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# An investigation for a second order Volterra-Fredholm integro-differential equation with two algebraic weakly singular kernels

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**Abstract.** This research focuses on the analytical and numerical analysis of a nonlinear second-order Volterra-Fredholm integro-differential equation with two algebraic weakly singular kernels. We rigorously establish the existence and uniqueness of the solution using Krasnoselskii's fixed-point theorem, which elegantly addresses the nonlinear structure of the equation. To approximate the solution, we employ the Nyström method combined with the product integration technique, specifically designed to overcome the challenges posed by the weak singularities.

We conduct extensive numerical experiments to demonstrate the performance and the accuracy of our proposed approach. The results not only validate our theoretical findings but also underscore the method's effectiveness in solving similar classes of integro-differential equations. This study advances our understanding of numerical methods for singular and nonlinear equations, offering valuable insights into their potential applications.

## 1. Introduction

Integral and integro-differential equations are highly effective for modeling a wide range of real-life scenarios, including physical processes [4], biological events [3, 15], population dynamics [6], financial issues [1], and various other fields. Their versatility and applicability make them an essential focus of mathematical research. However, the inherent complexity of these equations necessitates robust analytical and numerical methods to explore their solvability and behavior effectively.

A wide range of studies have made significant advancements in this field, introducing and developing an efficient methods for analyzing and solving a different kinds of integral and integro-differential equations in various contexts. For instance, collocation methods [7], hat functions [8], hybrid functions [9, 11, 13], hybrid modified block-pulse functions [14], the product integration method [17, 18], Taylor wavelet [10], Hermite wavelet [12], and the Galerkin-Chebyshev-wavelets method [5]. Each of these methods is carefully

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designed to address specific structural characteristics of the equations, such as singularities, nonlinearity, or regularity conditions, demonstrating the versatility and adaptability of numerical approaches in this domain.

In this work, we present a new class of Volterra-Fredholm integro-differential equation as follows:

$$u(t) = \int_0^1 |t-s|^\beta \phi_1(s,u(s),u'(s),u''(s))ds + \int_0^t (t-s)^\beta \phi_2(s,u(s),u'(s),u''(s))ds + f(t), \qquad t \in [0,1] \tag{1}$$

where $\phi_m \in C^2([0,1]\times\mathbb{R}^3)$, $m = 1,2$, $f \in C^2(0,1)$ are given functions, $1 < \beta < 2$ and the unknown is $u \in C^2(0,1)$.

The equation (1) features a specific structure, where the unknown $u$ and its derivatives $u'$ and $u''$ appear inside the nonlinear kernels $\phi_1$ and $\phi_2$. To fully characterize the solution $u$, we complement equation (1) with two additional equations. By differentiating equation (1), we obtain the first auxiliary equation:

$$u'(t) = \int_0^1 \gamma\beta|t-s|^{\beta-1} \phi_1(s,u(s),u'(s),u''(s))ds + \int_0^t \beta(t-s)^{\beta-1} \phi_2(s,u(s),u'(s),u''(s))ds + f'(t), \tag{2}$$

where $\gamma$ represents the signum function of $(t-s)$, defined as follow:

$$\gamma = sign(t-s) = \begin{cases} 1 & t > s, \\ -1 & t < s, \\ 0 & t = s. \end{cases}$$

Further differentiation of equation (2) yields the second auxiliary equation:

$$u''(t) = \int_0^1 \beta(\beta-1)|t-s|^{\beta-2} \phi_1(s,u(s),u'(s),u''(s))ds + \int_0^t \beta(\beta-1)(t-s)^{\beta-2} \phi_2(s,u(s),u'(s),u''(s))ds + f''(t). \tag{3}$$

Furthermore, we mention that the weak singularity originates from the second derivative equation (3), specifically, from terms $|t-s|^{\beta-2}$ and $(t-s)^{\beta-2}$ which lead to infinity as $t$ converges to $s$ for $1 < \beta < 2$. These singularities pose significant challenges for numerical approximation and necessitate tailored methods to ensure accuracy.

The main contribution of our work lies:

1. Investigating the existence and uniqueness of the solution to the system of equations (1)-(3).
2. Developing a numerical method adapted to the specific structure and regularity of the kernel.

Concerning the numerical strategy, the equations (1) and (2) where the kernel is of class $C^2(0,1)$, we employ the Nyström method. This method discretizes the integral equation by using quadrature rules to approximate the integral operator. Although the kernel is nonlinear, the method effectively reduces the problem to a nonlinear system of equations that can be solved iteratively.

For equation (3), the kernel is further complicated by being multiplied by a function in $\mathcal{L}^1(0,1)$, introducing a weakly algebraic singularity. To address this, we use the Sloan projection method with a first-degree Lagrange basis, which known as the Product Integration method. This method is specifically designed to handle algebraic weakly singular kernels, enabling accurate numerical integration while accommodating the singularity.

These numerical strategies are carefully selected to account for the nonlinearity of the kernel, its dependence on the unknown function and its derivatives, as well as the presence of singularities. This ensures the robustness and precision of the proposed approximation methods, providing a comprehensive framework for analyzing this new class of integro-differential equations.

## 2. Analytical study

In this section, we focus on analyzing equations (1)-(3) solvability. We explore practical conditions that lead to the existence and uniqueness of the solution.

We suppose that $\phi_1,\ \phi_2$, satisfy the following hypotheses:

$(\mathcal{H}_1):\quad \exists M_1, M_2 \in \mathbb{R}_+^*,\ \forall t \in [0,1],\ \forall x,y,z \in \mathbb{R}:\ \max\left|\phi_1(t,x,y,z)\right| \leq M_1,\ \max\left|\phi_2(t,x,y,z)\right| \leq M_2.$

$(\mathcal{H}_2):\quad \exists A_1, B_1, C_1 \in \mathbb{R}_+^*,\ \forall t \in [0,1],\ \forall x,y,z,x',y',z' \in \mathbb{R}:$
$\left|\phi_1(t,x,y,z) - \phi_1(t,x',y',z')\right| \leq A_1|x-x'| + B_1|y-y'| + C_1|z-z'|.$

$(\mathcal{H}_3):\quad \exists \rho \in \mathbb{R}_+^*:\ \rho = 2\max(A_1,B_1,C_1)\left(\frac{1}{\beta+1}+\beta+1\right) < 1.$

**Theorem 2.1.** *According to the hypotheses $(\mathcal{H}_1)-(\mathcal{H}_3)$, the equation (1) has at least a solution in $C^2(0,1)$.*

*Proof.* Consider the Banach space $C^2(0,1)$ equipped with the norm:

$$\forall \xi \in C^2(0,1),\ \|\xi\|_{C^2(0,1)} = \|\xi\|_\infty + \|\xi'\|_\infty + \|\xi''\|_\infty, \quad \textit{where}, \quad \|\xi\|_\infty = \sup_{t\in[0,1]} |\xi(t)|.$$

We define the operator $T: C^2(0,1) \to C^2(0,1)$ by:

$$\forall \xi \in C^2(0,1),\ \forall t \in [0,1],\ T\xi(t) = T_1\xi(t) + T_2\xi(t),$$

where

$$T_1\xi(t) = \int_0^1 |t-s|^\beta \phi_1(s,\xi(s),\xi'(s),\xi''(s))ds + f(t),$$

$$T_2\xi(t) = \int_0^t (t-s)^\beta \phi_2(s,\xi(s),\xi'(s),\xi''(s))ds.$$

The expressions for the operators $T'$ and $T''$ can be obtained by deriving the two members of the previous equation with respect to $t$ once and twice respectively.

