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Some applications related to the arithmetic-geometric mean inequality

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Abstract. In this paper, we introduce some refinements of the arithmetic-geometric-logarithmic mean inequality. The obtained results allow us to get new inequalities for the inverse sine function and refined inequalities for well-known hyperbolic inequalities.

Further, as a new track in this field, we present a possible arithmetic-geometric mean inequality in the complex plane, with applications towards complex inequalities for concave functions and their sub-additive behavior.

Moreover, the celebrated Jensen-Mercer inequality for convex functions is refined in a way that implies some new relations among the arithmetic, geometric, Heinz, and weighted logarithmic means.

1. Introduction

Given two positive numbers a, b, the arithmetic-geometric mean inequality (AM-GM inequality) states that $\sqrt{ab} \le \frac{a+b}{2}$. This inequality is a special case of its weighted version, which asserts, for $0 \le \nu \le 1$,

$$a\sharp_\nu b \le a\nabla_\nu b, \tag{1}$$

where $a\sharp_\nu b = a^{1-\nu}b^\nu$ and $a\nabla_\nu b = (1-\nu)a + \nu b$ are the weighted geometric and arithmetic means, respectively. The inequality (1) is usually referred to as Young's inequality in the literature. When $\nu = \frac{1}{2}$, we simply write $a\sharp b$ and $a\nabla b$ instead of $a\sharp_{\frac{1}{2}}$ and $a\nabla_{\frac{1}{2}} b$, respectively.

The $n-$tuple version of (1) states that

$$\prod_{i=1}^{n} x_i^{p_i} \le \sum_{i=1}^{n} p_i x_i, \tag{2}$$

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where $x_i, p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$.
Research in this field includes obtaining refinements, reverses, new insights, and new relations that govern the inequalities above. For example, (2) was refined and reversed in the following multiplicative forms

$$\left(\frac{\sum_{i=1}^{n}\frac{x_i}{n}}{\prod_{i=1}^{n} x_i^{1/n}}\right)^{np_{\min}} \prod_{i=1}^{n} x_i^{p_i} \le \sum_{i=1}^{n} p_i x_i \le \left(\frac{\sum_{i=1}^{n}\frac{x_i}{n}}{\prod_{i=1}^{n} x_i^{1/n}}\right)^{np_{\max}} \prod_{i=1}^{n} x_i^{p_i},$$

and in the following additive forms

$$np_{min}\left(\frac{\sum_{i=1}^{n} x_i}{n} - \prod_{i=1}^{n} x_i^{1/n}\right) \le \sum_{i=1}^{n} p_i x_i - \prod_{i=1}^{n} x_i^{p_i} \le np_{max}\left(\frac{\sum_{i=1}^{n} x_i}{n} - \prod_{i=1}^{n} x_i^{1/n}\right), \tag{3}$$

where $p_{\min} = \min\{p_1, \cdots, p_n\}$ and $p_{\max} = \max\{p_1, \cdots, p_n\}$. We refer the reader to [1, 7, 12] for further discussion, applications and related references.

When $n = 2$, the inequalities in (3) can be stated as follows [4, 5]:

$$a\sharp_v b \le a\nabla_v b - 2r\left(a\nabla b - a\sharp b\right), \tag{4}$$

and

$$a\nabla_v b \le a\sharp_v b + 2R\left(a\nabla b - a\sharp b\right), \tag{5}$$

where $r = \min\{v, 1-v\}$ and $R = \max\{v, 1-v\}$. We refer the reader to [9, 13] for further discussion of these forms.

For a general discussion of the AM-GM inequality and its applications, we refer the reader to [2, 3, 8, 10, 11].

Related to the geometric mean, the logarithmic mean of the positive numbers a, b is given by

$$\int_0^1 (a\sharp_t b)\, dt = \frac{b-a}{\log b - \log a}, \ \ (a, b > 0, a \ne b). \tag{6}$$

It is well known that

$$a\sharp b \le \int_0^1 (a\sharp_t b)\, dt \le a\nabla b. \tag{7}$$

In this paper, we first give an upper and a lower bound of the difference $a\nabla b - \int_0^1 (a\sharp_t b) dt$, as a refinement and a reverse of the inequality $\int_0^1 (a\sharp_t b) dt \le a\nabla b$, as follows.

Proposition 1.1. *Let $a, b > 0$. Then*

$$\frac{1}{2}\left(a\nabla b - a\sharp b\right) \le a\nabla b - \int_0^1 (a\sharp_t b)\, dt \le \frac{3}{2}\left(a\nabla b - a\sharp b\right).$$

Proof. This follows immediately upon integrating (4) and (5). □

This paper aims to introduce new refinements of the AM-GM inequality $a\sharp b \le a\nabla b$, which enable us to obtain new inequalities for the inverse trigonometric functions and the hyperbolic functions. A detailed discussion of the obtained inequality and its relations to the existing literature will be included. For example, our results will enable us to show that

$$\cosh y - \frac{3}{4e^y}(1-e^y)^2 \le \frac{\sinh y}{y} \le \cosh y - \frac{1}{4e^y}(1-e^y)^2$$

as a refinement and a reverse of the well-known inequality

$$\frac{\sinh y}{y} \leq \cosh y.$$

Further, we present a refinement of Stolarsky inequality [16]

$$\exp\left(\frac{x}{\tanh x} - 1\right) < \cosh x; \ \ (x > 0),$$

and the well-known inequality

$$\cosh x < \exp\left(\frac{x^2}{2}\right); \ \ (x > 0),$$

where the Specht ratio interferes.

Further, as a new investigation in this field, we present AM-GM inequalities in the complex plane, emphasizing how convex inequalities work in the complex plane. The significance of this discussion is due to the fact that the complex plane needs to be better ordered. This will be done using the modulus function.

Strongly related to our discussion, we present a new refinement of the Jensen-Mercer inequality for convex functions. The result will imply new relations among the arithmetic, geometric, Heinz, and weighted logarithmic means.

