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Pre-orders defined by weighted core–EP inverse

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Abstract. Our aim is to introduce new binary relations on the set of all Wg-Drazin invertible bounded linear operator between two Hilbert spaces in terms of the weighted core–EP inverse. Different characterizations of new binary relations are presented as well as block operator matrix forms for operators in these relations. We prove that three of four our new binary relations are pre-orders on the corresponding set and we state as an open problem to check is the fourth relation a pre-order on the same set.

1. Introduction

Let $\mathcal{B}(X, Y)$ be the set of all bounded linear operators from X to Y, where X and Y are arbitrary Hilbert spaces. The range, null space and adjoint of $A \in \mathcal{B}(X, Y)$ are denoted by $R(A)$, $N(A)$ and A^*, respectively. We state $\mathcal{B}(X) = \mathcal{B}(X, X)$. The sets of all invertible and quasinilpotent ($\sigma(Q) = \{0\}$) operators of $\mathcal{B}(X, Y)$ are represented as $\mathcal{B}(X, Y)^{-1}$ and $\mathcal{B}(X, Y)^{qnil}$, respectively.

Different questions in mechanics, physics, control theory, and so on, reduce to the solvability of systems of equations, and for practical reasons, generalized inverses are defined as solutions of systems of equations [3, 5, 14, 28, 30, 31].

For $W \in \mathcal{B}(Y, X)\backslash\{0\}$, an operator $A \in \mathcal{B}(X, Y)$ is Wg–Drazin invertible if there exists $B \in \mathcal{B}(X, Y)$ such that

$$AWB = BWA, \quad BWAWB = B \quad \text{and} \quad A - AWBWA \in \mathcal{B}(X, Y)^{qnil}.$$

When the Wg–Drazin inverse B of A exists, it is uniquely determined and marked by $A^{d,W}$ [6]. In the case that $A - AWBWA$ is nilpotent, $A^{d,W} = A^{D,W}$ is the W-weighted Drazin inverse of A [7]. If $X = Y$ and $W = I$, $A^d = A^{d,W}$ is the generalized Drazin inverse of A [16] and $A^D = A^{D,W}$ is the Drazin inverse of A [7]. The symbols $\mathcal{B}(X, Y)^{d,W}$ and $\mathcal{B}(X)^d$ denote the sets of all Wg–Drazin invertible operators of $\mathcal{B}(X, Y)$ and generalized Drazin invertible operators of $\mathcal{B}(X)$, respectively. Recall that $A \in \mathcal{B}(X, Y)^{d,W}$ if and only if $AW \in \mathcal{B}(Y)^d$ if and only if $WA \in \mathcal{B}(X)^d$ [6]. In addition, $A^{d,W} = A[(WA)^d]^2 = [(AW)^d]^2 A$. Weighted pre-orders and partial orders introduced for Wg–Drazin invertible operators can be found in [21].

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If $W \in \mathcal{B}(Y,X)\backslash\{0\}$ and $A \in \mathcal{B}(X,Y)^{d,W}$, there exists the unique W-weighted core–EP inverse B of A, denoted by $A^{\textcircled{\dagger},W}$, such that [23]

$$WAWB = P_{R((WA)^d)} \quad \text{and} \quad R(B) \subseteq R((AW)^d).$$

In particular, when $X = Y$ and $W = I$, $A^{\textcircled{\dagger}} = A^{\textcircled{\dagger},W}$ is the core–EP inverse of A [24]. It is known that $A^{\textcircled{\dagger},W} = A[(WA)^{\textcircled{\dagger}}]^2$ and $(WA)^{\textcircled{\dagger}} = WA^{\textcircled{\dagger},W}$ [23].

The core–EP inverse has attracted attention of many researches from defining. Properties and representations of the core–EP inverse were established in [9, 11, 19, 26, 29, 32, 33] Characterizations of the weighted core–EP inverse were studied in [1, 2, 10, 12, 15, 17, 23]. Some constrained approximation problems were solved in [18, 25] applying the weighted core–EP inverse.

