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On n-Hom-pre-Lie color algebras and associated Kupershmidt operators

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Abstract. In this paper, we introduce the Hom-color generalization of n-pre-Lie algebras called n-Hom-pre-Lie color algebras in which, we investigate their representation theory and we give some related results and structures based on Rota-Baxter operators, Kupershmidt operators and Nijenhuis operators. Moreover, we show that given $(n-2)$-linear form of a Hom-pre-Lie color algebra satisfying certain conditions, one can construct a structure of n-Hom-pre-Lie color algebra called induced n-Hom-pre-Lie color algebra. The same procedure is applied for the representation of n-Hom-pre-Lie color algebras.

1. Introduction

The notion of n-Lie algebras was introduced in 1985 by Filippov [31], so it was given a classification of the $(n+1)$-dimensional n-Lie algebras over an algebraically closed field of characteristic zero. The structure of n-Lie algebras is very different from that of Lie algebras due to the n-ary multilinear operations involved. The $n = 3$ case, i.e. 3-ary multilinear operation, first appeared in Nambu's work [47] in the description of simultaneous classical dynamics of three particles. Hom-(super) generalization of this structures called n-Hom-Lie (super)algebras was studied in [2, 5, 9, 59] (see also [43], for more details). These algebraic structures are considered as super-generalization of the structures of Hom-algebras which first appeared in 2003 in the work of Hartwig, Larsson and Silvestrov [35], in their investigation of the extension to general twisted derivations of q-deformations of the Witt and Virasoro algebras.

Pre-Lie algebras (called also left-symmetric algebras, Vinberg algebras, quasi associative algebras) are a class of a natural algebraic systems appearing in many fields in mathematics and mathematical physics. They were first mentioned by Cayley in 1890 [21] as a kind of rooted tree algebra and later arose again from the study of convex homogeneous cones [57], affine manifold and affine structures on Lie groups [38], and deformation of associative algebras [32]. They play an important role in the study of symplectic and complex structures on Lie groups and Lie algebras [6, 25, 27, 28, 41], phases spaces of Lie algebras [12, 39], certain integrable systems [19], classical and quantum Yang–Baxter equations [29], combinatorics [30], quantum field theory [26] and operads [22]. See [10], and the survey [20] and the references therein for more details.

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Recently, pre-Lie superalgebras, the $\mathbb{Z}_2$-graded version of pre-Lie algebras also appeared in many others fields; see, for example, [1, 22, 32, 56]. Classifications of complexes pre-Lie superalgebras in dimensions two and three have been given recently, by Zhang and Bai [14]. See [4, 23, 36, 37] about further results. The notion of Hom-pre-Lie algebras is a twisted analog of pre-Lie algebras, where the pre-Lie algebra identity is twisted by a self linear map, called the structure map. This notion was introduced in [44]. There is a close relationship between Hom-pre- Lie algebras and Hom-Lie algebras: a Hom-pre-Lie algebra $(\mathcal{N}, \circ, \alpha)$ gives rise to a Hom-Lie algebra $(\mathcal{N}, [\cdot,\cdot]^C, \alpha)$ via the commutator bracket, which is called the subadjacent Hom-Lie algebra and denoted by $\mathcal{N}^C$. Hom-pre-Lie algebras play several roles, among them and the most important are the problems related to the representations of Hom-Lie algebras. We can explain this in terms that the map $L : \mathcal{N} \to \mathfrak{gl}(\mathcal{N})$, defined by $L_x(y) = x \circ y$ for all $x, y \in \mathcal{N}$, gives rise to a representation of the subadjacent Hom-Lie algebra $\mathcal{N}^C$ on $\mathcal{N}$ with respect to $\alpha \in \mathfrak{gl}(\mathcal{N})$. Recently, Hom-pre-Lie algebras were studied from several aspects: The geometrization of Hom-pre-Lie algebras was studied in [60], universal α-central extensions of Hom-pre-Lie algebras were studied in [54] and the bialgebra theory of Hom-pre-Lie algebras was studied in [55]. Some generalizations of pre-lie algebra have been studied, among which are given in [45], as the authors introduce the notion of n-pre-Lie algebra, which gives a n-Lie algebra naturally and its left multiplication operator gives rise to a representation of this n-Lie algebra. An n-pre-Lie algebra can also be obtained through the action of a relative Rota-Baxter operator on an n-Lie algebra. For $(n = 3)$, see [13, 24] for more details. In [33], the authors describe the symplectic structures and phase spaces of 3-Hom–Lie algebras from the structures of 3-Hom–pre-Lie algebras. The same work has been introduced in the general case of n-pre-Lie algebras[34].

Lie color algebra is a generalization of Lie algebra introduced by Ree [49]. This class plays an important role in theoretical physics (see [53, 53] for more details). In [46], the author proved that Simple Lie color algebra can be obtained from associative graded algebra, while the Ado theorem and the PBW theorem of Lie color algebra were proven by Scheunert [51]. Lie color algebras have developed in the last twenty years as an interesting topic in Mathematics and Physics (see [7, 8, 15, 18, 40] for more details).

Representations theory of different algebraic structures is an important subject of study in algebra and diverse areas. They appear in many fields of mathematics and physics. In particular, they appear in deformation and cohomology theory among other areas. In this paper, we have to talk about the representation of algebraic structures of Lie and Hom-Lie type which are introduced in several works. The notion of representation introduced for Hom–Lie algebras in [52] (see also [16]). The extension of this work to the n-ary and n-ary-Hom-super cases has been given in [2, 5]. Some other extensions are given in several works such as the representation of Hom-Lie superalgebras, BiHom-Lie superalgebras, (Hom)-Lie Rinehart (super)algebras, (n)-(Hom)-Poisson (super)algebras... (see [17, 42, 48, 58], for more details). In this paper we base on the representation theory of Hom-pre-Lie color algebras. This notion was introduced in [55] in the study of Hom-pre-Lie bialgebras (Hom-left-symmetric bialgebras). In [50], the authors gave the natural formula of a dual representation, which is nontrivial and showed that there is well defined tensor product of two representations of a Hom-pre-Lie algebra. A generalization of the representation theory of pre-Lie algebras was introduced in [34], in which the authors defined the representation of n-pre-Lie algebras and gave some other associated results. In this paper, we generalize this notion in the Hom-color case and we give some related results.

This paper is organized as follows. In Section 2, we recall some definitions and known results about n-Lie color algebras and n-Hom-Lie color algebras. We also recall some examples for these structures. In Section 3, we introduce the notion of n-Hom-pre-Lie color algebras and their representations and we give some important results and related structures. In Section 4, we provide a construction procedure of n-Hom-pre-Lie color algebras starting from a Hom-pre-Lie color algebra and an $(n-2)$-linear form satisfies specific conditions. Moreover, we applied same procedure for the representations of n-Hom-pre-Lie color algebras.

Throughout this paper, we will for simplicity of exposition assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero, even though for most of the general definitions and results in the paper this assumption is not essential.

2. Preliminaries and Basics

In this section, we give some preliminaries and basic results on n-(Hom)-Lie color algebras and to give some examples, also we give some results using representations, Kupershmidt operators and Nijenhuis operators.

Definition 2.1. *Let Γ be an abelian group.*

1. *A linear space $\mathcal{N}$ is said to be Γ-graded if, there exists a family $(\mathcal{N}_a)_{a\in\Gamma}$ of linear subspaces of $\mathcal{N}$ such that*
$$\mathcal{N} = \bigoplus_{a\in\Gamma} \mathcal{N}_a.$$
2. *An element $x \in \mathcal{N}$ is said to be homogeneous of degree $a \in \Gamma$ if $x \in \mathcal{N}_a$. We denote $\mathcal{H}(\mathcal{N})$ the set of all homogeneous elements in $\mathcal{N}$.*
3. *Let $\mathcal{N} = \bigoplus_{a\in\Gamma} \mathcal{N}_a$ and $\mathcal{N}' = \bigoplus_{a\in\Gamma} \mathcal{N}'_a$ be two Γ-graded linear spaces. A linear mapping $f : \mathcal{N} \to \mathcal{N}'$ is said to be homogeneous of degree b if*
$$f(\mathcal{N}_a) \subseteq \mathcal{N}'_{a+b}, \quad \text{for all} \quad a \in \Gamma.$$
4. *Let $\mathcal{M}$ and $\mathcal{N}$ be two Γ-graded vector spaces. A structure of Γ-graded vector space on $\mathcal{M}\oplus\mathcal{N}$ can be introduced canonically:*
$$(\mathcal{M}\oplus\mathcal{N})_{\gamma\in\Gamma} = \mathcal{M}_\gamma \oplus \mathcal{N}_\gamma.$$

Remark 2.1. *If the base field is considered as a graded vector space, it is understood that the graduation of $\mathbb{K}$ is given by $\mathbb{K}_0 = \mathbb{K}$ and $\mathbb{K}_a = \{0\}$, if $a \in \Gamma \setminus \{0\}$.*

Definition 2.2.
1. *An algebra $(\mathcal{N}, \cdot)$ is said to be Γ-graded if its underlying linear space is Γ-graded i.e. $\mathcal{N} = \bigoplus_{a\in\Gamma} \mathcal{N}_a$, and if furthermore*
$$\mathcal{N}_a \cdot \mathcal{N}_b \subseteq \mathcal{N}_{a+b}, \quad \text{for all} \quad a, b \in \Gamma.$$
2. *A morphism $f : \mathcal{N} \to \mathcal{N}'$ of Γ-graded algebras $\mathcal{N}$ and $\mathcal{N}'$ is by definition an algebra morphism from $\mathcal{N}$ to $\mathcal{N}'$ which is, in addition an even mapping (i.e. f of degree zero).*

Definition 2.3. *Let Γ be an abelian group. A map $\epsilon : \Gamma\times\Gamma \to \mathbb{K}^*$ is called commutation factor (or a skew-symmetric bicharacter) on Γ if the following identities hold for all $a, a', a'' \in \Gamma$,*

$$\epsilon(a, a')\epsilon(a', a) = 1, \tag{1}$$
$$\epsilon(a, a' + a'') = \epsilon(a, a')\epsilon(a, a''), \tag{2}$$
$$\epsilon(a + a', a'') = \epsilon(a, a'')\epsilon(a', a''), \tag{3}$$

In particular, the definition above implies the following relations

$$\epsilon(a, 0) = \epsilon(0, a) = 1 \ \text{ and } \ \epsilon(a, a) = \pm 1 \text{ for all } a \in \Gamma.$$

If x and y are two homogeneous elements of degree a and b respectively and ϵ is a bicharacter, then we shorten the notation by writing $\epsilon(x, y)$ instead of $\epsilon(a, b)$.

Notations: For any $X = (x_1, \dots, x_n) \in \mathcal{N}^n$, we need the following notations

$$X_{i,j} = \sum_{k=i}^{j} x_k, \ \forall 1 \le i \le j \le n, \ \ X_i = \sum_{k=1}^{i} x_k \quad \text{and} \quad X_0 = 0.$$

Example 2.1. *Let $\Gamma = \mathbb{Z}$. We define the bicharacter $\epsilon : \Gamma \times \Gamma \to \mathbb{K}^*$ by*

$$\epsilon(m, n) = \cos(mn\pi).$$

Example 2.2 ([11]). *Some standard examples of bicharacters are:*

1. $\Gamma = \mathbb{Z}_2, \quad \epsilon(i, j) = (-1)^{ij}$, *or more generally*
$$\Gamma = \mathbb{Z}_2^n = \{(\alpha_1, \dots, \alpha_n) \mid \alpha_i \in \mathbb{Z}_2\},$$
$$\epsilon((\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)) := (-1)^{\alpha_1\beta_1+\cdots+\alpha_n\beta_n}.$$
2. $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \epsilon((i_1, i_2), (j_1, j_2) = (-1)^{i_1 j_2 - i_2 j_1}$,
3. $\Gamma = \mathbb{Z} \times \mathbb{Z}, \quad \epsilon((i_1, i_2), (j_1, j_2)) = (-1)^{(i_1+i_2)(j_1+j_2)}$,
4. $\Gamma = \{-1, +1\}, \quad \epsilon(i, j) = (-1)^{(i-1)(j-1)/4}$.

Definition 2.4. *An n-Lie color algebra is a triple $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon)$ consisting of a Γ-graded vector space $\mathcal{N} = \bigoplus_{\gamma\in\Gamma} \mathcal{N}_\gamma$, an n-linear map $[\cdot, \dots, \cdot] : \mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N} \to \mathcal{N}$ of degree zero and a bicharacter $\epsilon : \Gamma \times \Gamma \to \mathbb{K}^*$ satisfying*

$$[x_1, \dots, x_i, x_{i+1}, \dots, x_n] = -\epsilon(x_i, x_{i+1})[x_1, \dots, x_{i+1}, x_i, \dots, x_n], \tag{4}$$

$$\Big[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]\Big] = \sum_{i=1}^{n} \epsilon(X_{n-1}, Y_{i-1})\Big[y_1, \dots, y_{i-1}, [x_1, \dots, x_{n-1}, y_i], y_{i+1}, \dots, y_n\Big], \tag{5}$$

for any $x_i, y_j \in \mathcal{H}(\mathcal{N})$, $1 \le i, j \le n$.

Remark 2.2. *The condition* (4) *is equivalent to*

$$\Big[x_1, \dots, x_i, \dots, x_j, \dots, x_n\Big] = -\epsilon(X_{i+1,j-1}, x_i + x_j)\epsilon(x_i, x_j)\Big[x_1, \dots, x_j, \dots, x_i, \dots, x_n\Big], \ \forall\, 1 \le i < j \le n. \tag{6}$$

Example 2.3. *Let $\Gamma = \mathbb{Z}_2$, $\epsilon(i, j) = (-1)^{ij}$ and $\mathcal{N} = \mathcal{N}_{\bar 0} \oplus \mathcal{N}_{\bar 1} =< e_1, \dots, e_n > \oplus < e_{n+1} >$. Define the n-linear map $[\cdot, \dots, \cdot] : \wedge^n \mathcal{N} \to \mathcal{N}$ of degree zero by*

$$[e_1, \dots, \hat{e}_i, \dots, e_n, e_{n+1}] = e_{n+1}, \ \forall 1 \le i \le n,$$

where $\hat{e}_i$ means that the element e_i is omitted. Then $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon)$ is an n-Lie color algebra.

Definition 2.5. *An **n-Hom-Lie color algebra** is a graded vector space $\mathcal{N} = \bigoplus_{\gamma\in\Gamma} \mathcal{N}_\gamma$ with an homogeneous n-linear map $[\cdot, \dots, \cdot] : \mathcal{N} \times \cdots \times \mathcal{N} \longrightarrow \mathcal{N}$ of degree zero, a bicharacter $\epsilon : \Gamma \times \Gamma \to \mathbb{K} \setminus \{0\}$ and a linear map $\alpha : \mathfrak{g} \longrightarrow \mathfrak{g}$ of degree zero, such that the condition* (4) *is satisfied and for all $X = (x_1, \dots, x_{n-1}), Y = (y_1, \dots, y_n) \in \mathcal{H}(\mathcal{N})$, we have*

$$\Big[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [y_1, \dots, y_n]\Big] = \sum_{i=1}^{n} \epsilon(X_{n-1}, Y_{i-1})\Big[\alpha_1(y_1), \dots, \alpha_{i-1}(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha_{i+1}(y_{i+1}), \dots, \alpha_{n-1}(y_n)\Big]. \tag{7}$$

Remarks 2.1.

1. *When $\Gamma = \{0\}$, the trivial group, this is the ordinary n-Hom-Lie algebra (see [9] for more details).*
2. *When $\Gamma = \mathbb{Z}_2$, $\epsilon(x, y) = (-1)^{|x||y|}$, we recover Hom-Lie superalgebra introduced in [2].*
3. *If $\alpha = id_{\mathcal{N}}$, then $\mathcal{N}$ is just an n-Lie color algebra.*

Let $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ and $(\mathcal{N}', [\cdot, \dots, \cdot]', \epsilon, \alpha')$ be two n-Hom-Lie color algebras. A linear map of degree zero $f : \mathcal{N} \to \mathcal{N}'$ is an n-Hom-Lie color algebras **morphism** if its satisfies

$$f \circ \alpha = \alpha' \circ f,$$
$$f([x_1, \dots, x_n]) = [f(x_1), \dots, f(x_n)]'.$$

Definition 2.6.

1. *An n-Hom-Lie color algebra* $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ *is said to be multiplicative if* α *is an algebra endomorphism, (i.e. a linear map on* $\mathcal{N}$ *which is also a homomorphism with respect to the multiplication* $[\cdot, \dots, \cdot]$*).*
2. *An n-Hom-Lie color algebra* $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ *is said to be regular if* α *is an automorphism.*
3. *An n-Hom-Lie color algebra* $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ *is said to be involutive if* $\alpha^2 = id$.

