Filomat 39:32 (2025), 11531–11546
https://doi.org/10.2298/FIL2532531L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Further results on the m-weak core inverse

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Abstract. This work establishes multiple novel representations for the m-weak core inverse, accompanied by proofs of their validity. Furthermore, we derive perturbation bounds and analyze continuity properties for this generalized inverse. By utilizing the m-weak core inverse, we characterize the unique solution to a constrained minimization problem in the Frobenius norm framework: $\min \left\|M^{m+1}X - M^{2m}(M^m)^{\dagger}B\right\|_F^2$, subject to the range constraint $\mathcal{R}(X) \subseteq \mathcal{R}(M^k)$, where $m \in \mathbb{N}$, $B \in \mathbb{C}^{n\times q}$, $M \in \mathbb{C}^{n\times n}$ and $\mathrm{ind}(M) = k$.

1. Introduction

Typically, for any matrix $M \in \mathbb{C}^{n\times n}$, we denote by M^* is its conjugate transpose, rank(M) is its rank, $\mathcal{R}(M)$ is its range space, and $\mathcal{N}(M)$ is its null space. The symbol $\mathbb{C}^{p\times n}$ is the set of $p \times n$ matrices with complex entries. As always, $\mathbb{C}_r^{m\times n} = \{M \in \mathbb{C}^{m\times n} |\ \mathrm{rank}(M) = r\}$. When $\mathbb{C}^{m\times 1}$ is direct sum of subspaces G and H, we use $P_{G,H}$ to denote a projector onto G along H. Also, P_G stands for the orthogonal projector onto a subspace G.

We begin with a few important generalized inverses [1, 3, 5, 6, 8, 10, 24, 31]. For any matrix $M \in \mathbb{C}^{n\times m}$, its Moore-Penrose inverse $M^{\dagger}$ represents the unique solution satisfying the Penrose equations (see [1]):

$$XMX = X, MXM = M, (MX)^* = MX, (XM)^* = XM.$$

The matrix $X \in \mathbb{C}^{n\times m}$ is called an outer inverse (or also called {2}-inverses) of $M \in \mathbb{C}^{n\times m}$ if it satisfies the condition $XMX = X$. The {2}-inverse of M with the range T and null space S, denoted by $M^{(2)}_{T,S}$ [1] satisfies

$$XMX = X,\ R(M) = T,\ N(M) = S,$$

where T and S are the subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$ and with dimensions $s \le r$ and $m - s$, respectively.

In [3], Drazin first proposed the notion of pseudo inverse while studying the structure of combining rings and semigroups. Among these, the Drazin inverse emerged as an important outer inverse formulation

2020 *Mathematics Subject Classification*. Primary 15B99; Secondary 15A09, 15A10, 15A24.
Keywords. m-weak core inverse, m-weak group inverse, perturbation, Constrained matrix optimization problem.
Received: 27 June 2025; Revised: 28 August 2025; Accepted: 27 September 2025
Communicated by Dijana Mosić
The authors are thankful to Reviewers and Editor for their valuable suggestions on our manuscript. Additionally, this work was supported partially by the National Natural Science Foundation of China (No. 12061015) and the Guangxi Natural Science Foundation (No. 2024GXNSFAA010503).
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for square matrices. Let $M \in \mathbb{C}^{n\times n}$ and $k = \text{ind}(M)$, there exists a unique matrix $X \in \mathbb{C}^{n\times n}$ called the Drazin inverse (written M^D) such that

$$XMX = X,\ MX = XM,\ M^{k+1}X = M^k.$$

In the specific case where $\text{ind}(M) = 1$, the Drazin inverse coincides with the group inverse ($M^D = M^\#$). A matrix $M \in \mathbb{C}^{n\times n}$ is said to be range-Hermitian if its range coincides with that of its conjugate transpose $R(M) = R(M^*)$.

The core-EP inverse has gained considerable attention in current research as a fundamental generalized inverse. First presented in [10] for general square matrices, it represents a special class of outer inverses. For $M \in \mathbb{C}^{n\times n}$ having index k, the core-EP inverse exists uniquely as $X = M^{\oplus} \in \mathbb{C}^{n\times n}$ fulfilling these key equalities:

$$XMX = X,\ R\left(M^k\right) = R(X) = R(X^*).$$

According to [25], the core-EP inverse admits the representation $M^{\oplus} = M^D M^k \left(M^k\right)^{\dagger}$. Notably, when $k = 1$, the core-EP inverse becomes the core inverse $M^{\textcircled{\#}} = M^\# M M^\dagger$.