$$T_1'\xi(t) = \int_0^1 \gamma\beta|t-s|^{\beta-1}\phi_1(s,\xi(s),\xi'(s),\xi''(s))ds + f'(t),$$

$$T_2'\xi(t) = \int_0^t \beta(t-s)^{\beta-1}\phi_2(s,\xi(s),\xi'(s),\xi''(s))ds,$$

and

$$T_1''\xi(t) = \int_0^1 \beta(\beta-1)|t-s|^{\beta-2}\phi_1(s,\xi(s),\xi'(s),\xi''(s))ds + f''(t),$$

$$T_2''\xi(t) = \int_0^t \beta(\beta-1)(t-s)^{\beta-2}\phi_2(s,\xi(s),\xi'(s),\xi''(s))ds.$$

It is clear that if $T$ have a fixed point then the equation (1) have at least one solution. We prove that $T$ have a fixed point using the Krasnoselskii's theorem. The proof will be presented in several steps. First, we define

$$S := \left\{ u \in C^2(0,1),\ \|u\|_{C^2(0,1)} \leq (2M_1 + M_2)\left(\frac{1}{\beta+1} + \beta + 1\right) + \|f\|_{C^2(0,1)} \right\}.$$

• **Step 1:** We prove that $T_1u_1 + T_2u_2 \in S$ for every $u_1, u_2 \in S \subseteq C^2(0,1)$.
For all $u_1, u_2 \in S$ and $t \in [0,1]$, we have:

$$\begin{aligned}
|T_1u_1(t) + T_2u_2(t)| \leq & \int_0^t (t-s)^\beta \left|\phi_1(s, u_1(s), u_1'(s), u_1''(s))\right| ds + \int_t^1 (s-t)^\beta \left|\phi_1(s, u_1(s), u_1'(s), u_1''(s))\right| ds \\
& + \int_0^t (t-s)^\beta \left|\phi_2(s, u_2(s), u_2'(s), u_2''(s))\right| ds + |f(t)|, \\
& \leq M_1 \int_0^t (t-s)^\beta ds + M_1 \int_t^1 (s-t)^\beta ds + M_2 \int_0^t (t-s)^\beta ds + |f(t)|, \\
& \leq (M_1 + M_2)\frac{t^{\beta+1}}{\beta+1} + M_1 \frac{(1-t)^{\beta+1}}{\beta+1} + |f(t)|, \\
& \leq \frac{2M_1 + M_2}{\beta+1} + \|f\|_\infty.
\end{aligned}$$

Then

$$\|T_1u_1 + T_2u_2\|_\infty \leq \frac{2M_1 + M_2}{\beta+1} + \|f\|_\infty.$$

In the same way, we get

$$\|T_1'u_1 + T_2'u_2\|_\infty \leq 2M_1 + M_2 + \|f'\|_\infty,$$

and

$$\|T_1''u_1 + T_2''u_2\|_\infty \leq \beta(2M_1 + M_2) + \|f''\|_\infty.$$

Therefore, we deduce that

$$\|T_1u_1 + T_2u_2\|_{C^2(0,1)} \leq (2M_1 + M_2)\left(\frac{1}{\beta+1} + \beta + 1\right) + \|f\|_{C^2(0,1)}.$$

That is $T_1u_1 + T_2u_2 \in S$ for every $u_1, u_2 \in S$.
• **Step 2:** We prove that $T_1$ is a contraction on $S$. For all $u_1, u_2 \in S$ and $t \in [0,1]$ we have:

$$\begin{aligned}
|T_1u_1(t) - T_1u_2(t)| = & \left| \int_0^1 |t-s|^\beta \phi_1(s, u_1(s), u_1'(s), u_1''(s)) ds - \int_0^1 |t-s|^\beta \phi_1(s, u_2(s), u_2'(s), u_2''(s)) ds \right|, \\
& \leq \int_0^1 |t-s|^\beta \left| \phi_1(s, u_1(s), u_1'(s), u_1''(s)) ds - \phi_1(s, u_2(s), u_2'(s), u_2''(s)) \right| ds, \\
& \leq \int_0^1 |t-s|^\beta \left( A_1|u_1(s) - u_2(s)| + B_1|u_1'(s) - u_2'(s)| + C_1|u_1''(s) - u_2''(s)| \right) ds,
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big(A_1\|u_1-u_2\|_\infty + B_1\|u_1'-u_2'\|_\infty + C_1\|u_1''-u_2''\|_\infty\Big)\int_0^1 |t-s|^\beta ds,\\
&\leq \max(A_1,B_1,C_1)\Big(\|u_1-u_2\|_\infty + \|u_1'-u_2'\|_\infty + \|u_1''-u_2''\|_\infty\Big)\left(\frac{t^{\beta+1}}{\beta+1}+\frac{(1-t)^{\beta+1}}{\beta+1}\right),\\
&\leq \frac{2}{\beta+1}\max(A_1,B_1,C_1)\|u_1-u_2\|_{C^2(0,1)},
\end{aligned}
$$

which gives

$$\|T_1u_1 - T_1u_2\|_\infty \leq \frac{2}{\beta+1}\max(A_1,B_1,C_1)\|u_1-u_2\|_{C^2(0,1)}.$$

In the same way, we get

$$\|T_1'u_1 - T_1'u_2\|_\infty \leq 2\max(A_1,B_1,C_1)\|u_1-u_2\|_{C^2(0,1)},$$

and

$$\|T_1''u_1 - T_1''u_2\|_\infty \leq 2\beta\max(A_1,B_1,C_1)\|u_1-u_2\|_{C^2(0,1)}.$$

Finally we have

$$\|T_1u_1 - T_1u_2\|_{C^2(0,1)} \leq 2\max(A_1,B_1,C_1)\left(\frac{1}{\beta+1}+\beta+1\right)\|u_1-u_2\|_{C^2(0,1)}.$$

Therefore, from $(\mathcal{H}_3)$ we deduce that

$$\|T_1u_1 - T_1u_2\|_{C^2(0,1)} \leq \rho\|u_1-u_2\|_{C^2(0,1)},$$

since $\rho < 1$, $T_1$ is contraction mapping.

• **Step 3:** Now we prove that $T_2$ is continuous and $T_2(S)$ is relatively compact.

First, $\phi_2$ and $f$ are continuous then $T_2$ is continuous. Next, For proving that the set $T_2(S)$ is relatively compact, we prove that it is uniformly bounded and equicontinuous (using Arzela-Ascoli Theorem).

From step 1, we can write: (by taking $u_1 = 0$ and $u_2 = u$)

$$\|Tu\|_{C^2(0,1)} \leq M_2\left(\frac{1}{\beta+1}+\beta+1\right) < (2M_1+M_2)\left(\frac{1}{\beta+1}+\beta+1\right) + \|f\|_{C^2(0,1)},$$

then $T_2(S) \subset S$ and it is uniformly bounded.