Using the core–EP inverse and the weighted core–EP inverse, various kinds of pre-orders and partial orders were defined and investigated in [8, 22, 27]. For $A \in \mathcal{B}(X)^d$ and $B \in \mathcal{B}(X)$, the core–EP pre-order is defined by

$$A \leq^{\textcircled{\dagger}} B \Leftrightarrow AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}} \quad \text{and} \quad A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B.$$

In [23], the pre-orders $WA \leq^{\textcircled{\dagger}} WB$ and $AW \leq^{\textcircled{\dagger}} BW$ were considered in terms of the weighted core–EP inverse as well as the pre-order which is a conjunction of $WA \leq^{\textcircled{\dagger}} WB$ and $AW \leq^{\textcircled{\dagger}} BW$. Based on weighted core–EP inverse, new pre-orders were presented in [14] on the set of rectangular complex matrices.

Motivated by previously mentioned results about weighted pre-orders, we continue to develop this area. In particular, we define four new binary relations on the set of all Wg-Drazin invertible bounded linear operator between two Hilbert spaces, applying the weighted core–EP inverse. Two of four new binary relations are introduced through two equalities, that is, they are two-sided relations, and the rest two binary relations are defined by one equality, i.e. they are one-sided relations. We present characterizations of new binary relations and we establish block operator matrix forms of operators in these relations. We obtain new results and we extend some results given in [14] for rectangular matrices to more general settings. We verify that three of four our new binary relations are pre-orders on the set $\mathcal{B}(X,Y)^{d,W}$ and we state as an open problem to show is the fourth relation a pre-order on $\mathcal{B}(X,Y)^{d,W}$.

The organization of this paper follows. In Section 2, we introduce and characterize two binary relations which are two-sided. Section 3 contains definitions and properties of two one-sided binary relations. An open problem is also proposed in Section 3.

2. Pre-orders defined by weighted core–EP inverse

Using the weighted core–EP inverse, we introduce two new binary relations between two bounded linear operators on Hilbert spaces, extending corresponding definitions for rectangular matrices proposed in [14].

Definition 2.1. *Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$, $B \in \mathcal{B}(X,Y)$ and let $A \in \mathcal{B}(X,Y)$ be Wg-Drazin invertible. Then we say*

(i) $A \leq^{\textcircled{\dagger},W} B$ *if*
$$AWA^{\textcircled{\dagger},W} = BWA^{\textcircled{\dagger},W} \quad \text{and} \quad A^{\textcircled{\dagger},W}WA = A^{\textcircled{\dagger},W}WB;$$

(ii) $A \leq^{\textcircled{\dagger},W,R} B$ *if*
$$AWA^{\textcircled{\dagger},W}W = BWA^{\textcircled{\dagger},W}W \quad \text{and} \quad A^{\textcircled{\dagger},W}WAW = A^{\textcircled{\dagger},W}WBW.$$

Remark that, for $X = Y$ and $W = I$, $A \leq^{\textcircled{\dagger},W} B$ and $A \leq^{\textcircled{\dagger},W,R} B$ reduce to $A \leq^{\textcircled{\dagger}} B$. If we state that $A \leq^{\textcircled{\dagger},W,L} B$ holds when $WAWA^{\textcircled{\dagger},W} = WBWA^{\textcircled{\dagger},W}$ and $WA^{\textcircled{\dagger},W}WA = WA^{\textcircled{\dagger},W}WB$, by $(WA)^{\textcircled{\dagger}} = WA^{\textcircled{\dagger},W}$, we deduce that $A \leq^{\textcircled{\dagger},W,L} B$ is equivalent to $WA \leq^{\textcircled{\dagger}} WB$, which is the relation studied in [23]. Notice also that if $A \leq^{\textcircled{\dagger},W} B$ is satisfied, then $A \leq^{\textcircled{\dagger},W,R} B$ and $WA \leq^{\textcircled{\dagger}} WB$ hold.

In the first theorem of this section, we will present some characterizations of the new relation $\leq^{\textcircled{\dagger},W}$.

