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Pth moment stability and approximate controllability of NDEs with delayed impulses and time-varying delays driven by fBm

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Abstract. Pth moment stability and approximate controllability of neutral differential equations (NDEs) with delayed impulse and time-varying delays driven by fractional Brownian motion (fBm) with $H \in (0, 1/2)$ are studied. Unlike the fBm with Hurst parameter $H \in (1/2, 1)$ in most of the literature, the fBm with Hurst parameter $H \in (0, 1/2)$ in this work. The delayed impulse here means time delay in impulse, and the delay is no more than the minimum value of impulsive intervals. Based on Lyapunov stability theory and analytic semigroup theory, combined with stochastic analysis methods and impulse integral inequalities, some conditions guaranteeing the pth moment stability of the mild solution are established. Afterward, approximate controllability conditions of the system are acquired by the Lebesgue-dominated convergence theorem. At last, the validity of secured results is verified by an example.

1. Introdution

FBm as a central Gaussian stochastic process is proposed by Benoit Mandelbrot and Van Ness, which can model systems with self-similarity, non-smoothness and long dependence[1]. FBm significantly relies on Hurst parameter H with values that lie in the range (0, 1), denoted by $B_Q^H = \{B_Q^H(t), t \geq 0\}$. Furthermore, the Hurst parameter not only affects the roughness of the process but also determines the long-range correlation of the time series. When $H \in (0, 1/2)$, fBm is considered a process of short-term memory; when $H \in (1/2, 1)$, fBm is described as a process of long-term memory; fBm degenerates into the standard Brown motion when $H = 1/2$. Compared with standard Brown motion, it is long-term dependence and non-Markovian, and the classical stochastic analysis theory cannot be directly applied. Currently, random differential equations driven by fBm are widely used in control engineering[2], option pricing[3], environmental science[4], etc.

NDEs are equations whose state vectors are related not only to current and past state vectors but also to rates of change of past state vectors, which was first proposed by Hale and Meyer[5]. NDEs driven by

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fBm with $H \in (1/2, 1)$ have key results[6–11]. For example, In [6], Arthi et al. delved into the exponential stability of NDEs with integrodifferential terms and impulses driven by fBm. Based on [6], the mixed fBm was considered in [7–9], where global attractiveness and quasi-invariant sets of systems were examined in [7] and the existence and exponential stability of systems were investigated in [8, 9]. In [10], Xu et al. investigated the stabilization of Markovian jumping neutral systems with fBm, however, impulse effects were not taken into account in systems. Dung et al. discussed the existence, uniqueness and asymptotic stability of neutral impulsive differential equations driven by fBm with finite and infinite time delays, respectively[11]. However, literatures for the stability of NDEs driven by fBm with $H \in (0, 1/2)$ have not been retrieved.

The theories and applications of impulsive differential equations have been rapidly developed because many practical systems abruptly change in evolution. Impulsive differential equations driven by fBm can be better described as phenomena characterized by some features such as nonlinearity, nonsmoothness, long memory, and suddenness. For example, in [6–9, 11–14], the impulsive differential equations driven by fBm with $H \in (1/2, 1)$ were discussed, however, the impulses in the equations were instantaneous impulses. Nevertheless, impulsive transients are contingent upon both the current and the historical states of the system in practice, which are described as delayed impulses. Delayed impulses allow more accurate modeling of dynamical systems, are widely applied in many fields, such as chaotic systems[15], neural networks[16], multiagent systems[17], etc. In the past, many scholars considered delayed impulsive effects on deterministic systems[15–18]. Delayed impulsive differential equations driven by random terms have been gradually paid close attention to by scholars, for example, random terms are considered as standard Brownian motion in [19, 20] and fBm in [21–23]. Pth moment exponentially stable for delayed impulsive differential equations was investigated in [19], the time delays of the impulses are less-than the minimum value of impulsive intervals. Input-state stability for delayed impulsive differential equations was researched in [20], the size of the time delays were not restricted like in [19]. Zhou et al. provided the existence and uniqueness conditions of mild solution for stochastic Volterra integrodifferential equations with linear delayed impulses and fBm in [21]. Based on the [21], the mean-square asymptotic stability of such system was further investigated in [22]. What's more, the boundedness and exponential stability of differential equations with delayed impulses exposed to additive fBm and multiplicative fBm interference were examined in [23]. In [21–23], delayed impulsive differential equations driven by fBm with Hurst parameter taking values in $(1/2, 1)$ were studied. Delayed impulsive differential equations driven by fBm with $H \in (0, 1/2)$ have not been studied systematically.

Stability problems are one of the topical issues for deterministic and stochastic differential equations, some references therein [6, 8–11, 14–20, 22–28]. The exponential stability, the finite-time stability, and the asymptotic stability of deterministic equations were explored in [15], [17], and [18], respectively. The mean square stability, the Mittag-Leffler string stability, the exponential stability, stochastic stability, and input-to-state stability of differential equations driven by standard Brownian motion were explored in [25], [26], [19, 27], [28], and [20], respectively. The stability of differential equations driven by fBm was examined in [6, 8–11, 14, 22–24]. The stability of impulsive NDEs driven by fBm was studied in [6, 8, 9, 14]. The stability of differential equations with delayed impulses driven by fBm was studied in [22, 23].

Controllability implies that there exists at least one sequence of control inputs that can drive the state of the system from any arbitrary initial state to any desired final state within a finite amount of time. Controllability is an important index to evaluate the system's stability, and it is the core element of whether the system can be guided and managed by human beings. Approximate controllability means that the system can reach or approach the desired state within a certain error range by control. Approximate controllability is more popular because it is possible to steer the system to an arbitrarily small domain of the target state[29–37]. Because of existing random noise in the dynamic system, most scholars in the field have directed their research toward differential equations suffering from fBm ($H \in (1/2, 1)$) noise, revealing the approximate controllability problem[29–32]. However, as fBm with the Hurst parameter H in the range $(0, 1/2)$ exhibits more irregular and singular properties, the study of approximate controllability problems in differential equations driven by fBm presents unique challenges. Zhao et al. investigated fractional-order differential equations incorporated by multiple delay controls and Poisson jumps driven by fBm ($H \in (0, 1/2)$), and established approximate controllability conditions for these equations[36]. Liu et al.

explored the issue of approximate controllability of systems with fBm ($H \in (0, 1/2)$) and non-instantaneous impulses[37]. Nevertheless, impulses were not considered in [36], the impulses were not delayed impulses in [37], and the equations are non-neutral in [36, 37]. Approximate controllability of NDEs with delayed impulses driven by fBm ($H \in (0, 1/2)$) has not been surveyed.

Based on the above discussion, the issues of the pth moment stability and approximate controllability of NDEs driven by fBm ($H \in (0, 1/2)$) with delayed impulses and time-varying delays have not been reported. This problem will be addressed, and the following are the key contributions.

(1) NDEs with time-varying delays and delayed impulses driven by fBm ($H \in (0, 1/2)$) are investigated as a new try, and the conditions for pth moment stability and approximate controllability of the systems are provided.

(2) Unlike literatures [21–23], the Hurst parameter H of fBm in this paper takes the value range $(0, 1/2)$ instead of $(1/2, 1)$. In addition, the integral estimates for fBm with $H \in (0, 1/2)$ presented in this paper are better due to the inhomogeneity and singularity of fBm.

(3) Compared with literatures [6, 8–11, 13, 14, 24], the impulses in this paper contain time delay, while the impulses studied in the above paper are instantaneous.

The structure of this paper is summarised below. In Section 2, several notations, lemmas, and concepts will be presented. In Section 3, pth moment stability conditions of mild solution will be provided. In Section 4, approximate controllability of the systems will be demonstrated. In Section 5, the validity of the results will be verified by an example. Finally, conclusions will be drawn.

2. Preliminaries

In this paper, pth moment stability and approximate controllability problems are considered for delayed impulsive NDEs with time-varying delays driven by fBm with $H \in (0, 1/2)$ of the form

$$\begin{cases} \mathrm{d}[y(\mathrm{t}) - \mathcal{G}(\mathrm{t}, y(\mathrm{t} - \kappa(\mathrm{t})))] = [\mathscr{A} y(\mathrm{t}) + \mathfrak{h}(\mathrm{t}, y(\mathrm{t}), y(\mathrm{t} - \vartheta(\mathrm{t}))) + \mathscr{B} u(t)]\mathrm{d}\mathrm{t} + \mathfrak{I}(\mathrm{t})\mathrm{d}\mathrm{B}_Q^H(\mathrm{t}), \mathrm{t} \geq \mathrm{t}_0, \mathrm{t} \neq \mathrm{t}_\mathrm{r}, \\ \Delta y(\mathrm{t}_\mathrm{r}) = y(\mathrm{t}_\mathrm{r}^+) - y(\mathrm{t}_\mathrm{r}^-) = I_\mathrm{r} y(\mathrm{t}_\mathrm{r} - \varpi), \mathrm{r} \in \mathbb{N}, \\ y(\mathrm{t}_0 + \eta) = \psi(\eta), \eta \in [-\varsigma, 0], \end{cases} \tag{1}$$

where $y(\mathrm{t})$ is the state variable, $\mathcal{G} : [\mathrm{t}_0, \infty) \times \mathbb{X} \to \mathbb{X}$, $\mathbb{X}$ is a Hilbert space. The delays $\kappa(\mathrm{t}), \vartheta(\mathrm{t}) : [\mathrm{t}_0, \infty) \to [0, \tau]$ ($\tau > 0$) are continuous. $\mathscr{A}$ is an infinitesimal generator of the $(S(\mathrm{t}))_{\mathrm{t}\geq \mathrm{t}_0}$, $(S(\mathrm{t}))_{\mathrm{t}\geq \mathrm{t}_0}$ represents an analytic semigroup defined in $\mathbb{X}$. $\mathfrak{h} : [\mathrm{t}_0, \infty) \times B_1 \times \mathbb{X} \to \mathbb{X}$, phase space $B_1 = \{y : [-\varsigma, \infty) \to \mathbb{X}$: $y(\mathrm{t})$ is continuous everywhere besides $y(\mathrm{t}_\mathrm{r}^+)$ and $y(\mathrm{t}_\mathrm{r}^-)$, $\mathrm{r} = 1, 2, \cdots\}$, $y(\mathrm{t}_\mathrm{r}^+)$ and $y(\mathrm{t}_\mathrm{r}^-)$ correspond to the right-hand and the left-hand limits, respectively, of $y(\mathrm{t})$ at $\mathrm{t} = \mathrm{t}_\mathrm{r}$. The operator $\mathscr{B}$ is a bounded linear transformation mapping $\mathbb{U}$ into $\mathbb{X}$, and $\|\mathscr{B}\| \leq M_{\mathscr{B}}$, where $M_{\mathscr{B}}$ is a constant and $\mathbb{U}$ represents a Hilbert space comprising admissible control functions. The control function, denoted as $u(\cdot)$, is confined to take values within the space $\mathscr{L}_1^2([\mathrm{t}_0, \infty), \mathbb{U})$. $\mathfrak{I} : [\mathrm{t}_0, \infty) \to \mathscr{L}_2^0(\Upsilon, \mathbb{X})$ are suitable functions, $\mathscr{L}_2^0(\Upsilon, \mathbb{X})$ is the space containing all Q-Hilbert-Schmidt $\Upsilon \to \mathbb{X}$, Υ is a real, separable Hilbert space. $\mathrm{B}_Q^H(\mathrm{t})$ is designated as a fBm with $H \in (0, 1/2)$ which is defined on Υ. $\Delta y(\mathrm{t}_\mathrm{r}) = y(\mathrm{t}_\mathrm{r}^+) - y(\mathrm{t}_\mathrm{r}^-)$ quantifies the instantaneous change in the state due to the impulse at time t_r. I_r is positive real number, $\mathrm{r} = 1, 2, \cdots$. Denote $\overline{\gamma} = \max\{\mathrm{t}_\mathrm{r} - \mathrm{t}_{\mathrm{r}-1}\}$ and $\underline{\gamma} = \min\{\mathrm{t}_\mathrm{r} - \mathrm{t}_{\mathrm{r}-1}\}$, ϖ is the constant delay in the impulses satisfying $0 \leq \varpi < \underline{\gamma}$. The set $\mathbb{N}$ is defined to include all positive integers. The sequence of impulse moments is ordered such that $0 \leq \mathrm{t}_0 < \mathrm{t}_1 < \mathrm{t}_2 < \cdots < \mathrm{t}_\mathrm{r} < \cdots$, $\mathrm{r} \in \mathbb{N}$ and $\lim_{\mathrm{r}\to\infty} \mathrm{t}_\mathrm{r} = \infty$. For $\psi \in B_1$, $\|\psi\|_{B_1} = \sup_{\eta\in[-\varsigma,0]} \|\psi(\eta)\| < \infty$. The space $\mathscr{L}^2(\Omega, \mathscr{F}_{\mathrm{t}_{\mathrm{r}+1}}, \mathbb{X})$ denotes a Hilbert space comprising all random variables that are $\mathscr{F}_{\mathrm{t}_{\mathrm{r}+1}}$-measurable, take values in $\mathbb{X}$ and are square-integrable. $\mathscr{L}^2(\Omega, \mathbb{X})$ is the Hilbert space consisting of square-integrable random variables that are measurable of the probability space and take values in $\mathbb{X}$. $\mathscr{L}^2_{\mathscr{F}}((\mathrm{t}_\mathrm{r}, \mathrm{t}_{\mathrm{r}+1}], \mathscr{L}_2^0(\Upsilon, \mathbb{X}))$ is a Hilbert space, of which all square-integrable and $\mathscr{F}_\mathrm{t}$-measurable processes take values in $\mathscr{L}_2^0(\Upsilon, \mathbb{X})$.

