Filomat 39:31 (2025), 11061–11077
https://doi.org/10.2298/FIL2531061C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Asymptotically almost periodic type functions of several variables and applications

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Abstract. In this paper, we introduce and analyze several new classes of $\mathbb{D}$-asymptotically almost periodic type functions of the form $F : I \times X \to Y$, where $\emptyset \neq I \subseteq \mathbb{R}^n$, $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces. We present many structural results for the introduced classes of $\mathbb{D}$-asymptotically almost periodic type functions, providing also some illustrative examples and applications of the established results to the abstract Volterra integro-differential equations.

1. Introduction and preliminaries

The class of almost periodic functions was introduced and studied by the Danish mathematician H. Bohr around 1924–1926. Suppose that $(X, \|\cdot\|)$ is a complex Banach space and $F : \mathbb{R}^n \to X$ is a continuous function, where $n \in \mathbb{N}$. Then $F(\cdot)$ is called *almost periodic* if and only if for each $\epsilon > 0$ there exists $r > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists

$$\tau \in B(\mathbf{t}_0, r) \equiv \{\mathbf{a} \in \mathbb{R}^n : |\mathbf{a} - \mathbf{t}_0| \leq r\}$$

such that

$$\left\|F(\mathbf{t} + \tau) - F(\mathbf{t})\right\| \leq \epsilon, \quad \mathbf{t} \in \mathbb{R}^n,$$

where $|\cdot - \cdot|$ is the Euclidean distance in $\mathbb{R}^n$. This is also equivalent to saying that for any sequence $(\mathbf{b}_k)$ in $\mathbb{R}^n$ there exists a subsequence $(\mathbf{a}_k)$ of $(\mathbf{b}_k)$ such that the sequence of functions $(F(\cdot + \mathbf{a}_k))$ converges in

2020 *Mathematics Subject Classification*. Primary 42A75, 43A60; Secondary 47D06, 47D60, 47D99.
Keywords. $\mathbb{D}$-asymptotically almost periodic type functions, composition principles, convolution invariance, abstract Volterra integro-differential equations.
Received: 16 October 2023; Accepted: 30 September 2025
Communicated by Dragan S. Djordjević
Alan Chávez and Manuel Pinto were funded by CONCYTEC through the PROCIENCIA program under the E041-2023-01 competition, according to contract PE501082885-2023, Manuel Pinto was also partially supported by Fondecyt 1170466. Marko Kostić is partially supported by grant 451-03-68/2020/14/200156 of Ministry of Science and Technological Development, Republic of Serbia.
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$C_b(\mathbb{R}^n : X)$, the Banach space of all bounded continuous functions on $\mathbb{R}^n$, equipped with the sup-norm. Any trigonometric polynomial in $\mathbb{R}^n$ is almost periodic and any almost periodic function $F(\cdot)$ is bounded and uniformly continuous. It is also well known that a continuous function $F(\cdot)$ is almost periodic if and only if there exists a sequence of trigonometric polynomials in $\mathbb{R}^n$ which converges to $F(\cdot)$ in $C_b(\mathbb{R}^n : X)$.

The notion of asymptotical almost periodicity for scalar-valued functions of one real variable was introduced by A. S. Kovanko [21] in 1929; the usually employed definition of asymptotical almost periodicity was introduced later, by M. Fréchet [15] in 1941. Let us recall that a continuous function $f : I \rightarrow X$, where I is either $\mathbb{R}$ or $[0, \infty)$, is said to be asymptotically almost periodic if and only if there exist an almost periodic function $g : \mathbb{R} \rightarrow X$ and a continuous function $\phi : I \rightarrow X$ such that $\lim_{t \in I, |t| \rightarrow +\infty} q(t) = 0$ and $f(t) = g(t) + \phi(t)$ for all $t \in I$. The notion of asymptotical almost periodicity for a continuous function $f : \mathbb{R} \rightarrow X$ can be introduced in some other ways; for example, we may require that the function $q(\cdot)$ vanishes only at plus infinity or minus infinity. This fact has influenced us to consider the general notion of $\mathbb{D}$-asymptotical almost periodicity here, for the functions of the form $F : I \times X \rightarrow Y$, where $\emptyset \neq I \subseteq \mathbb{R}^n$ and $(Y, \| \cdot \|_Y)$ is another complex Banach space; in the one-dimensional setting, the above described notions of asymptotical almost periodicity are very special cases of $\mathbb{D}$-asymptotical almost periodicity with $\mathbb{D}$ being $\mathbb{R}$, $[0, +\infty)$ or $(-\infty, 0]$. But, the situation is much more complicated in the multi-dimensional setting; for example, under certain reasonable assumptions, the unique solution of the mixed initial value problem

$$u_t(x,t) = u_{xx}(x,t), \;\; x > 0, \; t > 0;$$
$$u(x,0) = u_0(x), \; x > 0, \;\; u(0,t) = g(t), \; t > 0$$

is given by

$$u(x,t) = \frac{1}{2} \int_{-x}^{x} \frac{\partial E_1}{\partial y}(y,t) u_0(x-y)\, dy - \int_0^t \frac{\partial E_1}{\partial t}(x, t-s) g(s)\, ds, \quad x > 0, \; t > 0,$$

where

$$E_1(x,t) := \Big(\pi t\Big)^{-1/2} \int_0^x e^{-y^2/4t}\, dy, \quad x \in \mathbb{R}, \; t > 0.$$

In our analysis of the existence and uniqueness of $\mathbb{D}$-asymptotically almost periodic type solutions of the above problem, we fix first a finite real number $T > 0$ and require that $\mathbb{D}$ is any unbounded subset of $[0, \infty)^2$ such that

$$\lim_{|(x,t)| \rightarrow +\infty, (x,t) \in \mathbb{D}} \min\left(\frac{x^2}{4(t+T)}, t \right) = +\infty;$$

see Example 3.1 for more details.

The organization and main ideas of this work can be briefly described as follows. After explaining the basic notation and terminology used throughout the paper, we recollect some definitions and results about multi-dimensional almost periodic type functions in Subsection 1.1. Our main results are given in Section 2, where we analyze several new classes of $\mathbb{D}$-asymptotically almost periodic type functions; here, $\mathbb{D}$ is a non-empty subset of $\mathbb{R}^n$ obeying certain properties. Definition 2.1 introduces the notion of space $C_{0,\mathbb{D}}(I \times X : Y)$, which is crucial for introducing the notions of various types of $\mathbb{D}$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity and $\mathbb{D}$-asymptotically Bohr $\mathcal{B}$-almost periodicity; see Definition 2.2. Some relations between the range of an asymptotically almost periodic function and its almost periodic part are given in Lemma 2.3 and Lemma 2.4. After that, we discuss the uniqueness of decomposition in Definition 2.2; see Proposition 2.5. Closure properties under uniform convergence and differentiation of $\mathbb{D}$-asymptotically almost periodic type functions are examined in the continuation; see Proposition 2.6, Proposition 2.7 and Proposition 2.8. Definition 2.9 introduces the notion of $\mathbb{D}$-asymptotical Bohr $(\mathcal{B}, I')$-almost periodicity of type 1 and $\mathbb{D}$-asymptotical $(\mathcal{B}, I')$-uniform recurrence of type 1. Relations between $\mathbb{D}$-asymptotically almost periodic functions and $\mathbb{D}$-asymptotically almost periodic functions of type 1 are analyzed in Subsection 2.1; the composition theorems for $\mathbb{D}$-asymptotically almost periodic type functions are analyzed in Subsection 2.2, while the invariance of $\mathbb{D}$-asymptotical almost type periodicity under the actions of convolution products are analyzed in Subsection 2.3. The final section of paper is reserved for some examples and applications of the introduced notion to the partial differential equations and the systems of abstract first-order differential equations.

Our main results are Theorem 2.11 and Theorem 2.13. Concerning the proof of Theorem 2.11, we would like to emphasize that H. Bart and S. Goldberg have proved in [3] that, for every function $f \in AP([0,\infty) : X)$, there exists a unique almost periodic function $Ef : \mathbb{R} \to X$ such that $Ef(t) = f(t)$ for all $t \geq 0$ (see also the paper [13] by S. Favarov and O. Udodova, where the authors have investigated the extensions of almost periodic functions defined on $\mathbb{R}^n$ to the tube domains in $\mathbb{C}^n$, and the paper [4] by J. F. Berglund, where the author has investigated the extensions of almost periodic functions in topological groups and semigroups). In Theorem 2.11 and Theorem 2.13, we follow the argumentation contained in the proof of the important theoretical result [25, Theorem 3.4], established by W. M. Ruess and W. H. Summers.

Notation and terminology.

We assume henceforth that $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are complex Banach spaces, $n \in \mathbb{N}$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $\mathcal{B}$ is a non-empty collection of non-empty subsets of X, R is a non-empty collection of sequences in $\mathbb{R}^n$ and R_X is a non-empty collection of sequences in $\mathbb{R}^n \times X$; usually, $\mathcal{B}$ denotes the collection of all bounded subsets of X or all compact subsets of X. Set $\mathcal{B}_X := \{y \in X : (\exists B \in \mathcal{B})\ y \in B\}$. We will always assume henceforth that $\mathcal{B}_X = X$, i.e., that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. By $L(X, Y)$ we denote the Banach algebra of all bounded linear operators from X into Y; $L(X, X) \equiv L(X)$. Further on, by B°, ∂B and $\overline{B}$ we denote the interior, the boundary and the closure of a subset B of a topological space, respectively. If $F : \mathcal{A} \to C$ is a function, then its range is denoted by $R(F)$. Set $\mathbb{N}_n := \{1, \cdots, n\}$.

The symbol $C(I : X)$ stands for the space of all X-valued continuous functions defined on the domain I. By $C_b(I : X)$ (respectively, $BUC(I : X)$) we denote the subspace of $C(I : X)$ consisting of all bounded (respectively, all bounded uniformly continuous functions). Both $C_b(I : X)$ and $BUC(I : X)$ are Banach spaces with the sup-norm. This also holds for the space $C_0(I : X)$ consisting of all continuous functions $f : I \to X$ such that $\lim_{t\in I, |t|\to+\infty} f(t) = 0$.