Theorem 2.2. *Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$, $B \in \mathcal{B}(X,Y)$ and let $A \in \mathcal{B}(X,Y)$ be Wg-Drazin invertible. Then the following statements are equivalent:*

(i) $A \leq^{\textcircled{d},W} B$;

(ii) $AWA^{\textcircled{d},W}W = BWA^{\textcircled{d},W}W$ and $WA^{\textcircled{d},W}WA = WA^{\textcircled{d},W}WB$;

(iii) $AW(AW)^{\textcircled{d}} = BW(AW)^{\textcircled{d}}$ and $(WA)^{\textcircled{d}}WA = (WA)^{\textcircled{d}}WB$;

(iv) $AWA^{d,W}W = BWA^{d,W}W$ and $WA^{\textcircled{d},W}WA = WA^{\textcircled{d},W}WB$;

(v) $AW(AW)^d = BW(AW)^d$ and $(WA)^{\textcircled{d}}WA = (WA)^{\textcircled{d}}WB$;

(vi) $A(WA)^d = B(WA)^d$ and $(WA)^{\textcircled{d}}WA = (WA)^{\textcircled{d}}WB$;

(vii) $A(WA)^{\textcircled{d}} = B(WA)^{\textcircled{d}}$ and $(WA)^{\textcircled{d}}WA = (WA)^{\textcircled{d}}WB$;

(viii) $AWA^{d,W} = BWA^{d,W}$ and $A^{\textcircled{d},W}WA = A^{\textcircled{d},W}WB$.

Proof. (i) ⇒ (ii): This is clear by the definition of the relation $\leq^{\textcircled{d},W}$.

(ii) ⇒ (iv): Using $A^{\textcircled{d},W}WAWA^{d,W} = A^{d,W}$ [22, 23] and $AWA^{\textcircled{d},W}W = BWA^{\textcircled{d},W}W$, we obtain

$$AWA^{d,W}W = (AWA^{\textcircled{d},W}W)AWA^{d,W}W = BW(A^{\textcircled{d},W}WAWA^{d,W})W = BWA^{d,W}W.$$

(iv) ⇒ (v): It follows by the equalities $A^{d,W}W = (AW)^d$ and $WA^{\textcircled{d},W} = (WA)^{\textcircled{d}}$ from [23].

(v) ⇒ (iii): The equalities $(AW)^dAW(AW)^{\textcircled{d}} = (AW)^{\textcircled{d}}$ and $AW(AW)^d = BW(AW)^d$ give us

$$AW(AW)^{\textcircled{d}} = [AW(AW)^d]AW(AW)^{\textcircled{d}} = BW[(AW)^dAW(AW)^{\textcircled{d}}] = BW(AW)^{\textcircled{d}}.$$

(iii) ⇒ (vi): From $(AW)^{\textcircled{d}}AW(AW)^d = (AW)^d$ and $AW(AW)^{\textcircled{d}} = BW(AW)^{\textcircled{d}}$, we have

$$AW(AW)^d = [AW(AW)^{\textcircled{d}}]AW(AW)^d = BW[(AW)^{\textcircled{d}}AW(AW)^d] = BW(AW)^d.$$

Therefore, by properties of the generalized Drazin inverse,

$$\begin{aligned} A(WA)^d &= A(WA)^dWA(WA)^d = [AW(AW)^d]A(WA)^d \\ &= BW(AW)^dA(WA)^d = B(WA)^dWA(WA)^d \\ &= B(WA)^d. \end{aligned}$$

(vi) ⇒ (vii): Applying $(WA)^dWA(WA)^{\textcircled{d}} = (WA)^{\textcircled{d}}$ and $A(WA)^d = B(WA)^d$, we obtain

$$A(WA)^{\textcircled{d}} = A(WA)^dWA(WA)^{\textcircled{d}} = B(WA)^dWA(WA)^{\textcircled{d}} = B(WA)^{\textcircled{d}}.$$

(vii) ⇒ (i): By the equalities $WA^{\textcircled{d},W} = (WA)^{\textcircled{d}}$ and $A(WA)^{\textcircled{d}} = B(WA)^{\textcircled{d}}$, we can conclude

$$AWA^{\textcircled{d},W} = A(WA)^{\textcircled{d}} = B(WA)^{\textcircled{d}} = BWA^{\textcircled{d},W}.$$