2. An inverse trigonometric refinement of the AM-GM inequality

In this section, we present a refinement of the AM-GM inequality in terms of the arc sine function. While this refinement is interesting, its application in obtaining an inequality for the arc sine is essential.

Theorem 2.1. *Let $a, b > 0$. Then*

$$1 \leq \frac{1}{2}\left\{1 + \left(\frac{a\nabla b}{a\sharp b}\right)\left(\frac{a+b}{|b-a|}\right)\arcsin\left(\frac{|b-a|}{a+b}\right)\right\} \leq \frac{a\nabla b}{a\sharp b}. \tag{8}$$

Proof. Let $0 \leq t \leq 1$. Then by applying (1) twice, we get

$$\begin{aligned} \frac{a+b}{2} &= \frac{(1-t)a + tb + (1-t)b + ta}{2} \\ &\geq \sqrt{((1-t)a + tb)((1-t)b + ta)} \\ &\geq \sqrt{(a^{1-t}b^t)(b^{1-t}a^t)} \\ &= \sqrt{ab}. \end{aligned} \tag{9}$$

Since

$$\begin{aligned} ((1-t)a + tb)((1-t)b + ta) &= (1-t)^2 ab + t(1-t)a^2 + t(1-t)b^2 + t^2 ab \\ &= \left((1-t)^2 + t^2\right)ab + t(1-t)\left(a^2 + b^2\right), \end{aligned}$$

we can write (9), in the following form

$$\sqrt{ab} \leq \sqrt{\left((1-t)^2 + t^2\right)ab + t(1-t)(a^2 + b^2)} \leq \frac{a+b}{2}.$$

Now, by integrating over $0 \le t \le 1$, we get

$$\int_0^1 \sqrt{\left((1-t)^2+t^2\right)ab+t(1-t)(a^2+b^2)}dt$$

$$= \left.\frac{(a+b)^2\arcsin\left(\frac{|b-a|(2t-1)}{a+b}\right)}{8|b-a|}+\frac{(2t-1)\sqrt{\left((1-t)^2+t^2\right)ab+t(1-t)(a^2+b^2)}}{4}\right|_0^1$$

$$= \frac{(a+b)^2}{8|b-a|}\left(\arcsin\left(\frac{|b-a|}{a+b}\right)-\arcsin\left(-\frac{|b-a|}{a+b}\right)\right)+\frac{\sqrt{ab}}{2}.$$

Therefore,

$$\sqrt{ab} \le \frac{(a+b)^2}{8|b-a|}\left(\arcsin\left(\frac{|b-a|}{a+b}\right)-\arcsin\left(-\frac{|b-a|}{a+b}\right)\right)+\frac{\sqrt{ab}}{2} \le \frac{a+b}{2}.$$

Since $\arcsin(-y) = -\arcsin y$, we get

$$\sqrt{ab} \le \frac{(a+b)^2}{4|b-a|}\arcsin\left(\frac{|b-a|}{a+b}\right)+\frac{\sqrt{ab}}{2} \le \frac{a+b}{2}.$$

This completes the proof of the theorem. □

Remark 2.2. *The second inequality in* (8) *is written by*

$$\xi(a,b) := \left(2-\frac{a+b}{|b-a|}\arcsin\left(\frac{|b-a|}{a+b}\right)\right)^{-1} \le \frac{a\nabla b}{a\sharp b}.$$

Since $1 < \dfrac{\arcsin(x)}{x} \le \dfrac{\pi}{2}$ *for* $0 < x \le 1$, *and* $0 < \dfrac{|b-a|}{a+b} \le 1$, *we also find* $1 < \xi(a,b) \le \dfrac{2}{4-\pi}$. *Therefore, the inequality* (8) *improves the arithmetic-geometric mean inequality.*

We also notice the following inequality for the arc sine function as a conclusion of Theorem 2.1.

Corollary 2.3. *Let* $a, b > 0$. *Then*

$$\frac{a\sharp b}{a\nabla b} \le \frac{a+b}{|b-a|}\arcsin\left(\frac{|b-a|}{a+b}\right) \le 2-\frac{a\sharp b}{a\nabla b}.$$

In particular, if $x > 0$, *then*

$$\frac{2\sqrt{x}}{x+1} \le \frac{x+1}{|x-1|}\arcsin\left(\frac{|x-1|}{x+1}\right) \le 2-\frac{2\sqrt{x}}{x+1}.$$

3. A hyperbolic inequality via the AM-GM inequality

In this section, we prove a refinement of the AM-GM inequality to enable us to obtain newly refined inequalities for the hyperbolic functions. First, we show the following refinement and a reverse for the second inequality in (7).

Theorem 3.1. *Let* $a, b > 0$. *Then*

$$\frac{1}{8}\left(\left(\sqrt[4]{ab}-\sqrt{a}\right)^2+\left(\sqrt[4]{ab}-\sqrt{b}\right)^2\right)+\frac{1}{4}\left(\sqrt{a}-\sqrt{b}\right)^2+\int_0^1 (a\sharp_t b)\,dt \le a\nabla b$$

and

$$a\nabla b \le \int_0^1 (a\sharp_t b)\, dt + \frac{3}{4}\left(\sqrt{a} - \sqrt{b}\right)^2 - \frac{1}{8}\left(\left(\sqrt[4]{ab} - \sqrt{b}\right)^2 + \left(\sqrt[4]{ab} - \sqrt{a}\right)^2\right).$$