1.1. *Multi-dimensional almost periodic type functions and supremum formula*

We open this subsection by recalling the following notion from [8]:

Definition 1.1. ($(\mathrm{R}, \mathcal{B})$**-multi-almost periodic**) *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function, and the following condition holds:*

$$\text{If } \mathbf{t} \in I,\ \mathbf{b} \in \mathrm{R} \text{ and } l \in \mathbb{N}, \text{ then we have } \mathbf{t} + \mathbf{b}(l) \in I. \tag{1}$$

Then we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \cdots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : I \times X \to Y$ such that

$$\lim_{l\to+\infty} F\Big(\mathbf{t} + (b_{k_l}^1, \cdots, b_{k_l}^n); x\Big) = F^*(\mathbf{t}; x) \tag{2}$$

uniformly for all $x \in B$ and $\mathbf{t} \in I$.

Definition 1.2. ($(\mathrm{R}_X, \mathcal{B})$**-multi-almost periodic**) *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function, and the following condition holds:*

$$\text{If } \mathbf{t} \in I,\ (\mathbf{b}; \mathbf{x}) \in \mathrm{R}_X \text{ and } l \in \mathbb{N}, \text{ then we have } \mathbf{t} + \mathbf{b}(l) \in I. \tag{3}$$

Then we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \cdots, b_k^n); x_k)_k) \in \mathrm{R}_X$ there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \cdots, b_{k_l}^n); x_{k_l})_{k_l})$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : I \times X \to Y$ such that

$$\lim_{l\to+\infty} F\Big(\mathbf{t} + (b_{k_l}^1, \cdots, b_{k_l}^n); x + x_{k_l}\Big) = F^*(\mathbf{t}; x) \tag{4}$$

uniformly for all $x \in B$ and $\mathbf{t} \in I$.

Definition 1.3. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I \subseteq I$. Then we say that:*

(i) (**Bohr $\mathcal{B}$-almost periodic**) *$F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I$ such that*

$$\left\| F(\mathbf{t} + \tau; x) - F(\mathbf{t}; x) \right\|_Y \leq \epsilon, \quad \mathbf{t} \in I,\ x \in B. \tag{5}$$

(ii) (**$\mathcal{B}$-uniformly recurrent**) *$F(\cdot;\cdot)$ is $\mathcal{B}$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_k) in I such that $\lim_{k\to+\infty} |\tau_k| = +\infty$ and*

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in I; x\in B} \left\| F(\mathbf{t} + \tau_k; x) - F(\mathbf{t}; x) \right\|_Y = 0. \tag{6}$$

If $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is Bohr almost periodic (uniformly recurrent).

If $F : I \times X \to Y$ is a Bohr $\mathcal{B}$-almost periodic function and the set I unbounded, then it can be simply shown that the function $F(\cdot;\cdot)$ is $\mathcal{B}$- uniformly recurrent. In the sequel, we will use the following supremum formula ([8]):

Proposition 1.4. *(Supremum formula for $(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions) Suppose that $F : I \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, $a \geq 0$ and $x \in X$. If there exists a sequence $b(\cdot)$ in R whose any subsequence is unbounded and for which we have $t - \mathbf{b}(l) \in I$ whenever $t \in I$ and l big enough, then we have*

$$\sup_{\mathbf{t}\in I} \left\| F(\mathbf{t}; x) \right\|_Y = \sup_{\mathbf{t}\in I, |\mathbf{t}|\geq a} \left\| F(\mathbf{t}; x) \right\|_Y. \tag{7}$$

2. $\mathbb{D}$-Asymptotically almost periodic type functions

We start this subsection by introducing the following definition (many other classes of weighted ergodic components in $\mathbb{R}^n$ and their metrical generalizations, introduced and analyzed recently in [19]-[20], can be used to slightly extend the notion from Definition 2.2 below):

Definition 2.1. *Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded. By $C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ we denote the vector space consisting of all continuous functions $Q : I \times X \to Y$ such that, for every $B \in \mathcal{B}$, we have $\lim_{t\in\mathbb{D}, |t|\to+\infty} Q(\mathbf{t}; x) = 0$, uniformly for $x \in B$.*

Now we are ready to introduce the following notion:

Definition 2.2. *Suppose that the set $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ is unbounded, and $F : I \times X \to Y$ is a continuous function. Then we say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. $\mathbb{D}$-asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic, if and only if there exist an $(\mathrm{R}, \mathcal{B})$-multi-almost periodic function $G : I \times X \to Y$, resp. an $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic function $G : I \times X \to Y$, and a function $Q \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ such that $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for all $\mathbf{t} \in I$ and $x \in X$.*

Let $I = \mathbb{R}^n$. Then it is said that $F(\cdot;\cdot)$ is asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic, if and only if $F(\cdot;\cdot)$ is $\mathbb{R}^n$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, resp. $\mathbb{R}^n$-asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic.

We similarly introduce the notions of ($\mathbb{D}$-)asymptotical Bohr $\mathcal{B}$-almost periodicity, ($\mathbb{D}$-)asymptotical $\mathcal{B}$-uniform recurrence, ($\mathbb{D}$-)asymptotical Bohr $(\mathcal{B}, I')$-almost periodicity and ($\mathbb{D}$-)asymptotical $(\mathcal{B}, I')$-uniform recurrence (cf. [8, Definition 2.14]). If $X \in \mathcal{B}$, then we omit the term $\mathcal{B}$ from the notation introduced, with the meaning clear.

The main strucural properties of multi-dimensional almost periodic type functions clarified in [8, Proposition 2.28] can be simply formulated for the corresponding classes of $\mathbb{D}$-asymptotically almost periodic type functions introduced above.

Relation between range of asymptotically almost periodic function and its almost periodic part. In Definition 2.2, the function $G(\cdot;\cdot)$ is usually called the *principal part* of $F(\cdot;\cdot)$ and the function $Q(\cdot;\cdot)$ is usually called the *ergodic part* of $F(\cdot;\cdot)$. We will prove here two auxiliary results which relates the range of $G(\cdot;\cdot)$ and the range of $F(\cdot;\cdot)$. The first result reads as follows:

Lemma 2.3. *Let I be unbounded and $I + I \subseteq I$. Assume that $F(\cdot;\cdot)$ is I-asymptotically $\mathcal{B}$-uniformly recurrent, $G : I \times X \to Y$, $Q \in C_{0,I,\mathcal{B}}(I \times X : Y)$ and $F(\mathbf{t};x) = G(\mathbf{t};x) + Q(\mathbf{t};x)$ for all $\mathbf{t} \in I$ and $x \in X$; then, we have*

$$\overline{\left\{G(\mathbf{t};x) : \mathbf{t} \in I,\ x \in X\right\}} \subseteq \overline{\left\{F(\mathbf{t};x) : \mathbf{t} \in I,\ x \in X\right\}}. \tag{8}$$

Proof. Since G is $\mathcal{B}$-uniformly recurrent, for every $B \in \mathcal{B}$, there exists a sequence $(\tau_k) \subseteq I$ such that $|\tau_k| \to \infty$ as $k \to \infty$ and:

$$\lim_{k\to\infty} \sup_{t\in I, x\in B} \|G(t + \tau_k, x) - G(t, x)\|_Y = 0.$$

Let $t_0 \in I$ and $x_0 \in X$ be fixed. Then we have

$$\begin{aligned} F(t_0 + \tau_k, x_0) &= G(t_0 + \tau_k, x_0) + Q(t_0 + \tau_k, x_0) \\ &= G(t_0 + \tau_k, x_0) - G(t_0, x_0) + G(t_0, x_0) + Q(t_0 + \tau_k, x_0); \end{aligned}$$

since $\|t_0 + \tau_k\| \to \infty$ as $k \to \infty$, from this equality we may conclude that $\lim_{k\to\infty} F(t_0 + \tau_k, x_0) = G(t_0, x_0)$. Therefore, (8) holds. □

It is worth noting that the inclusion (8) also holds for asymptotically Bohr $\mathcal{B}$-almost periodic functions, which can be simply shown. For the sequel, we need the following auxiliary lemma (see also [7, Lemma 2.12]):

Lemma 2.4. *Suppose that there exist an $(\mathrm{R}, \mathcal{B})$-multi-almost periodic function $G(\cdot;\cdot)$ and a function $Q \in C_{0,I,\mathcal{B}}(I\times X : Y)$ such that $F(\mathbf{t};x) = G(\mathbf{t};x) + Q(\mathbf{t};x)$ for all $\mathbf{t} \in I$ and $x \in X$. Then (8) holds provided that for each sequence $\mathbf{b} \in \mathrm{R}$ we have $I \pm \mathbf{b}(l) \in I$, $l \in \mathbb{N}$ and there exists a sequence in R whose any subsequence is unbounded.*

Proof. Let $(b_k) \in \mathrm{R}$ be a sequence such that any subsequence is unbounded. Since G is $(\mathrm{R}, \mathcal{B})$-multi almost periodic, given any $B \in \mathcal{B}$, there exist a subsequence $(b_{k_l}) \subseteq (b_k)$ and a function $G^* : I \times X \to Y$ such that the following limits hold:

$$\lim_{l\to\infty} G(t + b_{k_l}, x) = G^*(t, x) \tag{9}$$

$$\lim_{l\to\infty} G^*(t - b_{k_l}, x) = G(t, x), \tag{10}$$

and are uniformly for $t \in I$ and for $x \in B$.