The assumption $(WA)^{\textcircled{d}}WA = (WA)^{\textcircled{d}}WB$ gives

$$\begin{aligned} A^{\textcircled{d},W}WA &= A^{\textcircled{d},W}WAWA^{\textcircled{d},W}WA = A^{\textcircled{d},W}WA[(WA)^{\textcircled{d}}WA] \\ &= A^{\textcircled{d},W}WA(WA)^{\textcircled{d}}WB = A^{\textcircled{d},W}WB. \end{aligned}$$

(i) ⇔ (viii): This equivalence follows by $A^{\textcircled{d},W}WAWA^{d,W} = A^{d,W}$ and $A^{d,W}WAWA^{\textcircled{d},W} = A^{\textcircled{d},W}$ [22, 23]. □

Remark that the second equality in statements (i)–(viii) of Theorem 2.2 can be replaced with the equality $A^{\textcircled{d},W}WA = A^{\textcircled{d},W}WB$ from the definition.

As Theorem 2.2, we get characterizations for the relation $A \leq^{\textcircled{d},W,R} B$ to hold.

Theorem 2.3. *Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$, $B \in \mathcal{B}(X,Y)$ and let $A \in \mathcal{B}(X,Y)$ be Wg-Drazin invertible. Then the following statements are equivalent:*

(i) $A \preceq^{\textcircled{d},W,R} B$;

(ii) $AWA^{\textcircled{d},W}W = BWA^{\textcircled{d},W}W$ *and* $WA^{\textcircled{d},W}WAW = WA^{\textcircled{d},W}WBW$;

(iii) $AW(AW)^{\textcircled{d}} = BW(AW)^{\textcircled{d}}$ *and* $(WA)^{\textcircled{d}}WAW = (WA)^{\textcircled{d}}WBW$;

(iv) $AWA^{d,W}W = BWA^{d,W}W$ *and* $WA^{\textcircled{d},W}WAW = WA^{\textcircled{d},W}WBW$;

(v) $AW(AW)^{d} = BW(AW)^{d}$ *and* $(WA)^{\textcircled{d}}WAW = (WA)^{\textcircled{d}}WBW$;

(vi) $A(WA)^{d} = B(WA)^{d}$ *and* $(WA)^{\textcircled{d}}WAW = (WA)^{\textcircled{d}}WBW$;

(vii) $A(WA)^{\textcircled{d}} = B(WA)^{\textcircled{d}}$ *and* $(WA)^{\textcircled{d}}WAW = (WA)^{\textcircled{d}}WBW$;

(viii) $AWA^{d,W} = BWA^{d,W}$ *and* $A^{\textcircled{d},W}WAW = A^{\textcircled{d},W}WBW$;

(ix) $AWA^{\textcircled{d},W} = BWA^{\textcircled{d},W}$ *and* $WA^{\textcircled{d},W}WAW = WA^{\textcircled{d},W}WBW$.

Observe that $A^{\textcircled{d},W}WAW = A^{\textcircled{d},W}WBW$ can be stated instead of the second equality in statements (i)–(ix) of Theorem 2.3.

To prove new characterizations of the relations $\preceq^{\textcircled{d},W}$ and $\preceq^{\textcircled{d},W,R}$, we will use the next result from [23].

Lemma 2.4. [23] *Let $W \in \mathcal{B}(Y, X)\backslash\{0\}$ and let $A \in \mathcal{B}(X, Y)$ be Wg-Drazin invertible. Then*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \to \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix}$$

and

$$W = \begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix} : \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix} \to \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix},$$

where $A_1 \in B(R((WA)^d), R((AW)^d))^{-1}$, $W_1 \in B(R((AW)^d), R((WA)^d))^{-1}$, $A_3W_3 \in B(N[((AW)^d)^*])^{qnil}$ and $W_3A_3 \in B(N[((WA)^d)^*])^{qnil}$. In addition, we have

$$A^{\textcircled{d},W} = \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \to \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix}.$$

We firstly get block operator matrix forms for operators A, W and B when $A \preceq^{\textcircled{d},W} B$.