$\mathcal{K}_H(\mathrm{t}, s)$ is the kernel operator, and

$$\mathcal{K}_H(\mathrm{t}, s) = \begin{cases} D_H[(\frac{s}{\mathrm{t}(\mathrm{t}-s)})^{1/2-H} + (1/2 - H)s^{1/2-H} \int_s^\mathrm{t} (\upsilon - s)^{H-1/2} \upsilon^{H-3/2} \mathrm{d}\upsilon], & \mathrm{t} > s, \\ 0, & \mathrm{t} \leq s, \end{cases}$$

where $D_H = \sqrt{H/(\mathbb{B}(1-2H, H+1/2)(1-2H))}$, $\mathbb{B}(\cdot,\cdot)$ represents the Beta function. Referring [39], for deterministic function $\mu \in \mathscr{L}^2([t_0,\infty))$, the Wiener integral of μ with regard to $B_Q^H(t)$ is established as

$$\int_{t_0}^{t} \mu(s)dB_Q^H(s) = \int_{t_0}^{t} (\mathcal{K}_{H,t}^{*}\mu)(s)dB(s),$$

where $B(t)$ is a Wiener process and $(\mathcal{K}_{H,t}^{*}\mu)(s) = \mathcal{K}_H(t,s)\mu(s) + \int_s^t (\mu(v) - \mu(s))\frac{\partial \mathcal{K}_H}{\partial v}(v,s)dv$. Denote by $\mathscr{L}(\Upsilon, \mathbb{X})$ the space of all bounded linear operators that map from Υ into $\mathbb{X}$. Define Q in $\mathscr{L}(\Upsilon,\Upsilon)$ as a nonnegative self-adjoint operator with $Qx_m = \lambda_m x_m$, where real number $\lambda_m \geq 0$ $(m = 1, 2, \cdots)$, $\{x_m\}_{m\geq 1}$ is identified as constituting a complete set of orthonormal basis vectors in Υ. The identification of infinite-dimensional fBm is achieved by

$$B_Q^H(t) = \sum_{m=1}^{\infty} \sqrt{\lambda_m} x_m B_m^H(t), t \geq t_0,$$

where $\{B_m^H(t)\}_{m\in\mathbb{N}}$ represents collections with independent fBm. Consider $\mathscr{L}_2^0(\Upsilon,\mathbb{X})$ to be the space of all $\aleph \in \mathscr{L}(\Upsilon,\mathbb{X})$ of the form $\aleph\sqrt{Q}$, where $\aleph\sqrt{Q}$ is characterized as a Hilbert-Schmidt operator, with an associated norm such that

$$\|\aleph\|^2_{\mathscr{L}_2^0(\Upsilon,\mathbb{X})} = \sum_{m=1}^{\infty} \left\| \sqrt{\lambda_m \aleph(s) x_m} \right\|^2_{\mathbb{X}} = \operatorname{tr}(\aleph Q \aleph^*) < \infty.$$

Definition 2.1. *A stochastic process taking values in* $\mathbb{X}$, *denoted by* $\{y(t), t \in [-\varsigma, \infty)\}$, *is referred to as a mild solution of Eq.*(1), *when* $y(t_0 + \eta) = \psi(\eta)$ *on* $[-\varsigma, 0]$ *and the underneath conditions are fulfilled.*

(a) $y(\cdot)$ *exhibits continuity over the interval* $[t_0, t_1]$ *and maintains this continuity across each subsequent interval* $(t_r, t_{r+1}]$, $r = 1, 2, \cdots$,

(b) for arbitrary r, $y(t_r^+)$ *and* $y(t_r^-)$ *are well-defined, and it holds that* $y(t_r^-) = y(t_r)$,

(c) for each $t \geq t_0$,

$$y(t) = \begin{cases} S(t-t_0)\left(\psi(0) - \mathcal{G}(t_0, \psi(-\kappa(t_0)))\right) + \mathcal{G}(t, y(t-\kappa(t))) \\ + \int_{t_0}^{t} S(t-s)[\mathfrak{h}(s, y(s), y(s-\vartheta(s))) + \mathscr{B}u(s)]ds + \int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s), t \in [t_0, t_1], \\ \\ S(t-t_r)\left(y(t_r^-) - \mathcal{G}(t_r, y(t_r - \kappa(t_r))) + I_r y(t_r - \varpi)\right) + \mathcal{G}(t, y(t-\kappa(t))) \\ + \int_{t_r}^{t} S(t-s)[\mathfrak{h}(s, y(s), y(s-\vartheta(s))) + \mathscr{B}u(s)]ds + \int_{t_r}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s), \\ t \in (t_r, t_{r+1}], r = 1, 2, \cdots. \end{cases} \tag{2}$$

Lemma 2.2 ([37]). *Given any* $y_{t_{r+1}} \in \mathscr{L}^2(\Omega, \mathscr{F}_{t_{r+1}}, \mathbb{X})$, *there exists a corresponding* $\psi_r \in \mathscr{L}^2_{\mathscr{F}}\left((t_r, t_{r+1}], \mathscr{L}_2^0(\Upsilon, \mathbb{X})\right)$ *that makes*

$$y_{t_{r+1}} = \mathbb{E}y_{t_{r+1}} + \int_{t_r}^{t_{r+1}} \psi_r(s)dB(s).$$

Similar to [32], the controllability operator is defined by

$$\Xi_{t_r}^{t_{r+1}} = \int_{t_r}^{t_{r+1}} S(t_{r+1} - s)\mathscr{B}^*\mathscr{B}S^*(t_{r+1} - s)ds,$$

where S^* and $\mathscr{B}^*$ individually represent the adjoint of S and $\mathscr{B}$. Evidently, it can be ascertained that $\Xi_{t_r}^{t_{r+1}}$ is a bounded linear operator.

Inspired by Ref.[37], for any $a > 0$ and $y_{t_{r+1}} \in \mathscr{L}^2(\Omega, \mathscr{F}_{t_{r+1}}, \mathbb{X})$, the control function is constructed as follows

$$\begin{aligned}
u^a(t, y) = {} & \mathscr{B}^* S^*(t_{r+1} - t)(a\mathcal{I} + \Xi_{t_r}^{t_{r+1}})^{-1} \left[\mathbb{E}y_{t_{r+1}} - S(t_{r+1} - t_r)\left(y(t_r^-) - \mathcal{G}(t_r, y(t_r - \kappa(t_r)))\right.\right. \\
& \left.\left. + I_r y(t_r - \varpi)\right) - \mathcal{G}(t_{r+1}, y(t_{r+1} - \kappa(t_{r+1})))\right] \\
& - \mathscr{B}^* S^*(t_{r+1} - t) \int_{t_r}^{t_{r+1}} \left(a\mathcal{I} + \Xi_{t_r}^{t_{r+1}}\right)^{-1} S(t_{r+1} - s)\mathfrak{h}(s, y(s), y(s - \vartheta(s)))ds \\
& - \mathscr{B}^* S^*(t_{r+1} - t) \int_{t_r}^{t_{r+1}} \left(a\mathcal{I} + \Xi_{t_r}^{t_{r+1}}\right)^{-1} S(t_{r+1} - s)\mathfrak{I}(s)dB_Q^H(s) \\
& + \mathscr{B}^* S^*(t_{r+1} - t) \int_{t_r}^{t_{r+1}} \left(a\mathcal{I} + \Xi_{t_r}^{t_{r+1}}\right)^{-1} \psi_r(s)dB(s),
\end{aligned}$$

where $y_{t_{r+1}} = \mathbb{E}y_{t_{r+1}} + \int_{t_r}^{t_{r+1}} \psi_r(s)dB(s)$.

3. Pth moment Stability of Mild Solution

To guarantee the pth moment stability of the mild solution, some assumptions are established.

(H1) $\forall\, t \geq t_0$, $S(t)$ is compact. Under this circumstance, there are positive constants λ and $\mathcal{M}$ ensuring that

$$\|S(t)\| \leq \mathcal{M}e^{-\lambda t}.$$

(H2) $\forall\, t \geq t_0$, there exists a nonnegative real number R_1, both $y(t)$ and $y(t - v(t))$ belong to $\mathbb{X}$ such that

$$\|\mathfrak{h}(t, y(t), y(t - \vartheta(t)))\|^p \leq R_1(\|y(t)\|^p + \|y(t - \vartheta(t))\|^p), p \geq 2.$$

(H3) There exist nonnegative real numbers C_1, R_2, $\forall\, t \in [t_0, \infty)$, $y(t - \kappa(t)) \in \mathbb{X}$ such that

$$\|\mathcal{G}(t, y(t - \kappa(t)))\|^p \leq C_1 e^{-\lambda t} + R_2\|y(t - \kappa(t))\|^p, p \geq 2.$$

(H4) The function $\mathfrak{I}$: $[t_0, \infty) \to \mathscr{L}_2^0(\Upsilon, \mathbb{X})$ satisfies

$$\int_{t_0}^{\infty} e^{\lambda s}\|\mathfrak{I}(s)\|^p_{\mathscr{L}_2^0(\Upsilon,\mathbb{X})}ds < \infty, p \geq 2.$$

(H5) The impulses satisfy

$$\lim_{r\to\infty} \Gamma < \infty,$$

where

$$\begin{aligned}
\Gamma = {} & r^{p-1}\Big((C_r^1)^{p-1}\sum_{\ell_1=1}^{r} I_{\ell_1}^p e^{-\lambda p(r-\ell_1)\underline{\gamma}} + (C_r^2)^{p-1}\sum_{\ell_1=1}^{r-1}\sum_{\ell_2>\ell_1}^{r} I_{\ell_1}^p I_{\ell_2}^p e^{-\lambda p[(r-\ell_1)\underline{\gamma}-\varpi]} \\
& + \cdots + (C_r^r)^{p-1}\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_r>\ell_{r-1}}^{r} I_{\ell_1}^p I_{\ell_2}^p\cdots I_{\ell_r}^p\, e^{-\lambda p[(r-\ell_1)\underline{\gamma}-(r-1)\varpi]}\Big),
\end{aligned}$$

with binomial coefficient $C_r^j = \binom{j}{r}$, $j = 1, 2, \cdots, r$.

Remark 3.1. Regarding (H5), two examples are given here. If $I_{\ell_r} = 0$, $r \in \mathbb{N}$, then $\Gamma = 0$. If $I_{\ell_j} = \frac{1}{r^2}$, one gains

$$\begin{aligned}\Gamma &= r^{p-1}\Big((C_r^1)^{p-1}\sum_{\ell_1=1}^{r} I_{\ell_1}^p e^{-\lambda p(r-\ell_1)\underline{\gamma}} + (C_r^2)^{p-1}\sum_{\ell_1=1}^{r-1}\sum_{\ell_2>\ell_1}^{r} I_{\ell_1}^p I_{\ell_2}^p e^{-\lambda p[(r-\ell_1)\underline{\gamma}-\varpi]} \\ &\quad + \cdots + (C_r^r)^{p-1}\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_r>\ell_{r-1}}^{r} I_{\ell_1}^p I_{\ell_2}^p \cdots I_{\ell_r}^p e^{-\lambda p[(r-\ell_1)\underline{\gamma}-(r-1)\varpi]}\Big) \\ &\le r^{p-1}\Big((C_r^1)^{p-1}\sum_{\ell_1=1}^{r}\Big(\frac{1}{r^2}\Big)^p + (C_r^2)^{p-1}\sum_{\ell_1=1}^{r-1}\sum_{\ell_2>\ell_1}^{r}\Big(\frac{1}{r^{2\times 2}}\Big)^p \\ &\quad + \cdots + (C_r^r)^{p-1}\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_r>\ell_{r-1}}^{r}\Big(\frac{1}{r^{2\times r}}\Big)^p\Big).\end{aligned} \tag{3}$$

Simplification of the above equation gives

$$\begin{aligned}\Gamma &\le r^{p-1}\sum_{j=1}^{r}\Big(C_r^j\frac{1}{r^{2\times j}}\Big)^p \\ &\le r^{p-1}\Big(\sum_{j=1}^{r} C_r^j\frac{1}{r^{2\times j}}\Big)^p \\ &= r^{p-1}\Big(\Big(1+\frac{1}{r^2}\Big)^r - 1\Big)^p.\end{aligned} \tag{4}$$

By Taylor's Formula, one gets

$$\lim_{r\to\infty}\Gamma \le \lim_{r\to\infty}\frac{\{[(1+\frac{1}{r^2})^{r^2}]^{\frac{1}{r}} - 1\}^p}{r^{1-p}} = 0.$$

Consequently, $\lim_{r\to\infty}\Gamma = 0$.