Proof. It follows from [17, Lemma 1] that

$$r_0\left(\sqrt[4]{ab} - \sqrt{a}\right)^2 + t\left(\sqrt{a} - \sqrt{b}\right)^2 + a\sharp_t b \le a\nabla_t b;\ \left(0 \le t \le \frac{1}{2}\right)$$

and

$$r_0\left(\sqrt[4]{ab} - \sqrt{b}\right)^2 + (1-t)\left(\sqrt{a} - \sqrt{b}\right)^2 + a\sharp_t b \le a\nabla_t b;\ \left(\frac{1}{2} \le t \le 1\right)$$

where $r_0 = \min\{2r, 1-2r\}$ and $r = \min\{t, 1-t\}$. Since

$$\min\{a, b\} = \frac{a + b - |a-b|}{2};\ (a, b \in \mathbb{R}),$$

we get for $0 \le t \le 1/2$,

$$\begin{aligned}
&\min\{2r, 1-2r\}\left(\sqrt[4]{ab} - \sqrt{a}\right)^2 + t\left(\sqrt{a} - \sqrt{b}\right)^2 + a\sharp_t b \\
&= \left(\frac{1 - |4r-1|}{2}\right)\left(\sqrt[4]{ab} - \sqrt{a}\right)^2 + t\left(\sqrt{a} - \sqrt{b}\right)^2 + a\sharp_t b \\
&= \left(\frac{1 - |4\min\{t, 1-t\} - 1|}{2}\right)\left(\sqrt[4]{ab} - \sqrt{a}\right)^2 + t\left(\sqrt{a} - \sqrt{b}\right)^2 + a\sharp_t b \\
&= \left(\frac{1 - |1 - 2|2t-1||}{2}\right)\left(\sqrt[4]{ab} - \sqrt{a}\right)^2 + t\left(\sqrt{a} - \sqrt{b}\right)^2 + a\sharp_t b \\
&\le a\nabla_t b
\end{aligned}$$

i.e.,

$$\left(\frac{1 - |1 - 2|2t-1||}{2}\right)\left(\sqrt[4]{ab} - \sqrt{a}\right)^2 + t\left(\sqrt{a} - \sqrt{b}\right)^2 + a\sharp_t b \le a\nabla_t b.$$

Taking integral over $0 \le t \le 1/2$, we obtain

$$\frac{1}{8}\left(\left(\sqrt[4]{ab} - \sqrt{a}\right)^2 + \left(\sqrt{a} - \sqrt{b}\right)^2\right) + \int_0^{\frac{1}{2}} (a\sharp_t b)\, dt \le \frac{3}{8}a + \frac{1}{8}b. \tag{10}$$

In a similar way, we have

$$\left(\frac{1 - |1 - 2|2t-1||}{2}\right)\left(\sqrt[4]{ab} - \sqrt{b}\right)^2 + (1-t)\left(\sqrt{a} - \sqrt{b}\right)^2 + a\sharp_t b \le a\nabla_t b$$

for $1/2 \le t \le 1$. Taking integral over $1/2 \le t \le 1$, we reach

$$\frac{1}{8}\left(\left(\sqrt[4]{ab} - \sqrt{b}\right)^2 + \left(\sqrt{a} - \sqrt{b}\right)^2\right) + \int_{\frac{1}{2}}^1 (a\sharp_t b)\, dt \le \frac{1}{8}a + \frac{3}{8}b. \tag{11}$$

By combining inequalities (10) and (11), we have

$$\begin{aligned}
&\frac{1}{8}\left(\left(\sqrt[4]{ab}-\sqrt{a}\right)^2+\left(\sqrt[4]{ab}-\sqrt{b}\right)^2\right)+\frac{1}{4}\left(\sqrt{a}-\sqrt{b}\right)^2+\int_0^{\frac{1}{2}}(a\sharp_t b)\,dt+\int_{\frac{1}{2}}^{1}(a\sharp_t b)\,dt\\
&=\frac{1}{8}\left(\left(\sqrt[4]{ab}-\sqrt{a}\right)^2+\left(\sqrt[4]{ab}-\sqrt{b}\right)^2\right)+\frac{1}{4}\left(\sqrt{a}-\sqrt{b}\right)^2+\int_0^{1}(a\sharp_t b)\,dt\\
&\le a\nabla b
\end{aligned}$$

as desired.

The second inequality follows from the following inequalities [17, Lemma 2]:

$$a\nabla_t b\le a\sharp_t b+(1-t)\left(\sqrt{a}-\sqrt{b}\right)^2-r_0\left(\sqrt[4]{ab}-\sqrt{b}\right)^2;\left(0\le t\le\frac{1}{2}\right),$$

and

$$a\nabla_t b\le a\sharp_t b+t\left(\sqrt{a}-\sqrt{b}\right)^2-r_0\left(\sqrt[4]{ab}-\sqrt{a}\right)^2;\left(\frac{1}{2}\le t\le 1\right).$$

This completes the proof. □

Now we use Theorem 3.1 to obtain the following hyperbolic inequality.

Corollary 3.2. *Let $y\in\mathbb{R}$. Then*

$$\cosh y-\frac{3}{4e^y}(1-e^y)^2\le\frac{\sinh y}{y}\le\cosh y-\frac{1}{4e^y}(1-e^y)^2. \tag{12}$$

Proof. It follows from Theorem 3.1 that, for $x>0$,

$$\frac{1+x}{2}-\frac{3}{4}\left(1-\sqrt{x}\right)^2\le\frac{x-1}{\log x}\le\frac{1+x}{2}-\frac{1}{4}\left(1-\sqrt{x}\right)^2 \tag{13}$$

thanks to (6). In (13), replace x by e^{2y}, to get

$$\frac{1+e^{2y}}{2}-\frac{3}{4}(1-e^y)^2\le\frac{e^{2y}-1}{2y}\le\frac{1+e^{2y}}{2}-\frac{1}{4}(1-e^y)^2.$$

Dividing by e^y implies the desired inequalities. □

It is possible to express the Taylor series at zero of the sinh and cosh functions explicitly. Indeed, we can write

$$\frac{\sinh x}{x}=1+\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots,$$

$$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots,$$

and, of course,

$$1\le\frac{\sinh x}{x}\le\cosh x. \tag{14}$$

Clearly, (12) delivers a refinement and a reverse for the inequality (14).

We have the following remark related to the hyperbolic functions, which provides a refinement and a reverse of the simple inequality $\cosh x\ge 1$. This further illustrates the relation between hyperbolic and mean inequalities.