Now, we show that $T_2(S)$ is equicontinuous: For $u \in S,\ t_1, t_2 \in [0,1]$ with $t_1 < t_2$ we have

$$
\begin{aligned}
|T_2u(t_2) - T_2u(t_1)| &= \left|\int_0^{t_2}(t_2-s)^\beta\phi_2(s,u(s),u'(s),u''(s))ds - \int_0^{t_1}(t_1-s)^\beta\phi_2(s,u(s),u'(s),u''(s))ds\right|,\\
&\leq \left|\int_0^{t_1}(t_2-s)^\beta\phi_2(s,u(s),u'(s),u''(s))ds - \int_0^{t_1}(t_1-s)^\beta\phi_2(s,u(s),u'(s),u''(s))ds\right|\\
&+ \left|\int_{t_1}^{t_2}(t_2-s)^\beta\phi_2(s,u(s),u'(s),u''(s))ds\right|,\\
&\leq \int_0^{t_1}\Big[(t_2-s)^\beta-(t_1-s)^\beta\Big]\Big|\phi_2(s,u(s),u'(s),u''(s))\Big|ds + \int_{t_1}^{t_2}(t_2-s)^\beta\Big|\phi_2(s,u(s),u'(s),u''(s))\Big|ds,
\end{aligned}
$$

$$\leq M_2\int_0^{t_1}\left[(t_2-s)^\beta-(t_1-s)^\beta\right]ds+M_2\int_{t_1}^{t_2}(t_2-s)^\beta ds,$$
$$\leq \frac{M_2}{\beta+1}(t_2^{\beta+1}-t_1^{\beta+1}).$$

In a similar manner, we get

$$|T_2'u(t_2)-T_2'u(t_1)|\leq M_2(t_2^\beta-t_1^\beta),$$

and

$$|T_2''u(t_2)-T_2''u(t_1)|\leq \beta M_2(t_2^{\beta-1}-t_1^{\beta-1}).$$

It is clear that as $t_1\to t_2$, the right-hand sides of the last three inequalities are not depend on $u$ and tend to zero. So

$$|T_2u(t_2)-T_2u(t_1)|\to 0,\ \forall\quad |t_2-t_1|\to 0,\ u\in S.$$
$$\left|T_2'u(t_2)-T_2'u(t_1)\right|\to 0,\ \forall\quad |t_2-t_1|\to 0,\ u\in S.$$
$$\left|T_2''u(t_2)-T_2''u(t_1)\right|\to 0,\ \forall\quad |t_2-t_1|\to 0,\ u\in S.$$

Then $T_2(S)$ is equicontinuous. By applying the Arzela-Ascoli Theorem, it can be deduced that $T_2$ is relatively compact on $S$.

Consequently, all the conditions required by Krasnoselskii fixed point theorem are met. Hence, we can infer that the equation (1) possesses at least one solution on $C^2(0,1)$. □

Obviously, that the Krasnoselskii fixed point theorem guarantees just the existence of solution of the equation (1). So, before presenting the uniqueness result, we need to define the new following hypotheses:

$(\mathcal{H}_4)$ : $\exists A_2,B_2,C_2\in\mathbb{R}_+^*,\ \forall t\in[0,1],\ \forall x,y,z,x',y',z'\in\mathbb{R}$ :
$$\left|\phi_2(t,x,y,z)-\phi_2(t,x',y',z')\right|\leq A_2|x-x'|+B_2|y-y'|+C_2|z-z'|.$$

$(\mathcal{H}_5)$ : $\exists\tilde\rho,\theta\in\mathbb{R}_+^*,\ \tilde\rho=\max(A_2,B_2,C_2)\left(\frac{1}{\beta+1}+\beta+1\right)$ and $\theta=\rho+\tilde\rho<1$.

**Theorem 2.2.** *According to the hypotheses $(\mathcal{H}_1)-(\mathcal{H}_5)$, the equation (1) has a unique solution in $C^2(0,1)$.*

*Proof.* We suppose that $u(t),v(t)\in C^2(0,1)$ are two different solutions of the equation (1), for all $t\in[0,1]$ we have:

$$\begin{aligned}|u(t)-v(t)|\leq&\left|\int_0^1|t-s|^\beta\phi_1(s,u(s),u'(s),u''(s))ds-\int_0^1|t-s|^\beta\phi_1(s,v(s),v'(s),v''(s))ds\right|\\&+\left|\int_0^t(t-s)^\beta\phi_2(s,u(s),u'(s),u''(s))ds-\int_0^t(t-s)^\beta\phi_2(s,v(s),v'(s),v''(s))ds\right|,\\\leq&\int_0^1|t-s|^\beta\left|\phi_1(s,u(s),u'(s),u''(s))-\phi_1(s,v(s),v'(s),v''(s))\right|ds\\&+\int_0^t(t-s)^\beta\left|\phi_2(s,u(s),u'(s),u''(s))-\phi_2(s,v(s),v'(s),v''(s))\right|ds,\\\leq&\int_0^1|t-s|^\beta\left(A_1|u(s)-v(s)|+B_1|u'(s)-v'(s)|+C_1|u''(s)-v''(s)|\right)ds\end{aligned}$$

$$+\int_0^t (t-s)^\beta \left(A_2|u(s)-v(s)| + B_2|u'(s)-v'(s)| + C_2|u''(s)-v''(s)|\right) ds,$$

$$\leq \max(A_1,B_1,C_1)\|u-v\|_{C^2(0,1)} \int_0^1 |t-s|^\beta ds + \max(A_2,B_2,C_2)\|u-v\|_{C^2(0,1)} \int_0^t (t-s)^\beta ds,$$

$$\leq \max(A_1,B_1,C_1)\|u-v\|_{C^2(0,1)} \left(\frac{t^{\beta+1}}{\beta+1} + \frac{(1-t)^{\beta+1}}{\beta+1}\right) + \max(A_2,B_2,C_2)\|u-v\|_{C^2(0,1)} \frac{t^{\beta+1}}{\beta+1},$$

$$\leq \frac{2}{\beta+1}\max(A_1,B_1,C_1)\|u-v\|_{C^2(0,1)} + \frac{1}{\beta+1}\max(A_2,B_2,C_2)\|u-v\|_{C^2(0,1)},$$

$$\leq \frac{1}{\beta+1}\left(2\max(A_1,B_1,C_1) + \max(A_2,B_2,C_2)\right)\|u-v\|_{C^2(0,1)}.$$

Then

$$\|u-v\|_\infty \leq \frac{1}{\beta+1}\left(2\max(A_1,B_1,C_1) + \max(A_2,B_2,C_2)\right)\|u-v\|_{C^2(0,1)}.$$

In the same way, we get

$$\|u'-v'\|_\infty \leq \left(2\max(A_1,B_1,C_1) + \max(A_2,B_2,C_2)\right)\|u-v\|_{C^2(0,1)},$$

and

$$\|u''-v''\|_\infty \leq \beta\left(2\max(A_1,B_1,C_1) + \max(A_2,B_2,C_2)\right)\|u-v\|_{C^2(0,1)}.$$

Thus,

$$\|u-v\|_{C^2(0,1)} \leq \theta\|u-v\|_{C^2(0,1)},$$

therefore $\theta > 1$. This is a contradiction with $0 < \theta < 1$ and then $u(t) = v(t),\ u'(t) = v'(t),\ u''(t) = v''(t),\ \forall t \in [0,1]$. Consequently the equation (1) has a unique solution in $C^2(0,1)$. □

## 3. Numerical study

In this section, we develop a Nyström approach [2] to estimate the unique solution that was derived analytically in the previous section. Let $n \in \mathbb{N}^*$, we define the following subdivision $t_j = jh,\ h = \frac{1}{n}$, for $0 \leq j \leq n$,. The formula of the numerical integration is:

$$\forall \xi \in C^0(0,1),\ \int_0^1 \xi(t)dt \approx \sum_{j=0}^{n} \omega_j \xi(t_j),$$

where $\omega_j$ are real such that $\exists \Omega > 0, \forall n \in \mathbb{N}^*,\ \max\limits_{0\leq j\leq n} |\omega_j| \leq \Omega$.