Lemma 3.2 ([6]). *For certain positive constants $\tilde{\lambda} > 0$, $\xi_k > 0$ $(k = 1, 2, 3)$, and $\psi : [-\varsigma, \infty) \to [0, \infty)$, it follows that*

$$\psi(t) \le \begin{cases} \xi_1 e^{-\tilde{\lambda} t} + \xi_2 \sup_{\eta\in[-\varsigma,0]}\psi(t+\eta) + \xi_3\int_{t_0}^{t} e^{-\tilde{\lambda}(t-s)}\sup_{\eta\in[-\varsigma,0]}\psi(t+\eta)ds, & t \ge t_0, \\ \xi_1 e^{-\tilde{\lambda} t}, & t \in [-\varsigma, t_0]. \end{cases}$$

If $\xi_2 + \frac{\xi_3}{\tilde{\lambda}} < 1$, then there are positive constants σ and $\overline{\lambda}$ ensuring that

$$\psi(t) \le \sigma e^{-\overline{\lambda} t}, \forall t \ge -\varsigma.$$

Lemma 3.3 ([39]). *Since for arbitrary $t \ge t_0$, $\int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s)$ is a central Gaussian random variable, let C_H be a positive constant which allows*

$$\mathbb{E}\Big\|\int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s)\Big\|^p \le C_H\Big(\mathbb{E}\Big\|\int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s)\Big\|^2\Big)^{p/2}.$$

Lemma 3.4 ([40]). *There exist two positive constants $C_2, C_3 > 0$, depending only on $H \in (0, 1/2)$, $\alpha \in (2, \frac{2}{1-2H})$, $\delta \in (\frac{3}{2} - H - \frac{1}{\alpha}, 1)$, $\omega \in \Big(0, \min\{\frac{1}{\alpha}, \frac{\alpha-2}{2\alpha\delta}\}\Big)$ such that*

$$\mathbb{E}\Big\|\int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s)\Big\|^2 \le \Big(C_2 + C_3(t-t_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\Big)\int_{t_0}^{t} e^{-\lambda(t-s)}\|\mathfrak{I}(s)\|_{\mathscr{L}_2^0}^2 ds.$$

Lemma 3.5 ([38]). *For any stochastic variable χ_{r} such that*

$$\mathbb{E}\left\{\left\|\sum_{\mathrm{r}=1}^{m}\chi_{\mathrm{r}}\right\|^{p}\right\}\leq C_{p}\sum_{\mathrm{r}=1}^{m}\mathbb{E}\{\|\chi_{\mathrm{r}}\|^{p}\},$$

where

$$C_p=\begin{cases} m^{p-1}, & p\geq 1,\\ 1, & 0<p<1.\end{cases}$$

For convenience, one denotes $\mathfrak{h}(\mathrm{t},y(\mathrm{t}),y(\mathrm{t}-\vartheta(\mathrm{t})))=\widetilde{\mathfrak{h}}(\mathrm{t})$, $\mathcal{G}(\mathrm{t},y(\mathrm{t}-\kappa(\mathrm{t})))=\widetilde{\mathcal{G}}(\mathrm{t})$.

Theorem 3.6. *Assume that conditions (H1) to (H5) hold. Further, the mild solution of Eq.*(1) *is pth moment stability if*

$$\begin{aligned} &k^{1-p}R_2+\left(\frac{4}{1-k}\right)^{p-1}\left[N_1+e^{-p\lambda\underline{\gamma}}\mathcal{M}^pN_1\left(1+R_2+I_{\mathrm{r}}^p\right)+N_1R_2+\frac{2R_1\mathcal{M}^p\overline{\gamma}N_1}{(\lambda q)^{p-1}}\right]\\ &+2\left(\frac{4}{1-k}\right)^{p-1}\mathcal{M}^p\lambda^{-p}R_1<1 \end{aligned} \tag{5}$$

holds, where $N_1=\frac{7^{p-1}\mathcal{M}^{2p}\mathcal{M}_{\mathscr{B}}^{2p}}{p\lambda^p q^{p-1}a^p}$, $k\in(0,1)$, $\frac{1}{p}+\frac{1}{q}=1\,(p\geq 2, 1<q\leq 2)$.

Proof. Define the space $\mathscr{F}=\{y(\mathrm{t},\mho):[-\varsigma,\infty)\times\Omega\rightarrow\mathbb{X}$: $y(\mathrm{t})$ satisfies $y(\mathrm{t}_0+\eta)=\psi(\eta)$ for $\eta\in[-\varsigma,0]$, complies with conditions (a)-(b) as per Definition 2.1, and satisfies the criterion that $\lim_{\mathrm{t}\rightarrow\infty}\mathbb{E}\left\|y(\mathrm{t})\right\|^p=0\}$. Next, for all $y(\mathrm{t})\in\mathscr{F}$, $\mathbb{E}\|y(\mathrm{t})\|^p$ as $\mathrm{t}\rightarrow\infty$ will be estimated.

For $\mathrm{t}\in[\mathrm{t}_0,\mathrm{t}_1]$, one has

$$y(\mathrm{t})=S(\mathrm{t}-\mathrm{t}_0)\Big(\psi(0)-\widetilde{\mathcal{G}}(\mathrm{t}_0)\Big)+\widetilde{\mathcal{G}}(\mathrm{t})+\int_{\mathrm{t}_0}^{\mathrm{t}}S(\mathrm{t}-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]\mathrm{d}s+\int_{\mathrm{t}_0}^{\mathrm{t}}S(\mathrm{t}-s)\mathfrak{I}(s)\mathrm{dB}_Q^H(s), \tag{6}$$

denote $Q=\psi(0)-\widetilde{\mathcal{G}}(\mathrm{t}_0)$.

For $\mathrm{t}\in(\mathrm{t}_1,\mathrm{t}_2]$, by Definition 2.1 one obtains

$$\begin{aligned} y(\mathrm{t})=&S(\mathrm{t}-\mathrm{t}_1)\left(y(\mathrm{t}_1^{\,-})-\widetilde{\mathcal{G}}(\mathrm{t}_1)+I_1y(\mathrm{t}_1-\varpi)\right)+\widetilde{\mathcal{G}}(\mathrm{t})+\int_{\mathrm{t}_1}^{\mathrm{t}}S(\mathrm{t}-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]\mathrm{d}s\\ &+\int_{\mathrm{t}_1}^{\mathrm{t}}S(\mathrm{t}-s)\mathfrak{I}(s)\mathrm{dB}_Q^H(s). \end{aligned} \tag{7}$$

By (6), (7) and $y(\mathrm{t}_{\mathrm{r}}^-)=y(\mathrm{t}_{\mathrm{r}})$, for $\mathrm{t}\in(\mathrm{t}_1,\mathrm{t}_2]$,

$$\begin{aligned} y(\mathrm{t})=&S(\mathrm{t}-\mathrm{t}_1)\Big(S(\mathrm{t}_1-\mathrm{t}_0)Q+\widetilde{\mathcal{G}}(\mathrm{t}_1)+\int_{\mathrm{t}_0}^{\mathrm{t}_1}S(\mathrm{t}_1-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]\mathrm{d}s\\ &+\int_{\mathrm{t}_0}^{\mathrm{t}_1}S(\mathrm{t}_1-s)\mathfrak{I}(s)\mathrm{dB}_Q^H(s)\Big)-S(\mathrm{t}-\mathrm{t}_1)\widetilde{\mathcal{G}}(\mathrm{t}_1)+S(\mathrm{t}-\mathrm{t}_1)I_1\Big(S(\mathrm{t}_1-\mathrm{t}_0-\varpi)Q+\widetilde{\mathcal{G}}(\mathrm{t}_1-\varpi)\\ &+\int_{\mathrm{t}_0}^{\mathrm{t}_1-\varpi}S(\mathrm{t}_1-\varpi-s)\mathfrak{I}(s)\mathrm{dB}_Q^H(s)+\int_{\mathrm{t}_0}^{\mathrm{t}_1-\varpi}S(\mathrm{t}_1-\varpi-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]\mathrm{d}s\Big)+\widetilde{\mathcal{G}}(\mathrm{t})\\ &+\int_{\mathrm{t}_1}^{\mathrm{t}}S(\mathrm{t}-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]\mathrm{d}s+\int_{\mathrm{t}_1}^{\mathrm{t}}S(\mathrm{t}-s)\mathfrak{I}(s)\mathrm{dB}_Q^H(s). \end{aligned} \tag{8}$$

Using semigroup theory and properties of integrals, for $t \in (t_1, t_2]$, one gets

$$\begin{aligned} y(t) = {} & S(t-t_0)Q + \int_{t_0}^{t} S(t-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds + \int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s) \\ & + I_1 S(t-\varpi-t_0)Q + I_1 S(t-t_1)\widetilde{\mathcal{G}}(t_1-\varpi) \\ & + I_1 \int_{t_0}^{t_1-\varpi} S(t-\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds \\ & + I_1 \int_{t_0}^{t_1-\varpi} S(t-\varpi-s)\mathfrak{I}(s)dB_Q^H(s) + \widetilde{\mathcal{G}}(t). \end{aligned} \tag{9}$$

For $t \in (t_2, t_3]$, utilizing Definition 2.1, one acquires

$$\begin{aligned} y(t) = {} & S(t-t_2)\Big(y(t_2^-) - \widetilde{\mathcal{G}}(t_2) + I_2 y(t_2-\varpi)\Big) + \widetilde{\mathcal{G}}(t) \\ & + \int_{t_2}^{t} S(t-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds + \int_{t_2}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s). \end{aligned} \tag{10}$$

Combining (9) with (10), for $t \in (t_2, t_3]$, one receives

$$\begin{aligned} y(t) = {} & Q\Big(S(t-t_0) + (I_1+I_2)S(t-t_0-\varpi) + I_1 I_2 S(t-t_0-2\varpi)\Big) + I_1 S(t-t_1)\widetilde{\mathcal{G}}(t_1-\varpi) \\ & + I_2 S(t-t_2)\widetilde{\mathcal{G}}(t_2-\varpi) + I_1 I_2 S(t-t_1-\varpi)\widetilde{\mathcal{G}}(t_1-\varpi) + \widetilde{\mathcal{G}}(t) \\ & + \int_{t_0}^{t} S(t-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds + I_1 \int_{t_0}^{t_1-\varpi} S(t-\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds \\ & + I_2 \int_{t_0}^{t_2-\varpi} S(t-\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds + \int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s) \\ & + I_1 \int_{t_0}^{t_1-\varpi} S(t-\varpi-s)\mathfrak{I}(s)dB_Q^H(s) + I_2 \int_{t_0}^{t_2-\varpi} S(t-\varpi-s)\mathfrak{I}(s)dB_Q^H(s) \\ & + I_1 I_2 \int_{t_0}^{t_1-\varpi} S(t-2\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds + I_1 I_2 \int_{t_0}^{t_1-\varpi} S(t-2\varpi-s)\mathfrak{I}(s)dB_Q^H(s). \end{aligned} \tag{11}$$

Similarly to the calculation method of (10) and (11), for $t \in (t_3, t_4]$, one attains

$$\begin{aligned} y(t) = {} & Q(S(t-t_0) + (I_1+I_2+I_3)S(t-t_0-\varpi) + (I_1I_2 + I_1I_3 + I_2I_3)S(t-t_0-2\varpi) \\ & + I_1I_2I_3 S(t-t_0-3\varpi)) + I_1 S(t-t_1)\widetilde{\mathcal{G}}(t_1-\varpi) + I_2 S(t-t_2)\widetilde{\mathcal{G}}(t_2-\varpi) + I_3 S(t-t_3)\widetilde{\mathcal{G}}(t_3-\varpi) \\ & + I_1I_2 S(t-t_1-\varpi)\widetilde{\mathcal{G}}(t_1-\varpi) + I_1I_3 S(t-t_1-\varpi)\widetilde{\mathcal{G}}(t_1-\varpi) + I_2I_3 S(t-t_2-\varpi)\widetilde{\mathcal{G}}(t_2-\varpi) + \widetilde{\mathcal{G}}(t) \\ & + I_1I_2I_3 S(t-t_1-2\varpi)\widetilde{\mathcal{G}}(t_1-\varpi) + \int_{t_0}^{t} S(t-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds \\ & + I_1 \int_{t_0}^{t_1-\varpi} S(t-\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds + I_2 \int_{t_0}^{t_2-\varpi} S(t-\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds \\ & + I_3 \int_{t_0}^{t_3-\varpi} S(t-\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds + \int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s) \\ & + I_1 \int_{t_0}^{t_1-\varpi} S(t-\varpi-s)\mathfrak{I}(s)dB_Q^H(s) + I_2 \int_{t_0}^{t_2-\varpi} S(t-\varpi-s)\mathfrak{I}(s)dB_Q^H(s) \\ & + I_3 \int_{t_0}^{t_3-\varpi} S(t-\varpi-s)\mathfrak{I}(s)dB_Q^H(s) + I_1I_2 \int_{t_0}^{t_1-\varpi} S(t-2\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(s,y)]ds \end{aligned}$$

$$
\begin{aligned}
&+ I_1 I_3 \int_{t_0}^{t_1-\varpi} S(t-2\varpi-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]ds \\
&+ I_2 I_3 \int_{t_0}^{t_2-\varpi} S(t-2\varpi-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]ds \\
&+ I_1 I_2 I_3 \int_{t_0}^{t_1-\varpi} S(t-3\varpi-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]ds \\
&+ I_1 I_2 \int_{t_0}^{t_1-\varpi} S(t-2\varpi-s)\mathfrak{I}(s)dB_Q^H(s) + I_1 I_3 \int_{t_0}^{t_1-\varpi} S(t-2\varpi-s)\mathfrak{I}(s)dB_Q^H(s) \\
&+ I_2 I_3 \int_{t_0}^{t_2-\varpi} S(t-2\varpi-s)\mathfrak{I}(s)dB_Q^H(s) + I_1 I_2 I_3 \int_{t_0}^{t_1-\varpi} S(t-3\varpi-s)\mathfrak{I}(s)dB_Q^H(s).
\end{aligned}
\tag{12}
$$