Example 2.4. *Let* $\Gamma = \mathbb{Z}_2$, $\epsilon(i, j) = (-1)^{ij}$ *and* $\mathcal{N} = \mathcal{N}_{\bar{0}} \oplus \mathcal{N}_{\bar{1}} =< e_1, \dots, e_n > \oplus < e_{n+1} >$*. Define the super-skew-symmetric n-linear map* $[\cdot, \dots, \cdot] : \wedge^n \mathcal{N} \to \mathcal{N}$ *of degree zero by*

$$[e_1, \dots, \hat{e_i}, \dots, e_n, e_{n+1}] = (-1)^{i+1} e_{n+1}, \ \forall 1 \leq i \leq n,$$

where $\hat{e_i}$ *means that the element* e_i *is omitted and the linear map* $\alpha : \mathcal{N} \to \mathcal{N}$ *defines on the basis of* $\mathcal{N}$ *by*

$$\alpha(e_i) = e_i, \ 1 \leq i \leq n \ \text{ and } \ \alpha(e_{n+1}) = 0.$$

Then $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ *is a multiplicative n-Hom-Lie color algebra.*

Theorem 2.5. *(**Twist Theorem**) Let* $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon)$ *be an n-Lie color algebra and* $\alpha : \mathcal{N} \to \mathcal{N}$ *be an algebra endomorphism of degree zero. Then, the multilinear map* $[\cdot, \dots, \cdot]_\alpha = \alpha \circ [\cdot, \dots, \cdot]$ *defines on* $\mathcal{N}$ *a multiplicative n-Hom-Lie color algebra structure.*

Example 2.6. *([11]) Let* $\mathcal{N}$ *be a graded linear space*

$$\mathcal{N} = \mathcal{N}_{(0,0)} \oplus \mathcal{N}_{(0,1)} \oplus \mathcal{N}_{(1,0)} \oplus \mathcal{N}_{(1,1)}$$

with $\mathcal{N}_{(0,0)} =< e_1, e_2 >$, $\mathcal{N}_{(0,1)} =< e_3 >$, $\mathcal{N}_{(1,0)} =< e_4 >$, $\mathcal{N}_{(1,1)} =< e_5 >$ *. The* 4*-linear multiplication* $[\cdot, \cdot, \cdot, \cdot] : \mathcal{N} \times \mathcal{N} \times \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ *defined for basis* $\{e_i \mid i = 1, \dots, 5\}$ *by*

$$\begin{aligned}&[e_2, e_3, e_4, e_5] = e_1, \ [e_1, e_3, e_4, e_5] = e_2, \ [e_1, e_2, e_4, e_5] = e_3,\\ &[e_1, e_2, e_3, e_4] = 0 \text{ and } [e_1, e_2, e_3, e_5] = 0,\end{aligned}$$

makes $\mathcal{N}$ *into the five dimensional* 4*-Lie color algebra.*

Now define on $(\mathcal{N}, [\cdot, \cdot, \cdot, \cdot], \epsilon)$ *the endomorphism* $\alpha : \mathcal{N} \to \mathcal{N}$ *by*

$$\alpha(e_1) = e_2, \quad \alpha(e_2) = e_1, \quad \alpha(e_i) = e_i, \quad i = 3, 4, 5.$$

Then $\mathcal{N}_\alpha = (\mathcal{N}, [\cdot, \cdot, \cdot, \cdot]_\alpha, \epsilon, \alpha)$ *is a* 4*-Hom-Lie color algebra. Observe that* α *is involutive (hence bijective).*

Definition 2.7. *Let* $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ *be an n-Hom-Lie color algebras. Then*

1. *A* Γ*-graded subspace* $\mathfrak{h}$ *of* $\mathcal{N}$ *is a* ***color subalgebra*** *of* $\mathcal{N}$*, if for all* $\gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma$ *:*

$$\alpha(\mathfrak{h}_{\gamma_i}) \subseteq \mathfrak{h}_{\gamma_i}, \ \forall 1 \leq i \leq n \ \text{ and } \ [\mathfrak{h}_{\gamma_1}, \mathfrak{h}_{\gamma_2}, \dots, \mathfrak{h}_{\gamma_n}] \subseteq \mathfrak{h}_{\gamma_1 + \dots + \gamma_n}.$$

2. *A* Γ*-graded subspace* $\mathfrak{I}$ *of* $\mathcal{N}$ *is a* ***color ideal*** *of* $\mathcal{N}$ *if it satisfies:*

$$\alpha(\mathfrak{I}_{\gamma_i}) \subseteq \mathfrak{I}_{\gamma_i}, \ \forall 1 \leq i \leq n \ \text{ and } \ [\mathfrak{I}_{\gamma_1}, \mathfrak{g}_{\gamma_2}, \dots, \mathfrak{g}_{\gamma_n}] \subseteq \mathfrak{I}_{\gamma_1 + \dots + \gamma_n}, \ \forall \gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma$$

Definition 2.8. *([3]) A representation of an n-Hom-Lie color algebra* $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ *is a triple* (V, ρ, β) *consisting of a* Γ*-graded vector space* V*, an* ϵ*-skew-symmetric multilinear map* $\rho : \mathcal{N}^{n-1} \to End(V)$ *of degree zero and a linear map* $\beta : V \to V$ *of degree zero such that for all* $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in \mathcal{H}(\mathcal{N})$*, we have*

$$\begin{aligned}&\rho(\alpha(x_1), \dots, \alpha(x_{n-1}))\rho(y_1, \dots, y_{n-1}) - \epsilon(X_{n-1}, Y_{n-1})\rho(\alpha(y_1), \dots, \alpha(y_{n-1}))\rho(x_1, \dots, x_{n-1})\\ &= \sum_{i=1}^{n-1} \epsilon(X_{n-1}, Y_{i-1})\rho(\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha(y_{i+1}), \dots \alpha(y_{n-1}))\beta,\end{aligned} \tag{8}$$

$$\begin{aligned}&\rho(\alpha(x_1), \dots, \rho(x_{n-2}), [y_1, \dots, y_n])\beta\\ &= \sum_{i=1}^{n} (-1)^{n-i} \epsilon(X_{n-2}, Y + y_i)\epsilon(y_i, Y_{i+1,n})\rho(\alpha(y_1), \dots, \hat{y_i}, \dots, \alpha(y_n))\rho(x_1, \dots, x_{n-2}, y_i).\end{aligned} \tag{9}$$

Remark 2.3. *If (V, ρ, β) is a representation of a multiplicative n-Hom-Lie color algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$, then ρ satisfies the following condition*

$$\rho(\alpha(x_1), \dots, \alpha(x_{n-1}))\beta = \beta\rho(x_1, \dots, x_{n-1}). \tag{10}$$

Example 2.7. *Defining for any integer $s \geq 0$ the α^s-adjoint representation of an n-Hom-Lie color algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ on $\mathcal{N}^{\otimes n-1}$ as follows*

$$ad^s_{x_1,\dots,x_{n-1}}(x) = [\alpha^s(x_1), \dots, \alpha^s(x_{n-1}), x] \ \text{ for all } x_i, x \in \mathcal{H}(\mathcal{N}),\ 1 \leq i \leq n-1.$$

Let us denote the α^s-adjoint representation of the n-Hom-Lie color algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ by the triple $(\mathcal{N}, ad^s, \alpha)$. We also denote $ad^0_{x_1,\dots,x_{n-1}}$ simply by $ad_{x_1,\dots,x_{n-1}}$ for any $x_1, \dots, x_{n-1} \in \mathcal{H}(\mathcal{N})$.

Example 2.8. ***(Dual representation)*** *Let $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ be a regular n-Hom-Lie color algebra and (V, ρ, β) be an n-Hom-Lie color algebra representation with β being an invertible linear map. Define $\rho^* : \mathcal{N}^{n-1} \to End(V^*)$ as usual by*

$$< \rho^*(x_1, \dots, x_{n-1})(\xi), u >= - < \xi, \rho(x_1, \dots, x_{n-1})(u) >,\ \forall x_i \in \mathcal{H}(\mathcal{N}), 1 \leq i \leq n-1,\ u \in \mathcal{H}(V),\ \xi \in V^*.$$

However, in general ρ^* is not a representation of $\mathcal{N}$ anymore. Let us define the map $\rho^\star : \mathcal{N}^{n-1} \to End(V^*)$ by

$$\begin{aligned} < \rho^\star(x_1, \dots, x_{n-1})(\xi), v >: &=< \rho^*(\alpha(x_1), \dots, \alpha(x_{n-1}))((\beta^{-2})^*(\xi)), v > \\ &= - < \xi, \rho(\alpha^{-1}(x_1), \dots, \alpha^{-1}(x_{n-1}))(\beta^{-2}(v)) >, \end{aligned} \tag{11}$$

for all $\xi \in V^*$, $x_1, \dots, x_{n-1} \in \mathcal{H}(\mathcal{N})$ and $v \in V$. Then, the triple $(V^*, \rho^\star, (\beta^{-1})^*)$ is a representation of the n-Hom-Lie color algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ on the dual vector space V^* with respect to the map $(\beta^{-1})^*$. This is also known as the "dual representation" associated to (V, ρ, β).
In particular, let us also recall that the "coadjoint representation" of a regular n-Hom-Lie color algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ on $\mathcal{N}^*$ with respect to $(\alpha^{-1})^*$ is given by the triple $(\mathcal{N}^*, ad^\star, (\alpha^{-1})^*)$, where

$$< ad^\star(x_1, \dots, x_{n-1})(\xi), x > = - < \xi, ad(\alpha^{-1}(x_1), \dots, \alpha^{-1}(x_{n-1}))(\alpha^{-2}(x)) > \tag{12}$$

$$= - < \xi, [\alpha^{-1}(x_1), \dots, \alpha^{-1}(x_{n-1}), \alpha^{-2}(x)] >, \tag{13}$$

for all $x_i, x \in \mathcal{H}(\mathcal{N})$, $1 \leq i \leq n-1$, $\xi \in \mathcal{N}^*$.

Proposition 2.9. *([3]) Let $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ be an n-Hom-Lie color algebra, $\rho : \Lambda^{n-1}\mathcal{N} \to gl(V)$ and $\beta : V \to V$ are two linear maps of degree zero. Then $(\mathcal{N} \oplus V, [\cdot, \dots, \cdot]_{\mathcal{N}\oplus V}, \epsilon, \alpha + \beta)$ is an n-Hom-Lie color algebra if and only if (V, ρ, β) is a representation of $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$, where*

$$[x_1 + a_1, \dots, x_n + a_n]_{\mathcal{N}\oplus V} = [x_1, \dots, x_n] + \sum_{k=1}^{n} \epsilon(x_k, X_{k+1,n})\rho(x_1, \dots, \hat{x_k}, \dots, x_n)a_k, \tag{14}$$

for all $x_1, \dots, x_n \in \mathcal{H}(\mathcal{N})$ and $a_1, \dots, a_n \in \mathcal{H}(V)$.

Definition 2.9. *Let $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ be an n-Hom-Lie color algebra and s be a non-negative integer. Then, a linear operator $\mathcal{R} : \mathcal{N} \to \mathcal{N}$ of degree zero is called an s-Rota–Baxter operator of weight λ on $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ if $\mathcal{R} \circ \alpha = \alpha \circ \mathcal{R}$ and the following identity is satisfied:*

$$[R(x_1), \dots, R(x_n)] = R\Big(\sum_{\emptyset \neq I \subseteq [n]} \lambda^{|I|-1}[\hat{R}(x_1), \dots, \hat{R}(x_i), \dots, \hat{R}(x_n)]\Big), \tag{15}$$

where $\hat{R}(x_i) := \hat{R}_I(x_i) := \begin{cases} x_i, & i \in I, \\ \alpha^s R(x_i), & i \notin I \end{cases}$ *for all $x_1, \dots, x_n \in \mathcal{N}$.*

For $\alpha = Id$, then we recover the notion of Rota–Baxter operators on an n-Lie color algebra.

Definition 2.10. *Let $(\mathcal{N},[\cdot,\dots,\cdot],\epsilon,\alpha)$ be an n-Hom-Lie color algebra and (V,ρ,β) a representation of $\mathcal{N}$. A linear map $T:V\to\mathcal{N}$ of degree zero is called Kupershmidt operator (also called $\mathcal{O}$-operator) associated to (V,ρ,β) if T satisfies*

$$\alpha\circ T=T\circ\beta, \tag{16}$$

$$[T(u_1),\dots,T(u_n)]=T\Big(\sum_{i=1}^{n}(-1)^{n-i}\epsilon(u_i,U_{i+1,n})\rho(T(u_1),\dots,\widehat{T(u_i)},\dots,T(u_n))(u_i)\Big), \tag{17}$$

for all $u_i\in\mathcal{H}(V)$, $1\le i\le n$. A Kupershmidt operator associated to the adjoint representation (A,ad,α) is called Rota-Baxter operator of weight $\lambda=0$.

Remark 2.4. *Recall the α^s-adjoint representation $(\mathcal{N},ad^s,\alpha)$ of an n-Hom-Lie color algebra $(\mathcal{N},[\cdot,\dots,\cdot],\epsilon,\alpha)$ for any integer $s\ge 0$ given in Example 2.7. Then, an s-Rota–Baxter operator of weight 0 on the n-Hom-Lie color algebra $(\mathcal{N},[\cdot,\dots,\cdot],\epsilon,\alpha)$ is an Kupershmidt operator on $(\mathcal{N},[\cdot,\dots,\cdot],\epsilon,\alpha)$ with respect to the representation $(\mathcal{N},ad^s,\alpha)$. Thus, the notion of Kupershmidt operators is a generalization of Rota–Baxter operators and therefore also known as relative or generalized Rota–Baxter operators.*

Example 2.10. *Let (V,ρ) be a representation of an n-Lie color algebra $(\mathcal{N},[\cdot,\dots,\cdot],\epsilon)$ and $T:V\to\mathcal{N}$ be a Kupershmidt operator associated to (V,ρ). A pair $(\phi_{\mathcal{N}},\phi_V)$ is an endomorphism of the Kupershmidt operator T if*

$$T\circ\phi_V=\phi_{\mathcal{N}}\circ T\quad\text{and}$$
$$\rho(\phi_{\mathcal{N}}(x_1),\dots,\phi_{\mathcal{N}}(x_{n-1})))(\phi_V(v))=\phi_V(\rho(x_1,\dots,x_{n-1})(v)),\quad\text{for all } x_i\in\mathcal{N},\ 1\le i\le n-1,\ v\in V.$$

Let us consider the n-Hom-Lie color algebra $(\mathcal{N},[\cdot,\dots,\cdot,]_{\phi_{\mathcal{N}}},\epsilon,\phi_{\mathcal{N}})$ obtained by composition, where the n-Hom-Lie bracket is given by

$$[\cdot,\dots,\cdot,]_{\phi_{\mathcal{N}}}:=\phi_{\mathcal{N}}\circ[\cdot,\dots,\cdot].$$

If we consider the composition $\rho_{\phi_V}:=\phi_V\circ\rho$, then the triple (V,ρ_{ϕ_V},ϕ_V) is an n-Hom-Lie color algebra representation of $(\mathcal{N},[\cdot,\dots,\cdot]_{\phi_{\mathcal{N}}},\epsilon,\phi_{\mathcal{N}})$. Moreover,

$$\begin{aligned}[T(u_1),\dots,T(u_n)]_{\phi_{\mathcal{N}}}&=\phi_{\mathcal{N}}([T(u_1),\dots,T(u_n)])\\&=\phi_{\mathcal{N}}\Big(T\Big(\sum_{i=1}^{n}(-1)^{n-i}\epsilon(u_i,U_{i+1,n})\rho(T(u_1),\dots,\widehat{T(u_i)},\dots,T(u_n))(u_i)\Big)\Big)\\&=T\Big(\sum_{i=1}^{n}(-1)^{n-i}\epsilon(u_i,U_{i+1,n})\rho_{\phi_V}(T(u_1),\dots,\widehat{T(u_i)},\dots,T(u_n))(u_i)\Big),\end{aligned}$$

for all $u_i\in V$, $1\le i\le n$. Clearly, it follows that the map $T:V\to\mathcal{N}$ is a Kupershmidt operator on the n-Hom-Lie color algebra $(\mathcal{N},[\cdot,\dots,\cdot]_{\phi_{\mathcal{N}}},\epsilon,\phi_{\mathcal{N}})$ with respect to the n-Hom-Lie color algebra representation (V,ρ_{ϕ_V},ϕ_V).

In the following, we give a characterization of a Kupershmidt operator T in terms of an n-Hom-Lie subalgebra structure on the graph of T defined by

$$Gr(T)=\{(T(u),u)/\ u\in V\}.$$

Proposition 2.11. *A linear map $T:\mathcal{N}\to V$ of degree zero is a Kupershmidt operator on the n-Hom-Lie color algebra $(\mathcal{N},[\cdot,\dots,\cdot],\epsilon,\alpha)$ with respect to the representation (V,ρ,β) if and only if $Gr(T)$ is a color subalgebra of the semi-direct product n-Hom-Lie color algebra $(\mathcal{N}\oplus V,[\cdot,\dots,\cdot]_{\mathcal{N}\oplus V},\epsilon,\alpha+\beta)$, defined in Proposition 2.9.*

Proof. Let $(Tu,u)\in Gr(T)_\gamma$, $\gamma\in\Gamma$. By using (16) and the fact that α,β are of degree zero, we have $(\alpha+\beta)(Tu,u)=(\alpha(Tu),\beta(u))=(T(\beta(u)),\beta(u))\in Gr(T)_\gamma$.

Let $(Tu_k, u_k) \in Gr(T)_{\gamma_k},\ \gamma_k \in \Gamma,\ 1 \le k \le n$. Then, if T is a Kupershmidt operator on $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$, we have

$$\begin{aligned}[(Tu_1, u_1), \dots, (Tu_n, u_n)]_{\mathcal{N}\oplus V} &= [Tu_1, \dots, Tu_n], \sum_{k=1}^{n} \epsilon(u_k, U_{k+1,n})\rho(Tu_1, \dots, \widehat{Tu_k}, \dots, Tu_n)u_k \\ &= T\Big(\sum_{k=1}^{n} (-1)^{n-k}\epsilon(u_k, U_{k+1,n})\rho(Tu_1, \dots, \widehat{Tu_k}, \dots, Tu_n)u_k\Big), \\ &\sum_{k=1}^{n} \epsilon(u_k, U_{k+1,n})\rho(Tu_1, \dots, \widehat{Tu_k}, \dots, Tu_n)u_k \in Gr(T)_{\gamma_1+\dots+\gamma_n},\end{aligned}$$

which implies that $Gr(T)$ is a color subalgebra of the semi-direct product n-Hom-Lie color algebra $(\mathcal{N} \oplus V, [\cdot, \dots, \cdot]_{\mathcal{N}\oplus V}, \epsilon, \alpha + \beta)$.