Theorem 2.5. *Let $W \in \mathcal{B}(Y, X)\backslash\{0\}$, $B \in \mathcal{B}(X, Y)$ and let $A \in \mathcal{B}(X, Y)$ be Wg-Drazin invertible. Then the following statements are equivalent:*

(i) $A \preceq^{\textcircled{d},W} B$;

(ii) *there exist the following matrix representations with respect to the orthogonal sums $X = R((WA)^d) \oplus N[((WA)^d)^*]$ and $Y = R((AW)^d) \oplus N[((AW)^d)^*]$:*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \ W = \begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix}, \ B = \begin{bmatrix} A_1 & A_2 + W_1^{-1}W_2(A_3 - B_3) \\ 0 & B_3 \end{bmatrix},$$

where $A_1 \in \mathcal{B}(R((WA)^d), R((AW)^d))^{-1}$, $W_1 \in \mathcal{B}(R((AW)^d), R((WA)^d))^{-1}$, $A_3W_3 \in \mathcal{B}(N[((AW)^d)^*])^{qnil}$ and $W_3A_3 \in \mathcal{B}(N[((WA)^d)^*])^{qnil}$.

Proof. (i) ⇒ (ii): Let A and W be represented as in Lemma 2.4 and

$$B = \begin{bmatrix} B_1 & B_2 \\ B_4 & B_3 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \to \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix}.$$

We have, by

$$BWA^{\textcircled{d},W} = \begin{bmatrix} B_1W_1(W_1A_1W_1)^{-1} & 0 \\ B_4W_1(W_1A_1W_1)^{-1} & 0 \end{bmatrix} = AWA^{\textcircled{d},W} = \begin{bmatrix} W_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

that $B_1 = A_1$ and $B_4 = 0$.

On the other hand, from $A^{\textcircled{d},W}WA = A^{\textcircled{d},W}WB$ and also from

$$A^{\textcircled{d},W}WA = \begin{bmatrix} W_1^{-1} & (W_1A_1W_1)^{-1}(W_1A_2 + W_2A_3) \\ 0 & 0 \end{bmatrix},$$

$$A^{\textcircled{d},W}WB = \begin{bmatrix} (W_1A_1W_1)^{-1}(W_1B_1 + W_2B_4) & (W_1A_1W_1)^{-1}(W_1B_2 + W_2B_3) \\ 0 & 0 \end{bmatrix},$$

we obtain $W_1A_2 + W_2A_3 = W_1B_2 + W_2B_3$. Hence, $B_2 = A_2 + W_1^{-1}W_2(A_3 - B_3)$.

(ii) ⇒(i): This part follows by elementary computations. □

In the case that $A \leq^{\textcircled{d},W,R} B$, we develop the next block operator matrix forms of A, W and B.

Theorem 2.6. *Let $W \in \mathcal{B}(Y, X)\backslash\{0\}$, $B \in \mathcal{B}(X, Y)$ and let $A \in \mathcal{B}(X, Y)$ be Wg-Drazin invertible. Then the following statements are equivalent:*

(i) *$A \leq^{\textcircled{d},W,R} B$;*

(ii) *there exist the following matrix representations with respect to the orthogonal sums $X = R((WA)^d) \oplus N[((WA)^d)^*]$ and $Y = R((AW)^d) \oplus N[((AW)^d)^*]$:*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \ W = \begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix}, \ B = \begin{bmatrix} A_1 & B_2 \\ 0 & B_3 \end{bmatrix},$$

where $A_1 \in \mathcal{B}(R((WA)^d), R((AW)^d))^{-1}$, $W_1 \in \mathcal{B}(R((AW)^d), R((WA)^d))^{-1}$, $A_3W_3 \in \mathcal{B}(N[((AW)^d)^*])^{qnil}$, $W_3A_3 \in \mathcal{B}(N[((WA)^d)^*])^{qnil}$ and

$$(W_1A_2 + W_2A_3)W_3 = (W_1B_2 + W_2B_3)W_3.$$

Proof. (i) ⇒ (ii): Theorem 2.3(ix) gives that $A \leq^{\textcircled{d},W,R} B$ if and only if $AWA^{\textcircled{d},W} = BWA^{\textcircled{d},W}$ and $WA^{\textcircled{d},W}WAW = WA^{\textcircled{d},W}WBW$. Using the same notations as in the proof of Theorem 2.5, we have that $BWA^{\textcircled{d},W} = AWA^{\textcircled{d},W}$ implies $B_1 = A_1$ and $B_4 = 0$. Since