Employing the Nyström perspective to the equations (1) and (2) we obtain the following non linear algebraic system: for $0 \leq i \leq n$,

$$U_i = \sum_{j=0}^{n} \omega_j |t_i - t_j|^\beta \phi_1(t_j, U_j, V_j, W_j) + \sum_{j=0}^{i} \omega_j (t_i - t_j)^\beta \phi_2(t_j, U_j, V_j, W_j) + f(t_i), \tag{4}$$

$$V_i = \sum_{j=0}^{n} \omega_j \gamma_j \beta |t_i - t_j|^{\beta-1} \phi_1(t_j, U_j, V_j, W_j) + \sum_{j=0}^{i} \omega_j \beta (t_i - t_j)^{\beta-1} \phi_2(t_j, U_j, V_j, W_j) + f'(t_i), \tag{5}$$

where, $\gamma_j = sign(t_i - t_j)$ and $U_i,\ V_i,\ W_i$ are approximate values of $u(t_i),\ u'(t_i),\ u''(t_i)$ respectively.

On the other hand, if we want to apply the Nyström method on equation (3) the terms $|t_i - t_j|^{\beta-2}$ and $(t_i - t_j)^{\beta-2}$ become infinity when $i = j$, which renders that Nyström method is inappropriate to approximate this equation. To address this issue, we employ the Product Integration method [16], which is specifically designed to handle the weakly singular terms $|t - s|^{\beta-2}$ and $(t - s)^{\beta-2}$. The core idea of this method is to factor out the singular behavior and approximate only the smooth part of the integrand. In particular, we approximate the functions $\phi_1$ and $\phi_2$ using the piecewise linear interpolation with respect variable $s$, at nodes $t_j$, by the following formulas:

$$\phi_m(s, u(s), u'(s), u''(s)) \approx \left(\frac{s - t_j}{h}\right)\phi_m(t_{j+1}, u_{j+1}, u'_{j+1}, u''_{j+1}) + \left(\frac{t_{j+1} - s}{h}\right)\phi_m(t_j, u_j, u'_j, u''_j), \quad m = 1, 2,$$

for $t_j \le s \le t_{j+1}$, and $0 \le j \le n$, respectively.

Consequently, by employing the Integration Product method to (3) we obtain the following non linear algebraic system: for $0 \le i \le n$,

$$W_i = \beta(\beta - 1)\sum_{j=0}^{n} \mu_j\phi_1(t_j, U_j, V_j, W_j) + \beta(\beta - 1)\sum_{j=0}^{i} \nu_j\phi_2(t_j, U_j, V_j, W_j) + f''(t_i), \tag{6}$$

where, $\mu_j$ and $\nu_j$ are given by:

$$\begin{aligned}
\mu_0 &= \frac{1}{h}\int_0^{t_1}(t_1 - s)|t_i - s|^{\beta-2}ds,\\
\mu_j &= \frac{1}{h}\left(\int_{t_{j-1}}^{t_j}(s - t_{j-1})|t_i - s|^{\beta-2}ds + \int_{t_j}^{t_{j+1}}(t_{j+1} - s)|t_i - s|^{\beta-2}ds\right), \quad 1 \le j \le n - 1,\\
\mu_n &= \frac{1}{h}\int_{t_{n-1}}^{1}(s - t_{n-1})|t_i - s|^{\beta-2}ds,\\
\nu_0 &= \frac{1}{h}\int_0^{t_1}(t_1 - s)(t_i - s)^{\beta-2}ds,\\
\nu_j &= \frac{1}{h}\left(\int_{t_{j-1}}^{t_j}(s - t_{j-1})(t_i - s)^{\beta-2}ds + \int_{t_j}^{t_{j+1}}(t_{j+1} - s)(t_i - s)^{\beta-2}ds\right), \quad 1 \le j \le i - 1,\\
\nu_i &= \frac{1}{h}\int_{t_{i-1}}^{t_i}(s - t_{i-1})(t_i - s)^{\beta-2}ds.
\end{aligned}$$

An important question remains: Has the non linear algebraic system (4)-(6) a unique solution? That what we will see in the next theorem.

**Theorem 3.1.** *According to the hypotheses* $(\mathcal{H}_1) - (\mathcal{H}_5)$*, the system* (4)-(6) *has a unique solution.*

*Proof.* We define the operator $\Psi$ as follow:

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix} : (\mathbb{R}^{n+1})^3 \to (\mathbb{R}^{n+1})^3,$$

with

$$\Psi_1\begin{pmatrix}U\\ V\\ W\end{pmatrix} = U,\ \Psi_2\begin{pmatrix}U\\ V\\ W\end{pmatrix} = V,\ \Psi_3\begin{pmatrix}U\\ V\\ W\end{pmatrix} = W,$$

where $U = (U_0, \cdots, U_n)$, $V = (V_0, \cdots, V_n)$, $W = (W_0, \cdots, W_n)$ are vectors in $\mathbb{R}^{n+1}$.

Assume that the space $(\mathbb{R}^{n+1})^3$ has the following norm:

$$\forall \begin{pmatrix} U \\ V \\ W \end{pmatrix} \in (\mathbb{R}^{n+1})^3, \quad \left\| \begin{pmatrix} U \\ V \\ W \end{pmatrix} \right\|_1 = \max_{0\le i\le n}\{|U_i|\} + \max_{0\le i\le n}\{|V_i|\} + \max_{0\le i\le n}\{|W_i|\}.$$