Conversely, if $Gr(T)$ is a subalgebra of the semi-direct product n-Hom-Lie color algebra $(\mathcal{N}\oplus V, [\cdot, \dots, \cdot]_{\mathcal{N}\oplus V}, \epsilon, \alpha+\beta)$, then we have

$$[(Tu_1, u_1), \dots, (Tu_n, u_n)]_{\mathcal{N}\oplus V} = [Tu_1, \dots, Tu_n], \sum_{k=1}^{n} \epsilon(u_k, U_{k+1,n})\rho(Tu_1, \dots, \widehat{Tu_k}, \dots, Tu_n)u_k \in Gr(T),$$

which gives that $[Tu_1, \dots, Tu_n] = T\Big(\sum_{k=1}^{n} \epsilon(u_k, U_{k+1,n})\rho(Tu_1, \dots, \widehat{Tu_k}, \dots, Tu_n)u_k\Big)$. Therefore T is a Kupershmidt operator on $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$. □

It is of course that there are some other characterizations of the Kupershmidt operators on an n-Hom-Lie color algebras, among them and this most interesting the characterization in term of a Nijenhuis operators. In the following Proposition, we characterize Kupershmidt operators on n-Hom-Lie color algebras in terms of the Nijenhuis operators. In which, we need to define the Nijenhuis operator on an n-Hom-Lie color algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$, as a linear map $N : \mathcal{N} \to \mathcal{N}$ of degree zero which satisfies the following identity

$$[N(x_1), \dots, N(x_n)] = N\Big(\sum_{\emptyset \neq I \subseteq [n]} N^{|I|-1}[\hat{N}(x_1), \dots, \hat{N}(x_i), \dots, \hat{N}(x_n)]\Big), \tag{18}$$

where $\hat{N}(x_i) := \hat{N}_I(x_i) := \begin{cases} x_i, & i \in I, \\ N(x_i), & i \notin I \end{cases}$ for all $x_1, \dots, x_n \in \mathcal{H}(\mathcal{N})$.

Proposition 2.12. *Let (V, ρ, β) be a representation of an n-Hom-Lie color algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ and $T : V \to \mathcal{N}$ a linear map of degree zero. Then T is a Kupershmidt operator on $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ with respect to (V, ρ, β) if and only if the operator*

$$N_T = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} : \mathcal{N} \oplus V \to \mathcal{N} \oplus V$$

is a Nijenhuis operator on the semi-direct product n-Hom-Lie color algebra $(\mathcal{N} \oplus V, [\cdot, \dots, \cdot]_{\mathcal{N}\oplus V}, \epsilon, \alpha + \beta)$.

Proof. By using the Definition of the map N_T and the bracket $[\cdot, \dots, \cdot]_{\mathcal{N}\oplus V}$, we have

$$\begin{aligned}[N_T(x_1 + u_1), \dots, N_T(x_n + u_n)]_{\mathcal{N}\oplus V} &= [T(u_1) + 0, \dots, T(u_n) + 0] \\ &= [T(u_1), \dots, T(u_n)],\end{aligned}$$

and by the obvious result $N_T^k = 0,\ \forall k \ge 2$, we have

$$N_T\Big(\sum_{\emptyset \neq I \subseteq [n]} N_T^{|I|-1}[\hat{N}_T(x_1 + u_1), \dots, \hat{N}_T(x_i + u_i), \dots, \hat{N}_T(x_n + u_n)]_{\mathcal{N}\oplus V}\Big)$$

$$= N_T\Big(\sum_{i=1}^{n}[N_T(x_1+u_1),\dots,x_i+u_i,\dots,N_T(x_n+u_n)]_{\mathcal{N}\oplus V}\Big)$$
$$= N_T\Big(\sum_{i=1}^{n}[T(u_1),\dots,x_i+u_i,\dots,T(u_n)]_{\mathcal{N}\oplus V}\Big)$$
$$= N_T\Big(\sum_{i=1}^{n}[T(u_1),\dots,x_i,\dots,T(u_n)]+\sum_{i=1}^{n}\epsilon(u_i,U_{i+1,n})\rho(T(u_1),\dots,T(\hat{u}_i),\dots,T(u_n))(u_i)\Big)$$
$$= T\Big(\sum_{i=1}^{n}\epsilon(u_i,U_{i+1,n})\rho(T(u_1),\dots,T(\hat{u}_i),\dots,T(u_n))(u_i)\Big),$$

for all $x_i\in\mathcal{H}(\mathcal{N}),\ u_i\in\mathcal{H}(V)$. By a direct computation, we conclude that

$$N_T\Big(\sum_{\emptyset\neq I\subseteq[n]}N_T^{|I|-1}[\hat{N}_T(x_1+u_1),\dots,\hat{N}_T(x_i+u_i),\dots,\hat{N}_T(x_n+u_n)]_{\mathcal{N}\oplus V}\Big)$$
$$= N_T\Big(\sum_{\emptyset\neq I\subseteq[n]}N_T^{|I|-1}[\hat{N}_T(x_1+u_1),\dots,\hat{N}_T(x_i+u_i),\dots,\hat{N}_T(x_n+u_n)]_{\mathcal{N}\oplus V}\Big)$$

if and only if

$$[T(u_1),\dots,T(u_n)] = T\Big(\sum_{i=1}^{n}\epsilon(u_i,U_{i+1,n})\rho(T(u_1),\dots,T(\hat{u}_i),\dots,T(u_n))(u_i)\Big),$$

for all $x_i\in\mathcal{H}(\mathcal{N}),\ u_i\in\mathcal{H}(V)$, which gives the result. □

3. *n*-Hom-pre-Lie color algebras and their representations

In [20], the authors introduced the notion of pre-Lie algebras and given their representation, some other practical results are also studied, among those which are most interesting the cohomology and deformations of pre-Lie algebras. This notion has been extended in more general cases (for more details see [34]). In this section we introduce the notion of n-Hom-pre-Lie color algebras and define their representation also we give some algebraic structures and results concerning this notion.

3.1. n-Hom-pre-Lie color algebras

Definition 3.1. *An n-Hom-pre-Lie color algebra is a quadruple $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$ consisting of a Γ-graded vector space $\mathcal{N}$, a bicharacter $\epsilon:\Gamma\times\Gamma\to\mathbb{K}^*$, an n-linear map $\{\cdot,\dots,\cdot\}:\wedge^n\mathcal{N}\to\mathcal{N}$ of degree zero, ϵ-skew-symmetric on the first $(n-1)$ terms and a linear map $\alpha:\mathcal{N}\to\mathcal{N}$ of degree zero such that for all $x_i,y_i\in\mathcal{H}(\mathcal{N}),\ 1\le i\le n$, the following identities are satisfied:*

$$\{\alpha(x_1),\dots,\alpha(x_{n-1}),\{y_1,\dots,y_n\}\} = \sum_{i=1}^{n-1}\epsilon(X_{n-1},Y_{i-1})\{\alpha(y_1),\dots,\alpha(y_{i-1}),[x_1,\dots,x_{n-1},y_i]^C,\alpha(y_{i+1}),\dots,\alpha(y_n)\}$$
$$+\epsilon(X_{n-1},Y_{n-1})\{\alpha(y_1),\dots,\alpha(y_{n-1}),\{x_1,\dots,x_{n-1},y_n\}\}, \tag{19}$$
$$\{[x_1,\dots,x_n]^C,\alpha(y_1),\dots,\alpha(y_{n-1})\} = \sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i,X_{i+1,n})\{\alpha(x_1),\dots,\widehat{x_i},\dots,\alpha(x_n),\{x_i,y_1,\dots,y_{n-1}\}\}, \tag{20}$$

where $[\cdot,\dots,\cdot]^C$ is defined by

$$[x_1,\dots,x_n]^C=\sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i,X_{i+1,n})\{x_1,\dots,\widehat{x_i},\dots,x_n,x_i\},\quad \forall x_i\in\mathcal{H}(\mathcal{N}),1\le i\le n. \tag{21}$$

An n-Hom-pre-Lie color algebra $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$ is said to be multiplicative if α is an algebra endomorphism, (i.e. a linear map on $\mathcal{N}$ which is also a homomorphism with respect to multiplication $\{\cdot,\dots,\cdot\}$).

Proposition-Definition 3.1. *Let $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$ be an n-Hom-pre-Lie color algebra. Then, the quadruple $(\mathcal{N}, [\cdot, \dots, \cdot]^C, \epsilon, \alpha)$, where $[\cdot, \dots, \cdot]^C$ is given by Eq. (21), is an n-Hom-Lie color algebra called the sub-adjacent n-Hom-Lie color algebra of $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$ denoted by $\mathcal{N}^c$ and $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$ is called compatible n-Hom-pre-Lie color algebra of the n-Hom-Lie color algebra $\mathcal{N}^c$.*

Proof. Let $x_i, y_i \in \mathcal{H}(\mathcal{N}),\ 1 \le i \le n$. For all $1 \le k \le n-1$, then by using the definition of $[\cdot, \dots, \cdot]^C$, we have:

$$\begin{aligned}
[x_1, \dots, x_k, x_{k+1}, \dots, x_{n-1}, x_n]^C &= \sum_{i=1}^{n} (-1)^{n-i} \epsilon(x_i, X_{i+1,n}) \{x_1, \dots, \widehat{x_i} \dots, x_k, x_{k+1}, \dots, x_n, x_i\} \\
&+ \sum_{i=1}^{n} (-1)^{n-i} \epsilon(x_i, X_{i+1,n}) \{x_1, \dots, x_k, x_{k+1}, \dots, \widehat{x_i} \dots, x_n, x_i\} \\
&+ (-1)^{n-k} \epsilon(x_k, X_{k+1,n}) \{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n, x_k\} \\
&+ (-1)^{n-k-1} \epsilon(x_{k+1}, X_{k+2,n}) \{x_1, \dots, x_k, x_{k+2}, \dots, x_n, x_{k+1}\} \\
&= -\epsilon(x_k, x_{k+1}) \Big(\sum_{i=1}^{n} (-1)^{n-i} \epsilon(x_i, X_{i+1,n}) \{x_1, \dots, \widehat{x_i} \dots, x_{k+1}, x_k, \dots, x_n, x_i\} \\
&+ \sum_{i=1}^{n} (-1)^{n-i} \epsilon(x_i, X_{i+1,n}) \{x_1, \dots, x_{k+1}, x_k, \dots, \widehat{x_i} \dots, x_n, x_i\} \\
&+ (-1)^{n-k-1} \epsilon(x_k, X_{k+2,n}) \{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n, x_k\} \\
&+ (-1)^{n-k} \epsilon(x_{k+1}, X_{k+2,n} + x_k) \{x_1, \dots, x_k, x_{k+2}, \dots, x_n, x_{k+1}\} \Big) \\
&= -\epsilon(x_k, x_{k+1}) [x_1, \dots, x_{k+1}, x_k, \dots, x_{n-1}, x_n]^C,
\end{aligned}$$

which implies that $[\cdot, \dots, \cdot]^C$ is ϵ-skew-symmetric. It remains to show that $[\cdot, \dots, \cdot]^C$ satisfies condition (7). On the one hand, we have

$$\begin{aligned}
M &= [\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, \dots, y_n]^C]^C \\
&= \sum_{i=1}^{n-1} (-1)^{n-i} \epsilon(x_i, X_{i+1,n-1} + Y) \{\alpha(x_1), \dots, \alpha(\hat{x_i}), \dots, \alpha(x_{n-1}), [y_1, \dots, y_n]^C, \alpha(x_i)\} \\
&+ \{\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, \dots, y_n]^C\} \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (-1)^{i+j} \epsilon(x_i, X_{i+1,n-1} + Y) \epsilon(y_j, Y_{j+1,n}) \{\alpha(x_1), \dots, \alpha(\hat{x_i}), \dots, \alpha(x_{n-1}), \{y_1, \dots, \hat{y}_j, \dots, y_n, y_j\}, \alpha(x_i)\} \\
&+ \sum_{i=1}^{n-1} (-1)^{n-i} \epsilon(x_i, X_{i+1,n-1} + Y) \{\alpha(x_1), \dots, \alpha(\hat{x_i}), \dots, \alpha(x_{n-1}), \{y_1, \dots, \dots, y_n\}, \alpha(x_i)\} \\
&+ \sum_{j=1}^{n-1} (-1)^{n-j} \epsilon(y_j, Y_{j+1,n}) \{\alpha(x_1), \dots, \alpha(x_{n-1}), \{y_1, \dots, \hat{y}_j, \dots, y_n, y_j\}\} + \{\alpha(x_1), \dots, \alpha(x_{n-1}), \{y_1, \dots, \dots, y_n\}\}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
N &= \sum_{i=1}^{n} \epsilon(X_{n-1}, Y_{i-1}) [\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i]^C, \alpha(y_{i+1}), \dots, \alpha(y_n)]^C \\
&= \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{n-j} \epsilon(X_{n-1}, Y_{i-1}) \epsilon(y_j, X_{n-1} + Y_{j+1,n}) \{\alpha(y_1), \dots, \alpha(\hat{y}_j), \dots, [x_1, \dots, x_{n-1}, y_i]^C, \dots, \alpha(y_n), \alpha(y_j)\}
\end{aligned}$$

$$+\sum_{i=1}^{n}\sum_{j=i+1}^{n-1}(-1)^{n-j}\epsilon(X_{n-1},Y_{i-1})\epsilon(y_j,Y_{j+1,n})\{\alpha(y_1),\dots,[x_1,\dots,x_{n-1},y_i]^C,\dots,\alpha(\hat{y_j}),\dots,\alpha(y_n),\alpha(y_j)\}$$
$$+\sum_{i=1}^{n}(-1)^{n-i}\epsilon(X_{n-1},Y_{i-1})\epsilon(Y_{i+1,n},X_{n-1}+y_i)\{\alpha(y_1),\dots,\alpha(y_{i-1}),\alpha(y_{i+1}),\dots,\alpha(y_n),[x_1,\dots,x_{n-1},y_i]^C\}$$
$$+\sum_{i=1}^{n}\epsilon(X_{n-1},Y_{i-1})\{\alpha(y_1),\dots,\alpha(y_{i-1}),[x_1,\dots,x_{n-1},y_i]^C,\alpha(y_{i+1}),\dots,\alpha(y_n)\}$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{i-1}(-1)^{n-j}\epsilon(X_{n-1},Y_{i-1})\epsilon(y_j,X_{n-1}+Y_{j+1,n})\{\alpha(y_1),\dots,\alpha(\hat{y_j}),\dots,[x_1,\dots,x_{n-1},y_i]^C,\dots,\alpha(y_n),\alpha(y_j)\}$$
$$+\sum_{i=1}^{n}\sum_{j=i+1}^{n-1}(-1)^{n-j}\epsilon(X_{n-1},Y_{i-1})\epsilon(y_j,Y_{j+1,n})\{\alpha(y_1),\dots,[x_1,\dots,x_{n-1},y_i]^C,\dots,\alpha(\hat{y_j}),\dots,\alpha(y_n),\alpha(y_j)\}$$
$$+\sum_{i=1}^{n-1}(-1)^{n-i}\epsilon(X_{n-1},Y_{i-1})\epsilon(Y_{i+1,n},X_{n-1}+y_i)\{\alpha(y_1),\dots,\alpha(y_{i-1}),\alpha(y_{i+1}),\dots,\alpha(y_n),[x_1,\dots,x_{n-1},y_i]^C\}$$
$$+\sum_{i=1}^{n-1}(-1)^{n-i}\epsilon(X_{n-1},Y_{i-1})\epsilon(Y_{i+1,n},X_{n-1}+y_i)\{\alpha(y_1),\dots,\alpha(y_{n-1}),\{x_1,\dots,\hat{x_i}\dots,x_{n-1},y_n,x_i\}\}$$
$$+\{\alpha(y_1),\dots,\alpha(y_{n-1}),\{x_1,\dots,x_{n-1},y_n\}\}$$
$$+\sum_{i=1}^{n}\epsilon(y_i,Y_{i-1})\{[x_1,\dots,x_{n-1},y_i]^C,\alpha(y_1),\dots,\alpha(y_{i-1}),\alpha(y_{i+1}),\dots,\alpha(y_n)\}.$$

Using the identities (19)-(20) and by a direct computation, we find $M-N=0$, which implies that $[\cdot,\dots,\cdot]^C$ gives an n-Hom-Lie color algebra structure on $\mathcal{N}$. □

Let $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$ be an n-Hom-pre-Lie color algebra. Defining the two multiplications $L,R:\wedge^{n-1}\mathcal{N}\to End(\mathcal{N})$ by

$$L(x_1,\dots,x_{n-1})x_n=\{x_1,\dots,x_{n-1},x_n\}, \tag{22}$$

and

$$R(x_1,\dots,x_{n-1})x_n=\{x_n,x_1,\dots,x_{n-1}\}, \tag{23}$$

for all $x_i\in\mathcal{H}(\mathcal{N}),1\leq i\leq n$.

L is called left multiplication and R is called right multiplication. If there is an n-Hom-pre-Lie color algebra structure on its dual space $\mathcal{N}^*$, we denote the left multiplication and right multiplication by $\mathcal{L}$ and $\mathcal{R}$ respectively.

By the definitions of n-Hom-pre-Lie color algebra and representation of an n-Hom-Lie color algebra, we immediately obtain the following results:

Proposition 3.2. *With the above notations, (A,L,α) is a representation of the n-Hom-Lie color algebra $(\mathcal{N},[\cdot,\dots,\cdot]^C,\epsilon,\alpha)$. On the other hand, let $\mathcal{N}$ be a Γ-graded vector space with an n-linear map $\{\cdot,\dots,\cdot\}:(\wedge^{n-1}\mathcal{N})\otimes\mathcal{N}\to\mathcal{N}$ of degree zero . Then $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$ is an n-Hom-pre-Lie color algebra if $\mathcal{N}^C=(\mathcal{N},[\cdot,\dots,\cdot]^C,\epsilon,\alpha)$ is an n-Hom-Lie color algebra and the left multiplication L defined by Eq.* (22) *gives a representation of $\mathcal{N}^C$.*

Proof. We skip the straightforward proof. □

Proposition 3.3. *Let $T : V \to \mathcal{N}$ be an Kupershmidht operator on an n-Hom-Lie color algebra $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$ with respect to the representation (V, ρ, β) . Then there exists an n-Hom-pre-Lie color algebra structure on V given by*

$$\{u_1, \dots, u_n\}_T = \rho(Tu_1, \dots, Tu_{n-1})u_n, \quad \forall\, u_i \in \mathcal{H}(V), 1 \le i \le n. \tag{24}$$

In particular; If $V = \mathcal{N}$ and $P : \mathcal{N} \to \mathcal{N}$ be a Rota-Baxter operator of weight zero associated to $(\mathcal{N}, ad, \alpha)$. Then the compatible n-Hom-pre-Lie color algebra on $\mathcal{N}$ is given by

$$\{x_1, \dots, x_n\}_P = [P(x_1), \dots, P(x_{n-1}), x_n], \tag{25}$$

for any $x_1, \dots, x_n \in \mathcal{H}(\mathcal{N})$.