$$WA^{\textcircled{d},W}WAW = \begin{bmatrix} W_1 & W_2 + (W_1A_1)^{-1}(W_1A_2 + W_2A_3)W_3 \\ 0 & 0 \end{bmatrix}$$

and

$$WA^{\textcircled{d},W}WBW = \begin{bmatrix} W_1 & W_2 + (W_1A_1)^{-1}(W_1B_2 + W_2B_3)W_3 \\ 0 & 0 \end{bmatrix},$$

we see that $WA^{\textcircled{d},W}WAW = WA^{\textcircled{d},W}WBW$ is equivalent to $(W_1A_2 + W_2A_3)W_3 = (W_1B_2 + W_2B_3)W_3$.

(ii) ⇒ (i): This implication can be verified by basic calculations. □

Applying Theorem 2.5, we verify that $\leq^{\textcircled{d},W}$ is a pre-order on the corresponding set.

Theorem 2.7. *The binary relation $\leq^{\textcircled{d},W}$ is a pre-order on $\mathcal{B}(X, Y)^{d,W}$.*

Proof. To prove that some binary relation is a pre-order, we have to prove that it is reflexive and transitive. Here, it is obvious that the reflexivity of relation $\leq^{@,W}$ holds from the Definition 2.1.

In order to verify the transitivity, for $A, B, C \in \mathcal{B}(X,Y)^{d,W}$, assume that $A \leq^{@,W} B$ and $B \leq^{@,W} C$ hold. We can represent A, W and B as in Theorem 2.5(ii). Let

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \to \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix}.$$

From Theorem 2.2(vii), $B \leq^{@,W} C$ is equivalent to $B(WB)^@ = C(WB)^@$ and $(WB)^@WB = (WB)^@WC$. Because

$$WB = \begin{bmatrix} W_1A_1 & W_1A_2 + W_2A_3 \\ 0 & W_3B_3 \end{bmatrix}$$

is generalized Drazin invertible, by [4, Theorem 2.3] and [20, Lemma 2.3], we have that W_3B_3 is generalized Drazin invertible and

$$(WB)^@ = \begin{bmatrix} (W_1A_1)^{-1} & -(W_1A_1)^{-1}(W_1A_2 + W_2A_3)(W_3B_3)^@ \\ 0 & (W_3B_3)^@ \end{bmatrix}.$$

Since we know that $B(WB)^@ = C(WB)^@$ holds, from

$$B(WB)^@ = \begin{bmatrix} W_1^{-1} & * \\ 0 & B_3(W_3B_3)^@ \end{bmatrix}$$

and

$$C(WB)^@ = \begin{bmatrix} C_1(W_1A_1)^{-1} & * \\ C_3(W_1A_1)^{-1} & * \end{bmatrix},$$

we get $C_1 = A_1$ and $C_3 = 0$. The equalities

$$(WB)^@WB = \begin{bmatrix} I & (W_1A_1)^{-1}(W_1A_2 + W_2A_3 - (W_1A_2 + W_2A_3)(W_3B_3)^@W_3B_3) \\ 0 & (W_3B_3)^@W_3B_3 \end{bmatrix},$$

$$(WB)^@WC = \begin{bmatrix} I & (W_1A_1)^{-1}(W_1C_2 + W_2C_4 - (W_1A_2 + W_2A_3)(W_3B_3)^@W_3C_4) \\ 0 & (W_3B_3)^@W_3C_4 \end{bmatrix},$$

and $(WB)^@(WB) = (WB)^@(WC)$ give $(W_3B_3)^@W_3C_4 = (W_3B_3)^@W_3B_3$ and $W_1A_2 + W_2A_3 = W_1C_2 + W_2C_4$, that is, $C_2 = A_2 + W_1^{-1}W_2(A_3 - C_4)$. By Theorem 2.5, we observe that $A \leq^{@,W} C$ which means that the relation $\leq^{@,W}$ is transitive and hence a pre-order too. □

By [14, Example 2.6], we conclude that the relation $\leq^{@,W}$ is not antisymmetric.