Let us consider $t_0 \in I$ and $x_0 \in X$; then, from the equality:

$$F(t_0 + b_{k_l}, x_0) = G(t_0 + b_{k_l}, x_0) + Q(t_0 + b_{k_l}, x_0),$$

we have

$$\lim_{l\to\infty} F(t_0 + b_{k_l}, x_0) = G^*(t_0, x_0),$$

which means that

$$\overline{\left\{G^*(\mathbf{t};x) : \mathbf{t} \in I,\ x \in X\right\}} \subseteq \overline{\left\{F(\mathbf{t};x) : \mathbf{t} \in I,\ x \in X\right\}}. \tag{11}$$

From (10), we obtain

$$\overline{\left\{G(\mathbf{t};x) : \mathbf{t} \in I,\ x \in X\right\}} \subseteq \overline{\left\{G^*(\mathbf{t};x) : \mathbf{t} \in I,\ x \in X\right\}}. \tag{12}$$

Therefore, using (11) and (12) we conclude (8). □

Uniqueness of decomposition. Using the supremum formula clarified in Proposition 1.4, we can simply deduce that the decomposition in Definition 2.2 is unique:

Proposition 2.5. *(i) Suppose that there exist a function $G_i(\cdot;\cdot)$ which is $(\mathrm{R},\mathcal{B})$-multi-almost periodic and a function $Q_i \in C_{0,I,\mathcal{B}}(I\times X : Y)$ such that $F(\mathbf{t};x) = G_i(\mathbf{t};x) + Q_i(\mathbf{t};x)$ for all $\mathbf{t}\in I$ and $x \in X$ ($i = 1, 2$). Suppose that, for every sequence which belongs to R, any its subsequence belongs to R. If there exists a sequence $b(\cdot)$ in R whose any subsequence is unbounded and for which we have $\mathbf{T} - \mathbf{b}(l) \in I$ whenever $\mathbf{T} \in I$ and $l \in \mathbb{N}$, then $G_1 \equiv G_2$ and $Q_1 \equiv Q_2$.*

(ii) Suppose that $\mathcal{B}$ is any collection of compact subsets of X, there exist a Bohr $\mathcal{B}$-almost periodic function $G_i : \mathbb{R}^n \times X \to Y$ and a function $Q_i \in C_{0,I,\mathcal{B}}(I \times X : Y)$ such that $F(\mathbf{t};x) = G_i(\mathbf{t};x) + Q_i(\mathbf{t};x)$ for all $\mathbf{t} \in I$ and $x \in X$ ($i = 1, 2$). Then $G_1 \equiv G_2$ and $Q_1 \equiv Q_2$.

Closure properties under uniform convergence. Keeping in mind the inclusion (8) and the argumentation used in the proof of [12, Theorem 4.29], we can simply deduce the following result:

Proposition 2.6. *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is I-asymptotically Bohr $\mathcal{B}$-almost periodic (I-asymptotically $\mathcal{B}$-uniformly recurrent). If for each $B \in \mathcal{B}$ there exists $\epsilon_B > 0$ such that the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on the set $B^\circ \cup \bigcup_{x\in\partial B} B(x, \epsilon_B)$, then the function $F(\cdot;\cdot)$ is I-asymptotically Bohr $\mathcal{B}$-almost periodic (I-asymptotically $\mathcal{B}$-uniformly recurrent).*

Now we will state and prove the following result:

Proposition 2.7. *Suppose that, for every sequence $\mathbf{b}(\cdot)$ which belongs to R, any its subsequence belongs to R and $\mathbf{T} - \mathbf{b}(l) \in I$ whenever $\mathbf{T} \in I$ and $l \in \mathbb{N}$. Suppose, further, that there exists a sequence in R whose any subsequence is unbounded. If for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is I-asymptotically $(\mathrm{R},\mathcal{B})$-multi-almost periodic and for each $B \in \mathcal{B}$ there exists $\epsilon_B > 0$ such that the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on the set $B^\circ \cup \bigcup_{x\in\partial B} B(x, \epsilon_B)$, then the function $F(\cdot;\cdot)$ is I-asymptotically $(\mathrm{R},\mathcal{B})$-multi-almost periodic.*

Proof. Due to Proposition 2.5, we know that there exist a uniquely determined function $G(\cdot;\cdot)$ which is $(\mathrm{R},\mathcal{B})$-multi-almost periodic and a uniquely determined function $Q \in C_{0,I,\mathcal{B}}(I \times X : Y)$ such that $F(\mathbf{t};x) = G(\mathbf{t};x) + Q(\mathbf{t};x)$ for all $\mathbf{t} \in I$ and $x \in X$. Furthermore, we have

$$F_j(\mathbf{t};x) - F_m(\mathbf{t};x) = \Big[G_j(\mathbf{t};x) - G_m(\mathbf{t};x)\Big] + \Big[Q_j(\mathbf{t};x) - Q_m(\mathbf{t};x)\Big],$$

for all $\mathbf{t} \in I$, $x \in X$ and $j,\ m \in \mathbb{N}$. Due to [8, Proposition 2.28], we have that the function $G_j(\cdot;\cdot) - G_m(\cdot;\cdot)$ is $(\mathrm{R},\mathcal{B})$-multi-almost periodic ($j,\ m \in \mathbb{N}$). Keeping in mind this fact as well as Lemma 2.4 and the argumentation used in the proof of [12, Theorem 4.29], we get that

$$3 \sup_{\mathbf{t}\in I, x\in X} \Big\|F_j(\mathbf{t};x) - F_m(\mathbf{t};x)\Big\|_Y \geq \sup_{\mathbf{t}\in I, x\in X} \Big\|G_j(\mathbf{t};x) - G_m(\mathbf{t};x)\Big\|_Y + \sup_{\mathbf{t}\in I, x\in X} \Big\|Q_j(\mathbf{t};x) - Q_m(\mathbf{t};x)\Big\|_Y,$$

for any $j,\ m \in \mathbb{N}$. This implies that the sequences $(G_j(\cdot;\cdot))$ and $(Q_j(\cdot;\cdot))$ converge uniformly to the functions $G(\cdot;\cdot)$ and $Q(\cdot;\cdot)$, respectively. Due to [8, Proposition 2.9], we get that the function $G(\cdot;\cdot)$ is $(\mathrm{R},\mathcal{B})$-multi-almost periodic. The final conclusion follows from the obvious equality $F = G + Q$ and the fact that $C_{0,I,\mathcal{B}}(I \times X : Y)$ is a Banach space. □

Differentiation of $\mathbb{D}$-asymptotically almost periodic type functions. Concerning the partial derivatives of $\mathbb{D}$-asymptotically almost periodic type functions, we will only state the following result (by $(e_1, e_2, \cdots, e_n)$ we denote the standard basis of $\mathbb{R}^n$); the proof is similar to the proof of [8, Proposition 2.41] and therefore omitted:

Proposition 2.8. *Suppose that, for every sequence $\mathbf{b}(\cdot)$ which belongs to R, any its subsequence belongs to R and $\mathbf{T} - \mathbf{b}(l) \in I$ whenever $\mathbf{T} \in I$ and $l \in \mathbb{N}$. Suppose, further, that there exists a sequence in R whose any subsequence is unbounded as well as that the function $F(\cdot;\cdot)$ is I-asymptotically $(\mathrm{R},\mathcal{B})$-multi-almost periodic, the partial derivative $\frac{\partial F(\mathbf{t};x)}{\partial t_i}$ exists for all $\mathbf{t} \in I$, $x \in X$ and it is uniformly continuous on $\mathcal{B}$. Then the function $\frac{\partial F(\cdot;\cdot)}{\partial t_i}$ is I-asymptotically $(\mathrm{R},\mathcal{B})$-multi-almost periodic.*

For more details about the subject, see [8, Subsection 2.3].

2.1. Relations with $\mathbb{D}$-asymptotically almost periodic functions of type 1

In this subsection, we will first introduce the following general definition following the approach obeyed in [15]; for any set $\Lambda \subseteq \mathbb{R}^n$, we define $\Lambda_M := \{\lambda \in \Lambda ; |\lambda| \geq M\}$:

Definition 2.9. *Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I' \subseteq I$. Then we say that:*

- *(i) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, I')$-almost periodic of type 1 if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exist $l > 0$ and $M > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that*

$$\left\|F(\mathbf{t} + \tau; x) - F(\mathbf{t}; x)\right\|_Y \leq \epsilon, \text{ provided } \mathbf{t},\ \mathbf{t} + \tau \in \mathbb{D}_M,\ x \in B. \tag{13}$$

- *(ii) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically $(\mathcal{B}, I')$-uniformly recurrent of type 1 if and only if for every $B \in \mathcal{B}$ there exist a sequence (τ_k) in I' and a sequence (M_k) in $(0, \infty)$ such that $\lim_{k\to+\infty} |\tau_k| = \lim_{k\to+\infty} M_k = +\infty$ and*

$$\lim_{k\to+\infty} \sup_{\mathbf{t},\mathbf{t}+\tau_k \in \mathbb{D}_{M_k}; x\in B} \left\|F(\mathbf{t} + \tau_k; x) - F(\mathbf{t}; x)\right\|_Y = 0.$$

If $I' = I$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $\mathcal{B}$-almost periodic of type 1 ($\mathbb{D}$-asymptotically $\mathcal{B}$-uniformly recurrent of type 1); furthermore, if $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr I'-almost periodic of type 1 ($\mathbb{D}$-asymptotically I'-uniformly recurrent of type 1). If $I' = I$ and $X \in \mathcal{B}$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr almost periodic of type 1 ($\mathbb{D}$-asymptotically uniformly recurrent of type 1). As before, we remove the prefix "$\mathbb{D}$-" in the case that $\mathbb{D} = I$ and remove the prefix "$(\mathcal{B},)$" in the case that $X \in \mathcal{B}$.

The proof of following proposition is trivial and therefore omitted:

Proposition 2.10. *Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function and $I + I' \subseteq I$. If $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, I')$-almost periodic, resp. $\mathbb{D}$-asymptotically $(\mathcal{B}, I')$-uniformly recurrent, then $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, I')$-almost periodic of type 1, resp. $\mathbb{D}$-asymptotically $(\mathcal{B}, I')$-uniformly recurrent of type 1.*

Suppose now that the general assumptions from the preamble of Definition 2.9 hold true. Keeping in mind Proposition 2.10, it is natural to ask the following:

- (i) In which cases the $\mathbb{D}$-asymptotical Bohr $(\mathcal{B}, I')$-almost periodicity of type 1, resp. $\mathbb{D}$-asymptotical $(\mathcal{B}, I')$-uniform recurrence of type 1, implies the $\mathbb{D}$-asymptotical Bohr $(\mathcal{B}, I')$-almost periodicity, resp. $\mathbb{D}$-asymptotical $(\mathcal{B}, I')$-uniform recurrence of function $F(\cdot;\cdot)$?
- (ii) In which cases the asymptotical Bohr $\mathcal{B}$-almost periodicity (of type 1) implies the $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity of $F(\cdot;\cdot)$, where R denotes the collection of all sequences in I?
- (iii) In which cases the asymptotical Bohr $\mathcal{B}$-almost periodicity (of type 1) is a consequence of the $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity of $F(\cdot;\cdot)$, where R denotes the collection of all sequences in I?

