As its applications, we obtained some M-fuzzifying topological operators induced by an M-fuzzifying pseudo metric d in the sense of Morsi's fuzzy metric. Finally, it was shown that these M-fuzzifying topology are equal, that is, $\mathcal{T}^d = \mathcal{T}^{cl^d} = \mathcal{T}^{int^d} = \mathcal{T}^{der^d}$. In the future, we will explore the characterizations and constructions of M-fuzzifying topological operators induced by an M-fuzzifying partial pseudo metric.

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