Remark 3.3. *If, in* (5) *and* (4)*, we replace a and b by e^x and e^{-x}, we get*

$$\begin{aligned}&1+\frac{1}{2R}\left((1-\nu)\,e^x+\nu e^{-x}-(e^x)^{1-\nu}(e^{-x})^{\nu}\right)\\&\le\cosh x\\&\le 1+\frac{1}{2r}\left((1-\nu)\,e^x+\nu e^{-x}-(e^x)^{1-\nu}(e^{-x})^{\nu}\right),\end{aligned}$$

where $r=\min\{\nu,1-\nu\}$, $R=\max\{\nu,1-\nu\}$, and $0<\nu<1$.

In [16], Stolarsky obtained a possible bound for the exponential function whose power involves a hyperbolic function, as follows:

$$\exp\left(\frac{x}{\tanh x}-1\right)<\cosh x;\ \ (x>0)\,. \tag{15}$$

A well-known upper bound for $\cosh x$ is also given by

$$\cosh x<\exp\left(\frac{x^2}{2}\right);\ \ (x>0)\,, \tag{16}$$

which is confirmed by setting the function

$$f(x):=\frac{x^2}{2}-\log\cosh x,\quad (x>0).$$

Then we have $f'(x)=\dfrac{x+1+(x-1)e^{2x}}{e^{2x}+1}$ and $f''(x)=\left(\dfrac{e^{2x}-1}{e^{2x}+1}\right)^2>0$ for $x>0$. Therefore we have $f'(x)>f'(0)=0$ which implies $f(x)>f(0)=0$. The direct proof of (15) is not so tricky by setting the function

$$g(x):=\log\cosh x-\left(\frac{x}{\tanh x}-1\right),\quad (x>0).$$

Then we have $g'(x)=\dfrac{4e^{2x}h(x)}{(e^{2x}-1)^2}$ with $h(x):=\dfrac{x+1+(x-1)e^{2x}}{e^{2x}+1}$. Since the function $h(x)$, is just the same as $f'(x)$ above, we find $h(x)>0$. Therefore we have $g'(x)>0$ which implies $g(x)>g(0)=0$ since we have $\lim\limits_{x\downarrow 0}\dfrac{x}{\tanh x}=1$.

On the other hand, as a converse of the arithmetic-geometric mean inequality, Specht [15] estimated the ratio of the arithmetic mean to the geometric one in the following form

$$a\nabla b\le S\left(\frac{a}{b}\right)(a\sharp b)\,;\ S\,(h)=\frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}},h\ne 1. \tag{17}$$

Interestingly, the Specht ratio $S(h)$ can be used to obtain better bounds for the hyperbolic functions. In the following result, we obtain upper and lower bounds for $\cosh x$ in a way that refines (15) and (16).

Theorem 3.4. *Let $x>0$. Then*

$$\cosh x\le S\left(e^{2x}\right), \tag{18}$$

and

$$\frac{1}{2}\left(1+\frac{(\cosh x)\arcsin(\tanh x)}{\tanh x}\right)\le\cosh x. \tag{19}$$

Proof. It follows from the inequality (17) that

$$\frac{1+a}{2} \leq S(a)\sqrt{a}.$$

for any $a > 0$. Substituting $\sqrt{a} = t$, we conclude

$$\frac{1+t^2}{2} \leq S\left(t^2\right)t.$$

From this, we can write,

$$\frac{e^{\ln t} + e^{-\ln t}}{2} = \frac{1}{2}\left(t + \frac{1}{t}\right) = \frac{1+t^2}{2t} \leq S\left(t^2\right). \tag{20}$$

With the substitution t by e^x, (20) becomes

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{\ln e^x} + e^{-\ln e^x}}{2} \leq S\left((e^x)^2\right),$$

which shows the first desired result.

We reach the inequality (19) by applying the same method for the second inequality in (8). Indeed, if we set $a := e^x$ and $b := e^{-x}$ in the second inequality in (8), then we have the inequality (19) by simple calculations, noting that $e^x > e^{-x}$ for $x > 0$.

□

The following proposition shows that the inequalities (19) and (18) give sharper lower and upper bounds of $\cosh x$ than the inequalities (15) and (16), respectively.

Proposition 3.5. *For $x > 0$, the inequalities*

$$1 + \frac{(\cosh x)\arcsin(\tanh x)}{\tanh x} > 2\exp\left(\frac{x}{\tanh x} - 1\right) \tag{21}$$

and

$$\exp\left(\frac{x^2}{2}\right) > S(e^{2x}) \tag{22}$$

hold.

Proof. The inequality (21) is equivalent to

$$1 + \frac{t+1}{2\sqrt{t}} \cdot \frac{t+1}{t-1}\arcsin\left(\frac{t-1}{t+1}\right) > 2\exp\left(\frac{t+1}{t-1}\log\sqrt{t} - 1\right), \quad (t > 1), \tag{23}$$

by putting $e^{2x} := t > 1$ for $x > 0$. The inequality (23) is equivalent to

$$1 + \frac{\arcsin(s)}{s\sqrt{1-s^2}} > \frac{2}{e}\left(\frac{1+s}{1-s}\right)^{\frac{1}{2s}}, \quad (0 < s < 1),$$

by putting $\dfrac{t-1}{t+1} =: s$ for $t > 1$. Now we consider the function

$$f(s) := \log\left(1 + \frac{\arcsin(s)}{s\sqrt{1-s^2}}\right) - \log\left(\frac{2}{e}\left(\frac{1+s}{1-s}\right)^{\frac{1}{2s}}\right), \quad (0 < s < 1).$$

The proof of the first desired inequality will be accomplished upon showing the positivity of the function $f(s)$. We have

$$f'(s) = \frac{1}{2s^2}\left\{\log\left(\frac{1+s}{1-s}\right) - \frac{4s\arcsin(s)}{s\sqrt{1-s^2}+\arcsin(s)}\right\}.$$

Setting $s := \sin\theta$, we may set the following function also

$$g(\theta) := \log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) - \frac{4\theta\sin\theta}{\theta+\sin\theta\cos\theta}, \quad (0<\theta<\pi/2).$$