Concerning the item (ii), it is well known that the answer is negative provided that $X = \{0\}$, $\mathcal{B} = X$ and $I = \mathbb{R}$ because, in this case, the asymptotical Bohr $\mathcal{B}$-almost periodicity of $F : \mathbb{R} \to Y$ is equivalent with the asymptotical Bohr $\mathcal{B}$-almost periodicity of type 1 of $F(\cdot)$, i.e., the usual asymptotical almost periodicity of $F(\cdot)$, while the $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity of $F(\cdot)$ is equivalent in this case with the usual almost periodicity of $F(\cdot)$; see [30, Definition 2.2, Definition 2.3; Theorem 2.6] for the notion used. With the regards to the items (i) and (ii), we have the following statement which can be applied in the particular case $I = [0, \infty)^n$:

Theorem 2.11. *Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $I + I \subseteq I$, I is closed and $F : I \times X \to Y$ is a uniformly continuous, I-asymptotically Bohr $\mathcal{B}$-almost periodic function of type 1, where $\mathcal{B}$ is any family of compact subsets of X. If*

$$(\forall l > 0)\,(\forall M > 0)\,(\exists \mathbf{t_0} \in I)\,(\exists k > 0)\,(\forall \mathbf{t} \in I_{M+l})(\exists \mathbf{t}_0' \in I)$$
$$(\forall \mathbf{t}_0'' \in B(\mathbf{t}_0', l) \cap I)\ \mathbf{t} - \mathbf{t}_0'' \in B(\mathbf{t_0}, kl) \cap I_M, \tag{14}$$

there exists $L > 0$ *such that* $I_{kL} \setminus I_{(k+1)L} \neq \emptyset$ *for all* $k \in \mathbb{N}$ *and* $I_M + I \subseteq I_M$ *for all* $M > 0$, *then the function* $F(\cdot;\cdot)$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, where* R *denotes the collection of all sequences in* I. *Furthermore, if* $X = \{0\}$ *and* $\mathcal{B} = \{X\}$, *then* $F(\cdot)$ *is* I*-asymptotically Bohr almost periodic function.*

Proof. Let $F : \mathbb{R}^n \times X \to Y$ be a Bohr $\mathcal{B}$-almost periodic function, where $\mathcal{B}$ is any family of compact subsets of X. Let $B \in \mathcal{B}$ be fixed. We will consider the Banach space $l_\infty(B : Y)$ consisting of all bounded functions $f : B \to Y$, equipped with the sup-norm. Define the function $F_B : \mathbb{R}^n \to l_\infty(B : Y)$ by

$$\Big[F_B(\mathbf{t})\Big](x) := F(\mathbf{t}; x), \quad \mathbf{t} \in \mathbb{R}^n, \ x \in B. \tag{15}$$

The function $\mathbf{F}_B(\cdot)$ was introduced in our work [8]. Since $F(\cdot;\cdot)$ is uniformly continuous, we have that the function $\mathbf{F}_B(\cdot)$, given by (15), is likewise uniformly continuous. Arguing as in the proof of [8, Proposition 2.19], the assumption (14) enables one to deduce that the set $\{F(\mathbf{t}; x) : \mathbf{t} \in I, \ x \in B\}$ is relatively compact in Y as well as that the set $\{F_B(\mathbf{t}) : \mathbf{t} \in I\}$ is relatively compact in the Banach space $BUC(B : Y)$, consisting of all bounded, uniformly continuous functions from B into Y, equipped with the sup-norm. We know that there exist $l > 0$ and $M > 0$ such that for each $\mathbf{t}_0 \in I$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I$ such that (13) holds with $\mathbb{D} = I$. Using these facts, we can slightly modify the first part of the proof of [25, Theorem 3.3] (with the segment $[N, 3N]$ replaced therein with the set $I_N \setminus I_{3N}$, where $N = \max(L, l, M)$, and the number $\tau_k \in [kN, (k+1)N]$ replaced therein by the number $\tau_k \in I_{kL} \setminus I_{(k+1)L}$; we need condition $I_M + I \subseteq I_M$, $M > 0$ to see that the estimate given on [25, l. 2, p. 23] holds in our framework) in order to obtain that the set of translations $\{F_B(\cdot + \tau) : \tau \in I\}$ is relatively compact in $BUC(B : Y)$, which simply implies that $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, where R denotes the collection of all sequences in I. Suppose now that $X = \{0\}$ and $\mathcal{B} = \{X\}$. Then for each integer $k \in \mathbb{N}$ there exist $l_k > 0$ and $M_k > 0$ such that for each $\mathbf{t}_0 \in I$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I$ such that (13) holds with $\epsilon = 1/k$ and $\mathbb{D} = I$. Let τ_k be any fixed element of I such that $|\tau_k| > M_k + k^2$ and (13) holds with $\epsilon = 1/k$ and $\mathbb{D} = I$ ($k \in \mathbb{N}$). Then the first part of proof yields the existence of a subsequence (τ_{k_l}) of (τ_k) and a function $F^* : I \to Y$ such that $\lim_{l \to +\infty} F(\mathbf{t} + \tau_{k_l}) = F^*(\mathbf{t})$, uniformly for $t \in I$. The mapping $F^*(\cdot)$ is clearly continuous and now we will prove that $F^*(\cdot)$ is Bohr almost periodic. Let $\epsilon > 0$ be fixed, and let $l > 0$ and $M > 0$ be such that for each $\mathbf{t}_0 \in I$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I$ such that (13) holds with $\mathbb{D} = I$ and the number ϵ replaced therein by $\epsilon/3$. Let $\mathbf{t} \in I$ be fixed, and let $l_0 \in \mathbb{N}$ be such that $|\mathbf{t} + \tau_{k_{l_0}}| \geq M$ and $|\mathbf{t} + \tau + \tau_{k_{l_0}}| \geq M$. Then we have

$$\begin{aligned}
&\Big\| F^*(\mathbf{t} + \tau) - F^*(\mathbf{t}) \Big\| \\
&\leq \Big\| F^*(\mathbf{t} + \tau) - F^*\Big(\mathbf{t} + \tau + \tau_{k_{l_0}}\Big) \Big\| + \Big\| F^*\Big(\mathbf{t} + \tau + \tau_{k_{l_0}}\Big) - F^*\Big(\mathbf{t} + \tau_{k_{l_0}}\Big) \Big\| + \Big\| F^*\Big(\mathbf{t} + \tau_{k_{l_0}}\Big) - F^*(\mathbf{t}) \Big\| \leq 3 \cdot (\epsilon/3) = \epsilon,
\end{aligned}$$

as required. The fact that the function $\mathbf{t} \mapsto F(\mathbf{t}) - F^*(\mathbf{t})$, $\mathbf{t} \in I$ belongs to the space $C_{0,I}(I : Y)$ follows trivially by definition of $F^*(\cdot)$. The proof of theorem is thereby complete. □

Remark 2.12. *Suppose that the requirements of Theorem 2.11 hold with* $X = \{0\}$ *and* $\mathcal{B} = \{X\}$. *Suppose further that, for every* $\mathbf{t}' \in \mathbb{R}^n$, *there exist* $\delta > 0$ *and* $l_0 \in \mathbb{N}$ *such that the sequence* (τ_k) *from the above proof satisfies that* $\mathbf{t}'' + \tau_{k_l} \in I$ *for all* $l \in \mathbb{N}$ *with* $l \geq l_0$ *and* $\mathbf{t}'' \in B(\mathbf{t}', \delta)$. *Then the limit* $\lim_{l \to +\infty} F(\mathbf{t}' + \tau_{k_l}) := \tilde{F}^*(\mathbf{t}')$ *exists for all* $\mathbf{t}' \in \mathbb{R}^n$, *which can be easily seen from the estimate*

$$\begin{aligned}
&\Big\| F\Big(\mathbf{t}' + \tau_{k_{l_1}}\Big) - F\Big(\mathbf{t}' + \tau_{k_{l_2}}\Big) \Big\|_Y \\
&\leq \Big\| F\Big(\mathbf{t}' + \tau_{k_{l_1}}\Big) - F\Big(\mathbf{t}' + \tau_{k_{l_1}} + \tau\Big) \Big\|_Y + \Big\| F\Big(\mathbf{t}' + \tau_{k_{l_1}} + \tau\Big) - F\Big(\mathbf{t}' + \tau_{k_{l_2}} + \tau\Big) \Big\|_Y \\
&+ \Big\| F\Big(\mathbf{t}' + \tau_{k_{l_2}} + \tau\Big) - F\Big(\mathbf{t}' + \tau_{k_{l_2}}\Big) \Big\|_Y \\
&\leq 3 \cdot (\epsilon/3) = \epsilon,
\end{aligned} \tag{16}$$

which is valid for all numbers τ *such that there exist* $l > 0$ *and* $M > 0$ *such that for each* $\mathbf{t}_0 \in I$ *there exists* $\tau \in B(\mathbf{t}_0, l) \cap I$ *such that* (13) *holds with the number* ϵ *replaced therein with the number* $\epsilon/3$ *and* $\mathbb{D} = I$, *all sufficiently large natural numbers* l_1 *and* l_2 *depending on* τ, *where we have also applied the Cauchy criterion of convergence for the limit*

$\lim_{l\to+\infty} F(\mathbf{t}+\tau_{k_l}) = F^*(\mathbf{t})$, *uniform in* $t \in I$ *and our assumption* $I+I \subseteq I$. *The function* $\tilde{F}^*(\cdot)$ *is clearly continuous and it can be easily shown that it is Bohr* I*-almost periodic. Furthermore, if for every* $\mathbf{t}' \in \mathbb{R}^n$ *and* $M_1,\ M_2 > 0$ *there exists* $l_0 \in \mathbb{N}$ *such that* $\mathbf{t}' + \tau_{k_l} - \tau \in I_{M_2}$ *for all* $l \in \mathbb{N}$ *with* $l \geq l_0$*, then* $\tilde{F}^*(\cdot)$ *is Bohr* $(I \cup (-I))$*-almost periodic. Using a simple translation argument, the above gives an extension of [25, Theorem 3.4] in Banach spaces.*

Concerning the item (iii), we will clarify the following result:

Theorem 2.13. *Suppose that* $0 \in I \subseteq \mathbb{R}^n$*,* I *is closed,* $I + I \subseteq I$ *and* $\emptyset \neq I' \subseteq I$*. Suppose, further, that the set* $\mathbb{D} \subseteq I$ *is unbounded and condition* (MD) *holds, where:*

(MD) For each $M_0 > 0$ *there exists a finite real number* $M_1 > M_0$ *such that* $\mathbb{D}_{M_1} - \mathbf{t} \in I$ *and* $I'_{M_1} - \mathbf{t} \in I'$ *for all* $\mathbf{t} \in I \setminus I_{M_0}$.