Proof. Let $u_i, v_i \in \mathcal{H}(V),\ 1 \le i \le n$, then by using (8), (17) and (24), we have:

$$\begin{aligned}
&\{\beta(u_1), \dots, \beta(u_{n-1}), \{v_1, \dots, v_n\}_T\}_T - \epsilon(U_{n-1}, V_{n-1})\{\beta(v_1), \dots, \beta(v_{n-1}), \{u_1, \dots, u_n\}_T\}_T \\
&= \{\beta(u_1), \dots, \beta(u_{n-1}), \rho(Tv_1, \dots, Tv_{n-1})(v_n)\}_T - \epsilon(U_{n-1}, V_{n-1})\{\beta(v_1), \dots, \beta(v_{n-1}), \rho(Tu_1, \dots, Tu_{n-1})(u_n)\}_T \\
&= \rho\Big(T(\beta(u_1)), \dots, T(\beta(u_{n-1}))\Big)\rho\Big(Tv_1, \dots, Tv_{n-1}\Big)(v_n) \\
&- \epsilon(U_{n-1}, V_{n-1})\rho\Big(T(\beta(v_1)), \dots, T(\beta(v_{n-1}))\Big)\rho\Big(Tu_1, \dots, Tu_{n-1}\Big)(u_n) \\
&= \rho\Big(\alpha(Tu_1), \dots, \alpha(Tu_{n-1})\Big)\rho\Big(Tv_1, \dots, Tv_{n-1}\Big)(v_n) - \epsilon(U_{n-1}, V_{n-1})\rho\Big(\alpha(Tv_1), \dots, \alpha(Tv_{n-1})\Big)\rho\Big(Tu_1, \dots, Tu_{n-1}\Big)(u_n) \\
&= \sum_{i=1}^{n-1} \epsilon(U_{n-1}, V_{i-1})\rho\Big(\alpha(Tv_1), \dots, \alpha(Tv_{i-1}), [Tu_1, \dots, Tu_{n-1}, Tv_i], \dots, \alpha(Tv_{n-1})\Big)\beta \\
&= \sum_{i=1}^{n-1} \epsilon(U_{n-1}, V_{i-1})\rho\Big(\alpha(Tv_1), \dots, \alpha(Tv_{i-1}), T\Big(\sum_{j=1}^{n-1}(-1)^{n-i}\epsilon(u_j, U_{j+1,n-1} + v_i)\rho(Tu_1, \dots, \hat{Tu_j}, \dots Tu_{n-1}, Tv_i)u_j\Big), \\
&\qquad \dots, \alpha(Tv_{n-1})\Big)\beta + \sum_{i=1}^{n-1} \epsilon(U_{n-1}, V_{i-1})\rho\Big(\alpha(Tv_1), \dots, \alpha(Tv_{i-1}), T(\rho(Tu_1, \dots, Tu_{n-1})v_i), \dots, \alpha(Tv_{n-1})\Big)\beta \\
&= \sum_{i=1}^{n-1} \epsilon(U_{n-1}, V_{i-1})\rho\Big(T(\beta(v_1)), \dots, T(\beta(v_{i-1})), T([u_1, \dots, u_{n-1}, v_i]_V^C), \dots, T(\beta(v_{n-1}))\Big)\beta \\
&= \sum_{i=1}^{n-1} \epsilon(U_{n-1}, V_{i-1})\{\beta(v_1), \dots, \beta(v_{i-1}), [u_1, \dots, u_{n-1}, v_i]_V^C, \dots, \beta(v_{n-1})\}_T,
\end{aligned}$$

which gives that the identity (19) is satisfied on V. By the same way we show that the identity (20) is satisfied. Then (V, ρ, β) is an n-Hom-pre-Lie color algebra.
If $V = \mathcal{N}$, the result is obvious. □

Corollary 3.4. *With the above conditions, $(V, [\cdot, \dots, \cdot]_T^C, \epsilon, \beta)$ is an n-Hom-Lie color algebra as the sub-adjacent n-Hom-Lie color algebra of the n-Hom-pre-Lie color algebra given in Proposition 3.3, and T is an n-Hom-Lie color algebra morphism from $(V, [\cdot, \dots, \cdot]^C, \epsilon, \beta)$ to $(\mathcal{N}, [\cdot, \dots, \cdot], \epsilon, \alpha)$. Furthermore, $T(V) = \{Tv \mid v \in V\} \subset A$ is an n-Hom-Lie color subalgebra of $\mathcal{N}$ and there is an induced n-Hom-pre-Lie color algebra structure $\{\cdot, \dots, \cdot\}_{T(V)}$ on $T(V)$ given by*

$$\{Tu_1, \dots, Tu_n\}_{T(V)} := T\{u_1, \dots, u_n\}, \quad \forall u_i \in \mathcal{H}(V), 1 \le i \le n. \tag{26}$$

Proposition 3.5. *Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be an n-Hom-Lie color algebra. Then there exists a compatible n-Hom-pre-Lie color algebra if and only if there exists an invertible Kupershmidht operator $T : V \to \mathcal{N}$ with respect to a representation (V, ρ, β). Furthermore, the compatible n-Hom-pre-Lie color algebra structure on $\mathcal{N}$ is given by*

$$\{x_1, \dots, x_n\}_A = T\rho(x_1, \dots, x_{n-1})T^{-1}(x_n),\ \forall x_i \in \mathcal{H}(\mathcal{N}), 1 \le i \le n. \tag{27}$$

Proof. This is a direct computation, we apply Proposition 3.3 and Corollary 3.4 for $T(V) = \mathcal{N}$. □

3.2. Representations of n-Hom-pre-Lie color algebras

In [34], the authors introduce the notion of representation of n-pre-Lie algebras. In this subsection, we generalize this notion in the color case of Hom-type, so we give the construction of the corresponding semi-direct product n-Hom-pre-Lie color algebra and we give some other results related this notion.

Definition 3.2. *Let $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$ be an n-Hom-pre-Lie color algebra. A representation of $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$ on a Γ-graded vector space V is the given of a triple (l,r,β), where $l:\wedge^{n-1}\mathcal{N}\to End(V)$ is a representation of the n-Hom-Lie color algebra $\mathcal{N}^c$ on V, $r:\mathcal{N}\times\dots\times\mathcal{N}\to gl(V)$ is an $(n-1)$-linear map of degree zero, ϵ-skew-symmetric on the first $(n-2)$ terms and $\beta:V\to V$ is an linear maps of degree zero such that, for all $x_1,\dots,x_n,y_1,\dots,y_n\in\mathcal{H}(\mathcal{N})$, the following identities holds:*

$$\bullet\ l(\alpha(x_1),\dots,\alpha(x_{n-1}))r(y_1,\dots,y_{n-1})=\epsilon(X_{n-1},Y_{n-1})r(\alpha(y_1),\dots,\alpha(y_{n-1}))\mu(x_1,\dots,x_{n-1})$$
$$+\sum_{i=1}^{n-2}\epsilon(X_{n-1},Y_{i-1})r(\alpha(y_1),\dots,\alpha(y_{i-1}),[x_1,\dots,x_{n-1},y_i]^C,\alpha(y_{i+1}),\dots,\alpha(y_{n-1}))\beta \tag{28}$$
$$+\epsilon(X_{n-1},Y_{n-2})r(\alpha(y_1),\dots,\alpha(y_{n-2}),\{x_1,\dots,x_{n-1},y_{n-1}\})\beta,$$
$$\bullet\ r([x_1,\dots,x_n]^C,\alpha(y_1),\dots,\alpha(y_{n-2}))\beta= \tag{29}$$
$$\sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i,X_{i+1,n})l(\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\alpha(x_n))r(x_i,y_1,\dots,y_{n-2}),$$
$$\bullet\ r(\alpha(x_1),\dots,\alpha(x_{n-2}),\{y_1,\dots,y_n\})\beta=\epsilon(X_{n-2},Y_{n-1})l(\alpha(y_1),\dots,\alpha(y_{n-1}))r(x_1,\dots,x_{n-2},y_n)$$
$$+\sum_{i=1}^{n-1}(-1)^{i+1}\epsilon(X_{n-2}+y_i,Y_{i+1,n-1})\epsilon(X_{n-2},Y_{i-1})r(\alpha(y_1),\dots,\widehat{\alpha(y_i)},\dots,\alpha(y_n))\mu(x_1,\dots,x_{n-2},y_i), \tag{30}$$
$$\bullet\ r(\alpha(y_1),\dots,\alpha(y_{n-1}))\mu(x_1,\dots,x_{n-1})=\epsilon(X_{n-1},Y_{n-1})l(\alpha(x_1),\dots,\alpha(x_{n-1}))r(y_1,\dots,y_{n-1})$$
$$+\sum_{i=1}^{n-1}(-1)^{i}\epsilon(x_i,X_{i+1,n-1})r(\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\alpha(x_{n-1}),\{x_i,y_1,\dots,y_{n-1}\})\beta, \tag{31}$$

where $\mu(x_1,\dots,x_{n-1})=l(x_1,\dots,x_{n-1})+\sum_{i=1}^{n-1}(-1)^{i}\epsilon(x_i,X_{i+1,n-1})r(x_1,\dots,\widehat{x_i},\dots,x_{n-1},x_i).$

Remark 3.1. *If (V,l,r,β) is a representation of a multiplicative n-Hom-pre-Lie color algebra $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$, then r satisfies the following condition*

$$\beta r(x_1,\dots,x_{n-1})=r(\alpha(x_1),\dots,\alpha(x_{n-1}))\beta. \tag{32}$$

Let $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$ be an n-Hom-pre-Lie color algebra and (l,β) a representation of the sub-adjacent n-Hom-Lie color algebra $\mathcal{N}^c$ on V . Then $(l,0,\beta)$ is a representation of the n-Hom-pre-Lie color algebra $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$ on the Γ-graded vector space V. It is obvious that $(\mathcal{N},L,R,\alpha)$ is a representation of an n-Hom-pre-Lie color algebra on itself, which is called the adjoint representation.

Theorem 3.6. *Let (V,l,r) be a representation of an n-pre-Lie color algebra $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon)$, $\alpha:\mathcal{N}\to\mathcal{N}$ be a linear map of degree zero and $\beta\in End(V)$ be an endomorphism of degree zero such that*

$$\beta l(x_1,\dots,x_{n-1})=l(\alpha(x_1),\dots,\alpha(x_{n-1}))\beta,\quad \beta r(x_1,\dots,x_{n-1})=r(\alpha(x_1),\dots,\alpha(x_{n-1}))\beta,$$

for all $x_i\in\mathcal{H}(\mathcal{N})$, $1\le i\le n-1$. Then $(V,\widetilde{l},\widetilde{r},\beta)$ is a representation of the multiplicative n-Hom-pre-Lie color algebras $(\mathcal{N},\{\cdot,\dots,\cdot\}_\alpha^C,\epsilon,\alpha)$, where $\widetilde{l}=\beta\circ l$, $\widetilde{r}=\beta\circ r$ and $\{\cdot,\dots,\cdot\}_\alpha^C=\alpha\circ\{\cdot,\dots,\cdot\}^C$.

Proof. Let $x_i, y_i \in \mathcal{H}(\mathcal{N}),\ 1 \leq i \leq n$, then by condition (32), we have

$$\begin{aligned}\beta\widetilde{r}(x_1, \dots, x_{n-1}) &= \beta^2 r(x_1, \dots, x_{n-1}) = \beta r(\alpha(x_1), \dots, \alpha(x_{n-1}))\beta \\ &= \widetilde{r}(\alpha(x_1), \dots, \alpha(x_{n-1}))\beta.\end{aligned}$$

Then, the condition (32) is satisfied by $\widetilde{r}$. By the same way, we show that the conditions (28)-(31) hold. The theorem is proved. □

Proposition 3.7. *Let $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$ be an n-Hom-pre-Lie color algebra, V be a Γ-graded vector space and $l, r : \otimes^{n-1}\mathcal{N} \to End(V)$ two linear maps of degree zero. Then (V, l, r, β) is a representation of $\mathcal{N}$ if and only if there is an n-Hom-pre-Lie color algebra structure (called semi-direct product) on the direct sum $\mathcal{N} \oplus V$ of vector spaces, defined by*

$$(\alpha \oplus \beta)(x + u) = \alpha(x) + \beta(u), \tag{33}$$

$$\begin{aligned}\{x_1 + u_1, \dots, x_n + u_n\}_{\mathcal{N}\oplus V} =& \{x_1, \dots, x_n\} + l(x_1, \dots, x_{n-1})(u_n) \\ &+ \sum_{i=1}^{n-1}(-1)^{i+1}\epsilon(x_i, X_{i+1,n})r(x_1, \dots, \widehat{x_i}, \dots, x_n)(u_i),\end{aligned} \tag{34}$$

for $x, x_i \in \mathcal{H}(\mathcal{N}), u, u_i \in \mathcal{H}(V), 1 \leq i \leq n$. We denote this semi-direct product n-Hom-pre-Lie color algebra by $\mathcal{N} \ltimes^{\alpha,\beta}_{l,r} V$.

Proof. Let $x_i \in \mathcal{H}(\mathcal{N}),\ u_i \in \mathcal{H}(V),\ 1 \leq i \leq n$, then, for all $1 \leq j \leq n-2$, we have

$$\begin{aligned}&\{x_1 + u_1, \dots, x_j + u_j, x_{j+1} + u_{j+1}, \dots, x_n + u_n\}_{\mathcal{N}\oplus V} \\ =&\{x_1, \dots, x_j, x_{j+1}, \dots, x_n\} + l(x_1, \dots, x_j, x_{j+1}, \dots, x_{n-1})(u_n) \\ &+ \sum_{i=1}^{j-1}(-1)^{i+1}\epsilon(x_i, X_{i+1,n-1})r(x_1, \dots, \hat{x}_i, \dots, x_j, x_{j+1}, \dots, x_n)(u_i) \\ &+ \sum_{i=j+2}^{n-1}(-1)^{i+1}\epsilon(x_i, X_{i+1,n-1})r(x_1, \dots, x_j, x_{j+1}, \hat{x}_i, \dots, \dots, x_n)(u_i) \\ &+ (-1)^{j+1}\epsilon(x_j, X_{j+1,n-1})r(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)(u_j) \\ &+ (-1)^{j}\epsilon(x_{j+1}, X_{j+2,n-1})r(x_1, \dots, x_j, x_{j+2}, \dots, x_n)(u_{j+1}) \\ =& -\epsilon(x_j, x_{j+1})\Big(\{x_1, \dots, x_{j+1}, x_j, \dots, x_n\} + l(x_1, \dots, x_{j+1}, x_j, \dots, x_{n-1})(u_n) \\ &+ \sum_{i=1}^{j-1}(-1)^{i+1}\epsilon(x_i, X_{i+1,n-1})r(x_1, \dots, \hat{x}_i, \dots, x_{j+1}, x_j, \dots, x_n)(u_i) \\ &+ \sum_{i=j+2}^{n-1}(-1)^{i+1}\epsilon(x_i, X_{i+1,n-1})r(x_1, \dots, x_{j+1}, x_j, \hat{x}_i, \dots, \dots, x_n)(u_i) \\ &+ (-1)^{j}\epsilon(x_j, X_{j+2,n-1})r(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)(u_j) \\ &+ (-1)^{j+1}\epsilon(x_{j+1}, x_j + X_{j+1,n-1})r(x_1, \dots, x_j, x_{j+2}, \dots, x_n)(u_{j+1})\Big) \\ =&\epsilon(x_j + u_j, x_{j+1} + u_{j+1})\{x_1 + u_1, \dots, x_{j+1} + u_{j+1}, x_j + u_j, \dots, x_n + u_n\}_{\mathcal{N}\oplus V},\end{aligned}$$

which implies that $\{\cdot, \dots, \cdot\}_{\mathcal{N}\oplus V}$ is ϵ-skew-symmetric on the first $(n-1)$ terms.