Then we have for $0 \le i \le n$:

$$\begin{aligned}
&\left| \Psi_1 \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \Psi_1 \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right| \\
&\le \left( \max(A_1, B_1, C_1) \sum_{j=0}^{n} \omega_j |t_i - t_j|^{\beta} + \max(A_2, B_2, C_2) \sum_{j=0}^{i} \omega_j (t_i - t_j)^{\beta} \right) \left\| \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right\|_1, \\
&\le \left( \max(A_1, B_1, C_1) \int_0^1 |t_i - s|^{\beta} ds + \max(A_2, B_2, C_2) \int_0^{t_i} (t_i - s)^{\beta} ds \right) \left\| \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right\|_1, \\
&\le \frac{1}{\beta+1} \left( \max(A_1, B_1, C_1)(t_i^{\beta+1} + (1 - t_i)^{\beta+1}) + \max(A_2, B_2, C_2) t_i^{\beta+1} \right) \left\| \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right\|_1, \\
&\le \frac{1}{\beta+1} \left( 2\max(A_1, B_1, C_1) + \max(A_2, B_2, C_2) \right) \left\| \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right\|_1.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left| \Psi_2 \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \Psi_2 \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right| \\
&\le \left( \max(A_1, B_1, C_1) \sum_{j=0}^{n} \omega_j \beta |t_i - t_j|^{\beta-1} + \max(A_2, B_2, C_2) \sum_{j=0}^{i} \omega_j \beta (t_i - t_j)^{\beta-1} \right) \left\| \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right\|_1, \\
&\le \left( \max(A_1, B_1, C_1) \int_0^1 \beta |t_i - s|^{\beta-1} ds + \max(A_2, B_2, C_2) \int_0^{t_i} \beta (t_i - s)^{\beta-1} ds \right) \left\| \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right\|_1, \\
&\le \left( \max(A_1, B_1, C_1)(t_i^{\beta} + (1 - t_i)^{\beta}) + \max(A_2, B_2, C_2) t_i^{\beta} \right) \left\| \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right\|_1, \\
&\le \left( 2\max(A_1, B_1, C_1) + \max(A_2, B_2, C_2) \right) \left\| \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right\|_1.
\end{aligned}$$

And

$$\begin{aligned}
&\left| \Psi_3 \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \Psi_3 \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right| \\
&\le \left( \max(A_1, B_1, C_1)\beta(\beta-1) \sum_{j=0}^{n} \mu_j + \max(A_2, B_2, C_2)\beta(\beta-1) \sum_{j=0}^{i} \nu_j \right) \left\| \begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix} \right\|_1,
\end{aligned}$$

$$\leq \left(\max(A_1,B_1,C_1)\beta(\beta-1)\int_0^1 |t_i-s|^{\beta-2}ds\right)\left\|\begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix}\right\|_1$$

$$+\left(\max(A_2,B_2,C_2)\beta(\beta-1)\int_0^{t_i} (t_i-s)^{\beta-2}ds\right)\left\|\begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix}\right\|_1,$$

$$\leq \left(\max(A_1,B_1,C_1)\beta(t_i^{\beta-1}+(1-t_i)^{\beta-1})+\max(A_2,B_2,C_2)\beta t_i^{\beta-1}\right)\left\|\begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix}\right\|_1,$$

$$\leq \beta\left(2\max(A_1,B_1,C_1)+\max(A_2,B_2,C_2)\right)\left\|\begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix}\right\|_1.$$

Thus

$$\left\|\Psi\begin{pmatrix} U \\ V \\ W \end{pmatrix} - \Psi\begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix}\right\|_1 \leq \theta\left\|\begin{pmatrix} U \\ V \\ W \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{pmatrix}\right\|_1.$$

Using Banach's fixed point theorem and because $\theta < 1$, we get that operator $\Psi$ has a one fixed point, which means that system (4)-(6) has a unique solution. □

## 4. Convergence analysis

In this section, we will demonstrate that the solution of the system (4)-(6) converges to the solution of the equations (1)-(3) when $h$ is small enough. For this reason, we define

$$\varepsilon_h = \max_{0\leq i\leq n}\{|u(t_i)-U_i|\} + \max_{0\leq i\leq n}\{|u'(t_i)-V_i|\} + \max_{0\leq i\leq n}\{|u''(t_i)-W_i|\},$$

then we will prove that $\lim_{h\to 0}\varepsilon_h = 0$. Before beginning the proof, we point out the following lemma:

**Lemma 4.1.** *For all $0 \leq i \leq n$, $\xi \in C^2(0,1)$, we define the consistence errors of the Nyström method by:*

$$\delta_N(h,t_i) = \left|\int_0^1 |t_i-s|^{\beta}\phi_1(s,\xi(s),\xi'(s),\xi''(s))ds - \sum_{j=0}^{n}\omega_j|t_i-t_j|^{\beta}\phi_1(t_j,\xi(t_j),\xi'(t_j),\xi''(t_j)\right|$$
$$+\left|\int_0^{t_i} (t_i-s)^{\beta}\phi_2(s,\xi(s),\xi'(s),\xi''(s))ds - \sum_{j=0}^{i}\omega_j(t_i-t_j)^{\beta}\phi_2(t_j,\xi(t_j),\xi'(t_j),\xi''(t_j))\right|,$$

$$\tilde{\delta}_N(h,t_i) = \left|\int_0^1 \gamma\beta|t_i-s|^{\beta-1}\phi_1(s,\xi(s),\xi'(s),\xi''(s))ds - \sum_{j=0}^{n}\omega_j\gamma_j\beta|t_i-t_j|^{\beta-1}\phi_1(t_j,\xi(t_j),\xi'(t_j),\xi''(t_j)\right|$$
$$+\left|\int_0^{t_i} \beta(t_i-s)^{\beta-1}\phi_2(s,\xi(s),\xi'(s),\xi''(s))ds - \sum_{j=0}^{i}\omega_j\beta(t_i-t_j)^{\beta-1}\phi_2(t_j,\xi(t_j),\xi'(t_j),\xi''(t_j))\right|,$$

*and the consistence error of the integration product rule by:*

$$\delta_P(h,t_i) = \left|\int_0^1 \beta(\beta-1)|t_i-s|^{\beta-2}\phi_1(s,\xi(s),\xi'(s),\xi''(s))ds - \sum_{j=0}^{n}\mu_j\phi_1(t_j,\xi(t_j),\xi'(t_j),\xi''(t_j)\right|$$

$$+\left|\int_0^{t_i}\beta(\beta-1)(t_i-s)^{\beta-2}\phi_2(s,\xi(s),\xi'(s),\xi''(s))ds-\sum_{j=0}^{i}\nu_j\phi_2(t_j,\xi(t_j),\xi'(t_j),\xi''(t_j))\right|.$$

*Which satisfy*

$$\lim_{h\to 0}\left(\max_{0\le i\le n}\{\delta_N(h,t_i)\}\right)=0,\quad \lim_{h\to 0}\left(\max_{0\le i\le n}\{\tilde{\delta}_N(h,t_i)\}\right)=0 \;\; and \;\; \lim_{h\to 0}\left(\max_{0\le i\le n}\{\delta_P(h,t_i)\}\right)=0.$$

**Theorem 4.2.** *According to the hypotheses* $(\mathcal{H}_1)-(\mathcal{H}_5)$ *and when h is small enough, the solution of system* (4)-(6) *converges to the solution of equations* (1)-(3). *i.e.,* $\lim_{h\to 0}\varepsilon_h=0$.