Let R *denote the collection of all sequences in* I*, and let* $\mathcal{B}$ *denote any family of compact subsets of* X*. Then any* $(\mathrm{R},\mathcal{B})$*-multi-almost periodic function* $F : I \times X \to Y$ *is* $\mathbb{D}$*-asymptotically Bohr* $(\mathcal{B}, I')$*-almost periodic of type* 1.

Proof. Let $B \in \mathcal{B}$ and $\epsilon > 0$ be fixed. Since I is closed, we have that the restriction of function $F(\cdot;\cdot)$ to the set $I \times B$ is uniformly continuous, which easily implies that the function $F_B : I \to BUC(B : Y)$, given by (15), is well defined and uniformly continuous. Now we will prove that the function $F_B(\cdot)$ has a relatively compact range. Denote $K_k = [-k, k]^n$ for all integers $k \in \mathbb{N}$. Since the set $F_B(K_k \cap I)$ is relatively compact in $BUC(B : Y)$ for all integers $k \in \mathbb{N}$, it suffices to show that there exists $k \in \mathbb{N}$ such that, for every $\mathbf{t} \in I$, there exists a point $\mathbf{s} \in I \cap K_k$ such that $\|F(\mathbf{t};x) - F(\mathbf{s};x)\|_Y \leq \epsilon$ for all $x \in B$. Suppose the contrary. Then for each $k \in \mathbb{N}$ there exists $\mathbf{t}_k \in I$ such that, for every $\mathbf{s} \in I \cap K_k$, there exists $x \in B$ with $\|F(\mathbf{t}_k;x) - F(\mathbf{s};x)\|_Y > \epsilon$. Define $\mathbf{b}_k := \mathbf{t}_k$ for all $k \in \mathbb{N}$. Due to our assumption, there exists a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ such that (2) holds true. Since $0 \in I$, this implies the existence of a number $l_0(\epsilon) \in \mathbb{N}$ such that

$$\left\|F\Big(\mathbf{t}_{k_l};x\Big) - F\Big(\mathbf{t}_{k_m};x\Big)\right\|_Y \leq \epsilon, \quad l,\ m \in \mathbb{N},\ l,\ m \geq l_0(\epsilon),$$

uniformly for $x \in B$. In particular, we have

$$\left\|F\Big(\mathbf{t}_{k_l};x\Big) - F\Big(\mathbf{t}_{k_{l_0(\epsilon)}};x\Big)\right\|_Y \leq \epsilon, \quad l \in \mathbb{N},\ l \geq l_0(\epsilon),\ x \in B.$$

Therefore, $\mathbf{t}_{k_{l_0(\epsilon)}} \notin K_l$ for all $l \in \mathbb{N}$ with $l \geq l_0(\epsilon)$, which is a contradiction. Now it is quite simply to prove with the help of Cauchy criterion of convergence and the $(\mathrm{R},\mathcal{B})$-multi-almost periodicity of $F(\cdot;\cdot)$ that the set of translations $\{F_B(\cdot+\tau) : \tau \in I\}$ is relatively compact in $BUC(B : Y)$. Applying [25, Theorem 2.2; see 1. and 3.(ii)] (see also the second part of the proof of [25, Theorem 3.3]), we get that there exist a finite cover $(T_i)_{i=1}^k$ of the set I_1 and points $\mathbf{t}_i \in T_i$ $(1 \leq i \leq k)$ such that $\|F_B(\mathbf{t}+\omega) - F_B(\mathbf{t}_i+\omega)\|_{BUC(B:Y)} \leq \epsilon$ for all $\omega \in I$ and $t \in T_i$ $(1 \leq i \leq k)$. Let $M_0 := l := 1 + \max\{|\mathbf{t}_i| : 1 \leq i \leq k\}$, and let $M_1 > 0$ satisfy condition (MD) with this M_0. Set $M := 2M_1 + l$. Suppose that $\mathbf{t},\ \mathbf{t}+\tau \in \mathbb{D}_M$ and $\mathbf{t}_0 \in I'_M$. Then there exists $i \in \mathbb{N}_k$ such that $\mathbf{t}_0 \in T_i$ and $\tau = \mathbf{t}_0 - \mathbf{t}_i \in T_i - \mathbf{t}_i \in B(\mathbf{t}_0, l) \cap I'$ due to the first condition in (MD) and the obvious inequality $|\mathbf{t}_i| \leq l$. Furthermore, the second condition in (MD) implies $\mathbf{t} - t_i \in I$ and therefore

$$\begin{aligned}\left\|F_B(\mathbf{t}+\tau) - F_B(\mathbf{t})\right\|_{BUC(B:Y)} &= \left\|F_B(\mathbf{t}+\mathbf{t}_0-\mathbf{t}_i) - F_B(\mathbf{t})\right\|_{BUC(B:Y)}\\ &= \left\|F_B(\mathbf{t}_0 + [\mathbf{t}-\mathbf{t}_i]) - F_B(\mathbf{t}_i + [\mathbf{t}-\mathbf{t}_i])\right\|_{BUC(B:Y)} \leq \epsilon,\end{aligned}$$

which simply completes the proof. □

Remark 2.14. *(i) In [25, Theorem 3.3], W. M. Ruess and W. H. Summers have considered the situation in which* $I = [a,\infty)$*,* $X = \{0\}$ *and the set of all translations* $\{f(\cdot+\tau) : \tau \geq 0\}$ *is relatively compact in* $BUC(I : Y)$*. But, the obtained result is a simple consequence of the corresponding result with* $I = [0,\infty)$*, which follows from a simple translation argument. In Theorem 2.13, which therefore provides a proper extension of the corresponding result from [25, Theorem 3.3] with* $\mathbb{D} = I' = I = [0,\infty)$*, we have decided to consider the collection* R *of all sequences in* I*, only. The interested reader may try to further analyze the assumption in which the function* $F(\cdot;\cdot)$ *is* $(\mathrm{R},\mathcal{B})$*-multi-almost periodic with* R *being the collection of all sequences in a certain subset* I'' *of* $\mathbb{R}^n$ *which contains* 0 *and satisfies* $I + I'' \subseteq I$.

(ii) In the multi-dimensional framework, we cannot expect the situation in which $\mathbb{D} = I' = I$. The main problem lies in the fact that condition (MD) does not hold in this case; but, if $I = [0, \infty)^n$, for example, then the conclusion of Theorem 2.13 holds for any proper subsector I' of I, with the meaning clear, and $\mathbb{D} = I'$.

2.2. Composition theorems for $\mathbb{D}$-asymptotically almost periodic type functions

Suppose that $F : I \times X \to Y$ and $G : I \times Y \to Z$ are given functions. The main aim of this subsection is to analyze $\mathbb{D}$-asymptotically almost periodic properties of the multi-dimensional Nemytskii operator $W : I \times X \to Z$ given by

$$W(\mathbf{t}; x) := G\Big(\mathbf{t}; F(\mathbf{t}; x)\Big), \quad \mathbf{t} \in I, \ x \in X.$$

Our first result is in a close connection with [8, Theorem 2.37] and [12, Theorem 3.49]:

Theorem 2.15. *Suppose that the set $\mathbb{D} \subseteq \mathbb{R}^n$ is unbounded, $F_0 : I \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic, $Q_0 \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ and $F(\mathbf{t}; x) = F_0(\mathbf{t}; x) + Q_0(\mathbf{t}; x)$ for all $\mathbf{t} \in I$ and $x \in X$. Suppose further that $G_1 : I \times Y \to Z$ is $(\mathrm{R}', \mathcal{B}')$-multi-almost periodic, where R' is a collection of all sequences $b : \mathbb{N} \to \mathbb{R}^n$ from R and all their subsequences as well as $\mathcal{B}'$ is defined by*

$$\mathcal{B}' := \Big\{ \bigcup_{\mathbf{t} \in I} F_0(t; B) : B \in \mathcal{B} \Big\}. \tag{17}$$

Suppose also that $Q_1 \in C_{0,\mathbb{D},\mathcal{B}_1}(I \times Y : Z)$, where

$$\mathcal{B}_1 := \Big\{ \bigcup_{\mathbf{t} \in I} F(t; B) : B \in \mathcal{B} \Big\}, \tag{18}$$

and $G(\mathbf{t}; x) = G_1(\mathbf{t}; x) + Q_1(\mathbf{t}; x)$ for all $\mathbf{t} \in I$ and $x \in Y$. If there exists a finite constant $L > 0$ such that the estimate

$$\Big\| G_1(\mathbf{t}; x) - G_1(\mathbf{t}; y) \Big\|_Z \leq L \|x - y\|_Y, \quad \mathbf{t} \in I, \ x, \ y \in Y, \tag{19}$$

holds, then the function $W(\cdot; \cdot)$ is $\mathbb{D}$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost periodic.