For any $x_i, y_j \in \mathcal{H}(\mathcal{N}),\ u_i, v_j \in \mathcal{H}(V),\ 1 \leq i \leq n-1,\ 1 \leq j \leq n$ and by using Eq. (34), we have

$$\begin{aligned}&\{(\alpha \oplus \beta)(x_1 + u_1), \dots, (\alpha \oplus \beta)(x_{n-1} + u_{n-1}), \{y_1 + v_1, \dots, y_n + v_n\}_{\mathcal{N}\oplus V}\}_{\mathcal{N}\oplus V} \\ =&\{\alpha(x_1), \dots, \alpha(x_{n-1}), \{y_1, \dots, y_n\}\} + l(\alpha(x_1), \dots, \alpha(x_{n-1})l(y_1, \dots, y_{n-1})v_n\end{aligned}$$

$$+\sum_{j=1}^{n-1}(-1)^{j+1}\epsilon(y_j,Y_{j+1,n})l(x_1,\dots,x_{n-1})r(y_1,\dots,\widehat{y_j},\dots,y_n)v_j$$
$$+\sum_{i=1}^{n-1}(-1)^{i+1}\epsilon(x_i,X_{i+1,n-1}+Y)r(\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\alpha(x_{n-1}),\{y_1,\dots,y_n\})\alpha(u_i)$$

and

$$\sum_{i=1}^{n-1}\epsilon(X_{n-1},Y_{i-1})\{(\alpha\oplus\beta)(y_1+v_1),\dots,[x_1+u_1,\dots,x_{n-1}+u_{n-1},y_i+v_i]^C_{\mathcal{N}\oplus V},\dots,(\alpha\oplus\beta)(y_n+v_n)\}_{\mathcal{N}\oplus V}$$
$$+\epsilon(X_{n-1},Y_{n-1})\{(\alpha\oplus\beta)(y_1+v_1),\dots,(\alpha\oplus\beta)(y_{n-1}+v_{n-1}),\{x_1+u_1,\dots,x_{n-1}+u_{n-1},y_n+v_n\}_{\mathcal{N}\oplus V}\}_{\mathcal{N}\oplus V}$$
$$=\sum_{i=1}^{n-1}\epsilon(X_{n-1},Y_{i-1})\{\alpha(y_1),\dots,[x_1,\dots,x_{n-1},y_i]^C,\dots,\alpha(y_n)\}$$
$$+\sum_{i=1}^{n-1}\epsilon(X_{n-1},Y_{i-1})l(\alpha(y_1),\dots,[x_1,\dots,x_{n-1},y_i]^C,\dots,\alpha(y_{n-1}))\beta(v_n)$$
$$+\sum_{i=1}^{n-1}\sum_{j=1}^{i-1}(-1)^{j+1}\epsilon(X_{n-1},Y_{i-1})\epsilon(y_j,X_{n-1}+Y_{j+1,n})r(\alpha(y_1),\dots,\widehat{\alpha(y_j)},\dots,[x_1,\dots,x_{n-1},y_i]^C,\dots,\alpha(y_n))\beta(v_j)$$
$$+\sum_{i,k=1}^{n-1}(-1)^{i+1}\gamma_{i,j,k}r(\alpha(y_1),\dots,\widehat{\alpha(y_i)},\dots,\alpha(y_n))l(x_1,\dots,\widehat{(x_k)},\dots,x_{n-1},y_i)u_k$$
$$+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n-1}(-1)^{j+1}\epsilon(X_{n-1},Y_{i-1})\epsilon(y_j,X_{n-1}+Y_{j+1,n})r(\alpha(y_1),\dots,[x_1,\dots,x_{n-1},y_i]^C,\dots,\widehat{\alpha(y_j)},\dots,\alpha(y_n))\beta(v_j)$$
$$+\sum_{i=1}^{n-1}\epsilon(X_{n-1},Y_{i-1})\epsilon(y_i,X_{n-1}+Y_{i+1,n})r(\alpha(y_1),\dots,\widehat{\alpha(y_i)},\dots,\alpha(y_n))l(x_1,\dots,x_{n-1})v_i$$
$$+\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\sum_{j=1,\,j\neq k}^{n-1}(-1)^{n+j-k+1}\theta_{i,j,k}r(\alpha(y_1),\dots,\widehat{\alpha(y_i)},\dots,\alpha(y_n))r(x_1,\dots,\widehat{x_j},\dots,\widehat{x_k},\dots,x_{n-1},y_i,x_k)u_j$$
$$+\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}(-1)^{k}\epsilon(X_{n-1},Y_{i-1})\epsilon(x_k,X_{k+1,n-1}+y_i)r(\alpha(y_1),\dots,\widehat{\alpha(y_i)},\dots,\alpha(y_n))r(x_1,\dots,\widehat{x_k},\dots,x_{n-1},x_k)v_i$$
$$+\epsilon(X_{n-1},Y_{n-1})\{\alpha(y_1),\dots,\{x_1,\dots,x_{n-1},y_n\}\}+\epsilon(X_{n-1},Y_{n-1})l(\alpha(y_1),\dots,\alpha(y_{n-1}))l(x_1,\dots,x_{n-1})v_n$$
$$+\sum_{i=1}^{n-1}(-1)^{i+1}\epsilon(X_{n-1},Y_{n-1})\epsilon(y_i,X_{n-1}+Y_{i+1,n})r(\alpha(y_1),\dots,\widehat{\alpha(y_i)},\dots,\alpha(y_{n-1}),\{x_1,\dots,x_{n-1},y_n\})\beta(v_i)$$

where

$$\gamma_{i,j,k}=\epsilon(X_{n-1},Y_{i-1})\epsilon(y_i+X_{n-1},Y_{i+1,n-1})\epsilon(x_k,X_{k+1,n-1}+y_i),$$

and

$$\theta_{i,j,k}=\epsilon(X_{n-1},Y_{i-1})\epsilon(x_k,X_{k+1,n-1}+y_i)\epsilon(x_j,X_{j+1,n-1}+y_i).$$

Then, by using the fact that $(\mathcal{N},\{\cdot,\dots,\cdot\},\alpha)$ is an n-Hom-pre-Lie color algebra, a direct computation gives that, the n-product $\{\cdot,\dots,\cdot\}_{\mathcal{N}\oplus V}$ satisfying condition (19) on $\mathcal{N}\oplus V$, if and only if l satisfying condition (8) on $\mathcal{N}^C$ and conditions (28)-(30) hold. By the same way, we can show that, the n-product $\{\cdot,\dots,\cdot\}_{\mathcal{N}\oplus V}$ satisfying condition (20) on $\mathcal{N}\oplus V$ if and only if, l satisfying condition (9) on $\mathcal{N}^C$ and conditions (29)-(31) hold Which ends the proof. □

Let V be a Γ-graded vector space and (V, l, r, β) be a representation of the n-Hom-pre-Lie color algebra $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$ on V. Define $\widetilde{\rho} : \wedge^{n-1}\mathcal{N} \to End(V)$ by

$$\widetilde{\rho}(x_1, \dots, x_{n-1}) = l(x_1, \dots, x_{n-1}) + \sum_{i=1}^{n-1}(-1)^i\epsilon(x_i, X_{i+1,n-1})r(x_1, \dots, \widehat{x_i}, \dots, x_{n-1}, x_i), \tag{35}$$

for all $x_1, \dots, x_{n-1} \in \mathcal{H}(\mathcal{N})$.

Proposition 3.8. *With the above notations, $(V, \widetilde{\rho}, \beta)$ is a representation of the sub-adjacent n-Hom-Lie color algebra $(\mathcal{N}^c, [\cdot, \dots, \cdot]^C, \epsilon, \alpha)$ on the Γ-graded vector space V.*

Proof. By Proposition 3.7, we have the semi-direct product n-Hom-pre-Lie color algebra $\mathcal{N} \ltimes^{\alpha,\beta}_{l,r} V$. Consider its sub-adjacent n-Hom-Lie color algebra structure $[\cdot, \dots, \cdot]^C$, then for any $x_i \in \mathcal{H}(\mathcal{N}),\ u_i \in \mathcal{H}(V)$, we have

$$\begin{aligned}
&[x_1 + u_1, \dots, x_n + u_n]^C_{\mathcal{N}\oplus V} = \sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i, X_{i+1,n})\{x_1 + u_1, \dots, \widehat{x_i + u_i}, \dots, x_n + u_n, x_i + u_i\}_{\mathcal{N}\oplus V}\\
&= \sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i, X_{i+1,n})\{x_1, \dots, \widehat{x_i}, \dots, x_n, x_i\} + \sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i, X_{i+1,n})l(x_1, \dots, \widehat{x_i}, \dots, x_n)(u_i)\\
&+ \sum_{i=1}^{n}(-1)^{n-i}\Big(\sum_{1\le i<j\le n}(-1)^j\epsilon(X_{j+1,n}, x_i + x_j)\epsilon(x_i, x_j + X_{i+1,j-1})r(x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_n, x_i)(u_j)\\
&+ \sum_{1\le j<i\le n}(-1)^{j+1}\epsilon(X_{j+1,n}, x_i + x_j)\epsilon(x_j, x_i + X_{j+1,i-1})r(x_1, \dots, \widehat{x_j}, \dots, \widehat{x_i}, \dots, x_n, x_i)(u_j)\Big)\\
&= [x_1, \dots, x_n]^C + \sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i, X_{i+1,n})\Big(l(x_1, \dots, \widehat{x_i}, \dots, x_n)(u_i) \qquad (36)\\
&+ \sum_{1\le i<j\le n}(-1)^j(-1)^j\epsilon(X_{j+1,n}, x_i + x_j)\epsilon(x_i, x_j + X_{i+1,j-1})r(x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_n, x_i)(u_j)\\
&+ \sum_{1\le j<i\le n}(-1)^{j+1}\epsilon(X_{j+1,n}, x_i + x_j)\epsilon(x_j, x_i + X_{j+1,i-1})r(x_1, \dots, \widehat{x_j}, \dots, \widehat{x_i}, \dots, x_n, x_i)(u_j)\\
&= [x_1, \dots, x_n]^C + \sum_{k=1}^{n}(-1)^{n-k}\epsilon(x_k, X_{k+1,n})\ \widetilde{\rho}(x_1, \dots, \widehat{x_k}, \dots, x_n)(u_k). \qquad (37)
\end{aligned}$$

By Proposition 2.9, $(V, \widetilde{\rho}, \beta)$ is a representation of the sub-adjacent n-Hom-Lie color algebra $(\mathcal{N}^c, [\cdot, \dots, \cdot]^C, \epsilon, \alpha)$ on the Γ-vector space V. The proof is finished. □

If $(l, r, \beta) = (L, R, \beta)$ is a representation of an n-Hom-pre-Lie color algebra $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$, then $\widetilde{\rho} = ad$ is the adjoint representation of the sub-adjacent n-Hom-Lie color algebra $(\mathcal{N}^c, [\cdot, \dots, \cdot]^C, \epsilon, \alpha)$ on itself.

Corollary 3.9. *Let (V, l, r, β) be a representation of an n-Hom-pre-Lie color algebra $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$ on V. Then, the semi-product n-Hom-pre-Lie color algebras $\mathcal{N} \ltimes^{\alpha,\beta}_{l,r} V$ and $\mathcal{N} \ltimes^{\alpha,\beta}_{\widetilde{\rho}} V$ given by the representations (V, l, r, β) and $(V, \widetilde{\rho}, 0, \beta)$ respectively have the same sub-adjacent n-Hom-Lie color algebra $\mathcal{N}^c \ltimes^{\alpha,\beta}_{\widetilde{\rho}} V$ given by* (37).

Let (V, l, r, β) be a representation of an n-Hom-pre-Lie color algebra $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$. In the sequel, we always assume that β is invertible to study the dual representation. For all $x_1, \dots, x_{n-1} \in \mathcal{H}(\mathcal{N}),\ u \in \mathcal{H}(V),\ \xi \in V^*$, define $\widetilde{\rho}^*, r^* : \otimes^{n-1}\mathcal{N} \to gl(V^*)$ by

$$< \widetilde{\rho}^*(x_1, \dots, x_{n-1})(\xi), u >= - < \xi, \widetilde{\rho}(x_1, \dots, x_{n-1})(u) >,$$

and

$$< r^*(x_1, \dots, x_{n-1})(\xi), u >= - < \xi, r(x_1, \dots, x_{n-1})(u) > .$$

Then, define $\widetilde{\rho}^{\star}, r^{\star} : \otimes^{n-1}\mathcal{N} \to gl(V^*)$ by

$$\widetilde{\rho}^{\star}(x_1, \dots, x_{n-1})(\xi) := \widetilde{\rho}^{*}(\alpha(x_1), \dots, \alpha(x_{n-1}))((\beta^{-2})^*(\xi)), \tag{38}$$

$$r^{\star}(x_1, \dots, x_{n-1})(\xi) := r^*(\alpha(x_1), \dots, \alpha(x_{n-1}))((\beta^{-2})^*(\xi)). \tag{39}$$

Theorem 3.10. *Let (V, l, r, β) be a representation of an n-Hom-pre-Lie color algebra $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$ on V where β is invertible. Then $(V^*, \widetilde{\rho}^{\star}, -r^{\star}, (\beta^{-1})^*)$ is a representation of the n-Hom-pre-Lie color algebra $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$ on V^*, which is called the dual representation of (V, l, r, β).*

Proof. By Proposition 3.8, $(V, \widetilde{\rho}, \beta)$ is a representation of the sub-adjacent n-Hom-Lie color algebra $(\mathcal{N}^c, [\cdot, \dots, \cdot]^C, \epsilon, \alpha)$ on V. By Example 11, $(V^*, \widetilde{\rho}^{\star}, (\beta^{-1})^*)$ is a representation of $(\mathcal{N}^c, [\cdot, \dots, \cdot]^C, \epsilon, \alpha)$ on the dual vector space V^*. It is straightforward to deduce that other conditions of Definition 3.2 also holds. We leave details to readers. □

The tensor product of two representations of an n-Hom-pre-Lie color algebra is still a representation.

Theorem 3.11. *Let $(V_1, l_{V_1}, r_{V_1}, \beta_{V_1})$ and $(V_2, l_{V_2}, r_{V_2}, \beta_{V_2})$ be two representations of an n-Hom-pre-Lie color algebra $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$. Then $(V_1 \otimes V_2, l_{V_1} \otimes \beta_{V_2} + \beta_{V_1} \otimes (l_{V_2} - r_{V_2}), r_{V_1} \otimes \beta_{V_2}, \beta_{V_1} \otimes \beta_{V_2})$ is a representation of $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$.*

Proof. By using the fact that $(V_1, l_{V_1}, r_{V_1}, \beta_{V_1})$ and $(V_2, l_{V_2}, r_{V_2}, \beta_{V_2})$ are representations of $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$, then for all $x_1, \dots, x_n, y_1, \dots, y_{n-1} \in \mathcal{H}(\mathcal{N})$, we have:

$$\begin{aligned}
&\bullet(\beta_{V_1} \otimes \beta_{V_2}) \circ (r_{V_1} \otimes \beta_{V_2})(x_1, \dots, x_{n-1}) - (r_{V_1} \otimes \beta_{V_2}) \circ (\beta_{V_1} \otimes \beta_{V_2})(\alpha(x_1), \dots, \alpha(x_{n-1}))\\
&\quad = (\beta_{V_1} \circ r_{V_1}(x_1, \dots, x_n)) \otimes (\beta_{V_2} \otimes \beta_{V_2}) - (r_{V_1}(\alpha(x_1), \dots, \alpha(x_{n-1})) \circ \beta_{V_1}) \otimes (\beta_{V_2} \circ \beta_{V_2})\\
&\quad = 0
\end{aligned}$$

$$\begin{aligned}
&\bullet(l_{V_1} \otimes \beta_{V_2} + \beta_{V_1} \otimes (l_{V_2} - r_{V_2}))(\alpha(x_1), \dots, \alpha(x_{n-1})) \circ (r_{V_1} \otimes \beta_{V_2})(y_1, \dots, y_{n-1})\\
&\quad - \epsilon(X_{n-1}, Y_{n-1})(r_{V_1} \otimes \beta_{V_2})(\alpha(y_1), \dots, \alpha(y_{n-1})) \circ \mu_{V_1 \otimes V_2}(x_1, \dots, x_{n-1})\\
&\quad + \sum_{i=1}^{n-2} \epsilon(X_{n-1}, Y_{i-1})(r_{V_1} \otimes \beta_{V_2})(\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i]^C, \alpha(y_{i+1}), \dots, \alpha(y_{n-1})) \circ (\beta_{V_1} \otimes \beta_{V_2})\\
&\quad + \epsilon(X_{n-1}, Y_{n-2})(r_{V_1} \otimes \beta_{V_2})(\alpha(y_1), \dots, \alpha(y_{n-2}), \{x_1, \dots, x_{n-1}, y_{n-1}\}) \circ (\beta_{V_1} \otimes \beta_{V_2})\\
&\quad = (l_{V_1}(\alpha(x_1), \dots, \alpha(x_{n-1})) \circ r_{V_1}(y_1, \dots, y_{n-1})) \otimes (\beta_{V_2} \circ \beta_{V_2})\\
&\quad + ((\beta_{V_1} \circ r_{V_1})(y_1, \dots, y_{n-1})) \otimes (l_{V_2}(\alpha(x_1), \dots, \alpha(x_{n-1})) \circ \alpha_{V_2}\\
&\quad - ((\beta_{V_1} \circ r_{V_1})(y_1, \dots, y_{n-1})) \otimes (r_{V_2}(\alpha(x_1), \dots, \alpha(x_{n-1})) \circ \alpha_{V_2}\\
&\quad - \epsilon(X_{n-1}, Y_{n-1})(r_{V_1} \otimes \beta_{V_2})(\alpha(y_1), \dots, \alpha(y_{n-1}) \circ ((l_{V_1} \otimes \beta_{V_2} + \beta_{V_1} \otimes (l_{V_2} - r_{V_2}))(x_1, \dots, x_{n-1})\\
&\quad + \sum_{i=1}^{n-1} (-1)^i \epsilon(x_i, X_{i+1,n-1})(r_{V_1} \otimes \beta_{V_2})(x_1, \dots, \hat{x}_i, \dots, x_{n-1}, x_i))\\
&\quad + \sum_{i=1}^{n-2} \epsilon(X_{n-1}, Y_{i-1})(r_{V_1}(\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i]^C, \alpha(y_{i+1}), \dots, \alpha(y_{n-1})) \circ \beta_{V_1}) \otimes (\beta_{V_2} \circ \beta_{V_2})\\
&\quad + \epsilon(X_{n-1}, Y_{n-2})(r_{V_1}(\alpha(y_1), \dots, \alpha(y_{n-2}), \{x_1, \dots, x_{n-1}, y_{n-1}\}) \circ \beta_{V_1}) \otimes (\beta_{V_2} \circ \beta_{V_2})\\
&\quad = (l_{V_1}(\alpha(x_1), \dots, \alpha(x_{n-1})) \circ r_{V_1}(y_1, \dots, y_{n-1})) \otimes (\beta_{V_2} \circ \beta_{V_2})\\
&\quad + ((\beta_{V_1} \circ r_{V_1})(y_1, \dots, y_{n-1})) \otimes (l_{V_2}(\alpha(x_1), \dots, \alpha(x_{n-1})) \circ \alpha_{V_2}\\
&\quad - ((\beta_{V_1} \circ r_{V_1})(y_1, \dots, y_{n-1})) \otimes (r_{V_2}(\alpha(x_1), \dots, \alpha(x_{n-1})) \circ \alpha_{V_2}\\
&\quad - \epsilon(X_{n-1}, Y_{n-1})(r_{V_1}(\alpha(y_1), \dots, \alpha(y_{n-1}) \circ (l_{V_1}(x_1, \dots, x_{n-1})) \otimes (\beta_{V_2} \circ \beta_{V_2})\\
&\quad - \epsilon(X_{n-1}, Y_{n-1})(r_{V_1}(\alpha(y_1), \dots, \alpha(y_{n-1}) \circ \beta_{V_1})) \otimes (\beta_{V_2} \circ l_{V_2}(x_1, \dots, x_{n-1})
\end{aligned}$$

$$+\epsilon(X_{n-1},Y_{n-1})(r_{V_1}(\alpha(y_1),\dots,\alpha(y_{n-1})\circ\beta_{V_1}))\otimes(\beta_{V_2}\circ r_{V_2}(x_1,\dots,x_{n-1})$$
$$-\epsilon(X_{n-1},Y_{n-1})\sum_{i=1}^{n-1}(-1)^i\epsilon(x_i,X_{i+1,n-1})(r_{V_1}(\alpha(y_1),\dots,\alpha(y_{n-1})\circ r_{V_1}(x_1,\dots,\hat{x}_i,\dots,x_{n-1},x_i))\otimes(\beta_{V_2}\circ\beta_{V_2})$$
$$+\sum_{i=1}^{n-2}\epsilon(X_{n-1},Y_{i-1})(r_{V_1}(\alpha(y_1),\dots,\alpha(y_{i-1}),[x_1,\dots,x_{n-1},y_i]^C,\alpha(y_{i+1}),\dots,\alpha(y_{n-1}))\circ\beta_{V_1})\otimes(\beta_{V_2}\circ\beta_{V_2})$$
$$+\epsilon(X_{n-1},Y_{n-2})(r_{V_1}(\alpha(y_1),\dots,\alpha(y_{n-2}),\{x_1,\dots,x_{n-1},y_{n-1}\})\circ\beta_{V_1})\otimes(\beta_{V_2}\circ\beta_{V_2})$$
$$=0$$