$\forall\, \mathfrak{d} \in \mathbb{N}$, $t \in (t_{\mathfrak{d}}, t_{\mathfrak{d}+1}]$, one has

$$
\begin{aligned}
y(t) = {} & \mathcal{Q}\Big(S(t-t_0) + \sum_{\ell_1=1}^{\mathfrak{d}} I_{\ell_1} S(t-t_0-\varpi) + \sum_{\ell_1=1}^{\mathfrak{d}-1}\sum_{\ell_2>\ell_1}^{\mathfrak{d}} I_{\ell_1} I_{\ell_2} S(t-t_0-2\varpi) + \cdots \\
&+ \sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathfrak{d}}>\ell_{\mathfrak{d}-1}}^{\mathfrak{d}} I_{\ell_1} I_{\ell_2}\cdots I_{\ell_{\mathfrak{d}}} S(t-t_0-\mathfrak{d}\varpi)\Big) + \sum_{\ell_1=1}^{\mathfrak{d}} I_{\ell_1} S(t-t_{\ell_1})\widetilde{\mathscr{G}}(t_{\ell_1}-\varpi) \\
&+ \sum_{\ell_1=1}^{\mathfrak{d}-1}\sum_{\ell_2>\ell_1}^{\mathfrak{d}} I_{\ell_1} I_{\ell_2} S(t-t_{\ell_1}-\varpi)\widetilde{\mathscr{G}}(t_{\ell_1}-\varpi) + \cdots \\
&+ \sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathfrak{d}}>\ell_{\mathfrak{d}-1}}^{\mathfrak{d}} I_{\ell_1} I_{\ell_2}\cdots I_{\ell_{\mathfrak{d}}} S(t-t_{\ell_1}-(\mathfrak{d}-1)\varpi)\widetilde{\mathscr{G}}(t_{\ell_1}-\varpi) + \widetilde{\mathscr{G}}(t) \\
&+ \int_{t_0}^{t} S(t-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]ds + \int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s) \\
&+ \sum_{\ell_1=1}^{\mathfrak{d}} I_{\ell_1} \int_{t_0}^{t_{\ell_1}-\varpi} S(t-\varpi-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]ds \\
&+ \sum_{\ell_1=1}^{\mathfrak{d}} I_{\ell_1} \int_{t_0}^{t_{\ell_1}-\varpi} S(t-\varpi-s)\mathfrak{I}(s)dB_Q^H(s) \\
&+ \sum_{\ell_1=1}^{\mathfrak{d}-1}\sum_{\ell_2>\ell_1}^{\mathfrak{d}} I_{\ell_1} I_{\ell_2} \int_{t_0}^{t_{\ell_1}-\varpi} S(t-2\varpi-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]ds \\
&+ \sum_{\ell_1=1}^{\mathfrak{d}-1}\sum_{\ell_2>\ell_1}^{\mathfrak{d}} I_{\ell_1} I_{\ell_2} \int_{t_0}^{t_{\ell_1}-\varpi} S(t-2\varpi-s)\mathfrak{I}(s)dB_Q^H(s) + \cdots \\
&+ \sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathfrak{d}}>\ell_{\mathfrak{d}-1}}^{\mathfrak{d}} I_{\ell_1} I_{\ell_2}\cdots I_{\ell_{\mathfrak{d}}} \int_{t_0}^{t_{\ell_1}-\varpi} S(t-\mathfrak{d}\varpi-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]ds \\
&+ \sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathfrak{d}}>\ell_{\mathfrak{d}-1}}^{\mathfrak{d}} I_{\ell_1} I_{\ell_2}\cdots I_{\ell_{\mathfrak{d}}} \int_{t_0}^{t_{\ell_1}-\varpi} S(t-\mathfrak{d}\varpi-s)\mathfrak{I}(s)dB_Q^H(s).
\end{aligned}
\tag{13}
$$

For $\mathrm{t} \in (\mathrm{t}_{\mathfrak{d}+1}, \mathrm{t}_{\mathfrak{d}+2}]$, by (2) one gains

$$\begin{aligned} y(\mathrm{t}) &= S(\mathrm{t}-\mathrm{t}_{\mathfrak{d}+1})\Big(y(\mathrm{t}_{\mathfrak{d}+1}^{-}) - \widetilde{\mathscr{G}}(\mathrm{t}_{\mathfrak{d}+1}) + I_{\mathfrak{d}+1}y(\mathrm{t}_{\mathfrak{d}+1}-\varpi)\Big) + \widetilde{\mathscr{G}}(\mathrm{t}) \\ &\quad + \int_{\mathrm{t}_{\mathfrak{d}+1}}^{\mathrm{t}} S(\mathrm{t}-s)\Big[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^{a}(s,y)\Big]\mathrm{d}s + \int_{\mathrm{t}_{\mathfrak{d}+1}}^{\mathrm{t}} S(\mathrm{t}-s)\mathfrak{I}(s)\mathrm{dB}_{Q}^{H}(s). \end{aligned} \tag{14}$$

According to (13), (14), using semigroup theory and integral methods, for $\mathrm{t} \in (\mathrm{t}_{\mathfrak{d}+1}, \mathrm{t}_{\mathfrak{d}+2}]$, one obtains

$$\begin{aligned} y(\mathrm{t}) &= \mathcal{Q}\Big(S(\mathrm{t}-\mathrm{t}_0) + \sum_{\ell_1=1}^{\mathfrak{d}+1} I_{\ell_1}S(\mathrm{t}-\mathrm{t}_0-\varpi) + \sum_{\ell_1=1}^{\mathfrak{d}}\sum_{\ell_2>\ell_1}^{\mathfrak{d}+1} I_{\ell_1}I_{\ell_2}S(\mathrm{t}-\mathrm{t}_0-2\varpi) + \cdots \\ &\quad + \sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathfrak{d}+1}>\ell_{\mathfrak{d}}}^{\mathfrak{d}+1} I_{\ell_1}I_{\ell_2}\cdots I_{\ell_{\mathfrak{d}+1}}S(\mathrm{t}-\mathrm{t}_0-(\mathfrak{d}+1)\varpi)\Big) + \sum_{\ell_1=1}^{\mathfrak{d}+1} I_{\ell_1}S(\mathrm{t}-\mathrm{t}_{\ell_1})\widetilde{\mathscr{G}}(\mathrm{t}_{\ell_1}-\varpi) \\ &\quad + \sum_{\ell_1=1}^{\mathfrak{d}}\sum_{\ell_2>\ell_1}^{\mathfrak{d}+1} I_{\ell_1}I_{\ell_2}S(\mathrm{t}-\mathrm{t}_{\ell_1}-\varpi)\widetilde{\mathscr{G}}(\mathrm{t}_{\ell_1}-\varpi) + \cdots \\ &\quad + \sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathfrak{d}+1}>\ell_{\mathfrak{d}}}^{\mathfrak{d}+1} I_{\ell_1}I_{\ell_2}\cdots I_{\ell_{\mathfrak{d}+1}}S(\mathrm{t}-\mathrm{t}_{\ell_1}-\mathfrak{d}\varpi)\widetilde{\mathscr{G}}(\mathrm{t}_{\ell_1}-\varpi) \\ &\quad + \widetilde{\mathscr{G}}(\mathrm{t}) + \int_{\mathrm{t}_0}^{\mathrm{t}} S(\mathrm{t}-s)\mathfrak{I}(s)\mathrm{dB}_{Q}^{H}(s) + \int_{\mathrm{t}_0}^{\mathrm{t}} S(\mathrm{t}-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^{a}(s,y)]\mathrm{d}s \\ &\quad + \sum_{\ell_1=1}^{\mathfrak{d}+1} I_{\ell_1}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^{a}(s,y)]\mathrm{d}s \\ &\quad + \sum_{\ell_1=1}^{\mathfrak{d}+1} I_{\ell_1}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-\varpi-s)\mathfrak{I}(s)\mathrm{dB}_{Q}^{H}(s) \\ &\quad + \sum_{\ell_1=1}^{\mathfrak{d}}\sum_{\ell_2>\ell_1}^{\mathfrak{d}+1} I_{\ell_1}I_{\ell_2}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-2\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^{a}(s,y)]\mathrm{d}s \\ &\quad + \sum_{\ell_1=1}^{\mathfrak{d}}\sum_{\ell_2>\ell_1}^{\mathfrak{d}+1} I_{\ell_1}I_{\ell_2}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-2\varpi-s)\mathfrak{I}(s)\mathrm{dB}_{Q}^{H}(s) + \cdots \\ &\quad + \sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathfrak{d}+1}>\ell_{\mathfrak{d}}}^{\mathfrak{d}+1} I_{\ell_1}I_{\ell_2}\cdots I_{\ell_{\mathfrak{d}+1}}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-(\mathfrak{d}+1)\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^{a}(s,y)]\mathrm{d}s \\ &\quad + \sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathfrak{d}+1}>\ell_{\mathfrak{d}}}^{\mathfrak{d}+1} I_{\ell_1}I_{\ell_2}\cdots I_{\ell_{\mathfrak{d}+1}}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-(\mathfrak{d}+1)\varpi-s)\mathfrak{I}(s)\mathrm{dB}_{Q}^{H}(s). \end{aligned} \tag{15}$$

By mathematical induction, $\forall\ \mathfrak{r} \in \mathbb{N}$, $\mathrm{t} \in (\mathrm{t}_{\mathfrak{r}}, \mathrm{t}_{\mathfrak{r}+1}]$, one gets

$$\begin{aligned} y(\mathrm{t}) &= \mathcal{Q}\Big(S(\mathrm{t}-\mathrm{t}_0) + \sum_{\ell_1=1}^{\mathfrak{r}} I_{\ell_1}S(\mathrm{t}-\mathrm{t}_0-\varpi) + \sum_{\ell_1=1}^{\mathfrak{r}-1}\sum_{\ell_2>\ell_1}^{\mathfrak{r}} I_{\ell_1}I_{\ell_2}S(\mathrm{t}-\mathrm{t}_0-2\varpi) + \cdots \\ &\quad + \sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathfrak{r}}>\ell_{\mathfrak{r}-1}}^{\mathfrak{r}} I_{\ell_1}I_{\ell_2}\cdots I_{\ell_{\mathfrak{r}}}S(\mathrm{t}-\mathrm{t}_0-\mathfrak{r}\varpi)\Big) + \sum_{\ell_1=1}^{\mathfrak{r}} I_{\ell_1}S(\mathrm{t}-\mathrm{t}_{\ell_1})\widetilde{\mathscr{G}}(\mathrm{t}_{\ell_1}-\varpi) \\ &\quad + \sum_{\ell_1=1}^{\mathfrak{r}-1}\sum_{\ell_2>\ell_1}^{\mathfrak{r}} I_{\ell_1}I_{\ell_2}S(\mathrm{t}-\mathrm{t}_{\ell_1}-\varpi)\widetilde{\mathscr{G}}(\mathrm{t}_{\ell_1}-\varpi) + \cdots \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathrm{r}}>\ell_{\mathrm{r}-1}}^{\mathrm{r}} I_{\ell_1}I_{\ell_2}\cdots I_{\ell_{\mathrm{r}}}S(\mathrm{t}-\mathrm{t}_{\ell_1}-(\mathrm{r}-1)\varpi)\widetilde{\mathscr{G}}(\mathrm{t}_{\ell_1}-\varpi)\\
&+\widetilde{\mathscr{G}}(\mathrm{t})+\int_{\mathrm{t}_0}^{\mathrm{t}} S(\mathrm{t}-s)\mathfrak{I}(s)\mathrm{d}\mathrm{B}_Q^H(s)+\int_{\mathrm{t}_0}^{\mathrm{t}} S(\mathrm{t}-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]\mathrm{ds}\\
&+\sum_{\ell_1=1}^{\mathrm{r}} I_{\ell_1}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-\varpi-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]\mathrm{ds}\\
&+\sum_{\ell_1=1}^{\mathrm{r}} I_{\ell_1}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-\varpi-s)\mathfrak{I}(s)\mathrm{d}\mathrm{B}_Q^H(s)\\
&+\sum_{\ell_1=1}^{\mathrm{r}-1}\sum_{\ell_2>\ell_1}^{\mathrm{r}} I_{\ell_1}I_{\ell_2}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-2\varpi-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]\mathrm{ds}\\
&+\sum_{\ell_1=1}^{\mathrm{r}-1}\sum_{\ell_2>\ell_1}^{\mathrm{r}} I_{\ell_1}I_{\ell_2}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-2\varpi-s)\mathfrak{I}(s)\mathrm{d}\mathrm{B}_Q^H(s)+\cdots\\
&+\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathrm{r}}>\ell_{\mathrm{r}-1}}^{\mathrm{r}} I_{\ell_1}I_{\ell_2}\cdots I_{\ell_{\mathrm{r}}}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-\mathrm{r}\varpi-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]\mathrm{ds}\\
&+\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathrm{r}}>\ell_{\mathrm{r}-1}}^{\mathrm{r}} I_{\ell_1}I_{\ell_2}\cdots I_{\ell_{\mathrm{r}}}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-\mathrm{r}\varpi-s)\mathfrak{I}(s)\mathrm{d}\mathrm{B}_Q^H(s)\\
=&\sum_{\ell=1}^{4}\Theta_\ell. \qquad (16)
\end{aligned}
$$