Proof. By [8, Theorem 2.37], the function $(\mathbf{t}; x) \mapsto G_1(\mathbf{t}; F_0(\mathbf{t}; x))$, $\mathbf{t} \in I$, $x \in X$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic. Furthermore, we have the following decomposition

$$W(\mathbf{t}; x) = G_1(\mathbf{t}; F_0(\mathbf{t}; x)) + \Big[G_1(\mathbf{t}; F(\mathbf{t}; x)) - G_1(\mathbf{t}; F_0(\mathbf{t}; x)) \Big] + Q_1(\mathbf{t}; F(\mathbf{t}; x)),$$

for any $\mathbf{t} \in I$ and $x \in X$. Since

$$\Big\| G_1(\mathbf{t}; F(\mathbf{t}; x)) - G_1(\mathbf{t}; F_0(\mathbf{t}; x)) \Big\|_Z \leq L \Big\| Q_0(\mathbf{t}; x) \Big\|_Y, \quad \mathbf{t} \in I, \ x \in X,$$

we have that the function $(\mathbf{t}; x) \mapsto G_1(\mathbf{t}; F(\mathbf{t}; x)) - G_1(\mathbf{t}; F_0(\mathbf{t}; x))$, $\mathbf{t} \in I$, $x \in X$ belongs to the space $C_{0,\mathbb{D},\mathcal{B}}(I \times X : Z)$. The same holds for the function $(\mathbf{t}; x) \mapsto Q_1(\mathbf{t}; F(\mathbf{t}; x))$, $\mathbf{t} \in I$, $x \in X$ due to our choice of the collection $\mathcal{B}_1$ in (18). □

Corollary 2.16. *Suppose that $\mathcal{B}$ is any collection of compact subsets of X, the set $\mathbb{D} \subseteq \mathbb{R}^n$ is unbounded, $F_0 : \mathbb{R}^n \times X \to Y$ is Bohr $\mathcal{B}$-almost periodic, $Q_0 \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ and $F(\mathbf{t}; x) = F_0(\mathbf{t}; x) + Q_0(\mathbf{t}; x)$ for all $\mathbf{t} \in I$ and $x \in X$. Suppose further that $G_1 : I \times Y \to Z$ is Bohr $\mathcal{B}'$-almost periodic, where $\mathcal{B}'$ is defined by* (17), *$Q_1 \in C_{0,\mathbb{D},\mathcal{B}_1}(I \times Y : Z)$, where $\mathcal{B}_1$ is given by* (18) *and $G(\mathbf{t}; x) = G_1(\mathbf{t}; x) + Q_1(\mathbf{t}; x)$ for all $\mathbf{t} \in I$ and $x \in Y$. If there exists a finite constant $L > 0$ such that the estimate* (19) *holds, then the function $W(\cdot; \cdot)$ is $\mathbb{D}$-asymptotically Bohr $\mathcal{B}$-almost periodic.*

We can also prove the following result which corresponds to [8, Theorem 2.39] and [12, Theorem 3.50]:

Theorem 2.17. *Suppose that the set* $\mathbb{D} \subseteq \mathbb{R}^n$ *is unbounded,* $F_0 : I \times X \to Y$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic,* $Q_0 \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ *and* $F(\mathbf{t}; x) = F_0(\mathbf{t}; x) + Q_0(\mathbf{t}; x)$ *for all* $\mathbf{t} \in I$ *and* $x \in X$*. Suppose further that* $G_1 : I \times Y \to Z$ *is* $(\mathrm{R}', \mathcal{B}')$*-multi-almost periodic, where* R' *is a collection of all sequences* $b : \mathbb{N} \to \mathbb{R}^n$ *from* R *and all their subsequences as well as* $\mathcal{B}'$ *is defined by* (17), $Q_1 \in C_{0,\mathbb{D},\mathcal{B}_1}(I \times Y : Z)$*, where* $\mathcal{B}_1$ *is given through* (18)*, and* $G(\mathbf{t}; x) = G_1(\mathbf{t}; x) + Q_1(\mathbf{t}; x)$ *for all* $\mathbf{t} \in I$ *and* $x \in Y$*. Set*

$$\mathcal{B}_2 := \left\{ \bigcup_{\mathbf{t} \in I} F_0(t; B) : B \in \mathcal{B} \right\} \cup \bigcup_{(\mathbf{b}_k) \in \mathrm{R}; B \in \mathcal{B}} \left\{ F_0^*(t; B) : \mathbf{t} \in I \right\}.$$

If

$$(\forall B \in \mathcal{B})\ (\forall \epsilon > 0)\ (\exists \delta > 0)\Big(x,\ y \in \mathcal{B}_1 \cup \mathcal{B}_2 \text{ and } \left\|x - y\right\|_Y < \delta \Rightarrow \left\|G_1(\mathbf{t}; x) - G_1(\mathbf{t}; y)\right\|_Z < \epsilon,\ \mathbf{t} \in I\Big),$$

then the function $W(\cdot; \cdot)$ *is* $\mathbb{D}$*-asymptotically* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic.*

It is clear that Theorem 2.17 can be reformulated for Bohr $\mathcal{B}$-almost periodic functions with small terminological difficulties concerning the use of limit functions. Similar results can be established for the class of $\mathcal{B}$-uniformly recurrent functions ([19]).

2.3. *Invariance of* $\mathbb{D}$*-asymptotical almost type periodicity under actions of convolution products*

This subsection investigates the invariance of $(\mathrm{R}, \mathcal{B})$-multi-almost periodicity under the actions of convolution products. We will use the following notation: if any component of tuple $\mathbf{t} = (t_1, t_2, \cdots, t_n)$ is strictly positive, then we simply write $\mathbf{t} > \mathbf{0}$.

It seems that we must slightly strengthen the notion introduced in Definition 2.2 in order to investigate the invariance of $\mathbb{D}$-asymptotical almost periodicity under the actions of "finite" convolution products:

Definition 2.18. *Suppose that the set* $\mathbb{D} \subseteq \mathbb{R}^n$ *is unbounded, and* $F : I \times X \to Y$ *is a continuous function. Then we say that* $F(\cdot; \cdot)$ *is strongly* $\mathbb{D}$*-asymptotically* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp. strongly* $\mathbb{D}$*-asymptotically* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost periodic, if and only if there exist an* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic function* $G : \mathbb{R}^n \times X \to Y$*, resp. an* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost periodic function* $G : \mathbb{R}^n \times X \to Y$*, and a function* $Q \in C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ *such that* $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ *for all* $\mathbf{t} \in I$ *and* $x \in X$*.*

Let $I = \mathbb{R}^n$*. Then it is said that* $F(\cdot; \cdot)$ *is strongly asymptotically* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp. strongly asymptotically* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost periodic, if and only if* $F(\cdot; \cdot)$ *is strongly* $\mathbb{R}^n$*-asymptotically* $(\mathrm{R}, \mathcal{B})$*-multi-almost periodic, resp. strongly* $\mathbb{R}^n$*-asymptotically* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost periodic. Finally, if* $X = \{0\}$*, then we also say that the function* $F(\cdot)$ *is strongly asymptotically* R*-multi-almost periodic, and so on and so forth.*

Set, for brevity, $I_{\mathbf{t}} := (-\infty, t_1] \times (-\infty, t_2] \times \cdots \times (-\infty, t_n]$ and $\mathbb{D}_{\mathbf{t}} := I_{\mathbf{t}} \cap \mathbb{D}$ for any $\mathbf{t} = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n$. Now we are ready to formulate the following result:

Proposition 2.19. *Suppose that* $(R(\mathbf{t}))_{\mathbf{t} > \mathbf{0}} \subseteq L(X, Y)$ *is a strongly continuous operator family and* $\int_{(0,\infty)^n} \|R(\mathbf{t})\|\, d\mathbf{t} < \infty$*. If* $f : I \to X$ *is strongly* $\mathbb{D}$*-asymptotically almost periodic (bounded strongly* $\mathbb{D}$*-asymptotically* R*-multi-almost periodic),*

$$\lim_{|\mathbf{t}| \to \infty, \mathbf{t} \in \mathbb{D}} \int_{I_{\mathbf{t}} \cap \mathbb{D}^c} \|R(\mathbf{t} - \mathbf{s})\|\, d\mathbf{s} = 0 \tag{20}$$

and for each $r > 0$ *we have*

$$\lim_{|\mathbf{t}| \to \infty, \mathbf{t} \in \mathbb{D}} \int_{\mathbb{D}_{\mathbf{t}} \cap B(0,r)} \|R(\mathbf{t} - \mathbf{s})\|\, d\mathbf{s} = 0, \tag{21}$$

then the function

$$F(\mathbf{t}) := \int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t} - \mathbf{s}) f(\mathbf{s})\, ds, \quad \mathbf{t} \in I$$

is strongly $\mathbb{D}$*-asymptotically almost periodic (bounded strongly* $\mathbb{D}$*-asymptotically* R*-multi-almost periodic).*

Proof. We will consider only strong $\mathbb{D}$-asymptotical almost periodicity. By definition, we have the existence of an almost periodic function $g : \mathbb{R}^n \to X$ and a function $q \in C_{0,\mathbb{D}}(I : X)$ such that $f(\mathbf{t}) = g(\mathbf{t}) + q(\mathbf{t})$ for all $\mathbf{t} \in I$. Clearly, we have the decomposition

$$F(\mathbf{t}) = \int_{I_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})g(\mathbf{s})\,d\mathbf{s} + \Bigg[\int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} - \int_{I_{\mathbf{t}}\cap\mathbb{D}^c} R(\mathbf{t}-\mathbf{s})g(\mathbf{s})\,d\mathbf{s}\Bigg], \quad \mathbf{t} \in I.$$