Then we have

$$\begin{aligned} g'(\theta) &= \frac{2}{\cos\theta} - \frac{4\left(\theta\sin^3\theta+\cos\theta(\theta^2+\sin^2\theta)\right)}{(\theta+\cos\theta\sin\theta)^2} \\ &= \frac{2(\theta-\sin\theta\cos\theta)(\theta+\cos\theta\sin\theta-2\theta\cos^2\theta)}{\cos\theta(\theta+\cos\theta\sin\theta)^2}. \end{aligned}$$

For the function $h(\theta) := \theta - \sin\theta\cos\theta$, we have $h'(\theta) = 2\sin^2\theta > 0$ for $0<\theta<\pi/2$ so that we have $h(\theta) > h(0) = 0$. Also for the function $k(\theta) := \theta+\cos\theta\sin\theta-2\theta\cos^2\theta$, we have $k'(\theta) = 2\theta\sin(2\theta) > 0$ for $0<\theta<\pi/2$ so that we have $k(\theta) > k(0) = 0$. Therefore we have $g'(\theta) > 0$ which implies $g(\theta) > g(0) = 0$. This means $f'(s) > 0$ so that $f(s) > f(0) = 0$ since $\lim_{s\to 0}\left(\frac{1+s}{1-s}\right)^{\frac{1}{2s}} = e$ and $\lim_{s\to 0}\frac{\arcsin(s)}{s\sqrt{1-s^2}} = 1$. This shows that $f(s) > 0$, and the proof of (21) is complete.

To prove (22), we set the function

$$f(x) := \frac{1}{2}x^2 - \left(\frac{2x}{e^{2x}-1} - 1 - \log 2x + \log(e^{2x}-1)\right), \quad (x>0)$$

By a simple calculation, we have

$$f'(x) = \frac{\left\{x^{2x}(x-1)+x+1\right\}^2}{x(e^{2x}-1)^2} > 0.$$

So we have $f(x) > f(0) = \lim_{x\downarrow 0} f(x) = 0$, which means $\frac{x^2}{2} > \log S(e^{2x})$. Therefore we have $\exp\left(\frac{x^2}{2}\right) > S(e^{2x})$ for $x>0$. This completes the proof. □

From Theorem 3.4 and Proposition 3.5, we have for all $x \in \mathbb{R}$,

$$\exp\left(\frac{x}{\tanh x} - 1\right) \le \frac{1}{2}\left(1 + \frac{(\cosh x)\arcsin(\tanh x)}{\tanh x}\right) \le \cosh x \le S(e^{2x}) \le \exp\left(\frac{x^2}{2}\right),$$

since $\arcsin(-x) = -\arcsin x$, $\tanh(-x) = -\tanh x$, $\cosh(-x) = \cosh x$ and $S(1/h) = S(h)$ for $1 \ne h > 0$ with $S(1) =: \lim_{h\to 1} S(h) = 1$.

4. The AM-GM inequality in the complex plane

In this section, we present some inequalities in the complex plane. While the AM-GM inequality spins off first, we present other inequalities for convex functions. In this context, the modulus will be used in order to be able to compare complex quantities. Also, additional conditions on the arguments of the used complex numbers will be imposed. We begin with the following complex version of (2).

For $z_1,\dots,z_n \in \mathbb{C}$ and positive scalars $w_1,\dots w_n$ with $\sum_{i=1}^n w_i = 1$, we have

$$\left|\sum_{i=1}^n w_i z_i\right| \le \sum_{i=1}^n w_i|z_i|.$$

Assuming a condition for arguments, we have a reverse inequality above in the following.

Theorem 4.1. *Let $w_1,\dots w_n$ be positive scalars with $\sum_{i=1}^n w_i = 1$. If $z_1,\dots,z_n \in \mathbb{C}$ and $\theta \in [-\pi,\pi]$ satisfy*

$$|\arg z_i| \le |\theta|;\ \ (i = 1,\dots,n)\,,$$

then

$$\cos\theta\ \sum_{i=1}^n w_i\,|z_i| \le \left|\sum_{i=1}^n w_i z_i\right|.$$

Proof. We have

$$\begin{aligned}
&\left|\sum_{i=1}^n w_i z_i\right| \\
&\ge \Re\left(\sum_{i=1}^n w_i z_i\right) \\
&= \Re\left(w_1\,|z_1|\,e^{\mathrm{i}\arg z_1} + \cdots + w_n\,|z_n|\,e^{\mathrm{i}\arg z_n}\right) \\
&= \Re\left(w_1\,|z_1|\,(\cos(\arg z_1) + \mathrm{i}\sin(\arg z_1)) + \cdots + w_n\,|z_n|\,(\cos(\arg z_n) + \mathrm{i}\sin(\arg z_n))\right) \\
&= \cos(\arg z_1)\,w_1\,|z_1| + \cdots + \cos(\arg z_n)\,w_n\,|z_n| \\
&= \cos(|\arg z_1|)\,w_1\,|z_1| + \cdots + \cos(|\arg z_n|)\,w_n\,|z_n| \\
&\ge \cos|\theta|\ \sum_{i=1}^n w_i\,|z_i| \\
&= \cos\theta\ \sum_{i=1}^n w_i\,|z_i|,
\end{aligned} \tag{24}$$

where we have used the assumption $0 \le |\arg z_i| \le |\theta| \le \pi$ to obtain the second inequality, $\cos(-x) = \cos(x)$ and $\cos(x)$ is monotone decreasing in $x \in [0,\pi]$. □

Remark 4.2. *If we restrict $\theta \in [-\pi/2,\pi/2]$ in Theorem 4.1, we have*

$$\cos\theta \prod_{i=1}^n |z_i|^{w_i} \le \cos\theta\ \sum_{i=1}^n w_i\,|z_i| \le \left|\sum_{i=1}^n w_i z_i\right|,$$

since $\cos\theta \ge 0$ *for* $\theta \in [-\pi/2,\pi/2]$.

At this stage, we point out that when the z_i are all real numbers, one can take $\alpha = 0, \theta = 0$ in Remark 4.2. This implies (2). Another observation here is the use of (2) to obtain the first inequality in Remark 4.2. Notice that one can use a refined version of (2) to obtain a refined version of Remark 4.2. This will be left to the interested reader.