By Lemma 3.5, $\forall\ \mathrm{r}\in\mathbb{N}$, $\mathrm{t}\in(\mathrm{t}_\mathrm{r},\mathrm{t}_{\mathrm{r}+1}]$, one acquires

$$
\mathbb{E}\left\|y(\mathrm{t})\right\|^p=4^{p-1}\sum_{\ell=1}^{4}\mathbb{E}\|\Theta_\ell\|^p. \qquad (17)
$$

By fundamental inequality $|\zeta+v|^p\le\frac{|\zeta|^p}{k^{p-1}}+\frac{|v|^p}{(1-k)^{p-1}}$ $(k\in(0,1))$, one has

$$
\begin{aligned}
\mathbb{E}\|\Theta_1\|^p=&\ \mathbb{E}\left\|QS(\mathrm{t}-\mathrm{t}_0)+\widetilde{\mathscr{G}}(\mathrm{t})+\int_{\mathrm{t}_0}^{\mathrm{t}} S(\mathrm{t}-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(s,y)]\mathrm{ds}+\int_{\mathrm{t}_0}^{\mathrm{t}} S(\mathrm{t}-s)\mathfrak{I}(s)\mathrm{d}\mathrm{B}_Q^H(s)\right\|^p\\
\le&\ k^{1-p}\mathbb{E}\left\|\widetilde{\mathscr{G}}(\mathrm{t})\right\|^p+(\frac{4}{1-k})^{p-1}\Bigg(\|QS(\mathrm{t}-\mathrm{t}_0)\|^p+\mathbb{E}\left\|\int_{\mathrm{t}_0}^{\mathrm{t}} S(\mathrm{t}-s)\widetilde{\mathfrak{h}}(s)\mathrm{ds}\right\|^p\\
&+\mathbb{E}\left\|\int_{\mathrm{t}_0}^{\mathrm{t}} S(\mathrm{t}-s)\mathfrak{I}(s)\mathrm{d}\mathrm{B}_Q^H(s)\right\|^p+\mathbb{E}\left\|\int_{\mathrm{t}_0}^{\mathrm{t}} S(\mathrm{t}-s)\mathscr{B}u^a(s,y)\mathrm{ds}\right\|^p\Bigg). \qquad (18)
\end{aligned}
$$

By (H1), one gets

$$
\|QS(\mathrm{t}-\mathrm{t}_0)\|^p\le Q^p\mathcal{M}^pe^{-\lambda p(\mathrm{t}-\mathrm{t}_0)}=C_4e^{-\lambda\mathrm{t}}. \qquad (19)
$$

By Lemma 3.3 and Lemma 3.4, one secures

$$
\mathbb{E}\left\|\int_{\mathrm{t}_\mathrm{r}}^{\mathrm{t}_{\mathrm{r}+1}} S(\mathrm{t}_{\mathrm{r}+1}-s)\mathfrak{I}(s)\mathrm{d}\mathrm{B}_Q^H(s)\right\|^p
$$

$$\leq C_H\Big(C_2 + C_3(\mathrm{t}_{\mathrm{r}+1} - \mathrm{t}_{\mathrm{r}})^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\Big)^{\frac{p}{2}}\left(\int_{\mathrm{t}_{\mathrm{r}}}^{\mathrm{t}_{\mathrm{r}+1}} e^{-\lambda(\mathrm{t}_{\mathrm{r}+1}-s)}\left\|\mathfrak{I}(s)\right\|^2_{\mathscr{L}_2^0}\mathrm{d}s\right)^{\frac{p}{2}}$$
$$\leq C_H\Big(C_2 + C_3\overline{\gamma}^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\Big)^{\frac{p}{2}} e^{-\lambda \mathrm{t}}\left(\int_{\mathrm{t}_{\mathrm{r}}}^{\mathrm{t}_{\mathrm{r}+1}} e^{\lambda s}\left\|\mathfrak{I}(s)\right\|^2_{\mathscr{L}_2^0}\mathrm{d}s\right)^{\frac{p}{2}}. \tag{20}$$

According to (H4), there is a positive constant C_5 ensuring that

$$\mathbb{E}\left\|\int_{\mathrm{t}_{\mathrm{r}}}^{\mathrm{t}_{\mathrm{r}+1}} S(\mathrm{t}_{\mathrm{r}+1}-s)\mathfrak{I}(s)\mathrm{d}\mathrm{B}^H_Q(s)\right\|^p < C_5 e^{-\lambda \mathrm{t}}. \tag{21}$$

By (H2), (H3), and Hölder inequality, one gains

$$\begin{aligned}
&\mathbb{E}\left\|u^a(\mathrm{t},y)\right\|^p\\
&\leq 7^{p-1}\frac{\mathcal{M}^p_{\mathscr{B}}\mathcal{M}^p}{a^p}e^{-p\lambda(\mathrm{t}_{\mathrm{r}+1}-\mathrm{t})}\mathbb{E}\left\|y_{\mathrm{t}_{\mathrm{r}+1}}\right\|^p\\
&\quad+7^{p-1}\frac{\mathcal{M}^p_{\mathscr{B}}\mathcal{M}^p}{a^p}e^{-p\lambda(\mathrm{t}_{\mathrm{r}+1}-\mathrm{t})}e^{-p\lambda(\mathrm{t}_{\mathrm{r}+1}-\mathrm{t}_{\mathrm{r}})}\mathcal{M}^p\mathbb{E}\left\|y(\mathrm{t}_{\mathrm{r}}^-)\right\|^p\\
&\quad+7^{p-1}\frac{\mathcal{M}^p_{\mathscr{B}}\mathcal{M}^p}{a^p}e^{-p\lambda(\mathrm{t}_{\mathrm{r}+1}-\mathrm{t})}e^{-p\lambda(\mathrm{t}_{\mathrm{r}+1}-\mathrm{t}_{\mathrm{r}})}\mathcal{M}^p\left(C_1e^{-\lambda \mathrm{t}_{\mathrm{r}}}+R_2\mathbb{E}\left\|y(\mathrm{t}_{\mathrm{r}}-\kappa(\mathrm{t}_{\mathrm{r}}))\right\|^p\right)\\
&\quad+7^{p-1}\frac{\mathcal{M}^p_{\mathscr{B}}\mathcal{M}^p}{a^p}e^{-p\lambda(\mathrm{t}_{\mathrm{r}+1}-\mathrm{t})}e^{-p\lambda(\mathrm{t}_{\mathrm{r}+1}-\mathrm{t}_{\mathrm{r}})}\mathcal{M}^pI^p_{\mathrm{r}}\mathbb{E}\left\|y(\mathrm{t}_{\mathrm{r}}-\varpi)\right\|^p\\
&\quad+7^{p-1}\frac{\mathcal{M}^p_{\mathscr{B}}\mathcal{M}^p}{a^p}e^{-p\lambda(\mathrm{t}_{\mathrm{r}+1}-\mathrm{t})}\left(C_1e^{-\lambda \mathrm{t}_{\mathrm{r}+1}}+R_2\mathbb{E}\left\|y(\mathrm{t}_{\mathrm{r}+1}-\kappa(\mathrm{t}_{\mathrm{r}+1}))\right\|^p\right)\\
&\quad+7^{p-1}\frac{\mathcal{M}^p_{\mathscr{B}}\mathcal{M}^p}{a^p}e^{-p\lambda(\mathrm{t}_{\mathrm{r}+1}-\mathrm{t})}\frac{R_1\mathcal{M}^p}{(\lambda q)^{p-1}}\int_{\mathrm{t}_{\mathrm{r}}}^{\mathrm{t}_{\mathrm{r}+1}}\Big(\mathbb{E}\left\|y(s)\right\|^p+\mathbb{E}\left\|y(s-\mathrm{t}(s))\right\|^p\Big)\mathrm{d}s\\
&\quad+7^{p-1}\frac{\mathcal{M}^p_{\mathscr{B}}\mathcal{M}^p}{a^p}e^{-p\lambda(\mathrm{t}_{\mathrm{r}+1}-\mathrm{t})}C_5e^{-\lambda \mathrm{t}}.
\end{aligned}\tag{22}$$

After collating (22),

$$\begin{aligned}
&\mathbb{E}\left\|u^a(\mathrm{t},y)\right\|^p\\
&\leq 7^{p-1}\frac{\mathcal{M}^p_{\mathscr{B}}\mathcal{M}^p}{a^p}e^{-p\lambda(\mathrm{t}_{\mathrm{r}+1}-\mathrm{t})}\Bigg\{\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(\mathrm{t}+\eta)\right\|^p+e^{-p\lambda\underline{\gamma}}\mathcal{M}^p\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(\mathrm{t}+\eta)\right\|^p\\
&\quad+e^{-p\lambda\underline{\gamma}}\mathcal{M}^p\left(C_1e^{-\lambda \mathrm{t}_{\mathrm{r}}}+R_2\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(\mathrm{t}+\eta)\right\|^p\right)+e^{-p\lambda\underline{\gamma}}\mathcal{M}^pI^p_{\mathrm{r}}\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(\mathrm{t}+\eta)\right\|^p\\
&\quad+C_1e^{-\lambda \mathrm{t}_{\mathrm{r}+1}}+R_2\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(\mathrm{t}+\eta)\right\|^p+\frac{2R_1\mathcal{M}^p}{(\lambda q)^{p-1}}\int_{\mathrm{t}_{\mathrm{r}}}^{\mathrm{t}_{\mathrm{r}+1}}\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(s+\eta)\right\|^p\mathrm{d}s+\mathrm{C}_5\mathrm{e}^{-\lambda \mathrm{t}}\Bigg\}:=\hat{\mathcal{M}}.
\end{aligned}\tag{23}$$

Employing the Hölder inequality together with (23), one gets

$$\begin{aligned}
&\mathbb{E}\left\|\int_{t_0}^{t} S(t-s)\mathscr{B}u^a(s,y)ds\right\|^p \\
&\le N_1\Bigg(\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(t+\eta)\right\|^p + e^{-p\lambda\underline{\gamma}}\mathcal{M}^p\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(t+\eta)\right\|^p \\
&\quad + \mathcal{M}^p C_1 e^{-\lambda t} + e^{-p\lambda\underline{\gamma}}\mathcal{M}^p R_2\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(t+\eta)\right\|^p \\
&\quad + e^{-p\lambda\underline{\gamma}}\mathcal{M}^p I_r^p\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(t+\eta)\right\|^p \\
&\quad + R_2\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(t+\eta)\right\|^p + C_1e^{-\lambda t} \\
&\quad + \frac{2R_1\mathcal{M}^p\overline{\gamma}}{(\lambda q)^{p-1}}\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(t+\eta)\right\|^p + C_5e^{-\lambda t}\Bigg),
\end{aligned} \tag{24}$$

where $N_1 = \frac{7^{p-1}\mathcal{M}^{2p}\mathcal{M}_{\mathscr{B}}^{2p}}{p\lambda^p q^{p-1}a^p}$.

Utilizing (H1), one obtains

$$\mathbb{E}\left\|\int_{t_0}^{t} S(t-s)\widetilde{\mathfrak{h}}(s)ds\right\|^p \le \left(\int_{t_0}^{t}\mathcal{M}e^{-\lambda(t-s)}\mathbb{E}\left\|\widetilde{\mathfrak{h}}(s)\right\|ds\right)^p. \tag{25}$$

From the above equation by the Hölder inequality and (H2), one has

$$\begin{aligned}
&\mathbb{E}\left\|\int_{t_0}^{t} S(t-s)\mathscr{B}u^a(s,y)ds\right\|^p \\
&\le \mathcal{M}^p\left(\int_{t_0}^{t} e^{-\lambda(t-s)}ds\right)^{p-1}\left(\int_{t_0}^{t} e^{-\lambda(t-s)}\mathbb{E}\left\|\widetilde{\mathfrak{h}}(s)\right\|^p ds\right) \\
&\le 2\mathcal{M}^p\lambda^{1-p}R_1\int_{t_0}^{t} e^{-\lambda(t-s)}\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(s+\eta)\right\|^p ds.
\end{aligned} \tag{26}$$

By Lemma 3.3 and Lemma 3.4, there exists an arbitrarily small $\varepsilon > 0$ such that $\tilde{\lambda} = \lambda - \varepsilon$, one has

$$\begin{aligned}
&\left\|\int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s)\right\|^p \\
&\le C_H\left(C_2 + C_3(t-t_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\right)^{\frac{p}{2}}\left(\int_{t_0}^{t} e^{-\lambda(t-s)}\left\|\mathfrak{I}(s)\right\|_{\mathscr{L}_2^0}^2 ds\right)^{\frac{p}{2}} \\
&= C_H\left(C_2 + C_3(t-t_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\right)^{\frac{p}{2}} e^{-\varepsilon t}e^{-\tilde{\lambda}t}\left(\int_{t_0}^{t} e^{\lambda s}\left\|\mathfrak{I}(s)\right\|_{\mathscr{L}_2^0}^2 ds\right)^{\frac{p}{2}}.
\end{aligned} \tag{27}$$

By (H4), a particular constant $C_6 > 0$ can be found such that

$$C_H\left(C_2 + C_3(t-t_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\right)^{\frac{p}{2}} e^{-\varepsilon t}\left(\int_{t_0}^{t} e^{\lambda s}\left\|\mathfrak{I}(s)\right\|_{\mathscr{L}_2^0}^2 ds\right)^{\frac{p}{2}} \le C_6. \tag{28}$$