- $$(r_{V_1}\otimes\beta_{V_2})([x_1,\dots,x_n]^C,\alpha(y_1),\dots,\alpha(y_{n-2}))\circ(\beta_{V_1}\otimes\beta_{V_2})$$

$$-\sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i,X_{i+1,n})(l_{V_1}\otimes\beta_{V_2}+\beta_{V_1}\otimes(l_{V_2}-r_{V_2}))(\alpha(x_1),\dots,\widehat{\alpha(x_i)}),\dots,\alpha(x_n))\circ(r_{V_1}\otimes\beta_{V_2})(x_i,y_1,\dots,y_{n-2})$$
$$=(r_{V_1}([x_1,\dots,x_n]^C,\alpha(y_1),\dots,\alpha(y_{n-2}))\circ\beta_{V_1})\otimes(\beta_{V_2}\circ\beta_{V_2})$$
$$-\sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i,X_{i+1,n})(l_{V_1}(\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\alpha(x_n))\circ(r_{V_1}(x_i,y_1,\dots,y_{n-2}))\otimes(\beta_{V_2}\circ\beta_{V_2})$$
$$-\sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i,X_{i+1,n})(\beta_{V_1}\circ r_{V_1}(x_i,y_1,\dots,y_{n-2}))\otimes(l_{V_2}(\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\alpha(x_n))\circ\beta_{V_2})$$
$$+\sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i,X_{i+1,n})(\beta_{V_1}\circ r_{V_1}(x_i,y_1,\dots,y_{n-2}))\otimes(r_{V_2}(\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\alpha(x_n))\circ\beta_{V_2})$$
$$=0.$$

By the same way, we show that the conditions (30) and (31) hold. The proposition is proved. □

Lemma 3.12. *Let (V,l,r,β) be a representation of an n-Hom-pre-Lie color algebra $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$. Then we have*

$$(\widetilde{\rho}^\star)^\star=\widetilde{\rho}\ \text{ and }\ (r^\star)^\star=r,$$

where $\widetilde{\rho}^\star$ and $r^\star$ defined by (38) *and* (39) *respectively.*

Proof. Let $x_i\in\mathcal{H}(\mathcal{N}),\ 1\le i\le n-1,\ \xi\in V^*$ and $u\in V$, then we have

$$\begin{aligned}<(\widetilde{\rho}^\star)^\star(x_1,\dots,x_{n-1})(u),\xi> &=<(\widetilde{\rho}^\star)^*(\alpha(x_1),\dots,\alpha(x_{n-1}))(\beta^2(u)),\xi>\\ &=-<\beta^2(u),\widetilde{\rho}^\star(\alpha(x_1),\dots,\alpha(x_{n-1}))(\xi)>\\ &=-<\beta^2(u),\widetilde{\rho}^*(\alpha^2(x_1),\dots,\alpha^2(x_{n-1}))((\beta^{-2})^*(\xi))>\\ &=<\widetilde{\rho}(\alpha^2(x_1),\dots,\alpha^2(x_{n-1}))(\beta^2(u)),((\beta^{-2})^*(\xi))>\\ &=<\widetilde{\rho}(x_1,\dots,x_{n-1})(u),\xi>,\end{aligned}$$

Then $(\widetilde{\rho}^\star)^\star=\widetilde{\rho}$. Similarly, we have $(r^\star)^\star=r$. □

Proposition 3.13. *Let (V,l,r,β) be a representation of an n-Hom-pre-Lie color algebra $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$, where β is invertible. Then the dual representation of $(V^*,\widetilde{\rho}^\star,-r^\star,(\beta^{-1})^*)$ is $(V,\widetilde{\rho},-r,\beta)$.*

Proof. It is obviously that $(V^*)^*=V$ and $(((\beta^{-1})^*)^{-1})^*=\beta$. Using also Lemma 3.12, we obtain the result. □

Proposition 3.14. *Let (V,l,r,β) be a representation of an n-Hom-pre-Lie color algebra $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$, where β is invertible. Then the following conditions are equivalent:*

1. *The quadruple $(V,\widetilde{\rho},-r,\beta)$ is a representation of the n-Hom-pre-Lie color algebra $(\mathcal{N},\{\cdot,\dots,\cdot\},\epsilon,\alpha)$.*

2. *The quadruple* $(V^*, \widetilde{\rho}^\star + r^\star, r^\star, (\beta^{-1})^*)$ *is a representation of the n-Hom-pre-Lie color algebra* $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$.
3. $r(\alpha(x_1), \dots, \alpha(x_{n-1}))r(y_1, \dots, y_{n-1}) = -\epsilon(X_{n-1}, Y_{n-1})r(\alpha(y_1), \dots, \alpha(y_{n-1}))r(x_1, \dots, x_{n-1})$, *for all* $x_i, y_i \in \mathcal{H}(\mathcal{N})$, $1 \leq i \leq n-1$.

Proof. The equivalence of conditions 1 and 2 is deduced directly from the Theorem 3.10 and Proposition 3.13. By using the fact that (V, l, r, β) is a representation of an n-Hom-pre-Lie color algebra $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$, we deduce that conditions 1 and 3 are equivalent. □

Corollary 3.15. *Let* $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$ *be a regular n-Hom-pre-Lie color algebra. Then* $(\mathcal{N}^*, ad^\star, -R^\star, (\beta^{-1})^*)$ *is a representation of* $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$.

Definition 3.3. *Let* $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$ *be an n-Hom-pre-Lie color algebra and* (V, l, r, β) *be a representation of* $\mathcal{N}$*. A linear map* $T : V \to \mathcal{N}$ *of degree zero is called Kupershmidht operator (called also* $\mathcal{O}$*-operator) associated to* (V, l, r, β) *if it satisfies*

$$\alpha \circ T = T \circ \beta, \tag{40}$$

$$\{Tu_1, \dots, Tu_n\} = T\Big(l(Tu_1, \dots, Tu_{n-1})(u_n) + \sum_{i=1}^{n-1}(-1)^{i+1}\epsilon(u_i, U_{i+1,n})r(Tu_1, \dots, \widehat{Tu_i}, \dots, Tu_n)(u_i)\Big), \tag{41}$$

$\forall u_i \in \mathcal{H}(V)$, $1 \leq i \leq n$*. If* $(V, l, r, \beta) = (\mathcal{N}, L, R, \alpha)$*, then T is called a Rota-Baxter operator on* $\mathcal{N}$ *of weight zero denoted by P.*

Proposition 3.16. *Let P be a Rota-Baxter operator of weight zero on an n-Hom-pre-Lie color algebra* $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$*. Then* $(\mathcal{N}, \{\cdot, \dots, \cdot\}_P, \epsilon, \alpha)$ *is an n-Hom-pre-Lie color algebra where* $\{\cdot, \dots, \cdot\}_P$ *is defined by*

$$\{x_1, \dots, x_n\}_P = \sum_{i=1}^{n}\{P(x_1), \dots, P(x_{i-1}), x_i, P(x_{i+1}), \dots, P(x_n)\},$$

for all $x_i \in \mathcal{H}(\mathcal{N})$, $1 \leq i \leq n$.

To show this proposition we need the following lemma.

Lemma 3.17. *Let P be a Rota-Baxter operator of weight zero on an n-Hom-pre-Lie color algebra* $(\mathcal{N}, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$*. Then P is a Rota-Baxter operator of weight zero on the sub-adjacent n-Hom-Lie color algebra* $\mathcal{N}^C$.

Proof. It is obvious that $\alpha \circ P = P \circ \alpha$.
Let $x_i \in \mathcal{H}(\mathcal{N})$, $1 \leq i \leq n$, we have

$$\begin{aligned}
[P(x_1), \dots, P(x_n)]^C &= \sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i, X_{i+1,n})\{P(x_1), \dots, \hat{P(x_i)}, \dots, P(x_n), P(x_i)\} \\
&= \sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i, X_{i+1,n})P\Big(\sum_{j=1}^{n}\{P(x_1), \dots, x_j, \dots, \hat{P(x_i)}, \dots, P(x_n), P(x_i)\}\Big) \\
&= P\Big(\sum_{j=1}^{n}\sum_{i=1}^{n}(-1)^{n-i}\epsilon(x_i, X_{i+1,n})\{P(x_1), \dots, x_j, \dots, \hat{P(x_i)}, \dots, P(x_n), P(x_i)\}\Big) \\
&= P\Big(\sum_{j=1}^{n}[P(x_1), \dots, x_j, \dots, P(x_n)]^C\Big).
\end{aligned}$$

Then P is a Rota-Baxter operator of weight zero on $(\mathcal{N}^C, [\cdot, \dots, \cdot]^C, \epsilon, \alpha)$ □

Remark 3.2. $P(\{x_1, \dots, x_n\}_p) = \{P(x_1), \dots, P(x_n)\}$, $\forall x_i \in \mathcal{H}(\mathcal{N})$, $1 \leq i \leq n$.

Proof. **Proof of Proposition** 3.16 Let $x_i, y_i \in \mathcal{H}(\mathcal{N}),\ 1 \leq i \leq n-1.$ By using Remark 3.2 and condition (19), we have

$$\begin{aligned}
&\{\alpha(x_1),\dots,\alpha(x_{n-1}),\{y_1,\dots,y_n\}_P\}_P = \sum_{i=1}^{n-1}\{P(\alpha(x_1)),\dots,\alpha(x_i),\dots,P(\alpha(x_{n-1})),P(\{y_1,\dots,y_n\}_P)\}\\
&+\{P(\alpha(x_1)),\dots,P(\alpha(x_{n-1})),\{y_1,\dots,y_n\}_P\}\\
&=\sum_{i=1}^{n-1}\{P(\alpha(x_1)),\dots,\alpha(x_i),\dots,P(\alpha(x_{n-1})),\{P(y_1),\dots,P(y_n)\}\}\\
&+\sum_{i=1}^{n}\{P(\alpha(x_1)),\dots,P(\alpha(x_{n-1})),\{P(y_1),\dots,y_i,\dots,P(y_n)\}\}\\
&=\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\epsilon(X_{n-1},Y_{j-1})\{\alpha(P(y_1)),\dots,[P(x_1),\dots,x_i,\dots,P(x_{n-1}),P(y_j)]^C,\dots,\alpha(P(y_n))\}\\
&+\epsilon(X_{n-1},Y_{n-1})\{\alpha(P(y_1)),\dots,\alpha(P(y_{n-1})),\{P(x_1),\dots,x_i,\dots,P(x_i),P(y_n)\}\}\\
&+\sum_{i=1}^{n}\sum_{j=1}^{i-1}\epsilon(X_{n-1},Y_{j-1})\{\alpha(P(y_1)),\dots,[P(x_1),\dots,P(x_{n-1}),P(y_j)]^C,\dots,\alpha(y_i),\dots,\alpha(P(y_n))\}\\
&+\sum_{i=1}^{n}\sum_{j=i+1}^{n-1}\epsilon(X_{n-1},Y_{j-1})\{\alpha(P(y_1)),\dots,\alpha(y_i),\dots,[P(x_1),\dots,P(x_{n-1}),P(y_j)]^C,\dots,\alpha(P(y_n))\}\\
&+\sum_{i=1}^{n}\epsilon(X_{n-1},Y_{i-1})\{\alpha(P(y_1)),\dots,[P(x_1),\dots,P(x_{n-1}),y_i]^C,\dots,\alpha(P(y_n))\}\\
&+\epsilon(X_{n-1},Y_{n-1})\{\alpha(P(y_1)),\dots,\alpha(y_i),\dots,\alpha(P(y_{n-1})),\{P(x_1),\dots,P(x_{n-1}),P(y_n)\}\}.
\end{aligned}$$

By using (19), a direct computation gives that

$$\begin{aligned}
\{\alpha(x_1),\dots,\alpha(x_{n-1}),\{y_1,\dots,y_n\}_P\}_P &= \sum_{i=1}^{n-1}\epsilon(X_{n-1},Y_{i-1})\{\alpha(y_1),\dots,\alpha(y_{i-1}),\{x_1,\dots,x_{n-1},y_i\}_P,\alpha(y_{i+1}),\dots,\alpha(y_n)\}_P\\
&+\epsilon(X_{n-1},Y_{n-1})\{\alpha(y_1),\dots,\alpha(y_{n-1}),\{x_1,\dots,x_{n-1},y_n\}_P\}_P,
\end{aligned}$$

which implies that $\{\cdot,\dots,\cdot\}_P$ satisfies the condition (19). By the same way, we show that the condition (20) satisfies by $\{\cdot,\dots,\cdot\}_P$. Then $\{\cdot,\dots,\cdot\}_P$ gives an n-Hom-pre-Lie color algebra structure on $\mathcal{N}$. □

Proposition 3.18. *Let (P_1, P_2) be a pair of commuting Rota-Baxter operators (of weight zero) on an n-Hom-Lie color algebra $(\mathcal{N},[\cdot,\dots,\cdot],\epsilon,\alpha)$. Then P_2 is a Rota-Baxter operator (of weight zero) on the associated n-Hom-pre-Lie color algebra defined by*

$$\{x_1,\dots,x_n\} = [P_1(x_1),\dots,P_1(x_{n-1}),x_n],\ \forall x_i \in \mathcal{H}(\mathcal{N}), 1 \leq i \leq n.$$

Proof. For any $x_i \in \mathcal{H}(\mathcal{N}),\ 1 \leq i \leq n$, we have

$$\begin{aligned}
&\alpha\circ P_2 = P_2\circ\alpha\\
&\{P_2(x_1),\dots,P_2(x_n)\} = [P_1(P_2(x_1)),\dots,P_1(P_2(x_{n-1})),P_2(x_n)]\\
&=[P_2(P_1(x_1)),\dots,P_2(P_1(x_{n-1})),P_2(x_n)]\\
&=P_2\Big([P_2(P_1(x_1)),\dots,P_2(P_1(x_{n-1})),x_n]\\
&+\sum_{i=1}^{n-1}(-1)^{n-i}\epsilon(x_i,X_{i+1,n})[P_2(P_1(x_1)),\dots,P_2(\widehat{P_1(x_i)}),\dots,P_2(P_1(x_{n-1})),P_1(x_i)]\Big)\\
&=P_2\Big(\{P_2(x_1),\dots,P_2(x_{n-1}),x_n\}\\
&+\sum_{i=1}^{n-1}(-1)^{i+1}\epsilon(x_i,X_{i+1,n})\{x_i,P_2(x_1),\dots,\widehat{P_2(x_i)},\dots,P_2(x_{n-1}),P_2(x_n)\}\Big).
\end{aligned}$$

Then P_2 is a Rota-Baxter operator (of weight zero) on the n-Hom-pre-Lie color algebra $(A, \{\cdot, \dots, \cdot\}, \epsilon, \alpha)$. □

4. n-Hom-pre-Lie color algebras induced by Hom-pre-Lie color algebras

In [34], the authors introduced the construction of an $(n+1)$-pre-Lie algebra from an n-pre-Lie algebra using the trace map. In this section we generalize this construction to the color case of Hom-type by a new approach which is the construction of an n-Hom-pre-Lie color algebra from a Hom-pre-Lie color algebra, the same work has been studied in the Hom-Lie super case (see [43]). We start with the data of an homogeneous ϵ-skew-symmetric $(n-2)$-linear form $\Phi : \mathcal{N} \times \cdots \times \mathcal{N} \to \mathbb{K}$ of degree zero (i.e. $\Phi(x_1, \dots, x_{n-2}) = 0,\ \forall x_i \in \mathcal{H}(\mathcal{N}),\ 1 \leq i \leq n-2$, where $|x_1| + \cdots + |x_{n-2}| \neq 0$) and we define from this form an n-linear map ϵ-skew-symmetric on the first $(n-1)$ variables. A Hom-pre-Lie color algebra is a quadruple $(\mathcal{N}, \circ, \epsilon, \alpha)$ consisting of a Γ-graded vector space $\mathcal{N}$, an homogeneous bilinear map $\circ : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ of degree zero and an homogeneous linear map $\alpha : \mathcal{N} \to \mathcal{N}$ of degree zero such that, the following condition hold:

$$\mathfrak{ass}(x, y, z) - \epsilon(x, y)\mathfrak{ass}(y, x, z) = 0,\ \forall x, y, z \in \mathcal{H}(\mathcal{N}), \tag{42}$$

where, $\mathfrak{ass}(x, y, z) = \alpha(x) \circ (y \circ z) - (x \circ y) \circ \alpha(z)$.