Having presented the AM-GM inequality for complex numbers, we proceed with further complex inequalities. The significance of this inequality is due to the fact that a concave function $f : [0,\infty) \to [0,\infty)$ satisfies the Jensen inequality

$$\sum_{i=1}^n w_i f(x_i) \le f\left(\sum_{i=1}^n w_i x_i\right),$$

where $x_i, w_i > 0$ and $\sum_{i=1}^n w_i = 1$. We also recall that such a concave function $f : [0,\infty) \to [0,\infty)$ satisfies the simple inequality

$$rf(x) \le f(rx), 0 \le r \le 1. \tag{25}$$

Corollary 4.3. *Let $f:[0,\infty)\to[0,\infty)$ be a concave function, and let $w_1,\dots w_n$ be positive scalars with $\sum_{i=1}^n w_i=1$. If $z_1,\dots,z_n\in\mathbb{C}$ and $\theta\in[-\pi/2,\pi/2]$ satisfy*

$$|\arg z_i|\le|\theta|;\ \ (i=1,\dots,n),$$

then

$$\cos\theta\ \sum_{i=1}^n w_i f(|z_i|)\le f\left(\left|\sum_{i=1}^n w_i z_i\right|\right).$$

Proof. We notice that if $f:[0,\infty)\to[0,\infty)$ is concave, then it is necessarily increasing. This, together with (24), implies the first inequality below

$$f\left(\left|\sum_{i=1}^n w_i z_i\right|\right)\ge f\left(\cos\theta\sum_{i=1}^n w_i|z_i|\right)\ge\cos\theta\ \sum_{i=1}^n w_i f(|z_i|)$$

where we have used (25) with $0\le w_i\cos\theta\le 1$ in the last inequality. This completes the proof. □

A non-negative function $f:\mathbb{R}\to(0,\infty)$ is logarithmically concave (or log-concave for brief) if $\log f$ is concave; that is

$$(1-t)\log f(a)+t\log f(b)\le\log f((1-t)a+tb);\ \ (0\le t\le 1)$$

for all $a,b\in\mathbb{R}$. This is equivalent to stating that

$$f^{1-t}(a)\,f^t(b)\le f((1-t)a+tb).$$

Proposition 4.4. *Let $f:(0,\infty)\to(0,\infty)$ be an increasing log-concave function, and let $w_1,\dots w_n$ be positive scalars with $\sum_{i=1}^n w_i=1$. If $z_1,\dots,z_n\in\mathbb{C}$ and $\theta\in[-\pi/2,\pi/2]$ satisfy*

$$|\arg z_i|\le|\theta|;\ \ (i=1,\dots,n),$$

then

$$\prod_{i=1}^n\{f(|z_i|)\}^{w_i\cos\theta}\le f\left(\left|\sum_{i=1}^n w_i z_i\right|\right).$$

Since the function $f(x)=x^r\ (x,r>0)$ is an increasing log-concave function, we get the following result from Proposition 4.4.

Corollary 4.5. *Let $w_1,\dots w_n$ be positive scalars with $\sum_{i=1}^n w_i=1$. If $z_1,\dots,z_n\in\mathbb{C}$ and $\theta\in[-\pi/2,\pi/2]$ satisfy*

$$|\arg z_i|\le|\theta|;\ \ (i=1,\dots,n),$$

then

$$\prod_{i=1}^n|z_i|^{rw_i\cos\theta}\le\left|\sum_{i=1}^n w_i z_i\right|^r;\ \ (r>0).$$

In particular,

$$\sqrt[n]{|z_1\cdots z_n|^{\cos\theta}}\le\frac{|z_1+\cdots+z_n|}{n}.$$

5. AM-GM Inequality Through Convex Functions

The weighted version of (7) has been given in [14, Theorem 2.2] in the following form

$$a\sharp_t b \le L_t(a,b) \le a\nabla_t b, \tag{26}$$

where the weighted logarithmic mean was defined by

$$L_t(a,b) := \frac{1}{\log b - \log a}\left(\frac{1-t}{t}\left(a^{1-t}b^t - a\right) + \frac{t}{1-t}\left(b - a^{1-t}b^t\right)\right). \tag{27}$$

To obtain (26), the authors in [14] first proved that if $f : [a,b] \to \mathbb{R}$ is a convex function, then for any $0 \le t \le 1$

$$\begin{aligned} & f\left((1-t)\,a + tb\right) \\ & \le (1-t)\int_0^1 f\left(t\alpha\,(b-a) + a\right)d\alpha + t\int_0^1 f\left((1-t)\,\alpha\,(b-a) + (1-t)\,a + tb\right)d\alpha \\ & \le (1-t)\,f\,(a) + tf\,(b)\,, \end{aligned} \tag{28}$$

holds. In the following, we utilize (28) to improve the celebrated Jensen-Mercer's inequality [6], i.e.,

$$f\left(M + m - \sum_{i=1}^n w_i t_i\right) \le f\,(M) + f\,(m) - \sum_{i=1}^n w_i f\,(t_i),$$

where $f : [m,M] \to \mathbb{R}$ is a convex function, $m \le t_i \le M, (i = 1,2,\ldots,n)$, and $w_1, w_2, \ldots, w_n$ are positive real numbers such that $\sum_{i=1}^n w_i = 1$.