By (27) and (28), one obtains

$$\left\|\int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s)\right\|^p \le C_6 e^{-\tilde{\lambda}t}. \tag{29}$$

Combining (18), (19), (24), (25), (29), and (H3), one can infer that

$$\begin{aligned}
&\mathbb{E}\|\Theta_1\|^p \\
&\leq \left[\left(\frac{4}{1-k}\right)^{p-1}(C_4 + C_6 + C_1\mathcal{M}^p N_1 + N_1C_1 + N_1C_5) + k^{1-p}C_1\right]e^{-\tilde{\lambda}\mathrm{t}} \\
&\quad + \left\{k^{1-p}R_2 + \left(\frac{4}{1-k}\right)^{p-1}\left[N_1 + e^{-p\lambda\underline{\gamma}}\mathcal{M}^p N_1\left(1 + R_2 + I_{\mathrm{r}}^p\right) + N_1R_2 + \frac{2R_1\mathcal{M}^p\overline{\gamma}N_1}{(\lambda q)^{p-1}}\right]\right\}\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(\mathrm{t}+\eta)\right\|^p \quad (30)\\
&\quad + 2\left(\frac{4}{1-k}\right)^{p-1}\mathcal{M}^p\tilde{\lambda}^{1-p}R_1\int_{\mathrm{t}_0}^{\mathrm{t}} e^{-\tilde{\lambda}(\mathrm{t}-s)}\sup_{\eta\in[-\varsigma,0]}\mathbb{E}\left\|y(s+\eta)\right\|^p\mathrm{d}s.
\end{aligned}$$

Hence, applying Lemma 3.2, positive constants $\overline{N}$ and $\overline{\lambda}$ can be determined such that

$$\mathbb{E}\|\Theta_1\|^p \leq \overline{N}e^{-\overline{\lambda}\mathrm{t}}, \forall \mathrm{t} \geq \mathrm{t}_0. \quad (31)$$

By Lemma 3.5, one obtains

$$\begin{aligned}
\|\Theta_2\|^p &= \mathcal{Q}\Bigg\|\sum_{\ell_1=1}^{\mathrm{r}} I_{\ell_1}\, S(\mathrm{t}-\mathrm{t}_0-\varpi) + \sum_{\ell_1=1}^{\mathrm{r}-1}\sum_{\ell_2>\ell_1}^{\mathrm{r}} I_{\ell_1}I_{\ell_2}S(\mathrm{t}-\mathrm{t}_0-2\varpi) + \cdots \\
&\quad + \sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathrm{r}}>\ell_{\mathrm{r}-1}}^{\mathrm{r}} I_{\ell_1}I_{\ell_2}\cdots I_{\ell_{\mathrm{r}}}S(\mathrm{t}-\mathrm{t}_0-\mathrm{r}\varpi)\Bigg\|^p \\
&\leq \mathcal{Q}\mathrm{r}^{p-1}\Bigg[\left(\sum_{\ell_1=1}^{\mathrm{r}} I_{\ell_1}\,\|S(\mathrm{t}-\mathrm{t}_0-\varpi)\|\right)^p + \left(\sum_{\ell_1=1}^{\mathrm{r}-1}\sum_{\ell_2>\ell_1}^{\mathrm{r}} I_{\ell_1}I_{\ell_2}\,\|S(\mathrm{t}-\mathrm{t}_0-2\varpi)\|\right)^p + \cdots \\
&\quad + \left(\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathrm{r}}>\ell_{\mathrm{r}-1}}^{\mathrm{r}} I_{\ell_1}I_{\ell_2}\cdots I_{\ell_{\mathrm{r}}}\left\|S(\mathrm{t}-\mathrm{t}_0-\mathrm{r}\varpi)\right\|\right)^p\Bigg]. \quad (32)
\end{aligned}$$

From (H1), the above equation can be calculated as follows

$$\begin{aligned}
\|\Theta_2\|^p &\leq \mathcal{Q}\mathcal{M}^p\mathrm{r}^{p-1}\Bigg((C_{\mathrm{r}}^1)^{p-1}\sum_{\ell_1=1}^{\mathrm{r}} I_{\ell_1}^p e^{-\lambda p(\mathrm{t}-\mathrm{t}_0-\varpi)} + (C_{\mathrm{r}}^2)^{p-1}\sum_{\ell_1=1}^{\mathrm{r}-1}\sum_{\ell_2>\ell_1}^{\mathrm{r}} I_{\ell_1}^p I_{\ell_2}^p e^{-\lambda p(\mathrm{t}-\mathrm{t}_0-2\varpi)} + \cdots \\
&\quad + (C_{\mathrm{r}}^{\mathrm{r}})^{p-1}\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathrm{r}}>\ell_{\mathrm{r}-1}}^{\mathrm{r}} I_{\ell_1}^p I_{\ell_2}^p\cdots I_{\ell_{\mathrm{r}}}^p e^{-\lambda p(\mathrm{t}-\mathrm{t}_0-\mathrm{r}\varpi)}\Bigg). \quad (33)
\end{aligned}$$

By means of (H5), one gets

$$\lim_{\mathrm{r}\to\infty}\|\Theta_2\|^p = 0.$$

By (H1), Hölder inequality and Lemma 3.5, one derives

$$\begin{aligned}
\mathbb{E}\|\Theta_{31}\|^p &= \mathbb{E}\left\|\sum_{\ell_1=1}^{\mathrm{r}} I_{\ell_1}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(\mathrm{t},y)]\mathrm{d}s\right\|^p \\
&\leq (C_{\mathrm{r}}^1)^{p-1}\sum_{\ell_1=1}^{\mathrm{r}} I_{\ell_1}^p\mathbb{E}\left\|\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-\varpi-s)[\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(\mathrm{t},y)]\mathrm{d}s\right\|^p \\
&\leq (C_{\mathrm{r}}^1)^{p-1}\sum_{\ell_1=1}^{\mathrm{r}} I_{\ell_1}^p\mathcal{M}^p\left(\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} e^{-\lambda(\mathrm{t}-\varpi-s)}\mathrm{d}s\right)^{p-1}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} e^{-\lambda(\mathrm{t}-\varpi-s)}\mathbb{E}\left\|\widetilde{\mathfrak{h}}(s) + \mathscr{B}u^a(\mathrm{t},y)\right\|^p\mathrm{d}s. \quad (34)
\end{aligned}$$

By (H2), (34) and Lemma 3.5, one obtains

$$\begin{aligned}
&\mathbb{E}\|\Theta_{31}\|^p \\
&\le 2^{p-1}(C_{\mathrm{r}}^1)^{p-1}\sum_{\ell_1=1}^{\mathrm{r}} I_{\ell_1}^p \mathcal{M}^p \lambda^{1-p} e^{-\lambda(p-1)(\mathrm{t}-\mathrm{t}_{\ell_1})}(1-e^{-\lambda(\mathrm{t}_{\ell_1}-\mathrm{t}_0-\varpi)})^{p-1} \\
&\quad \times \int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} e^{-\lambda(\mathrm{t}-\varpi-s)}\left[\mathbb{E}\left\|\widetilde{\mathfrak{h}}(s)\right\|^p + \mathcal{M}_{\mathscr{B}}^p \mathbb{E}\left\|u^a(\mathrm{t},y)\right\|^p\right]\mathrm{d}s \\
&\le 2^{p-1}(C_{\mathrm{r}}^1)^{p-1}\sum_{\ell_1=1}^{\mathrm{r}} I_{\ell_1}^p \mathcal{M}^p \lambda^{1-p} e^{-\lambda p(\mathrm{t}-\mathrm{t}_{\ell_1})} \\
&\quad \times \int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} e^{-\lambda(\mathrm{t}_{\ell_1}-\varpi-s)}\left[R_1(\mathbb{E}\left\|y(s)\right\|^p + \mathbb{E}\left\|y(s-\vartheta(s))\right\|^p) + \mathcal{M}_{\mathscr{B}}^p \hat{\mathcal{M}}\right]\mathrm{d}s.
\end{aligned} \tag{35}$$

Doing the same calculation as (34)~(35), one acquires

$$\begin{aligned}
\mathbb{E}\|\Theta_{32}\|^p &= \mathbb{E}\left\|\sum_{\ell_1=1}^{\mathrm{r}-1}\sum_{\ell_2>\ell_1}^{\mathrm{r}} I_{\ell_1} I_{\ell_2}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-2\varpi-s)[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(\mathrm{t},y)]\mathrm{d}s\right\|^p \\
&\le 2^{p-1}(C_{\mathrm{r}}^2)^{p-1}\sum_{\ell_1=1}^{\mathrm{r}-1}\sum_{\ell_2>\ell_1}^{\mathrm{r}} I_{\ell_1}^p I_{\ell_2}^p \mathcal{M}^p \lambda^{1-p} e^{-\lambda p(\mathrm{t}-\mathrm{t}_{\ell_1}-\varpi)} \\
&\quad \times \int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} e^{-\lambda(\mathrm{t}_{\ell_1}-\varpi-s)}\left[R_1\left(\mathbb{E}\left\|y(s)\right\|^p + \mathbb{E}\left\|y(s-\vartheta(s))\right\|^p\right) + \mathcal{M}_{\mathscr{B}}^p \hat{\mathcal{M}}\right]\mathrm{d}s.
\end{aligned} \tag{36}$$

Similar ways as (36), one gains

$$\begin{aligned}
\mathbb{E}\,\|\Theta_{3\mathrm{r}}\|^p &= \mathbb{E}\left\|\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathrm{r}}>\ell_{\mathrm{r}-1}}^{\mathrm{r}} I_{\ell_1} I_{\ell_2}\cdots I_{\ell_{\mathrm{r}}}\int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} S(\mathrm{t}-\mathrm{r}\varpi-s)\left[\widetilde{\mathfrak{h}}(s)+\mathscr{B}u^a(\mathrm{t},y)\right]\mathrm{d}s\right\|^p \\
&\le 2^{p-1}(C_{\mathrm{r}}^{\mathrm{r}})^{p-1}\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathrm{r}}>\ell_{\mathrm{r}-1}}^{\mathrm{r}} I_{\ell_1}^p I_{\ell_2}^p\cdots I_{\ell_{\mathrm{r}}}^p \mathcal{M}^p \lambda^{1-p} e^{-\lambda p(\mathrm{t}-\mathrm{t}_{\ell_1}-(\mathrm{r}-1)\varpi)} \\
&\quad \times \int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} e^{-\lambda(\mathrm{t}_{\ell_1}-\varpi-s)}\left[R_1\left(\mathbb{E}\left\|y(s)\right\|^p + \mathbb{E}\left\|y(s-\vartheta(s))\right\|^p\right) + \mathcal{M}_{\mathscr{B}}^p \hat{\mathcal{M}}\right]\mathrm{d}s.
\end{aligned} \tag{37}$$

By (35)~(37), one gets

$$\begin{aligned}
\mathbb{E}\|\Theta_3\|^p &= \mathbb{E}\|\Theta_{31}+\Theta_{32}+\cdots+\Theta_{3\mathrm{r}}\|^p \\
&\le 2^{p-1}\mathrm{r}^{p-1}\mathcal{M}^p\lambda^{1-p}\Bigg((C_{\mathrm{r}}^1)^{p-1}\sum_{\ell_1=1}^{\mathrm{r}} I_{\ell_1}^p e^{-\lambda p(\mathrm{t}-\mathrm{t}_{\ell_1})} \\
&\quad \times \int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} e^{-\lambda(\mathrm{t}_{\ell_1}-\varpi-s)}\left[R_1\left(\mathbb{E}\left\|y(s)\right\|^p + \mathbb{E}\left\|y(s-\vartheta(s))\right\|^p\right) + \mathcal{M}_{\mathscr{B}}^p \hat{\mathcal{M}}\right]\mathrm{d}s \\
&\quad + (C_{\mathrm{r}}^2)^{p-1}\sum_{\ell_1=1}^{\mathrm{r}-1}\sum_{\ell_2>\ell_1}^{\mathrm{r}} I_{\ell_1}^p I_{\ell_2}^p e^{-\lambda p(\mathrm{t}-\mathrm{t}_{\ell_1}-\varpi)} \\
&\quad \times \int_{\mathrm{t}_0}^{\mathrm{t}_{\ell_1}-\varpi} e^{-\lambda(\mathrm{t}_{\ell_1}-\varpi-s)}\left[R_1\left(\mathbb{E}\left\|y(s)\right\|^p + \mathbb{E}\left\|y(s-\vartheta(s))\right\|^p\right) + \mathcal{M}_{\mathscr{B}}^p \hat{\mathcal{M}}\right]\mathrm{d}s \\
&\quad + \cdots + (C_{\mathrm{r}}^{\mathrm{r}})^{p-1}\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathrm{r}}>\ell_{\mathrm{r}-1}}^{\mathrm{r}} I_{\ell_1}^p I_{\ell_2}^p\cdots I_{\ell_{\mathrm{r}}}^p e^{-\lambda p(\mathrm{t}-\mathrm{t}_{\ell_1}-(\mathrm{r}-1)\varpi)}
\end{aligned}$$

$$\times \int_{t_0}^{t_{\ell_1}-\varpi} e^{-\lambda(t_{\ell_1}-\varpi-s)} \left[R_1\left(\mathbb{E}\left\|y(s)\right\|^p + \mathbb{E}\left\|y(s-\vartheta(s))\right\|^p\right) + \mathcal{M}_{\mathscr{B}}^p \hat{\mathcal{M}}\right] ds\Bigg). \tag{38}$$