Keeping in mind [8, Theorem 2.44], it suffices to show that the function

$$\mathbf{t} \mapsto \int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} - \int_{I_{\mathbf{t}}\cap\mathbb{D}^c} R(\mathbf{t}-\mathbf{s})g(\mathbf{s})\,d\mathbf{s}, \quad \mathbf{t} \in I$$

belongs to the class $C_{0,\mathbb{D}}(I : X)$. For the second addend, this immediately follows from the boundedness of function $g(\cdot)$ and condition (20). In order to show this for the first addend, fix a number $\epsilon > 0$. Then there exists $r > 0$ such that, for every $\mathbf{t} \in \mathbb{D}$ with $|\mathbf{t}| > r$, we have $\|q(\mathbf{t})\| < \epsilon$. Furthermore, we have

$$\int_{\mathbb{D}_{\mathbf{t}}} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} = \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s} + \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)^c} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s}, \quad \mathbf{t} \in I.$$

Clearly, $M := \sup_{\mathbf{t}\in\mathbb{D}} \|q(\mathbf{t})\| < \infty$ and

$$\left\|\int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s}\right\|_Y \leq M \int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)} \|R(\mathbf{t}-\mathbf{s})\|\,d\mathbf{s}, \quad \mathbf{t} \in I,$$

so that the first addend in the above sum belongs to the class $C_{0,\mathbb{D}}(I : X)$ due to condition (21). This is also clear for the second addend since

$$\left\|\int_{\mathbb{D}_{\mathbf{t}}\cap B(0,r)} R(\mathbf{t}-\mathbf{s})q(\mathbf{s})\,d\mathbf{s}\right\|_Y \leq \epsilon \int_{(0,\infty)^n} \|R(\mathbf{s})\|\,d\mathbf{s}, \quad \mathbf{t} \in I.$$

□

If $\mathbb{D} = [\alpha_1,\infty)\times[\alpha_2,\infty)\times\cdots\times[\alpha_n,\infty)$ for some real numbers $\alpha_1,\ \alpha_2, \cdots,\ \alpha_n$, then $\mathbb{D}_{\mathbf{t}} = [\alpha_1,t_1]\times[\alpha_2,t_2]\times\cdots\times[\alpha_n,t_n]$ and conditions (20)-(21) hold, as easily shown, which implies that the function $F(\mathbf{t}) = \int_{\mathbf{t}}^{\alpha} R(\mathbf{t}-\mathbf{s})f(\mathbf{s})\,ds$, $\mathbf{t} \in I$ is strongly $\mathbb{D}$-asymptotically almost periodic, where we accept the notation

$$\int_{\mathbf{t}}^{\alpha} \cdot = \int_{\alpha_1}^{t_1}\int_{\alpha_2}^{t_2}\cdots\int_{\alpha_n}^{t_n}.$$

Using composition principles established in the previous subsection, the Banach contraction principle and Proposition 2.19, we can analyze the existence and uniqueness of asymptotically almost periodic type solutions for the abstract semilinear integral equations of the form

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_0^{\mathbf{t}} R(\mathbf{t}-\mathbf{s})F(\mathbf{s},u(\mathbf{s}))\,d\mathbf{s}, \quad t \in [0,\infty)^n.$$

Details can be left to the interested readers; see also the fourth application given in [9, Section 6].

3. Some examples and applications

We start this section by examing the existence and uniqueness of $\mathbb{D}$-asymptotically almost periodic type solutions for the heat equation on the first quadrant (see also the second application given in [9, Section 6]):

Example 3.1. *Set*

$$E_1(x,t) := \left(\pi t\right)^{-1/2} \int_0^x e^{-y^2/4t}\, dy, \quad x \in \mathbb{R},\ t > 0.$$

Concerning the homogeneous solutions of the heat equation on domain $I := \{(x,t) : x > 0,\ t > 0\}$, we would like to recall that F. Trèves [28, p. 433] has proposed the following formula:

$$u(x,t) = \frac{1}{2}\int_{-x}^{x} \frac{\partial E_1}{\partial y}(y,t)u_0(x-y)\, dy - \int_0^t \frac{\partial E_1}{\partial t}(x,t-s)g(s)\, ds, \quad x > 0,\ t > 0, \tag{22}$$

for the solution of the following mixed initial value problem:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t),\ \ x > 0,\ t > 0;\\ &u(x,0) = u_0(x),\ x > 0,\ \ u(0,t) = g(t),\ t > 0 \end{aligned} \tag{23}$$

(for simplicity, we will not consider here the evolution analogues of (22) *and the generation of various classes of operator semigroups with the help of this formula). Concerning the existence and uniqueness of multi-dimensional almost periodic type solution of* (23)*, we will present only one result which exploits the formula* (22) *with $g(t) \equiv 0$. Suppose that $0 < T < \infty$ and the function $u_0 : [0,\infty) \to \mathbb{C}$ is bounded Bohr I_0-almost periodic, resp. bounded I_0-uniformly recurrent, for a certain non-empty subset I_0 of $[0,\infty)$. Set $I' := I_0 \times (0,T)$. If $\mathbb{D}$ is any unbounded subset of I which has the property that*

$$\lim_{|(x,t)|\to+\infty,(x,t)\in\mathbb{D}} \min\left(\frac{x^2}{4(t+T)}, t\right) = +\infty,$$

then the solution $u(x,t)$ of (23) *is $\mathbb{D}$-asymptotically I'-almost periodic of type* 1*, resp. $\mathbb{D}$-asymptotically I'-uniformly recurrent of type* 1 *(see Definition 2.9). In order to see that, observe that the formula* (22)*, in our concrete situation, reads as follows*

$$u(x,t) = \frac{1}{2}\int_{-x}^{x} \left(\pi t\right)^{-1/2} e^{-y^2/4t} u_0(x-y)\, dy, \quad x > 0,\ t > 0$$

as well as that for any $(x,t) \in I$ and $(\tau_1,\tau_2) \in I$ we have:

$$\begin{aligned} &\left|u\left(x+\tau_1, t+\tau_2\right) - u(x,t)\right| \leq \frac{\|u_0\|_\infty}{2}\int_x^{x+\tau_1} \left(\pi(t+\tau_2)\right)^{-1/2} e^{-y^2/4(t+\tau_2)}\, dy\\ &+ \frac{\|u_0\|_\infty}{2}\int_{-(x+\tau_1)}^{-x} \left(\pi(t+\tau_2)\right)^{-1/2} e^{-y^2/4(t+\tau_2)}\, dy\\ &+ \frac{1}{2}\int_{-x}^{x} \left|\left(\pi(t+\tau_2)\right)^{-1/2} e^{-y^2/4(t+\tau_2)} u_0\left(x+\tau_1-y\right) - \left(\pi t\right)^{-1/2} e^{-y^2/4t} u_0(x-y)\right| dy. \end{aligned} \tag{24}$$

The consideration for both classes is similar and we will analyze the class of $\mathbb{D}$-asymptotically I'-almost periodic functions of type 1 *below, only. Let $\epsilon > 0$ be given. Then we know that there exists $l > 0$ such that for each $x_0 \in I_0$ there exists $\tau_1 \in (x_0 - l, x_0 + l) \cap I_0$ such that*

$$\left|u_0(x+\tau_1) - u_0(x)\right| \leq \epsilon, \quad x \geq 0. \tag{25}$$

Furthermore, there exists a finite real number $M_0 > 0$ such that $\int_v^{+\infty} e^{-x^2}\, dx < \epsilon$ for all $v \geq M_0$. Let $M > 0$ be such that

$$\min\left(\frac{x^2}{4(t+T)}, t\right) > M_0^2 + \frac{1}{\epsilon},\ \text{provided } (x,t) \in \mathbb{D} \text{ and } |(x,t)| > M. \tag{26}$$

So, let $(x,t) \in \mathbb{D}$ and $|(x,t)| > M$. For the first addend in (24)*, we can use the estimates*

$$\begin{aligned}
&\frac{\|u_0\|_\infty}{2}\int_x^{x+\tau_1}\Big(\pi(t+\tau_2)\Big)^{-1/2} e^{-y^2/4(t+\tau_2)}\,dy \\
&\quad= \pi^{-1/2}\|u_0\|_\infty \int_{x/2\sqrt{t+\tau_2}}^{(x+\tau_1)/2\sqrt{t+\tau_2}} e^{-v^2}\,dv \\
&\quad\le \pi^{-1/2}\|u_0\|_\infty \int_{x/2\sqrt{t+\tau_2}}^{+\infty} e^{-v^2}\,dv \\
&\quad\le \pi^{-1/2}\|u_0\|_\infty \int_{x/2\sqrt{t+T}}^{+\infty} e^{-v^2}\,dv \le \epsilon\pi^{-1/2}\|u_0\|_\infty;
\end{aligned}$$

the same estimate can be used for the second addend in (24)*. For the third addend in* (24)*, we can use the decomposition (see* (25)*)*

$$\begin{aligned}
&\frac{1}{2}\int_{-x}^{x}\Big|\Big(\pi(t+\tau_2)\Big)^{-1/2} e^{-y^2/4(t+\tau_2)} u_0\Big(x+\tau_1-y\Big) - \Big(\pi t\Big)^{-1/2} e^{-y^2/4t} u_0(x-y)\Big|\,dy \\
&\le \frac{1}{2}\int_{-x}^{x}\Big(\pi(t+\tau_2)\Big)^{-1/2} e^{-y^2/4(t+\tau_2)}\Big|u_0\Big(x+\tau_1-y\Big) - u_0(x-y)\Big|\,dy, \\
&+\frac{1}{2}\int_{-x}^{x}\Big|\Big(\pi(t+\tau_2)\Big)^{-1/2} e^{-y^2/4(t+\tau_2)} u_0\Big(x-y\Big) - \Big(\pi t\Big)^{-1/2} e^{-y^2/4t} u_0(x-y)\Big|\,dy,
\end{aligned}$$

which enables one to further continue the computation as follows:

$$\begin{aligned}
&\le \frac{\epsilon}{2}\int_{-x}^{x}\Big(\pi(t+\tau_2)\Big)^{-1/2} e^{-y^2/4(t+\tau_2)}\,dy + \frac{\|u_0\|_\infty}{2}\int_{-x}^{x}\Big|\Big(\pi(t+\tau_2)\Big)^{-1/2} e^{-y^2/4(t+\tau_2)} - \Big(\pi t\Big)^{-1/2} e^{-y^2/4t}\Big|\,dy \\
&\le \epsilon\pi^{-1/2}\int_{-\infty}^{+\infty} e^{-v^2}\,dv + \frac{\|u_0\|_\infty}{2}\int_{-x}^{x}\Big|\Big(\pi(t+\tau_2)\Big)^{-1/2} e^{-y^2/4(t+\tau_2)} - \Big(\pi t\Big)^{-1/2} e^{-y^2/4t}\Big|\,dy.
\end{aligned}$$