Theorem 5.1. *Let $f : [m,M] \to \mathbb{R}$ be a convex function, let $m \le t_i \le M, (i = 1,2,\ldots,n)$, and let $w_1, w_2, \ldots, w_n$ be positive real numbers such that $\sum_{i=1}^n w_i = 1$. Then*

$$\begin{aligned} & \sum_{i=1}^n w_i f\,(t_i) + f\left(M + m - \sum_{i=1}^n w_i t_i\right) \\ & \le \sum_{i=1}^n \frac{w_i\,(M - t_i)}{M-m}\int_0^1 \left\{f\left((t_i - m)\,\alpha + m\right) + f\left((m - t_i)\,\alpha + M\right)\right\}d\alpha \\ & \quad + \sum_{i=1}^n \frac{w_i\,(t_i - m)}{M-m}\int_0^1 \left\{f\left((t_i - M)\,\alpha + M\right) + f\left((M - t_i)\,\alpha + m\right)\right\}d\alpha \\ & \le f\,(m) + f\,(M)\,. \end{aligned}$$

Proof. Let $m \le t_i \le M, (i = 1,2,\ldots n)$. If we replace $1 - t = \frac{M-t_i}{M-m}$, $t = \frac{t_i - m}{M-m}$, $a = m$, and $b = M$, in (28), we obtain

$$\begin{aligned} f\,(t_i) & \le \frac{M-t_i}{M-m}\int_0^1 f\left((t_i - m)\,\alpha + m\right)d\alpha + \frac{t_i - m}{M-m}\int_0^1 f\left((M-t)\,\alpha + t_i\right)d\alpha \\ & \le \frac{M-t_i}{M-m}f\,(m) + \frac{t_i - m}{M-m}f\,(M)\,. \end{aligned} \tag{29}$$

From this, we have

$$\begin{aligned}
&\sum_{i=1}^{n} w_i f(t_i) \\
&\le \sum_{i=1}^{n} w_i \left(\frac{M-t_i}{M-m} \int_0^1 f((t_i-m)\alpha+m)\,d\alpha \right) + \sum_{i=1}^{n} w_i \left(\frac{t_i-m}{M-m} \int_0^1 f((M-t_i)\alpha+t_i)\,d\alpha \right) \\
&\le \frac{M-\sum_{i=1}^{n} w_i t_i}{M-m} f(m) + \frac{\sum_{i=1}^{n} w_i t_i - m}{M-m} f(M).
\end{aligned} \tag{30}$$

Since $m \le t_i \le M$, we infer that $m \le M+m-t_i \le M$. Of course, $m \le \sum_{i=1}^{n} w_i (M+m-t_i) \le M$. Thus, we can replace t_i by $\sum_{i=1}^{n} w_i (M+m-t_i)$, in (29), to get

$$\begin{aligned}
&f\left(M+m-\sum_{i=1}^{n} w_i t_i\right) \\
&\le \frac{\sum_{i=1}^{n} w_i t_i - m}{M-m} \int_0^1 f\left(\sum_{i=1}^{n} w_i ((M-t_i)\alpha+m)\right) d\alpha \\
&\quad + \frac{M-\sum_{i=1}^{n} w_i t_i}{M-m} \int_0^1 f\left(\sum_{i=1}^{n} w_i ((t_i-m)\alpha+M+m-t_i)\right) d\alpha \\
&\le \frac{\sum_{i=1}^{n} w_i t_i - m}{M-m} f(m) + \frac{M-\sum_{i=1}^{n} w_i t_i}{M-m} f(M).
\end{aligned} \tag{31}$$

Combining (30) and (31), we infer that

$$\begin{aligned}
&f\left(M+m-\sum_{i=1}^{n} w_i t_i\right) + \sum_{i=1}^{n} w_i f(t_i) \\
&\le \frac{\sum_{i=1}^{n} w_i t_i - m}{M-m} \int_0^1 f\left(\sum_{i=1}^{n} w_i ((M-t_i)\alpha+m)\right) d\alpha \\
&\quad + \frac{M-\sum_{i=1}^{n} w_i t_i}{M-m} \int_0^1 f\left(\sum_{i=1}^{n} w_i ((t_i-m)\alpha+M+m-t_i)\right) d\alpha \\
&\quad + \sum_{i=1}^{n} w_i \left(\frac{M-t_i}{M-m} \int_0^1 f((t_i-m)\alpha+m)\,d\alpha \right) + \sum_{i=1}^{n} w_i \left(\frac{t_i-m}{M-m} \int_0^1 f((M-t_i)\alpha+t_i)\,d\alpha \right) \\
&\le f(M) + f(m).
\end{aligned}$$

Since

$$\int_0^1 f((M-t_i)\alpha+t_i)\,d\alpha = \int_0^1 f((t_i-M)\alpha+M)\,d\alpha,$$

we get

$$\begin{aligned}
&\sum_{i=1}^{n} w_i f(t_i) + f\left(M + m - \sum_{i=1}^{n} w_i t_i\right) \\
&\le \sum_{i=1}^{n} \frac{w_i (M - t_i)}{M - m} \int_0^1 \{f((t_i - m)\alpha + m) + f((m - t_i)\alpha + M)\} d\alpha \\
&\quad + \sum_{i=1}^{n} \frac{w_i (t_i - m)}{M - m} \int_0^1 \{f((t_i - M)\alpha + M) + f((M - t_i)\alpha + m)\} d\alpha \\
&\le f(m) + f(M),
\end{aligned}$$

as desired. □

In Theorem 5.1, if f is additive (linear), then both sides become the same $f(m) + f(M)$.

Remark 5.2. *The case $n = 1$ in Theorem 5.1 reduces to*

$$\begin{aligned}
&f(M + m - t) + f(t) \\
&\le \frac{M - t}{M - m} \left(\int_0^1 f((t - m)\alpha + m) d\alpha + \int_0^1 f((t - m)\alpha + M + m - t) d\alpha \right) \\
&\quad + \frac{t - m}{M - m} \left(\int_0^1 f((M - t)\alpha + t) d\alpha + \int_0^1 f((M - t)\alpha + m) d\alpha \right) \\
&\le f(m) + f(M).
\end{aligned} \tag{32}$$

Since both sides are trivially equal to $f(m) + f(M)$ for special cases such as $t := m$ or $t := M$, we consider the case $m < t < M$. Changing variables of the integrals in (32)*, we have for $m < t < M$,*

$$\begin{aligned}
&f(M + m - t) + f(t) \\
&\le \frac{M - t}{M - m} \left(\frac{1}{t - m} \int_m^t f(x)dx + \frac{1}{t - m} \int_{M+m-t}^{M} f(x)dx \right) \\
&\quad + \frac{t - m}{M - m} \left(\frac{1}{M - t} \int_m^{M+m-t} f(x)dx + \frac{1}{M - t} \int_t^M f(x)dx \right) \\
&\le f(m) + f(M).
\end{aligned} \tag{33}$$