Now, we show that $\leq^{@,W,R}$ is also a pre-order on $\mathcal{B}(X,Y)^{d,W}$.

Theorem 2.8. *The binary relation $\leq^{@,W,R}$ is a pre-order on $\mathcal{B}(X,Y)^{d,W}$.*

Proof. Clearly, $\leq^{@,W,R}$ is reflexive.

To check transitivity of the relation $\leq^{@,W,R}$, let $A, B, C \in \mathcal{B}(X,Y)^{d,W}$ satisfy $A \leq^{@,W,R} B$ and $B \leq^{@,W,R} C$. We express A, W and B as in Theorem 2.6(ii) and set

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \to \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix}.$$

Applying Theorem 2.3, $B \leq^{@,W,R} C$ if and only if $B(WB)^@ = C(WB)^@$ and $(WB)^@WBW = (WB)^@WCW$. Similarly as in the proof of Theorem 2.7, we have

$$(WB)^@ = \begin{bmatrix} (W_1A_1)^{-1} & -(W_1A_1)^{-1}(W_1B_2 + W_2B_3)(W_3B_3)^@ \\ 0 & (W_3B_3)^@ \end{bmatrix}$$

and $B(WB)^{\circledast} = C(WB)^{\circledast}$ yields $C_1 = A_1$ and $C_3 = 0$. Note that

$$(WB)^{\circledast}WBW = \begin{bmatrix} W_1 & W_2 + (W_1A_1)^{-1}(W_1B_2 + W_2B_3)(I - (W_3B_3)^{\circledast}W_3B_3)W_3 \\ 0 & (W_3B_3)^{\circledast}W_3B_3W_3 \end{bmatrix}$$

and

$$(WB)^{\circledast}WCW = \begin{bmatrix} W_1 & W_2 + (W_1A_1)^{-1}(W_1C_2 + W_2C_4 - (W_1B_2 + W_2B_3)(W_3B_3)^{\circledast}W_3C_4)W_3) \\ 0 & (W_3B_3)^{\circledast}W_3C_4W_3 \end{bmatrix}.$$

Because $(WB)^{\circledast}WBW = (WB)^{\circledast}WCW$, we obtain $(W_3B_3)^{\circledast}W_3C_4W_3 = (W_3B_3)^{\circledast}W_3B_3W_3$ and, by $(W_1A_2 + W_2A_3)W_3 = (W_1B_2 + W_2B_3)W_3$, we get $(W_1A_2 + W_2A_3)W_3 = (W_1C_2 + W_2C_4)W_3$. Theorem 2.6 implies that $A \preceq^{\circledast,W,R} C$. □

By Theorem 2.7 or Theorem 2.8, we obtain the next known result.

Corollary 2.9. *The binary relation $\preceq^{\circledast}$ is a pre-order on $\mathcal{B}(X)^d$.*

3. One-sided binary relations

Inspired by Definition 2.1, we define one-sided binary relations in terms of the weighted core–EP inverse.

Definition 3.1. *Let $W \in \mathcal{B}(Y, X)\backslash\{0\}$, $B \in \mathcal{B}(X, Y)$ and let $A \in \mathcal{B}(X, Y)$ be Wg-Drazin invertible. Then we say*

(i) *$A \preceq^{\circledast,W,l} B$ if*

$$A^{\circledast,W}WA = A^{\circledast,W}WB;$$

(ii) *$A \preceq^{\circledast,W,r} B$ if*

$$AWA^{\circledast,W} = BWA^{\circledast,W}.$$

From Definition 2.1 and Definition 3.1, we deduce that $A \preceq^{\circledast,W,l} B$ and $A \preceq^{\circledast,W,r} B$ is equivalent to $A \preceq^{\circledast,W} B$. Also, $A \preceq^{\circledast,W,R} B$ implies $A \preceq^{\circledast,W,r} B$.

By Theorem 2.2, we have the next characterizations for the relations $\preceq^{\circledast,W,l}$ and $\preceq^{\circledast,W,r}$.