Applying the substitution $v^2 = y^2/4t$, we get that

$$\begin{aligned}
&\frac{\|u_0\|_\infty}{2}\int_{-x}^{x}\Big|\Big(\pi(t+\tau_2)\Big)^{-1/2} e^{-y^2/4(t+\tau_2)} - \Big(\pi t\Big)^{-1/2} e^{-y^2/4t}\Big|\,dy \\
&\le \pi^{-1/2}\|u_0\|_\infty \int_{-\infty}^{+\infty}\left|\sqrt{\frac{t}{t+\tau_2}}\, e^{-v^2\cdot\frac{t}{t+\tau_2}} - e^{-v^2}\right|\,dv.
\end{aligned}$$

Applying the Lagrange mean value theorem for the function $x \mapsto xe^{-v^2x^2}$, $x \in [\sqrt{\frac{t}{t+\tau_2}}, 1]$ ($v \in \mathbb{R}$ is fixed), we obtain

$$\begin{aligned}
&\pi^{-1/2}\|u_0\|_\infty \int_{-\infty}^{+\infty}\left|\sqrt{\frac{t}{t+\tau_2}}\, e^{-v^2\cdot\frac{t}{t+\tau_2}} - e^{-v^2}\right|\,dv \\
&\le \pi^{-1/2}\|u_0\|_\infty \int_{-\infty}^{+\infty}\left|\sqrt{\frac{t}{t+\tau_2}} - 1\right| \max_{\zeta\in[\sqrt{\frac{t}{t+\tau_2}},1]} e^{-v^2\zeta^2}\Big(1+2\zeta^2 v^2\Big)\,dv \\
&\le \pi^{-1/2}\|u_0\|_\infty \int_{-\infty}^{+\infty}\left|\sqrt{\frac{t}{t+\tau_2}} - 1\right| e^{-\frac{t}{t+\tau_2}v^2}\Big(1+2v^2\Big)\,dv \\
&\le \pi^{-1/2}\|u_0\|_\infty \left|\sqrt{\frac{t}{t+\tau_2}} - 1\right| \int_{-\infty}^{+\infty} e^{-\frac{M_0^2}{M_0^2+T}v^2}\Big(1+2v^2\Big)\,dv.
\end{aligned}$$

The final conclusion now follows from the estimate (26)*, by observing that*

$$\left|\sqrt{\frac{t}{t+\tau_2}}-1\right| = \frac{\tau_2}{t+\tau_2+\sqrt{t^2+t\tau_2}} \leq \frac{T}{t}.$$

The following observation should be also made: If $u_0 : [0,\infty) \to \mathbb{C}$ is an essentially bounded function, then it can be easily shown that for each $x > 0$ the function $t \mapsto u(x,t)$, $t \geq 0$ is bounded and continuous. Furthermore, the calculus established above enables one to see that for each $x > 0$ the function $t \mapsto u(x,t)$, $t \geq 0$ is S-asymptotically ω-periodic for any positive real number $\omega > 0$ (see H. R. Henríquez, M. Pierri, P. Táboas [17] for the notion).

It is our strong belief that the existence and uniqueness of $\mathbb{D}$-asymptotically almost periodic type solutions of the mixed initial value problems on quadrants will receive considerable attention of the authors in the near future.

Example 3.2. *Consider the system of abstract partial differential equations*

$$u_s(s,t) = Au(s,t) + f_1(s,t), \quad u_t(s,t) = Bu(s,t) + f_2(s,t), \tag{27}$$

for $(s,t) \in [0,\infty)^2$. In this part, we would like to note that some partial results on the existence and uniqueness of $\mathbb{D}$-asymptotically almost periodic type solutions of this problem can be obtained by using the results from [1, Section 2.1] and some additional analyses. For simplicity, let us assume that A and B are two complex matrices of format $n \times n$, $AB = BA$, and A, resp. B, generate an exponentially decaying, strongly continuous semigroup $(T_1(s))_{s\geq 0}$, resp. $(T_2(t))_{t\geq 0}$. Let the functions $f_1(s,t)$ and $f_2(s,t)$ be continuously differentiable, let the compatibility condition $(f_2)_s - Af_2 = (f_1)_t - Bf_1$ hold ($s,\ t \geq 0$), $\mathbb{D} := \{(s,t) \in [0,\infty)^2 : c_1 s \leq t \leq c_2 s$ for some positive real numbers c_1 and $c_2\}$, and let the following conditions hold true:

- *(i) There exists a finite real constant $M > 0$ such that $|f_1(v,0)| + |f_2(0,\omega)| \leq M$, provided that $v,\ \omega \geq 0$ (here and hereafter, $|(z_1,\cdots,z_n)| := (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$ if $z_i \in \mathbb{C}$ for all $i \in \mathbb{N}_n$);*
- *(ii) The mappings $g_i : \mathbb{R}^2 \to \mathbb{C}^n$ are continuous, bounded ($i = 1,2$) and satisfy that, for every $\epsilon > 0$, there exists $l > 0$ such that any subinterval I of $\mathbb{R}$ of length $l > 0$ contains a number $\tau \in I$ such that, for every $s,\ t \geq 0$, we have $|g_1(s+\tau,t) - g_1(s,t)| \leq \epsilon$ and $|g_2(s,t+\tau) - g_2(s,t)| \leq \epsilon$;*
- *(iii) We have that the function $q_i : [0,\infty)^2 \to \mathbb{C}^n$ is bounded, $q_i \in C_{0,\mathbb{D}}([0,\infty)^2 : \mathbb{C}^n)$ and $f_i(s,t) = g_i(s,t) + q_i(s,t)$ for $(s,t) \in [0,\infty)^2$ and $i = 1,2$.*

Then there exists a unique classical solution $u(s,t)$ of (27) *(see [1, Definition 2.13]), and moreover, there exist a continuous function $u_{ap}(s,t)$ on $[0,\infty)^2$ and a function $u_0 \in C_{0,\mathbb{D}}([0,\infty)^2 : \mathbb{C}^n)$ such that $u(s,t) = u_{ap}(s,t) + u_0(s,t)$ for all $(s,t) \in [0,\infty)^2$, as well as for every $\epsilon > 0$, there exists $l > 0$ such that any subinterval I of $[0,\infty)$ of length $l > 0$ contains a number $\tau \in I$ such that, for every $s,\ t \geq 0$, we have $|u_{ap}(s+\tau,t) - u_{ap}(s,t)| \leq \epsilon$ and $|u_{ap}(s,t+\tau) - u_{ap}(s,t)| \leq \epsilon$. Keeping in mind [1, Theorem 2.6, Theorem 2.16], all that we need is to prove that the above conclusion holds for the function*

$$\begin{aligned} u(s,t) &= T_1(s)T_2(t)x + T_1(s)\int_0^t T_2(t-\omega)f_2(0,\omega)\,d\omega + \int_0^s T_1(s-v)f_1(v,t)\,dv \\ &= T_1(s)T_2(t)x + T_2(t)\int_0^s T_1(s-v)f_1(v,0)\,dv + \int_0^t T_2(t-\omega)f_2(s,\omega)\,d\omega, \quad s,\ t \geq 0. \end{aligned}$$

Since the quantities $s,\ t$ and $|(s,t)|$ are equivalent on $\mathbb{D}$, with the meaning clear, our assumption (i) and the exponential decaying of $(T_1(s))_{s\geq 0}$ ($(T_2(t))_{t\geq 0}$) together imply that:

$$\begin{aligned} &\lim_{(s,t)\in\mathbb{D}, |(s,t)|\to\infty} \left[T_1(s)T_2(t)x + T_1(s)\int_0^t T_2(t-\omega)f_2(0,\omega)\,d\omega\right] \\ &= \lim_{(s,t)\in\mathbb{D}, |(s,t)|\to\infty} \left[T_1(s)T_2(t)x + T_2(t)\int_0^s T_1(s-v)f_1(v,0)\,dv\right] = 0. \end{aligned}$$

Using the decomposition ($s,\ t \geq 0$)

$$\int_0^s T_1(s-v) f_1(v,t)\,dv = \int_{-\infty}^s T_1(s-v) g_1(v,t)\,dv + \left[\int_0^s T_1(s-v) q(v,t)\,dv - \int_{-\infty}^0 T_1(s-v) g_1(v,t)\,dv\right],$$

the corresponding decomposition for the term $t \mapsto \int_0^t T_2(t-\omega) f_2(s,\omega)\,d\omega$, $t \geq 0$, *our assumptions (ii)-(iii) and the argumentation contained in the proofs of [18, Proposition 2.6.11, Proposition 2.6.13; Remark 2.6.14], the required conclusion simply follows. Let us note, finally, that there exists a great number of concrete situations where the above assumptions are really satisfied. Suppose, for example, that* $n = 1$, $A = B = [-1]$,

$$f_1(s,t) = \sin s + \cos s + \int_0^t \frac{e^{\xi - t}}{1+\xi^2}\,d\,\xi, \quad s,\ t \geq 0$$

and

$$f_2(s,t) = \sin s + \frac{1}{1+t^2}, \quad s,\ t \geq 0;$$

see also [2, Proposition 1.3.5(d)]. Then the above requirements hold.

Finally, we would like to note that the Stepanov multi-dimensional almost periodic type functions and their applications have been analyzed in [9] as well as that the results established in this paper and [9] have been published in the research monograph [19].

Sadly, our coauthor Prof. Manuel Pinto, a renowned researcher in the field of differential equations and a mentor to several generations of mathematicians, has recently passed away. We would like to dedicate this paper to the memory of our dear friend and master. Conference on ordinary and functional differential equations "Celebrating the life and legacy of Professor Manuel Pinto J." will be held at Facultad de Ciencias, Universidad de Chile, Chile (November 20-21, 2025).

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