By letting $f(x) = e^x$ in (33)*, we get*

$$\begin{aligned}
&e^{M+m-t} + e^t \\
&\le \frac{M - t}{M - m} \left(\frac{e^t - e^m}{t - m} + \frac{e^M - e^{M+m-t}}{t - m} \right) + \frac{t - m}{M - m} \left(\frac{e^t - e^M}{t - M} + \frac{e^m - e^{M+m-t}}{t - M} \right) \\
&\le e^m + e^M.
\end{aligned}$$

Now, by substituting m, M, t by $\ln m, \ln M, \ln t$, respectively, we have for $m < t < M$

$$\begin{aligned}
&\frac{Mm}{t} + t \\
&\le \frac{\ln M - \ln t}{\ln M - \ln m} \left(\frac{t - m}{\ln t - \ln m} + \frac{M - Mm/t}{\ln t - \ln m} \right) + \frac{\ln t - \ln m}{\ln M - \ln m} \left(\frac{t - M}{\ln t - \ln M} + \frac{m - Mm/t}{\ln t - \ln M} \right) \\
&\le M + m.
\end{aligned}$$

If we take $t := \sqrt{Mm}$, then the above inequalities imply

$$\sqrt{Mm} \le \frac{M-m}{\log M - \log m} \le \frac{M+m}{2},$$

by elementary calculations.

Remark 5.3. *Putting $m := a$, $M := b$ and $t := \dfrac{a+b}{2}$ in (33), we obtain the famous Hermite–Hadamard inequality:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2},$$

by elementary calculations. Putting $m := 0$ and $M := 1$ in (33), we also have for $0 < t < 1$,

$$\begin{aligned} &f(1-t) + f(t) \\ &\le \frac{1-t}{t}\left(\int_0^t f(x)dx + \int_{1-t}^1 f(x)dx\right) + \frac{t}{1-t}\left(\int_0^{1-t} f(x)dx + \int_t^1 f(x)dx\right) \\ &\le f(0) + f(1). \end{aligned} \tag{34}$$

In particular,

$$f\left(\frac{1}{2}\right) \le \int_0^1 f(x)dx \le \frac{f(0)+f(1)}{2}. \tag{35}$$

Integrating (34) with respect to t from 0 to 1, we have a refinement of the second inequality in (35):

$$\begin{aligned} &\int_0^1 f(x)dx \\ &\le \frac{1}{2}\left(\int_0^1 \frac{1-t}{t}\left(\int_0^t f(x)dx + \int_{1-t}^1 f(x)dx\right)dt + \int_0^1 \frac{t}{1-t}\left(\int_0^{1-t} f(x)dx + \int_t^1 f(x)dx\right)dt\right) \\ &\le \frac{f(0)+f(1)}{2}. \end{aligned} \tag{36}$$

For example, if we take $f(x) := -\log(1+x)$ in (35), then we have $\log\dfrac{2}{3} \le \log\dfrac{e}{4} \le \log\sqrt{\dfrac{1}{2}}$, which gives rough bounds for Napier's e as $\dfrac{8}{3} \le e \le 2\sqrt{2}$. If we also take $f(x) := -\log(1+x)$ in (36), we have

$$\log\frac{e}{4} \le 3 - \frac{\pi^2}{4} + (\log 2)^2 - \log 4 \le \log\sqrt{\frac{1}{2}}.$$

The inequalities (34) give interesting inequalities related to means. We recall the Heinz mean defined by

$$Hz_t(a,b) := \frac{a\sharp_t b + a\sharp_{1-t} b}{2}, \quad (a,b>0,\ 0 \le t \le 1).$$

It is well known that $a\sharp b \le H_{z_t}(a,b) \le a\nabla b$. In the following, we present a refinement of the inequality $H_{z_t}(a,b) \le a\nabla b$ in terms of the weighted logarithmic mean.

Theorem 5.4. *Let $a, b > 0$ and $0 \le t \le 1$. Then*

$$Hz_t(a,b) \le \frac{L_t(a,b) + L_t(b,a)}{2} \le \frac{a+b}{2}, \tag{37}$$

where $L_t(a,b)$ is defined as in (27).

Proof. Putting $f(x) := u^x$, $(u > 0,\ 0 \le x \le 1)$ in (34), we have

$$u^{1-t} + u^t \le L_t(u,1) + \hat{L}_t(u,1) \le 1 + u, \quad (u > 0,\ 0 \le t \le 1) \tag{38}$$

where

$$\hat{L}_t(u,1) := \frac{1}{\ln u}\left\{\frac{1-t}{t}\left(u - u^{1-t}\right) + \frac{t}{1-t}\left(u^{1-t} - 1\right)\right\}.$$

Since we have a relation $\hat{L}_t\left(\frac{1}{u},1\right) = \frac{1}{u}L_t(u,1)$, the inequalities (38) is written as

$$u^{1-t} + u^t \le L_t(u,1) + uL_t\left(\frac{1}{u},1\right) \le 1 + u, \quad (u > 0,\ 0 \le t \le 1)$$

Putting $u := b/a$ in the above and multiplying $\frac{a}{2} > 0$ to both sides, we have the inequalities (37) by some calculations. □

If we take $t := \frac{1}{2}$ in Theorem 5.4, then the inequalities (37) recover the famous ordering between three means:

$$\sqrt{ab} \le \frac{b-a}{\log b - \log a} \le \frac{a+b}{2}.$$

If we also take $f(x) := u^x$, $(u > 0,\ 0 \le x \le 1)$ in (35) and (36), then we have

$$\sqrt{ab} \le \frac{b-a}{\log b - \log a} \le \frac{1}{2}\int_0^1 (L_t(a,b) + L_t(b,a))\, dt \le \frac{a+b}{2}.$$

However, the integral $\int_0^1 (L_t(a,b) + L_t(b,a))\, dt$ is not given by a simple form.

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