Let $(\mathcal{N}, \circ, \epsilon, \alpha)$ be a Hom-pre-Lie color algebra. Define the n-ary product as follows:

$$\{x_1, \dots, x_{n-1}, x_n\}_\Phi = \sum_{k=1}^{n-1} (-1)^{k+1} \epsilon(x_k, X_{k+1,n-1}) \Phi(x_1, \dots, \hat{x_k}, \dots, x_{n-1})(x_k \circ x_n), \tag{43}$$

for all $x_k \in \mathcal{H}(\mathcal{N}),\ 1 \leq k \leq n$.

It is clear that $\{\cdot, \dots, \cdot\}_\Phi$ is an homogeneous n-linear map of degree zero.

Proposition 4.1. *The n-ary product $\{\cdot, \dots, \cdot\}_\Phi$ is ϵ-skew-symmetric on the first $(n-1)$ variables.*

Proof. Let $x_1, \dots, x_n \in \mathcal{H}(\mathcal{N})$, then for all $i \in \{1, \dots, n-2\}$, we have

$$\begin{aligned}
&\{x_1, \dots, x_i, x_{i+1}, \dots, x_{n-1}, x_n\}_\Phi \\
=& \sum_{1 \leq k \neq i, i+1 < n-1} (-1)^{k+1} \epsilon(x_k, X_{k+1,n-1}) \Phi(x_1, \dots, x_i, x_{i+1}, \dots, \hat{x_k}, \dots, x_{n-1})(x_k \circ x_n) \\
&+ (-1)^{i+1} \epsilon(x_i, X_{i+1,n-1}) \Phi(x_1, \dots, \hat{x_i}, x_{i+1}, \dots, x_{n-1})(x_i \circ x_n) \\
&+ (-1)^{i} \epsilon(x_{i+1}, X_{i+2,n-1}) \Phi(x_1, \dots, x_i, \widehat{x_{i+1}}, \dots, x_{n-1})(x_i \circ x_n) \\
=& -\epsilon(x_i, x_{i+1})\Big(\sum_{1 \leq k \neq i, i+1 < n-1} (-1)^{k+1} \epsilon(x_k, X_{k+1,n-1}) \Phi(x_1, \dots, x_{i+1}, x_i, \dots, \hat{x_k}, \dots, x_{n-1})(x_k \circ x_n) \\
&+ (-1)^{i} \epsilon(x_i, X_{i+2,n-1}) \Phi(x_1, \dots, \hat{x_i}, x_{i+1}, \dots, x_{n-1})(_{i+1} \circ x_n) \\
&+ (-1)^{i+1} \epsilon(x_i, X_{i+1,n-1}) \Phi(x_1, \dots, x_i, \widehat{x_{i+1}}, \dots, x_{n-1})_{i+1} \circ x_n) \Big) \\
=& -\epsilon(x_i, x_{i+1}) \{x_1, \dots, x_{i+1}, x_i, \dots, x_{n-1}, x_n\}_\Phi,
\end{aligned}$$

which gives that $\{\cdot, \dots, \cdot\}_\Phi$ is ϵ-skew-symmetric on the first $(n-1)$ terms. □

Theorem 4.2. *Let $(\mathcal{N}, \circ, \alpha)$ be a Hom-pre-Lie color algebra and $\Phi : \mathcal{N} \times \cdots \times \mathcal{N} \to \mathbb{K}$ be an $(n-2)$-linear ϵ-skew-symmetric form of degree zero. Then $(\mathcal{N}, \{\cdot, \dots, \cdot\}_\Phi, \epsilon, \alpha)$ is an n-Hom-pre-Lie color algebra if and only if:*

$$\Phi(x_1 \dots, x_i, y \circ z, x_{i+1}, \dots, x_{n-3}) = 0,\ \forall x_i, y, z \in \mathcal{H}(\mathcal{N}),\ 1 \leq i \leq n-3, \tag{44}$$

$$\Phi \circ (\alpha \otimes Id \otimes \cdots \otimes Id) = \Phi, \tag{45}$$

$$\Phi \wedge \delta\Phi_X = 0,\ \forall X = (x_1, \dots, x_{n-3}) \in \wedge^{n-3}\mathcal{H}(\mathcal{N}), \tag{46}$$

where

$$\Phi\wedge\delta\Phi_X(Y)=\sum_{k=1}^{n-1}(-1)^{k+1}\epsilon(y_k,Y_{k+1,n-1})\Phi(y_1,\dots,\hat{y_k},\dots,y_{n-1})\Phi(X,y_k),\ \forall Y=(y_1,\dots,y_{n-1})\in\wedge^{n-1}\mathcal{H}(\mathcal{N}),$$

and $\{\cdot,\dots,\cdot\}_\Phi$ *is defined by Eq.*(43)*. We shall say that* $(\mathcal{N},\{\cdot,\dots,\cdot\}_\Phi,\epsilon,\alpha)$ *is induced by* $(\mathcal{N},\circ,\epsilon,\alpha)$*.*

Lemma 4.3. *If an* ϵ*-skew-symmetric* $(n-2)$*-linear form* $\Phi:\mathcal{N}\times\cdots\times\mathcal{N}\to\mathbb{K}$ *of degree zero satisfies condition* (44)*, then it satisfies the following condition*

$$\Phi(x_1,\dots,x_i,\{y_1,\dots,y_n\}_\Phi,x_{i+1},\dots,x_{n-3})=0,\tag{47}$$

for all $x_i,y_j\in\mathcal{H}(\mathcal{N}),\ (i,j)\in\{1,\dots,n-3\}\times\{1,\dots,n\}$*.*

Proof. This is a direct computation, by using the expression of $\{\cdot,\dots,\cdot\}_\Phi$. □

Proof. [Proof of Theorem 4.2]
Let $x_i,y_i\in\mathcal{H}(\mathcal{N}),\ 1\le i\le n$.

On the one hand, we have:

$$\begin{aligned}M&=\{\alpha(x_1),\dots,\alpha(x_{n-1}),\{y_1,\dots,y_n\}_\Phi\}_\Phi\\&=\sum_{i=1}^{n-1}(-1)^{i+1}\epsilon(x_i,X_{i+1,n-1})\Phi(\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\alpha(x_{n-1}))(\alpha(x_i)\circ\{y_1,\dots,y_n\}_\Phi)\\&=\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}\epsilon(x_i,X_{i+1,n-1})\epsilon(y_j,Y_{j+1,n-1})\Phi(\alpha(x_1),\dots,\widehat{\alpha(x_i)},\dots,\alpha(x_{n-1}))\Phi(y_1,\dots,\hat{y_j},\dots,y_{n-1})(\alpha(x_i)\circ(y_j\circ y_n)).\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}N&=\sum_{j=1}^{n-1}\epsilon(X_{n-1},Y_{j-1})\{\alpha(y_1),\dots,\alpha(y_{j-1}),[x_1,\dots,x_{n-1},y_j]^C_\Phi,\alpha(y_{j+1}),\dots,\alpha(y_n)\}_\Phi\\&+\epsilon(X_{n-1},Y_{n-1})\{\alpha(y_1),\dots,\alpha(y_{n-1}),\{x_1,\dots,x_{n-1},y_n\}_\Phi\}_\Phi\\&=\sum_{i,j=1}^{n-1}(-1)^{n-i}\epsilon(X_{n-1},Y_{j-1})\epsilon(x_i,X_{i+1,n-1}+y_j)\{\alpha(y_1),\dots,\alpha(y_{j-1}),\{x_1,\dots,\hat{x_i},\dots,x_{n-1},y_j,x_i\}_\Phi,\alpha(y_{j+1}),\dots,\alpha(y_n)\}_\Phi\\&+\sum_{j=1}^{n-1}\epsilon(X_{n-1},Y_{j-1})\{\alpha(y_1),\dots,\alpha(y_{j-1}),\{x_1,\dots,x_{n-1},y_j\}_\Phi,\alpha(y_{j+1}),\dots,\alpha(y_n)\}_\Phi\\&+\epsilon(X_{n-1},Y_{n-1})\{\alpha(y_1),\dots,\alpha(y_{n-1}),\{x_1,\dots,x_{n-1},y_n\}_\Phi\}_\Phi\\&=N_1+N_2+N_3,\end{aligned}$$

where

$$\begin{aligned}N_1&=\sum_{i,j=1}^{n-1}(-1)^{n-i}\epsilon(X_{n-1},Y_{j-1})\epsilon(x_i,X_{i+1,n-1}+y_j)\{\alpha(y_1),\dots,\alpha(y_{j-1}),\{x_1,\dots,\hat{x_i},\dots,x_{n-1},y_j,x_i\}_\Phi,\alpha(y_{j+1}),\dots,\alpha(y_n)\}_\Phi\\&=\sum_{i,j=1}^{n-1}\sum_{k=1}^{j-1}(-1)^{n-i+k+1}\gamma_{ij}\Phi(\alpha(y_1),\dots,\widehat{\alpha(y_k)},\dots,\alpha(y_{j-1}),\{x_1,\dots,\hat{x_i},\dots,x_{n-1},y_j,x_i\}_\Phi,\dots,\alpha(y_{n-1}))(\alpha(y_k)\circ\alpha(y_n))\\&+\sum_{i,j=1}^{n-1}\sum_{k=j+1}^{n-1}(-1)^{n-i+k+1}\gamma_{ij}\epsilon(y_k,X_{n-1})\Phi\Big(\alpha(y_1),\dots,\alpha(y_{j-1}),\{x_1,\dots,\hat{x_i},\dots,x_{n-1},y_j,x_i\}_\Phi,\dots\\&\qquad\dots,\widehat{\alpha(y_k)},\dots,\alpha(y_{n-1})\Big)(\alpha(y_k)\circ\alpha(y_n))\end{aligned}$$

$$+\sum_{i,j=1}^{n-1}(-1)^{n-i+j+1}\theta_{ij}\Phi(\alpha(y_1),\ldots,\widehat{\alpha(y_j)},\ldots,\alpha(y_{n-1}))(\{x_1,\ldots,\hat{x}_i,\ldots,x_{n-1},y_j,x_i\}_\Phi\circ\alpha(y_n))$$

where

$$\gamma_{ij}=\epsilon(X_{n-1},Y_{j-1})\epsilon(x_i,X_{i+1,n-1}+y_j)\epsilon(y_k,X_{n-1}+Y_{k+1,n-1}),$$

and

$$\theta_{ij}=\epsilon(X_{n-1},Y_{j-1})\epsilon(x_i,X_{i+1,n-1}+y_j)\epsilon(Y_{j+1,n-1},X_{n-1}+y_j).$$

Using Eq. (47), we notice that the first two terms of the second equality of N_1 are zero, which gives that

$$\begin{aligned}
N_1&=\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(-1)^{n-i+j+1}\theta_{ij}\Phi(\alpha(y_1),\ldots,\widehat{\alpha(y_j)},\ldots,\alpha(y_{n-1}))(\{x_1,\ldots,\hat{x}_i,\ldots,x_{n-1},y_j,x_i\}_\Phi\circ\alpha(y_n))\\
&=\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\sum_{\substack{k=1\\k\neq i}}^{n-1}(-1)^{n-i+j+1}\theta_{ijk}\Phi(\alpha(y_1),\ldots,\widehat{\alpha(y_j)},\ldots,\alpha(y_{n-1}))\Phi(x_1,\ldots,\hat{x}_k,\ldots,\hat{x}_i,\ldots,x_{n-1},y_j)((x_k\circ x_i)\circ\alpha(y_n))\\
&+\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(-1)^{i+j}\theta_{ij}\Phi((\alpha(y_1),\ldots,\widehat{\alpha(y_j)},\ldots,\alpha(y_{n-1}))\Phi(x_1,\ldots,\hat{x}_i,\ldots,x_{n-1})((y_j\circ x_i)\circ\alpha(y_n))\\
&=N_1'+N_2',
\end{aligned}$$

where $\theta_{ijk}=\theta_{ij}\epsilon(x_k,X_{k+1,n-1}+x_i+y_j)$.

By the same way, we have:

$$\begin{aligned}
N_2&=\sum_{j=1}^{n-1}\epsilon(X_{n-1},Y_{j-1})\{\alpha(y_1),\ldots,\alpha(y_{j-1}),\{x_1,\ldots,x_{n-1},y_j\}_\Phi,\alpha(y_{j+1}),\ldots,\alpha(y_n)\}_\Phi\\
&=\sum_{j=1}^{n-1}(-1)^{j+1}\epsilon(X_{n-1},Y_{j-1})\epsilon(Y_{j+1,n-1}+y_j)\Phi(\alpha(y_1),\ldots,\widehat{\alpha(y_j)},\ldots,\alpha(y_{n-1}))(\{x_1,\ldots,x_{n-1},y_j\}_\Phi\circ\alpha(y_n))\\
&=\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(-1)^{i+j}\lambda_{ij}\Phi(\alpha(y_1),\ldots,\widehat{\alpha(y_j)},\ldots,\alpha(y_{n-1}))\Phi(x_1,\ldots,\hat{x}_i,\ldots,x_{n-1})((x_i\circ y_j)\circ\alpha(y_n)),
\end{aligned}$$

where $\lambda_{ij}=\epsilon(X_{n-1}Y_{j-1})\epsilon(Y_{j+1,n-1},X_{n-1}+y_j)\epsilon(x_i,X_{i+1,n-1})=\theta_{ij}\epsilon(x_i,y_j)$,

and

$$\begin{aligned}
N_3&=\epsilon(X_{n-1},Y_{n-1})\{\alpha(y_1),\ldots,\alpha(y_{n-1}),\{x_1,\ldots,x_{n-1},y_n\}_\Phi\}_\Phi\\
&=\sum_{j=1}^{n-1}(-1)^{j+1}\epsilon(X_{n-1},Y_{n-1})\epsilon(y_j,Y_{j+1,n-1})\Phi(\alpha(y_1),\ldots,\widehat{\alpha(y_j)},\ldots,\alpha(y_{n-1}))(\alpha(y_j)\circ\{x_1,\ldots,x_{n-1},y_n\}_\Phi)\\
&=\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(-1)^{i+j}\nu_{ij}\Phi(\alpha(y_1),\ldots,\widehat{\alpha(y_j)},\ldots,\alpha(y_{n-1}))\Phi(x_1,\ldots,\hat{x}_i,\ldots,x_{n-1})(\alpha(y_j)\circ(x_i\circ y_n)),
\end{aligned}$$

where $\nu_{ij}=\epsilon(X_{n-1},Y_{n-1})\epsilon(y_j,Y_{j+1,n-1})\epsilon(x_i,X_{i+1,n-1})=\epsilon(x_i,y_j)\epsilon(y_j,Y_{j+1,n-1})\epsilon(x_i,X_{i+1,n-1})$, since Φ is of degree zero.

If we fixed i,k in the expression of N_1', then , we get:

$$N_1'=\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\sum_{\substack{k=1\\k\neq i}}^{n-1}(-1)^{n-i+j+1}\theta_{ijk}\Phi(\alpha(y_1),\ldots,\alpha(\hat{y}_j),\ldots,\alpha(y_{n-1}))\Phi(x_1,\ldots,\hat{x}_k,\ldots,\hat{x}_i,\ldots,x_{n-1},y_j)((x_k\circ x_i)\circ\alpha(y_n))$$

$$= \sum_{i=1}^{n-1}\sum_{\substack{k=1\\k\neq i}}^{n-1}(-1)^{n-i}\Big(\sum_{j=1}^{n-1}(-1)^{j+1}\theta_{ijk}\Phi(\alpha(y_1),\ldots,\alpha(\hat{y}_j),\ldots,\alpha(y_{n-1}))\Phi(x_1,\ldots,\hat{x}_k,\ldots,\hat{x}_i,\ldots,x_{n-1},y_j)\Big)((x_k\circ x_i)\circ\alpha(y_n))$$
$$= 0,$$

this from Eq. (46) and the fact that Φ is of degree zero. Moreover, if we apply condition (45), we find

$$\begin{aligned}M - N &= M - N_2' - N_2 - N_3\\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}\epsilon(x_i, X_{i+1,n-1}\epsilon(y_j, Y_{j+1,n-1}\Phi(x_1,\ldots,\widehat{\alpha(x_i)},\ldots,x_{n-1})\Phi(y_1,\ldots,\hat{y}_j,\ldots,y_{n-1})\\ &\qquad\Big(\alpha(x_i)\circ(y_j\circ y_n)-(x_i\circ y_j)\circ\alpha(y_n))-\epsilon(x_i,y_j)(\alpha(y_j)\circ(x_i\circ y_n)-(y_j\circ x_i)\circ\alpha(y_n)))\Big)\\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}\epsilon(x_i, X_{i+1,n-1})\epsilon(y_j, Y_{j+1,n-1}\Phi(x_1,\ldots,\widehat{\alpha(x_i)},\ldots,x_{n-1})\Phi(y_1,\ldots,\hat{y}_j,\ldots,y_{n-1})\\ &\qquad\Big(\mathfrak{ass}(x_i,y_j,y_n)-\epsilon(x_i,y_j)\mathfrak{ass}(y_j,x_i,y_n)\Big)\\ &= 0,\end{aligned}$$

The last equality follows from the fact that $(\mathcal{N},\circ,\epsilon,\alpha)$ is a Hom-pre-Lie color algebra. Then $\{\cdot,\ldots,\cdot\}_\Phi$ satisfies condition (19) on $\mathcal{N}$. Similarly, we show that $\{\cdot,\ldots,\cdot\}_\Phi$ satisfies condition (20) on $\mathcal{N}$. The theorem is proved. □

Let us given a Hom-pre-Lie color algebra $(\mathcal{N},\circ,\epsilon,\alpha)$ and a bilinear form $\Phi:\mathcal{N}\times\mathcal{N}\to\mathbb{K}$ of degree zero satisfying conditions (44), (45) and (46). Then by Theorem 4.2, the quadruple $(\mathcal{N},\{\cdot,\cdot,\cdot,\cdot\}_\Phi,\epsilon,\alpha)$ is a 4-Hom-pre-Lie color algebra.