By (H5), one concludes that

$$\lim_{r\to\infty} \mathbb{E}\|\Theta_3\|^p = 0.$$

According to Lemma 3.5, one has

$$\begin{aligned}
\mathbb{E}\|\Theta_{41}\|^p &= \mathbb{E}\left\|\sum_{\ell_1=1}^{r} I_{\ell_1} \int_{t_0}^{t_{\ell_1}-\varpi} S(t-\varpi-s)\mathfrak{I}(s)dB_Q^H(s)\right\|^p \\
&\le \left(C_r^1\right)^{p-1} \sum_{\ell_1=1}^{r} I_{\ell_1}^p \mathbb{E}\left\|\int_{t_0}^{t_{\ell_1}-\varpi} S(t-\varpi-s)\mathfrak{I}(s)dB_Q^H(s)\right\|^p.
\end{aligned} \tag{39}$$

Based on Lemma 3.3 and Lemma 3.4, the aforementioned equation can be computed as follows

$$\begin{aligned}
\mathbb{E}\|\Theta_{41}\|^p \le{}& \left(C_r^1\right)^{p-1} C_H \sum_{\ell_1=1}^{r} I_{\ell_1}^p \left(C_2 + C_3(t_{\ell_1}-\varpi-t_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\right)^{\frac{p}{2}} \\
&\times \left(\int_{t_0}^{t_{\ell_1}-\varpi} e^{-\lambda(t-\varpi-s)} \left\|\mathfrak{I}(s)\right\|_{\mathscr{L}_2^0}^2 ds\right)^{\frac{p}{2}}.
\end{aligned} \tag{40}$$

In conformity with (H1) and the Hölder inequality, the preceding equation can be calculated as follows

$$\begin{aligned}
\mathbb{E}\|\Theta_{41}\|^p \le{}& \left(C_r^1\right)^{p-1} C_H \sum_{\ell_1=1}^{r} I_{\ell_1}^p \left(C_2 + C_3(t_{\ell_1}-\varpi-t_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\right)^{\frac{p}{2}} \\
&\times \left(\int_{t_0}^{t_{\ell_1}-\varpi} e^{-\lambda(t-\varpi-s)} ds\right)^{\frac{p}{2}-1} \left(\int_{t_0}^{t_{\ell_1}-\varpi} e^{-\lambda(t-\varpi-s)} \left\|\mathfrak{I}(s)\right\|_{\mathscr{L}_2^0}^p ds\right).
\end{aligned} \tag{41}$$

After simplifying, one gains

$$\begin{aligned}
&\mathbb{E}\|\Theta_{41}\|^p \\
&\le \left(C_r^1\right)^{p-1} C_H \sum_{\ell_1=1}^{r} I_{\ell_1}^p \left(C_2 + C_3(t_{\ell_1}-\varpi-t_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\right)^{\frac{p}{2}} \lambda^{1-\frac{p}{2}} e^{-(\frac{p}{2}-1)\lambda(t-t_{\ell_1})} \\
&\quad \times \left(1-e^{-\lambda(t_{\ell_1}-t_0-\varpi)}\right)^{\frac{p}{2}-1} \left(\int_{t_0}^{t} e^{-\lambda(t-\varpi-s)} \left\|\mathfrak{I}(s)\right\|_{\mathscr{L}_2^0}^p ds\right) \\
&\le \left(C_r^1\right)^{p-1} C_H \sum_{\ell_1=1}^{r} I_{\ell_1}^p \left(C_2 + C_3(t_{\ell_1}-\varpi-t_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\right)^{\frac{p}{2}} e^{-\lambda(t_{\ell_1}-t_0-\varpi)} \\
&\quad \times \lambda^{1-\frac{p}{2}} e^{-\frac{p}{2}\lambda(t-t_{\ell_1})} \int_{t_0}^{t} e^{\lambda s} \left\|\mathfrak{I}(s)\right\|_{\mathscr{L}_2^0}^p ds.
\end{aligned} \tag{42}$$

Utilizing the same calculation methods as (41) and (42), one gets

$$\begin{aligned}
&\mathbb{E}\|\Theta_{42}\|^p \\
&\le \left(C_r^2\right)^{p-1} C_H \sum_{\ell_1=1}^{r-1} \sum_{\ell_2>\ell_1}^{r} I_{\ell_1}^p I_{\ell_2}^p \left(C_2 + C_3(t_{\ell_1}-\varpi-t_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\right)^{\frac{p}{2}} e^{-\lambda(t_{\ell_1}-t_0-\varpi)}
\end{aligned}$$

$$\times \lambda^{1-\frac{p}{2}} e^{-\frac{p}{2}\lambda(\mathrm{t}-\mathrm{t}_{\ell_1}-\varpi)} \int_{\mathrm{t}_0}^{\mathrm{t}} e^{\lambda s} \left\|\Im(s)\right\|^p_{\mathscr{L}_2^0} \mathrm{ds}. \tag{43}$$

Using the same calculation approach as (41) and (42), one receives

$$\begin{aligned}
&\mathbb{E}\|\Theta_{4\mathrm{r}}\|^p \\
&\le C_H \sum_{\ell_1=1}^{1} \sum_{\ell_2>\ell_1}^{2} \cdots \sum_{\ell_\mathrm{r}>\ell_{\mathrm{r}-1}}^{\mathrm{r}} I^p_{\ell_1} I^p_{\ell_2} \cdots I^p_{\ell_\mathrm{r}} \Big(C_2 + C_3(\mathrm{t}_{\ell_1} - \varpi - \mathrm{t}_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\Big)^{\frac{p}{2}} \\
&\quad \times e^{-\lambda(\mathrm{t}_{\ell_1}-\mathrm{t}_0-\varpi)} \lambda^{1-\frac{p}{2}} e^{-\frac{p}{2}\lambda(\mathrm{t}-\mathrm{t}_{\ell_1}-(\mathrm{r}-1)\varpi)} \int_{\mathrm{t}_0}^{\mathrm{t}} e^{\lambda s} \left\|\Im(s)\right\|^p_{\mathscr{L}_2^0} \mathrm{ds}.
\end{aligned} \tag{44}$$

By (41) ~ (44), one obtains that

$$\begin{aligned}
\mathbb{E}\|\Theta_4\|^p &= \mathbb{E}\|\Theta_{41} + \Theta_{42} + \cdots + \Theta_{4\mathrm{r}}\|^p \\
&\le \mathrm{r}^{p-1} C_H \lambda^{1-\frac{p}{2}} \int_{\mathrm{t}_0}^{\mathrm{t}} e^{\lambda s} \|\sigma(s)\|^p_{\mathscr{L}_2^0} \mathrm{ds} \Bigg[\left(C^1_\mathrm{r}\right)^{p-1} \sum_{\ell_1=1}^{\mathrm{r}} I^p_{\ell_1} e^{-\frac{p}{2}\lambda(\mathrm{t}-\mathrm{t}_{\ell_1})} \\
&\quad \times \Big(C_2 + C_3(\mathrm{t}_{\ell_1} - \varpi - \mathrm{t}_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\Big)^{\frac{p}{2}} e^{-\lambda(\mathrm{t}_{\ell_1}-\mathrm{t}_0-\varpi)} \\
&\quad + \left(C^2_\mathrm{r}\right)^{p-1} \sum_{\ell_1=1}^{\mathrm{r}-1} \sum_{\ell_2>\ell_1}^{\mathrm{r}} I^p_{\ell_1} I^p_{\ell_2} e^{-\frac{p}{2}\lambda(\mathrm{t}-\mathrm{t}_{\ell_1}-\varpi)} \\
&\quad \times \Big(C_2 + C_3(\mathrm{t}_{\ell_1} - \varpi - \mathrm{t}_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\Big)^{\frac{p}{2}} e^{-\lambda(\mathrm{t}_{\ell_1}-\mathrm{t}_0-\varpi)} \\
&\quad + \cdots + \sum_{\ell_1=1}^{1} \sum_{\ell_2>\ell_1}^{2} \cdots \sum_{\ell_\mathrm{r}>\ell_{\mathrm{r}-1}}^{\mathrm{r}} I^p_{\ell_1} I^p_{\ell_2} \cdots I^p_{\ell_\mathrm{r}} e^{-\frac{p}{2}\lambda(\mathrm{t}-\mathrm{t}_{\ell_1}-(\mathrm{r}-1)\varpi)} \\
&\quad \times \Big(C_2 + C_3(\mathrm{t}_{\ell_1} - \varpi - \mathrm{t}_0)^{2H+2\delta-2+\frac{2}{\alpha}-2\omega\delta}\Big)^{\frac{p}{2}} e^{-\lambda(\mathrm{t}_{\ell_1}-\mathrm{t}_0-\varpi)} \Bigg].
\end{aligned} \tag{45}$$

Thus, by (H5), it follows that

$$\lim_{\mathrm{r}\to\infty} \mathbb{E}\|\Theta_4\|^p = 0.$$

Finally, for all $y(\mathrm{t}) \in \mathscr{F}$, $\lim_{\mathrm{t}\to\infty} \mathbb{E}\left\|y(\mathrm{t})\right\|^p = 0$. The Theorem 3.6 is confirmed.

Remark 3.7. *In [6, 8, 9, 14], the stability of NDEs driven by fBm with $H \in (1/2, 1)$ was examined, however, the impulses considered were transient impulses rather than delayed impulses. In [22, 23], the stability of delayed impulsive differential equations driven by fBm with $H \in (1/2, 1)$ was investigated, however, the equations were non-neutral type.*

Remark 3.8. *There are little literatures on stability of NDEs with time-vary delays and delayed impulses driven by fBm ($H \in (0, 1/2)$). When delayed impulses items $I_\mathrm{r} y(\mathrm{t}_\mathrm{r} - \varpi) = 0$, $\mathrm{r} \in \mathbb{N}$, then the Eq.(1) reduces to NDEs with time-varying delays and fBm ($H \in (0, 1/2)$), which was investigated in [41]. When $\mathcal{G}(\mathrm{t}, y(\mathrm{t} - \kappa(\mathrm{t}))) = 0$, then the Eq.(1) reduces to differential equation with time-vary delays driven by fBm ($H \in (0, 1/2)$), which was studied in [36, 37], in [36], impulses were not considered and the impulses in [37] were not delayed impulses.*

4. Approximate Controllability of Mild Solution

In this segment, the time interval of Eq.(1) is finite. Then, Eq.(1) reduces to the following form,

$$\begin{cases} d[y(t) - \mathcal{G}(t, y(t-\kappa(t)))] = [\mathscr{A}y(t) + \mathfrak{h}(t, y(t), y(t-\vartheta(t))) + \mathscr{B}u(t)]dt \\ \qquad\qquad + \mathfrak{I}(t)dB_Q^H(t), t \in [0, \mathcal{T}], t \neq t_r, \\ \Delta y(t_r) = y(t_r^+) - y(t_r^-) = I_r y(t_r - \varpi), t = t_r, r = 1, 2, \cdots, m, \\ y(t_0) = \psi(0), \end{cases} \tag{46}$$

where $u(\cdot)$ takes values in $\mathscr{L}^2_{\mathscr{F}}([0, \mathcal{T}], \mathbb{U})$, and $0 \le t_i < t_j \le \mathcal{T}$ $(0 \le i < j \le m)$.

The state value of Eq.(46) at the terminal moment $\mathcal{T}$, denoted $y(\mathcal{T}; y(t_0), u)$, is determined by the initial value $y(t_0)$ and the control input u. Let $\mathcal{R}(\mathcal{T}, y(t_0)) = \{y(\mathcal{T}; y(t_0), u) : u \in \mathscr{L}^2([0, \mathcal{T}], \mathbb{U})\}$ denote the reachable set of states by Eq.(46) at the terminal moment $\mathcal{T}$. The closure of this set is represented by $\overline{\mathcal{R}(\mathcal{T}, y(t_0))}$.

Definition 4.1 ([37]). *The Eq.*(46) *is deemed to exhibit approximate controllability within the interval [0,$\mathcal{T}$] when* $\overline{\mathcal{R}(\mathcal{T}, y(t_0))} = \mathscr{L}^2(\Omega, \mathbb{X})$.

Lemma 4.2 ([30]). *Approximate controllability for the deterministic equation given of Eq.*(46) *on [0,$\mathcal{T}$] is achieved if and only if* $a(a\mathcal{I} + \Xi_{t_r}^{t_{r+1}})^{-1} \to 0$ *when* $a \to 0$.