*Proof.* For all $0\le i\le n$, we have

$$\begin{aligned}
|u(t_i)-U_i| &\le \left|\int_0^1|t_i-s|^\beta\phi_1(s,u(s),u'(s),u''(s))ds-\sum_{j=0}^{n}\omega_j|t_i-t_j|^\beta\phi_1(t_j,U_j,V_j,W_j)\right|\\
&\quad+\left|\int_0^{t_i}(t_i-s)^\beta\phi_2(s,u(s),u'(s),u''(s))ds-\sum_{j=0}^{i}\omega_j(t_i-t_j)^\beta\phi_2(t_j,U_j,V_j,W_j)\right|,\\
&\le \left|\int_0^1|t_i-s|^\beta\phi_1(s,u(s),u'(s),u''(s))ds-\sum_{j=0}^{n}\omega_j|t_i-t_j|^\beta\phi_1(t_j,u(t_j),u'(t_j),u''(t_j))\right|\\
&\quad+\left|\int_0^{t_i}(t_i-s)^\beta\phi_2(s,u(s),u'(s),u''(s))ds-\sum_{j=0}^{i}\omega_j(t_i-t_j)^\beta\phi_2(t_j,u(t_j),u'(t_j),u''(t_j))\right|\\
&\quad+\left|\sum_{j=0}^{n}\omega_j|t_i-t_j|^\beta\phi_1(t_j,u(t_j),u'(t_j),u''(t_j))-\sum_{j=0}^{n}\omega_j|t_i-t_j|^\beta\phi_1(t_j,U_j,V_j,W_j)\right|\\
&\quad+\left|\sum_{j=0}^{i}\omega_j(t_i-t_j)^\beta\phi_2(t_j,u(t_j),u'(t_j),u''(t_j))-\sum_{j=0}^{i}\omega_j(t_i-t_j)^\beta\phi_2(t_j,U_j,V_j,W_j)\right|.
\end{aligned}$$

According to hypotheses $(\mathcal{H}_2)$, $(\mathcal{H}_4)$ and by using Lemma 4.1 we obtain:

$$\begin{aligned}
|u(t_i)-U_i| &\le \delta_N(h,t_i)+(\max(A_1,B_1,C_1)\sum_{j=0}^{n}\omega_j|t_i-t_j|^\beta\\
&\quad+\max(A_2,B_2,C_2)\sum_{j=0}^{i}\omega_j(t_i-t_j)^\beta)(|u(t_i)-U_i|+|u'(t_i)-V_i|+|u''(t_i)-W_i|).
\end{aligned}$$

Therefore

$$|u(t_i)-U_i|\le\delta_N(h,t_i)+\frac{1}{\beta+1}(2\max(A_1,B_1,C_1)+\max(A_2,B_2,C_2))\varepsilon_h.$$

In the same way we get

$$\begin{aligned}
|u'(t_i)-V_i| &\le \tilde{\delta}_N(h,t_i)+(2\max(A_1,B_1,C_1)+\max(A_2,B_2,C_2))\,\varepsilon_h,\\
|u''(t_i)-W_i| &\le \delta_P(h,t_i)+\beta\,(2\max(A_1,B_1,C_1)+\max(A_2,B_2,C_2))\,\varepsilon_h.
\end{aligned}$$

Then

$$\varepsilon_h \leq \frac{\max\limits_{0\leq i\leq n}\{\delta_N(h,t_i)\} + \max\limits_{0\leq i\leq n}\{\tilde{\delta}_N(h,t_i)\} + \max\limits_{0\leq i\leq n}\{\delta_P(h,t_i)\}}{1-\theta}.$$

Finally,

$$\lim_{h\to 0}\varepsilon_h = 0.$$

□

## 5. Numerical Examples

In this section, we provide illustrative examples to validate the effectiveness of the numerical approach proposed in this work. First, we apply the Trapezoidal rule within the Nyström technique. After formulating the nonlinear algebraic system (4)-(6), we solve it iteratively. Once the approximate solutions are obtained, we compare them with the exact solutions for different values of $n$ using the following error functions:

$$E^1 = \max_{0\leq i\leq n}(|u(t_i) - U_i|),\ \ E^2 = \max_{0\leq i\leq n}(|u'(t_i) - V_i|),\ \ E^3 = \max_{0\leq i\leq n}(|u''(t_i) - W_i|).$$

All these steps will be addressed through the described algorithm 1, which also generates comparative graphs of the exact and approximate solutions. The results presented below, including the generated graphs, are obtained using MATLAB software.

**Algorithm 1:** The used Algorithm for solving examples.

**Data:** $n$, $\beta$, $u_{ext}$
**Result:** $E^1_\beta$, $E^2_\beta$, $E^3_\beta$, $plot(U^k_n, u_{ext})$, $plot(V^k_n, u'_{ext})$, $plot(W^k_n, u''_{ext})$
**Initialization:** $U^0_n \leftarrow 0_n, V^0_n \leftarrow 0_n, W^0_n \leftarrow 0_n, k \leftarrow 1, Tol \leftarrow 10^{-7}, E^{1,k} \leftarrow 1, E^{2,k} \leftarrow 1, E^{3,k} \leftarrow 1$
**while** $\max(E^{1,k}, E^{2,k}, E^{3,k}) > Tol$ **do**
  **for** $i \leftarrow 0$ **to** $n$ **do**
    **for** $j \leftarrow 0$ **to** $n$ **do**
      Calculate the left side of expressions (4), (5) and (6)
    **if** $i > 0$ **then**
      **for** $j \leftarrow 0$ **to** $i$ **do**
        Calculate the right side of expressions (4), (5) and (6)
    Compute $U^k_n$, $V^k_n$ and $W^k_n$
  Compute $E^{1,k} \leftarrow |U^k_n - U^{k-1}_n|, E^{2,k} \leftarrow |V^k_n - V^{k-1}_n|, E^{3,k} \leftarrow |W^k_n - W^{k-1}_n|$;
  $k \leftarrow k+1$
Save $U^k_n$, $V^k_n$ and $W^k_n$;
Find $E^1_\beta \leftarrow |U^k_n - u_{ext}|, E^2_\beta \leftarrow |V^k_n - u'_{ext}|, E^3_\beta \leftarrow |W^k_n - u''_{ext}|$

**Example 5.1.** *Consider the following nonlinear integro-differential equation:*

$$\begin{aligned} u(t) = {} & \frac{1}{2}\int_0^1 (|\,t-s\,|)^{1.5}\cos\left(u(s) - \frac{s}{2}u'(s) + u''(s) + \frac{s}{2} - 2\right)ds \\ & + \frac{1}{2}\int_0^t (t-s)^{1.5}\sin\left(u(s) + u'(s) + u''(s) - s^2 - s - 1 + \frac{\pi}{2}\right)ds + t^2 - t - \frac{1}{5}(1-t)^{2.5} - \frac{2}{5}t^{2.5}, \end{aligned}$$

*where, the exact solution of this example and its derivatives are:*

$$u(t) = t^2 - t,\ \ u'(t) = 2t-1,\ \ u''(t) = 2.$$

Table 1 shows that when the values of $n$ increases the error functions $E^1$, $E^2$ and $E^3$ close to zero. Which is equivalent to saying that the approximate solution with its first and second derivative converge to the exact solution with its first and second derivative, respectively. Moreover, Figures 1-3 show that for a simple value $n = 20$, the exact and approximate solutions with their derivatives are almost identical.

| $n$ | $E^1$ | $E^2$ | $E^3$ | Iterations | Time (sec) |
|---|---|---|---|---|---|
| 5 | 4.90E-3 | 2.54E-2 | 1.19E-4 | 17 | 0.659 |
| 10 | 1.27E-3 | 9.23E-3 | 1.36E-5 | 19 | 2.539 |
| 50 | 5.37E-5 | 8.57E-4 | 1.44E-7 | 20 | 130.44 |
| 100 | 1.35E-5 | 3.05E-4 | 1.16E-8 | 21 | 577.246 |
| 250 | 2.19E-6 | 7.78E-5 | 7.27E-10 | 21 | 2420.599 |

Table 1: The error functions $E^1$, $E^2$ and $E^3$ of example 1 by varying $n$.