Theorem 3.2. *Let $W \in \mathcal{B}(Y, X)\backslash\{0\}$, $B \in \mathcal{B}(X, Y)$ and let $A \in \mathcal{B}(X, Y)$ be Wg-Drazin invertible. Then the following statements are equivalent:*

(i) *$A \preceq^{\circledast,W,l} B$;*

(ii) *$WA^{\circledast,W}WA = WA^{\circledast,W}WB$;*

(iii) *$(WA)^{\circledast}WA = (WA)^{\circledast}WB$.*

Theorem 3.3. *Let $W \in \mathcal{B}(Y, X)\backslash\{0\}$, $B \in \mathcal{B}(X, Y)$ and let $A \in \mathcal{B}(X, Y)$ be Wg-Drazin invertible. Then the following statements are equivalent:*

(i) *$A \preceq^{\circledast,W,r} B$;*

(ii) *$AWA^{\circledast,W}W = BWA^{\circledast,W}W$;*

(iii) *$AW(AW)^{\circledast} = BW(AW)^{\circledast}$;*

(iv) *$AWA^{d,W}W = BWA^{d,W}W$;*

(v) *$AW(AW)^d = BW(AW)^d$;*

(vi) *$A(WA)^d = B(WA)^d$;*

(vii) *$A(WA)^{\circledast} = B(WA)^{\circledast}$;*

(viii) $AWA^{d,W} = BWA^{d,W}$.

Similarly as in the proof of Theorem 2.5, we obtain block operator matrix forms for operators satisfying $A \leq^{ⓓ,W,l} B$ or $A \leq^{ⓓ,W,r} B$.

Theorem 3.4. *Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$, $B \in \mathcal{B}(X,Y)$ and let $A \in \mathcal{B}(X,Y)$ be Wg-Drazin invertible. Then the following statements are equivalent:*

(i) $A \leq^{ⓓ,W,l} B$;

(ii) *there exist the following matrix representations with respect to the orthogonal sums $X = R((WA)^d) \oplus N[((WA)^d)^*]$ and $Y = R((AW)^d) \oplus N[((AW)^d)^*]$:*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \ W = \begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix}, \ B = \begin{bmatrix} A_1 - W_1^{-1}W_2B_4 & A_2 + W_1^{-1}W_2(A_3 - B_3) \\ B_4 & B_3 \end{bmatrix},$$

where $A_1 \in \mathcal{B}(R((WA)^d), R((AW)^d))^{-1}$, $W_1 \in \mathcal{B}(R((AW)^d), R((WA)^d))^{-1}$, $A_3W_3 \in \mathcal{B}(N[((AW)^d)^*])^{qnil}$, $W_3A_3 \in \mathcal{B}(N[((WA)^d)^*])^{qnil}$.

Theorem 3.5. *Let $W \in \mathcal{B}(Y,X)\backslash\{0\}$, $B \in \mathcal{B}(X,Y)$ and let $A \in \mathcal{B}(X,Y)$ be Wg-Drazin invertible. Then the following statements are equivalent:*

(i) $A \leq^{ⓓ,W,r} B$;

(ii) *there exist the following matrix representations with respect to the orthogonal sums $X = R((WA)^d) \oplus N[((WA)^d)^*]$ and $Y = R((AW)^d) \oplus N[((AW)^d)^*]$:*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \ W = \begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix}, \ B = \begin{bmatrix} A_1 & B_2 \\ 0 & B_3 \end{bmatrix},$$

where $A_1 \in \mathcal{B}(R((WA)^d), R((AW)^d))^{-1}$, $W_1 \in \mathcal{B}(R((AW)^d), R((WA)^d))^{-1}$, $A_3W_3 \in \mathcal{B}(N[((AW)^d)^*])^{qnil}$, $W_3A_3 \in \mathcal{B}(N[((WA)^d)^*])^{qnil}$.

As Theorem 2.7, we can show the following result.

Theorem 3.6. *The binary relation $\leq^{ⓓ,W,r}$ is a pre-order on $\mathcal{B}(X,Y)^{d,W}$.*

In this paper, we do not prove that $\leq^{ⓓ,W,l}$ is a pre-order on $\mathcal{B}(X,Y)^{d,W}$ and thus we state it as a conjecture.

Conjecture. The binary relation $\leq^{ⓓ,W,l}$ is a pre-order on $\mathcal{B}(X,Y)^{d,W}$.

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