Theorem 4.3. *Assume that there is a mild solution of Eq.*(46) *on [0,$\mathcal{T}$], and the function* $\mathfrak{h}$ *is uniformly bounded, then Eq.*(46) *can be characterized as approximately controllable on [0,$\mathcal{T}$].*

Proof. For $\forall\, q > 0$, set $D_q = \{y \in B_1, \|y\|_{B_1} \le q\} \subseteq B_1$. Consider the following operator Φ on D_q of the form

$$(\Phi y)(t) = \begin{cases} S(t-t_0)\Big(\psi(0) - \mathcal{G}(t_0, \psi(-\kappa(t_0)))\Big) + \mathcal{G}(t, y(t-\kappa(t))) \\ + \int_{t_0}^{t} S(t-s)[\mathfrak{h}(s, y(s), y(s-\vartheta(s))) + \mathscr{B}u(s)]ds + \int_{t_0}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s), t \in [t_0, t_1], \\ \\ S(t-t_r)\Big(y(t_r^-) - \mathcal{G}(t_r, y(t_r - \kappa(t_r))) + I_r y(t_r - \varpi)\Big) + \int_{t_r}^{t} S(t-s)[\mathfrak{h}(s, y(s), y(s-\vartheta(s))) \\ + \mathcal{G}(t, y(t-\kappa(t))) + \mathscr{B}u(s)]ds + \int_{t_r}^{t} S(t-s)\mathfrak{I}(s)dB_Q^H(s), t \in (t_r, t_{r+1}], r = 1, 2, ..., m. \end{cases} \tag{47}$$

Assume that y^a is a fixed point of Φ. Utilizing the stochastic Fubini theorem, one can observe

$$\begin{aligned} y^a(t_{r+1}) = {} & y_{t_{r+1}} - a(a\mathcal{I} + \Xi_{t_r}^{t_{r+1}})^{-1}[\mathbb{E}y_{t_{r+1}} - S(t_{r+1} - t_r)(y(t_r^-) - \mathcal{G}(t_r, y^a(t_r - \kappa(t_r))) \\ & + I_r y^a(t_r - \varpi)) - \mathcal{G}(t_{r+1}, y^a(t_{r+1} - \kappa(t_{r+1})))] \\ & + \int_{t_r}^{t_{r+1}} a(a\mathcal{I} + \Xi_{t_r}^{t_{r+1}})^{-1} S(t_{r+1} - s)\mathfrak{h}(s, y^a(s), y^a(s - \vartheta(s)))ds \\ & + \int_{t_r}^{t_{r+1}} a(a\mathcal{I} + \Xi_{t_r}^{t_{r+1}})^{-1} S(t_{r+1} - s)\mathfrak{I}(s)dB_Q^H(s) \\ & - \int_{t_r}^{t_{r+1}} a(a\mathcal{I} + \Xi_{t_r}^{t_{r+1}})^{-1} \psi_r dB(s). \end{aligned} \tag{48}$$

By (H2), a particular constant $G > 0$ can be found such that

$$\left\|\mathfrak{h}(t, y^a(t), y^a(t - \vartheta(t)))\right\|^2 \le G,$$

so $\{\mathfrak{h}(s, y^a(s), y^a(s - \vartheta(s)))\}$ weakly converges to $\{\mathfrak{h}(s)\}$ in $\mathbb{X}$. From (48), one has

$$\mathbb{E}\left\|y^a(t_{r+1}) - y_{t_{r+1}}\right\|^2$$

$$
\begin{aligned}
&\leq 9\mathbb{E}\left\|a(a\mathcal{I}+\Xi_{t_r}^{t_{r+1}})^{-1}\mathbb{E}\|y_{t_{r+1}}\|\right\|^2\\
&+9\mathbb{E}\left\|a(a\mathcal{I}+\Xi_{t_r}^{t_{r+1}})^{-1}S(t_{r+1}-t_r)\mathbb{E}\|y(t_r)\|\right\|^2\\
&+9\mathbb{E}\left\|a(a\mathcal{I}+\Xi_{t_r}^{t_{r+1}})^{-1}S(t_{r+1}-t_r)\mathcal{G}(t_r,y^a(t_r-\kappa(t_r)))\right\|^2\\
&+9\mathbb{E}\left\|a(a\mathcal{I}+\Xi_{t_r}^{t_{r+1}})^{-1}S(t_{r+1}-t_r)I_r y^a(t_r-\varpi)\right\|^2\\
&+9\mathbb{E}\left\|a(a\mathcal{I}+\Xi_{t_r}^{t_{r+1}})^{-1}\mathcal{G}(t_{r+1},y^a(t_{r+1}-\kappa(t_{r+1})))\right\|^2\\
&+9\mathbb{E}\left\|\int_{t_r}^{t_{r+1}}a(a\mathcal{I}+\Xi_{t_r}^{t_{r+1}})^{-1}S(t_{r+1}-s)\left[\mathfrak{h}(s,y^a(s),y^a(s-\vartheta(s)))-\mathfrak{h}(s)\right]ds\right\|^2\\
&+9\mathbb{E}\left\|\int_{t_r}^{t_{r+1}}a(a\mathcal{I}+\Xi_{t_r}^{t_{r+1}})^{-1}S(t_{r+1}-s)\mathfrak{h}(s)ds\right\|^2\\
&+9\mathbb{E}\left\|\int_{t_r}^{t_{r+1}}a(a\mathcal{I}+\Xi_{t_r}^{t_{r+1}})^{-1}S(t_{r+1}-s)\mathfrak{I}(s)dB_Q^H(s)\right\|^2\\
&+9\mathbb{E}\left\|\int_{t_r}^{t_{r+1}}a(a\mathcal{I}+\Xi_{t_r}^{t_{r+1}})^{-1}\psi_r dB(s)\right\|^2.
\end{aligned}
\tag{49}
$$

By Lemma 4.2, when $a \to 0^+$, then $a(a\mathcal{I}+\Xi_{t_r}^{t_{r+1}})^{-1} \to 0$. Besides, $\left\|a(a\mathcal{I}+\Xi_{t_r}^{t_{r+1}})^{-1}\right\| \leq 1$, by (29) and the Lebesgue-dominated convergence theorem, one deduces that

$$
\mathbb{E}\left\|y^a(t_{r+1})-y_{t_{r+1}}\right\|^2 \to 0,
\tag{50}
$$

when $a \to 0$. Based on Definition 4.1, this demonstration confirms the approximate controllability of the Eq.(46) on $[0,\mathcal{T}]$.

Remark 4.4. *The approximate controllability of impulsive differential equations driven by fBm was studied in [30, 32, 37] , however, the impulses considered in [30, 32, 37] were not delayed impulses and the equations were not neutral type. In [29–32], the approximate controllability of differential equations driven by fBm was studied, but the Hurst parameter considered takes values range in (1/2,1) instead of (0,1/2).*

5. Example

Consider NDEs driven by fBm ($H \in (0, \frac{1}{2})$) with delayed impulses as follows

$$
\begin{cases}
d\left[z(t,\zeta)-\rho_2(t)\frac{z\left(t-\frac{\varsigma}{2}(1+\sin t)\right)}{1+\left[z\left(t-\frac{\varsigma}{2}(1+\sin t)\right)\right]^2}\right]=\left[\frac{\partial^2}{\partial\zeta^2}z(t,\zeta)+\rho_1(t)\frac{z(t)}{1+[z(t)]^2}\frac{z\left(t-\frac{\varsigma}{2}(1+\cos t)\right)}{1+\left[z\left(t-\frac{\varsigma}{2}(1+\cos t)\right)\right]^2}\right]dt\\
\qquad\qquad +\mathscr{B}u(t)(\zeta)dt+e^{-\lambda t}dB_Q^H(t), t \geq t_0, t \neq t_r, 0 \leq \zeta \leq \pi,\\
\Delta z(t_r,\zeta)=\frac{\overline{\rho_3}}{r^2}z(t_r-\varpi), r \in \mathbb{N},\\
z(t,0)=z(t,\pi)=0, t \in [-\varsigma,\infty),\\
z(t_0+\eta,\zeta)=\psi(\eta,\zeta), \eta \in [-\varsigma,0],
\end{cases}
\tag{51}
$$

where $\rho_1(t),\rho_2(t) : [t_0,\infty) \to R^+$ are bounded continuous functions, and $\sup_{t\in[t_0,\infty)} \rho_1(t) = \overline{\rho_1} > 0$, $\sup_{t\in[t_0,\infty)} \rho_2(t) = \overline{\rho_2} > 0$. Let constant $\overline{\rho_3} > 0$, and $\mathbb{X} = \mathbb{Y} = \mathscr{L}^2[0,\pi]$. Let $\mathscr{A} = \frac{\partial^2}{\partial\zeta^2}$ denote the infinitesimal generator with domain $D(\mathscr{A}) := \mathbb{H}_0^1(0,\pi) \cap \mathbb{H}^2(0,\pi)$. $e_k(\zeta) = \sqrt{\frac{2}{\pi}}\sin(k\zeta), k \in \mathbb{N}$ are complete orthogonal set, which are eigenvectors of $\mathscr{A}$. Afterward,

$$\mathscr{A}x = -\sum_{k=1}^{\infty} k^2 \langle x, e_k\rangle e_k, x \in \mathrm{D}(\mathscr{A}),$$

and

$$S(\mathrm{t})x = \sum_{k=1}^{\infty} e^{-k^2\mathrm{t}} \langle x, e_k\rangle e_k, x \in \mathbb{X}, \mathrm{t} \geq \mathrm{t}_0.$$

One defines the operator $\mathscr{B}$ from

$$\mathbb{U} = \left\{ u = \sum_{k=2}^{\infty} u_k e_k : \|u\|_{\mathbb{U}}^2 := \sum_{k=2}^{\infty} u_k^2 < \infty \right\}$$

to $\mathbb{X}$:

$$\mathscr{B}u = 2u_2 e_1 + \sum_{k=2}^{\infty} u_k e_k.$$

Furthermore, one gets $\|S(\mathrm{t})\| \leq e^{-\pi^2\mathrm{t}}(\mathrm{t} \geq \mathrm{t}_0)$, when choosing $\lambda = \pi^2$, $\mathcal{M} = 1$ in (H1). Since

$$\mathcal{G}(\mathrm{t}, \mathcal{Z}_2) = \rho_2(\mathrm{t})\frac{\mathcal{Z}_2}{1+\mathcal{Z}_2^2}, \qquad \mathfrak{h}(\mathrm{t}, \mathcal{W}, Z_1) = \rho_1(\mathrm{t})\frac{\mathcal{W}}{1+\mathcal{W}^2}\frac{\mathcal{Z}_1}{1+\mathcal{Z}_1^2},$$

one chooses $R_1 = \overline{\rho_1}$ in (H2), $R_2 = \overline{\rho_2}$ in (H3) so that (H2) and (H3) hold. Choosing $\mathfrak{I}(\mathrm{t}) = e^{-\lambda\mathrm{t}}$, (H4) holds. Further,

$$\begin{aligned}\Gamma \leq \mathrm{r}\Bigg(&(C_{\mathrm{r}}^1)\sum_{\ell_1=1}^{\mathrm{r}}\left(\frac{c}{\mathrm{r}^2}\right)^2 + (C_{\mathrm{r}}^2)\sum_{\ell_1=1}^{\mathrm{r}-1}\sum_{\ell_2>\ell_1}^{\mathrm{r}}\left(\frac{c^2}{\mathrm{r}^{2\times 2}}\right)^2 + \cdots \\ &+ (C_{\mathrm{r}}^{\mathrm{r}})\sum_{\ell_1=1}^{1}\sum_{\ell_2>\ell_1}^{2}\cdots\sum_{\ell_{\mathrm{r}}>\ell_{\mathrm{r}-1}}^{\mathrm{r}}\left(\frac{c^{\mathrm{r}}}{\mathrm{r}^{2\times\mathrm{r}}}\right)^2\Bigg).\end{aligned} \tag{52}$$

By the Binomial Theorem, one gets

$$\Gamma \leq \mathrm{r}\sum_{j=1}^{\mathrm{r}}\left(C_{\mathrm{r}}^j\frac{c^j}{\mathrm{r}^{2j}}\right)^2 \leq \mathrm{r}\left(\sum_{j=1}^{\mathrm{r}}C_{\mathrm{r}}^j\frac{c^j}{\mathrm{r}^{2j}}\right)^2 = \mathrm{r}\left(\left(1+\frac{c}{\mathrm{r}^2}\right)^{\mathrm{r}} - 1\right)^2.$$

Thus, $\lim_{\mathrm{r}\to\infty}\Gamma = 0$, that is to say (H5) holds. Choose $\mathcal{M}_{\mathscr{B}} = 1, a = 1, \Delta\mathrm{t}_{\mathrm{r}} = 1, \varpi = 0.1, k = 1/2, p = 2, q = 2$, and $\overline{\gamma} = \underline{\gamma} = 1$. Further, the pth moment stability of the mild solution for Eq.(1) is affirmed if the subsequent inequality holds

$$2\overline{\rho_2} + \frac{14}{\pi^4} + \frac{14}{\pi^4 e^{2\pi^2}}(1 + \overline{\rho_2} + \frac{\overline{\rho_3}^2}{\mathrm{r}^4}) + \frac{16\overline{\rho_1} + 14\overline{\rho_2}}{\pi^4} + \frac{14\overline{\rho_1}}{\pi^6} < 1.$$

6. Conclusion

Pth moment stability and approximate controllability of NDEs with delayed impulses and time-varying delays driven by fBm with $H \in (0, 1/2)$ have been investigated. By means of semigroup theory and impulsive integral inequality techniques, sufficient conditions have been secured to ensure pth moment stability of the mild solution. Additionally, the result of approximate controllability of the system has been obtained on the basis of Lebesgue dominated convergence theorem. The validity of the result has been verified by an example.

Afterward, stability of NDEs incorporating mixed delays (discrete delays and distributed delays) and delayed impulses driven by fBm with $H \in (0, 1/2)$ will be focused. On the other hand, stability of neutral delayed impulsive differential equations with Markovian jumps driven by fBm with $H \in (0, 1/2)$ will also be investigated.

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