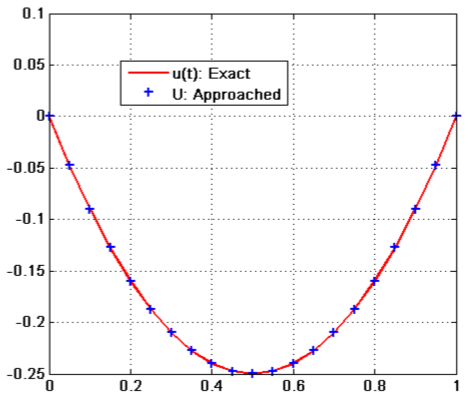


Figure 1: Graph of exact and approached solution of example 1 for $n = 20$.

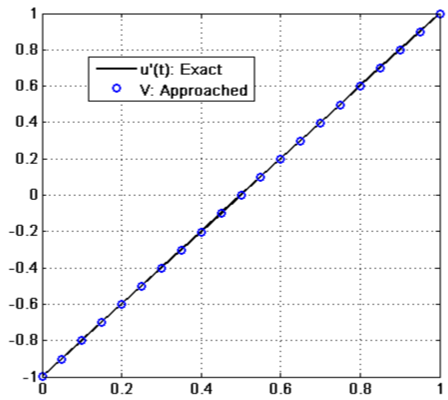


Figure 2: Graph of exact and approached derivative of solution of example 1 for $n = 20$.

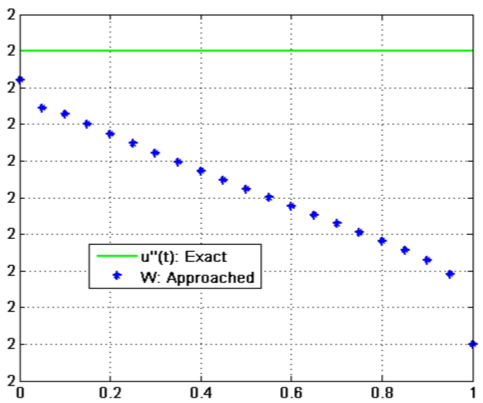


Figure 3: Graph of exact and approached second derivative of solution of example 1 for $n = 20$.

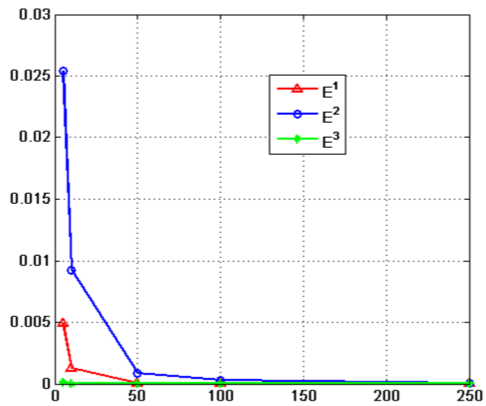


Figure 4: Graph of the error functions $E^1$, $E^2$ and $E^3$ of example 1.

**Example 5.2.** *Consider the following equation:*

$$
\begin{aligned}
u(t) = & \int_0^1 (| t-s |)^\beta \frac{9s(1+4\cos^2(s)\sin^2(s))}{u^2(s)+2(u'(s))^2+(u''(s))^2+1} ds \\
& + \frac{1}{2}\int_0^t (t-s)^\beta s\cos\left(4u(s)+u'(s)+u''(s)-2\cos(2s)\right) ds \\
& + f(t), \quad \beta \in ]1,2[, \ \forall t \in [0,1],
\end{aligned}
$$

*where,*

$$
f(t) = \sin(2t) - \frac{3t^{\beta+2}}{2(\beta^2+3\beta+2)} - \frac{(\beta+t+1)(1-t)^{\beta+1}}{(\beta^2+3\beta+2)}.
$$

*The exact solution of this example and its derivatives are:*

$$
u(t) = \sin(2t), \ \ u'(t) = 2\cos(2t), \ \ u''(t) = -4\sin(2t).
$$

Regarding to this example we vary the value of $\beta$ and the value of $n$ as follow:

$$
\begin{aligned}
\beta &\in \{1.01, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.99\}, \\
n &\in \{5, 10, 50, 100\},
\end{aligned}
$$

then we define the new error functions according to the value of $\beta$ and $n$ by:

$$
E_\beta^1 = \max_{0\le i\le n}(|u(t_i)-U_{i,\beta}|), \ \ E_\beta^2 = \max_{0\le i\le n}(|u'(t_i)-V_{i,\beta}|), \ \ E_\beta^3 = \max_{0\le i\le n}(|u''(t_i)-W_{i,\beta}|).
$$

Tables 2-4 show that for any value of $\beta$, the error functions $E_\beta^1, E_\beta^2$ and $E_\beta^3$ diminish as $n$ decreases, this confirms us the performance and the accuracy of our numerical method. Additionally, we can observe that the effectiveness of our numerical method is heavily influenced by the kernel's regularity which is associated with the value of $\beta$. Specifically, when $\beta$ is near to 1, indicating low regularity, our method converges slowly. Conversely, when $\beta$ approaches to 2, the convergence is much quicker.

| $\beta\backslash n$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| 1.01 | 8.88E-3 | 1.97E-3 | 2.46E-4 | 1.43E-4 |
| 1.1 | 7.85E-3 | 1.71E-3 | 6.37E-5 | 3.41E-5 |
| 1.2 | 6.86E-3 | 1.50E-3 | 2.72E-5 | 6.46E-6 |
| 1.3 | 6.10E-3 | 1.50E-3 | 5.41E-5 | 1.15E-5 |
| 1.4 | 6.71E-3 | 1.75E-3 | 6.96E-5 | 1.68E-5 |
| 1.5 | 7.53E-3 | 1.95E-3 | 7.91E-5 | 1.95E-5 |
| 1.6 | 8.22E-3 | 2.11E-3 | 8.58E-5 | 2.14E-5 |
| 1.7 | 8.81E-3 | 2.25E-3 | 9.10E-5 | 2.27E-5 |
| 1.8 | 9.33E-3 | 2.37E-3 | 9.56E-5 | 2.38E-5 |
| 1.9 | 9.79E-3 | 2.48E-3 | 9.97E-5 | 2.49E-5 |
| 1.99 | 1.01E-2 | 2.57E-3 | 1.03E-4 | 2.58E-5 |

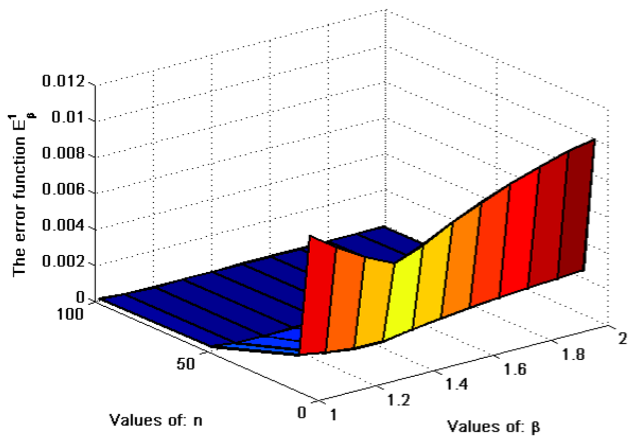


Table 2: The error function $E_\beta^1$ of example 2 by varying the values of $\beta$ and $n$.