Example 4.4. *Let $\Gamma=\mathbb{Z}_2$ and $\epsilon(i,j)=(-1)^{ij}$. Let $\mathcal{N}=\mathcal{N}_{\bar 0}\oplus\mathcal{N}_{\bar 1}$ be a two dimensional $\mathbb{Z}_2$-vector space with a basis $\{e_1,e_2\}$, where $\mathcal{N}_{\bar 0}=<e_1>$ and $\mathcal{N}_{\bar 1}=<e_2>$. Define on the basis of $\mathcal{N}$ the bilinear map $\circ:\mathcal{N}\times\mathcal{N}\to\mathcal{N}$ of degree zero by:*

$$e_2\circ e_2=e_1,$$

and the linear map $\alpha:\mathcal{N}\to\mathcal{N}$ by:

$$\alpha(e_1)=0,\quad \alpha(e_2)=e_2.$$

Then $(\mathcal{N},\circ,\epsilon,\alpha)$ is a Hom-pre-Lie color algebra.

Now, we define the ϵ-skew-symmetric bilinear form $\Phi:\mathcal{N}\times\mathcal{N}\to\mathbb{K}$ by

$$\Phi(e_2,e_2)=\lambda,\ \lambda\in\mathbb{K}^*.$$

It is obvious that Φ satisfies the conditions (44)-(46). Then, by Theorem 4.2, the quadruple $(\mathcal{N},\{\cdot,\cdot,\cdot,\cdot\}_\Phi,\epsilon,\alpha)$ is a 4-Hom-pre-Lie color algebra, where $\{\cdot,\cdot,\cdot,\cdot\}_\Phi$ defined on the basis of $\mathcal{N}$ by

$$\{e_2,e_2,e_2,e_2\}_\Phi=\lambda e_1.$$

Quite normal and thanks to the importance of the representation theory, any reader asks the following question: Can we also extend this work to representation, i.e. is it possible to construct an n-Hom-pre-Lie color algebra representation from a Hom-pre-Lie color algebra representation? The answer is yes, but the question requires us to define the representation of a Hom-pre-Lie color algebra $(\mathcal{N},\circ,\epsilon,\alpha)$ which is defined as a quadruple (V,l,r,β) consisting of a Γ-graded vector space V, two linear maps $l,r:\mathcal{N}\to End(V)$ of degree zero and a linear map $\beta:V\to V$ of degree zero such that the following conditions hold:

$$\beta l(x)=l(\alpha(x))\beta,\ \forall x\in\mathcal{H}(\mathcal{N}),\tag{48}$$
$$\beta r(x)=r(\alpha(x))\beta,\ \forall x\in\mathcal{H}(\mathcal{N}),\tag{49}$$
$$l([x,y]^C)\beta=l(\alpha(x))l(y)-\epsilon(x,y)l(\alpha(y))l(x),\ \forall x,y\in\mathcal{H}(\mathcal{N}),\tag{50}$$

$$r(\alpha(y))r(x) - r(x \circ y)\beta = r(\alpha(y))l(x) - \epsilon(x,y)r(\alpha(x))l(y), \ \forall x, y \in \mathcal{H}(\mathcal{N}), \tag{51}$$

where $[x,y]^C = x \circ y - \epsilon(x,y)y \circ x, \ \forall x, y \in \mathcal{H}(\mathcal{N})$, which is defined a Hom-Lie color algebra on $\mathcal{N}$ called the subadjacent Hom-Lie color algebra of $(\mathcal{N}, \circ, \epsilon, \alpha)$.

Remark 4.1. *Conditions* (48) *and* (50) *are equivalent to saying that l is a representation of the Hom-Lie color algebra $(\mathcal{N}, [\cdot,\cdot]^C, \epsilon, \alpha)$ with respect to β.*

In the sequel, we allow to answer the previous question which is summarized by the following proposition.

Proposition 4.5. *Let (V, l, r, β) be a representation of a Hom-pre-Lie color algebra $(\mathcal{N}, \circ, \epsilon, \alpha)$ and Φ be an ϵ-skew-symmetric $(n-2)$-linear form of degree zero satisfying conditions* (44)-(46). *Then $(V, l_\Phi, r_\Phi, \beta)$ is a representation of the induced n-Hom-pre-Lie color algebra $(\mathcal{N}, \{\cdot,\dots,\cdot\}_\Phi, \epsilon, \alpha)$, where $l_\Phi, r_\Phi : \mathcal{N} \times \cdots \times \mathcal{N} \to End(V)$ are two $(n-1)$-linear maps of degree zero defined by*

$$l_\Phi(x_1,\dots,x_{n-1}) = \sum_{k=1}^{n-1} (-1)^{k+1} \epsilon(x_k, X_{k+1,n-1}\Phi(x_1,\dots,\hat{x_k},\dots,x_{n-1})l(x_k), \tag{52}$$

$$r_\Phi(x_1,\dots,x_{n-2},x_{n-1}) = \Phi(x_1,\dots,x_{n-2})r(x_{n-1}), \tag{53}$$

for al $x_k \in \mathcal{H}(\mathcal{N}),\ 1 \le k \le n-1$.

Proof. Let (V, l, r, β) be a representation of $(\mathcal{N}, \circ, \epsilon, \alpha)$. Defining the map $\Phi_{\mathcal{N}\oplus V} : \mathcal{N} \oplus V \times \cdots \times \mathcal{N} \oplus V \to \mathbb{K}$ by

$$\Phi_{\mathcal{N}\oplus V}(x_1 + u_1, \dots, x_{n-2} + u_{n-2}) = \Phi(x_1,\dots,x_{n-2}),\ x_i \in \mathcal{H}(\mathcal{N}),\ u_i \in \mathcal{H}(V),\ 1 \le i \le n-2.$$

Then $\Phi_{\mathcal{N}\oplus V}$ satisfying the conditions (44)-(46) on the semi-direct product n-Hom-pre-Lie color algebra $\mathcal{N} \ltimes^{\alpha,\beta}_{l,r} V$. Then by (43), we have an n-Hom-pre-Lie color algebra structure on $\mathcal{N} \oplus V$ given by

$$\begin{aligned}
&\{x_1 + u_1, \dots, x_n + u_n\}_{\Phi_{\mathcal{N}\oplus V}} \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} \epsilon(x_k, X_{k+1,n-1}) \Phi_{\mathcal{N}\oplus V}(x_1 + u_1, \dots, \widehat{x_k + u_k}, \dots, x_{n-1} + u_{n-1}) \Big((x_k + u_k) \circ_{\mathcal{N}\oplus V} (x_n + u_n) \Big) \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} \epsilon(x_k, X_{k+1,n-1}) \Phi(x_1,\dots,\hat{x_k},\dots,x_{n-1}) \Big(x_k \circ x_n + l(x_k)(u_n) + (-1)^{|x_k||x_n|} r(x_n)(u_k) \Big) \\
&= \{x_1,\dots,x_n\}_\Phi + l_\Phi(x_1,\dots,x_{n-1})(u_n) + \sum_{k=1}^{n-1} (-1)^{k+1} \epsilon(x_k, X_{k+1,n-1}) r_\Phi(x_1,\dots,\hat{x_k},\dots,x_n)(u_k).
\end{aligned}$$

By applying Proposition 3.7, we deduce that $(V, l_\Phi, r_\Phi, \beta)$ is a representation of $(\mathcal{N}, \{\cdot,\dots,\cdot\}_\Phi, \epsilon, \alpha)$. □

References

[1] K. Abdaoui, S. Mabrouk and A. Makhlouf, *Rota–Baxter operators on pre-Lie superalgebras*, Bull. Malays. Math. Sci. Soc. **42**(4), 1567–1606 (2019).

[2] K. Abdaoui, S. Mabrouk and A. Makhlouf, *Cohomology of Hom-Leibniz and n-ary Hom-Nambu-Lie superalgebras*. arXiv:1406.3776v1 (2014).

[3] K. Abdaoui, R. Gharbi, S. Mabrouk and A. Makhlouf, *Cohomology and Formal Deformations of n-Hom–Lie color algebras*. Ukr. Math. J. **75**(9), 1313–1339 (2024).

[4] M. Aguiar and J. L. Loday, *Quadri-algebras*. J. Pure. Appl. Alg. vol. **191**: 251–221 (2004).

[5] F. Ammar, S. Mabrouk and A. Makhlouf, *Representations and cohomology of n-ary multiplicative Hom–Nambu–Lie algebras*. J. Geo. Phy. **61** , 1898–1913 (2011).

[6] A. Andrada and S. Salamon, *Complex product structure on Lie algebras*. Forum. Math. **17** , 261–295 (2005).

[7] A. Armakan, S. Silvestrov, M. Farhangdoost, *Enveloping algebras of color hom-Lie algebras*. Turk. J. Math. **43**, 316–339 (2019).

[8] A. Armakan, S. Silvestrov, M. R. Farhangdoost, *Extensions of hom- Lie color algebras*. Geor. Math. J. **28**(1), 15-27 (2021) (arXiv:1709.08620 [math.QA])
[9] H. Ataguema ,A. Makhlouf and S. Silvestrov , *Generalization of n-ary Nambu algebras and beyond*. J. Math. Phy. **50**(1) (2009).
[10] B. Bakalov and V. Kac, *Field algebras*. Int. Math. Res. Not. **3** 123–159 (2003).
[11] Bakayoko, I., Silvestrov, S., *Multiplicative n-Hom-Lie color algebras*. In: Silvestrov, S., Malyarenko, A., Rančić, M. (Eds.), Algebraic Structures and Applications, Springer Proceedings in Mathematics and Statistics **317**, Springer, Ch. 7, 159–187 (2020).
[12] C. M. Bai, *A further study on non-abelian phase spaces: Left-symmetric algebraic approach and related geometry* . Rev. Math. Phys, vol. **18** , 545–564 (2006).
[13] C. Bai, G. Li and Y. Sheng, *Bialgebras, the classical Yang-Baxter equation and Manin triples for 3-Lie algebras*. Adv. Theor. Math. Phys. **(23)** 1: 27–74 (2019).
[14] C. M. Bai and R. Zhang, *On some left-symmetric superalgebras*. J. Alg. Appl. **11** (5) (2012).
[15] P. D. Beites, I. Kaygorodov, Y. Popov. *Generalized derivations of multiplicative n-ary Hom-w color algebras*. Bull. Malays. Math. Sci. Soc. **42**(1), 315–335 (2019).
[16] S. Benayadi and A. Makhlouf, *Hom–Lie algebras with symmetric invariant nondegenerate bilinear forms*. J. Geo. Phy. **76** 38–60 (2014).
[17] A. Ben Hassine,T. Chtioui, S. Mabrouk and S. Silvestrov, *Structure and cohomology of* 3*-Lie-Rinehart superalgebras*. Comm. Alg. **49**(11), 4883–4904 (2021).
[18] J. Bergen, D. S. Passman. *Delta ideal of Lie color algebras*. J. Alg. **177** (3), 740–754 (1995).
[19] M. Bordemann, *Generalized Lax pairs, the modified classical Yang-Baxter equations, and affine geometry of Lie groups*. Comm. Math. Phys. **135** (1), 201–216 (1990).
[20] D. Burde, *Left-symmetric algebras, or pre-Lie algebras in geometry and physics*. Cent. Eur. J. Math. **4** (3), 323–357 (2006).
[21] Cayley, A. *On the theory of the analytic forms called trees*. Collected Mathematical Papers of Arthur Cayley Notices, **3**, pp. 242–246 (1890). Camb. Univ. Press, Cambridge.
[22] F. Chapoton, M. Livernet, *Pre-Lie algebras and the rooted trees operad*. Int. Math. Res. Notices **8** , 395–408 (2001).
[23] H. Chen and J. Li, *Left-symmetric algebra structures on the W-algebra* $W(2,2)$. Lin. Alg. Appl. **437**(7) , 1821–1834 (2012).
[24] T. Chtioui, and S. Mabrouk. 3*-L-dendriform algebras and generalized derivations*. Filomat. **35**(6), 1949–1961 (2021).
[25] B. Y. Chu, *Symplectic homogeneous spaces*. Trans. Amer. Math. Soc **197** , 145–159 (1974).
[26] A. Connes and D. Kreimer, *Hopf algebras, renormalization and noncommutative geometry*. Comm. Math. Phys. **199**, 203–242 (1998).
[27] J. M. Dardié and A. Médina, *Algèbres de Lie Kahlériennes et double extension*. J. Alg. **185**, 744–795 (1996).
[28] J. M. Dardié and A. Médina, *Double extension symplectique d'un groupe de Lie symplectique*. Adv. Math. **117**, 208–227 (1996).
[29] A. Diatta and A. Medina, *Classical Yang-Baxter equation and left-invariant affine geometry on Lie groups*. Manus. Math **114**, 477–486 (2004).
[30] K. Ebrahimi-Fard, *Loday-type algebras and the Rota-Baxter relation*. Lett. Math. Phys. **61**, 139–147 (2002).
[31] V. T. Filippov, *n-Lie algebras*. Sib. Mat. Zh. **26**(6), 126–140 (1985).
[32] Gerstenhaber, M, *The cohomology structure of associative ring*. Ann. Math. **78** (2), 267–288 (1963).
[33] Shuangjian Guo, Xiaohui Zhang and Shengxiang Wang, *The constructions of* 3*-Hom-Lie bialgebras*. arXiv:1902.03917v1 (2019).
[34] A. Hajjaji, T. Chtioui, S. Mabrouk and A. Makhlouf, *On n-pre-Lie algebras and dendrification of n-Lie algebras*. arXiv:2112.14290v1 (2021).
[35] J. T. Hartwig, D. Larsson, S. D. Silvestrov, *Deformations of Lie algebras using σ-derivations*. J. Alg. **295** (2), 314–361 (2006).
[36] X. Kong, H. Chen and C. Bai, *Classification of graded left-symmetric algebra structures on Witt and Virasoro algebras*. Int. J. Math. **22** (2) (2011).
[37] X. L. Kong and C. M. Bai, *Left-symmetric superalgebra structures on the super-Virasoro algebras*. Pac. J. Math. **235** (1) , 43–55 (2008).
[38] Koszul, J-L. *Domaines bornés homogènes et orbites de groupes de transformations affines*. Bull. Soc. Math. Fr. **89**, 515–533 (1961).
[39] B. A. Kupershmidt, *Non-abelian phase spaces*, J. Phys. A: Math. Gen, vol. **27** , 2801–2809 (1994).
[40] I. Laraiedh, S. Silvestrov, *Admissible Hom-Novikov-Poisson and Hom- Gelfand-Dorfman Color Hom-Algebras*. **Non-commutative and Non-associative Algebra and Analysis Structures**, Springer Proceedings in Mathematics and Statistics book series (PROMS,volume 426), (2023).
[41] A. Lichnerowicz and A. Medina, *On Lie group with left-invariant symplectic or Kählerian*. Lett. Math. Phys. **16** (3), 225–235 (1988) .
[42] Juan Li, Liangyun Chen and Yongsheng Cheng, *Representations of Bihom-Lie superalgebras*. Lin. Mult. Alg. **67** (2) 299–326 (2019).
[43] S. Mabrouk, O. Ncib and S. Silvestrov, *Generalized Derivations and Rota-Baxter Operators of n-ary Hom-Nambu superalgebras*. Adv. Appl. Cliff. Alg. **32** (31), (2021).
[44] A. Makhlouf and S. Silvestrov, *Hom-algebra structures*. J. Gen. Lie Theory Appl. **2**(2) 51–64 (2008).
[45] C. Ming, J. Liu, and Y. Ma. *Lie n-algebras and cohomologies of relative Rota-Baxter operators on n-Lie algebras*. J. Geo. Phy. doi.org/10.1016/j.geomphys.2023.104785 (2023).
[46] S. Montgomery. Constructing *simple Lie superalgebras from associative graded algebras*. J. Alg. **195** (2), 558–579 (1997).
[47] Y. Nambu, *Generalized Hamiltonian Dynamics*. Phys. Rev. , **7** (8), 2405–2412 (1973).
[48] Othmen Ncib, *On Hom-Nambu-Poisson superalgebras structures and their representations*. Asian-Eur. J. Math. **17** (8) (2024).
[49] R. Ree, *Generalized Lie elements*. Can. J. Math. **12**, 493–502 (1960).
[50] Shanshan Liu, Lina Song and Rong Tang, *Representations and cohomologies of regular Hom-pre-Lie algebras*. J. Alg. App. **19**(8) 2050149 (2020).
[51] M. Scheunert. *Generalized lie algebras*. J. Math. Phy. **20** (4), 712–720 (1979).
[52] Y. Sheng, *Representations of Hom-Lie Algebras*. Alg. Rep. Th. **15** (6), 1081–1098 (2012).
[53] Y. Su, K. Zhao, L. Zhu. *Classification of derivation-simple color algebras related to locally finite derivations*. J. Math. Phys. **45** (1), 525–536 (2004).
[54] B. Sun, L. Chen and X. Zhou, *On universal α-central extensions of Hom-pre-Lie algebras*. arXiv:1810.09848 (2018).
[55] Q. Sun and H. Li, *On parakähler Hom-Lie algebras and hom-left-symmetric bialgebras*. Comm. Alg. **45** (1), 105–120 (2017).

[56] E. A. Vasilieva and A. A. Mikhalev, *Free left-symmetric superalgebras*. Fund. Appl. Math. **2**, 611–613 (1996).
[57] Vinberg, E.B. *The theory of homogeneous convex cones*. Tr. Mos. Mat. Obsc. **12**, 303–358 (1963).
[58] L. Vitagliano,. *Representation of homotopy Lie-Rinehart algebras*. Math. Proc. Camb. Philos. Soc. **158** (1) 155–191 (2015).
[59] Yau D., *On n-ary Hom-Nambu and Hom-Nambu-Lie algebras*. J. Geo. Phy. **62** (2) 137–566 (2012).
[60] Q. Zhang, H. Yu and C. Wang, *Hom-Lie algebroids and hom-left-symmetric algebroids*. J. Geo. Phy. **116** 187–203 (2017).