| $\beta\backslash n$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| 1.01 | 1.44E-1 | 7.25E-2 | 1.44E-2 | 7.20E-3 |
| 1.1 | 1.18E-1 | 5.49E-2 | 9.32E-3 | 4.34E-3 |
| 1.2 | 9.35E-2 | 4.02E-2 | 5.75E-3 | 2.50E-3 |
| 1.3 | 7.41E-2 | 2.95E-2 | 3.55E-3 | 1.43E-3 |
| 1.4 | 5.88E-2 | 2.17E-2 | 2.20E-3 | 8.27E-4 |
| 1.5 | 4.69E-2 | 1.60E-2 | 1.36E-3 | 4.77E-4 |
| 1.6 | 3.77E-2 | 1.19E-2 | 8.55E-4 | 2.77E-4 |
| 1.7 | 3.06E-2 | 9.03E-3 | 5.44E-4 | 1.63E-4 |
| 1.8 | 2.52E-2 | 6.96E-3 | 3.56E-4 | 9.98E-5 |
| 1.9 | 2.11E-2 | 5.50E-3 | 2.45E-4 | 6.43E-5 |
| 1.99 | 1.83E-2 | 4.58E-3 | 1.84E-4 | 4.63E-5 |

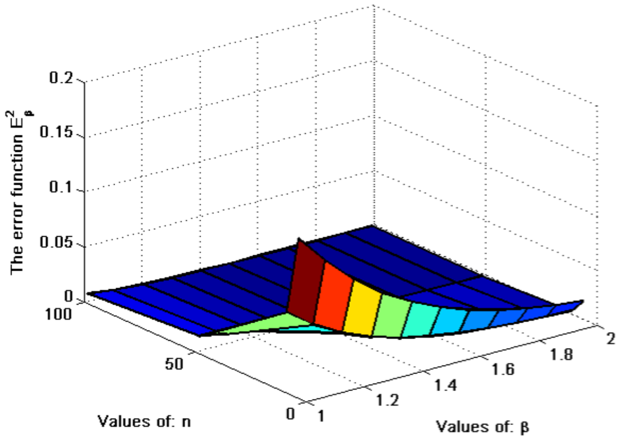


Table 3: The error function $E_\beta^2$ of example 2 by varying the values of $\beta$ and $n$.

| $\beta\backslash n$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| 1.01 | 7.60E-2 | 3.01E-2 | 1.06E-2 | 6.00E-3 |
| 1.1 | 4.43E-2 | 1.60E-2 | 2.89E-3 | 1.35E-3 |
| 1.2 | 2.55E-2 | 8.32E-3 | 1.04E-3 | 4.31E-4 |
| 1.3 | 1.49E-2 | 4.34E-3 | 4.15E-4 | 1.55E-4 |
| 1.4 | 8.55E-3 | 2.19E-3 | 1.67E-4 | 5.69E-5 |
| 1.5 | 4.51E-3 | 9.76E-4 | 6.97E-5 | 2.22E-5 |
| 1.6 | 2.58E-3 | 9.02E-4 | 5.20E-5 | 1.50E-5 |
| 1.7 | 2.72E-3 | 8.72E-4 | 4.30E-5 | 1.14E-5 |
| 1.8 | 2.86E-3 | 8.61E-4 | 3.85E-5 | 9.86E-6 |
| 1.9 | 2.98E-3 | 8.58E-4 | 3.64E-5 | 9.10E-6 |
| 1.99 | 3.06E-3 | 8.58E-4 | 3.54E-5 | 8.84E-6 |

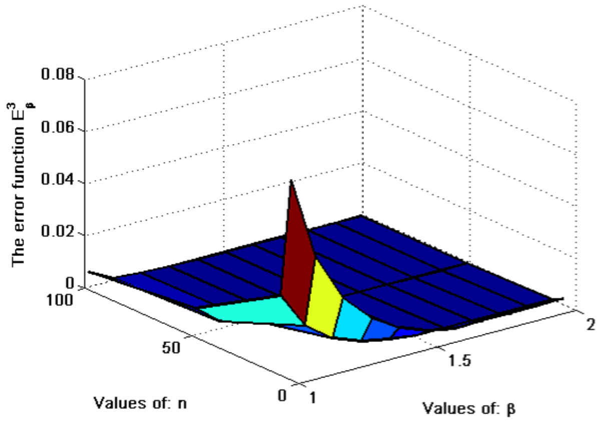


Table 4: The error function $E_\beta^3$ of example 2 by varying the values of $\beta$ and $n$.

| $\beta\backslash n$ | 5 | 10 | 50 | 100 |
|---|---|---|---|---|
| 1.01 | 54 | 73 | 99 | 103 |
| 1.1 | 43 | 49 | 54 | 54 |
| 1.2 | 38 | 40 | 42 | 43 |
| 1.3 | 35 | 36 | 37 | 37 |
| 1.4 | 32 | 33 | 34 | 34 |
| 1.5 | 30 | 31 | 31 | 31 |
| 1.6 | 29 | 29 | 30 | 30 |
| 1.7 | 28 | 28 | 28 | 28 |
| 1.8 | 28 | 27 | 27 | 27 |
| 1.9 | 27 | 26 | 26 | 26 |
| 1.99 | 26 | 26 | 26 | 26 |

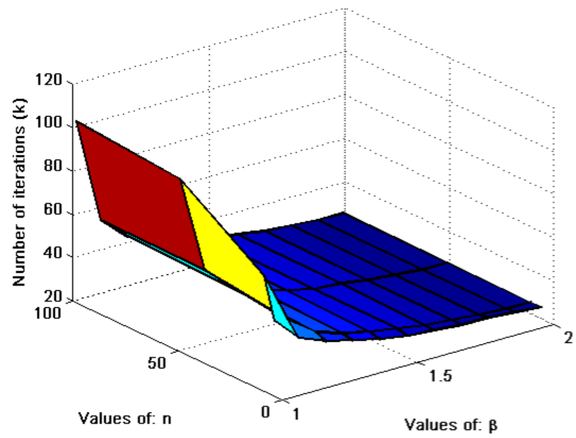


Table 5: The number of iterations ($k$) by varying the values of $\beta$ and $n$.

## 6. Conclusion

In this paper, we have presented the analytical and numerical treatment of a nonlinear second-order Volterra-Fredholm integro-differential equation with weakly singular kernels, establishing the existence and uniqueness of its solution. To approximate the solution, we employed the Product Integration and Nyström methods, which are well-suited to the specific structure of the kernels. The effectiveness of these methods has been demonstrated through numerical examples. While the results are promising, future work will aim to improve convergence rates by incorporating orthogonal polynomials and wavelets in $\mathcal{L}^2$-spaces. Moreover, we plan to extend this study to tackle equations with more general types of kernels, further broadening the applicability of our methods.

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