In 2018, Wang et al. [25] generalized the group inverse concept by introducing the weak group inverse for square matrices of arbitrary index, utilizing core-EP inverse theory. For $M \in \mathbb{C}^{n\times n}$, the weak group inverse refers to a matrix $X \in \mathbb{C}^{n\times n}$ satisfying:

$$MX^2 = X,\ MX = M^{\oplus}M,$$

such a matrix X is termed the weak group inverse (WGI) of M, denoted by $M^{\overline{W}}$ and when it exists, this inverse is unique. Zhou et al. [31] later expanded this concept to ring theory. Current developments regarding the weak group inverse appear in [4, 11].

According to the findings in [8], the m-weak group inverse of matrix M can be characterized through the following system of matrix equations employing the core-EP inverse:

$$MX^2 = X,\ MX = \left(M^{\oplus}\right)^m M^m,$$

such a matrix X is termed the m-weak group inverse (m-WGI) of M, denoted by $M^{\overline{W}_m}$. Furthermore, the authors established an explicit representation of the m-weak group inverse using the core-EP inverse:

$$X = \left(M^{\oplus}\right)^{m+1} M^m.$$

In recent times, Ferreyra et al. [5] proposed a new generalization of the core inverse. They utilized the WGI to define the weak core inverse (WCI), which is represented by the following matrix:

$$M^{\overline{W},\dagger} = M^{\overline{W}} P_M.$$

Ferreyra et al. [6] proposed a new type of generalized matrix inverse applicable to matrices with any index. This new inverse is named the m-weak core inverse (m-WCI). It serves as a generalization of the core-EP inverse, the weak core inverse, and by extension, the core inverse. For any $m \in \mathbb{N}$, the m-weak core inverse of a matrix $M \in \mathbb{C}^{n\times n}$ is defined as

$$X = M^{\overline{W}_m} P_{M^m},$$

such a matrix X is termed the m-weak core inverse of M, denoted by $M^{\textcircled{\#}_m}$ and when it exists, this inverse is unique. In scenarios where $m \geq k$, the m-weak core inverse and the core-EP inverse are equal, satisfying $M^{\textcircled{\#}_m} = M^{\oplus}$. A key finding in the work reveals that this inverse can be explicitly constructed from the core-EP inverse via the relation

$$M^{\textcircled{\#}_m} = \left(M^{\oplus}\right)^{m+1} M^m P_{M^m}.$$

Let M be an $n \times n$ complex matrix with index k. If $b \in \mathcal{R}\left(M^k\right)$, the solution $x = M^D b$ is known to be the unique Drazin inverse solution to the constrained linear system [2]

$$Mx = b, \quad x \in \mathcal{R}\left(M^k\right).$$

Specifically, if $\operatorname{ind}(M) = 1$ and $b \in \mathcal{R}(M)$, then $x = M^{\#} b$ (equivalently, $M^{\textcircled{\#}} b$) is the unique solution to $Mx = b$.

In [26], Wang et al. investigated the constrained matrix approximation problem in the Frobenius norm by employing the core inverse:

$$\|Mx - b\|_F = \min \quad \text{subject to} \quad x \in \mathcal{R}(M), \tag{1}$$

and provided the unique solution

$$x = M^{\textcircled{\#}} b, \tag{2}$$

where $b \in \mathbb{C}^n$, $\operatorname{ind}(M) = k$ and $M \in \mathbb{C}^{n\times n}$, omitting the condition $b \in \mathcal{R}(M)$.

Ji, Mosić et al. [9, 12] examined the constrained matrix approximation problem:

$$\min \|Mx - b\|_F \quad \text{subject to} \quad x \in \mathcal{R}(M^k), \tag{3}$$

and provided the unique solution

$$x = M^{\oplus} b, \tag{4}$$

where $b \in \mathbb{C}^n$, $\operatorname{ind}(M) = k$ and $M \in \mathbb{C}^{n\times n}$. Note that this formulation differs from [2] by omitting the condition $b \in \mathcal{R}(M^k)$.

Wang et al. [27] provided the solution to the matrix minimization problem under the Frobenius norm for the WGI:

$$\min \|M^2 X - MB\|_F \quad \text{subject to} \quad \mathcal{R}(X) \subseteq \mathcal{R}(M^k), \tag{5}$$

then $X = M^{\overline{W}} B$ is the unique solution to equation (5), where $B \in \mathbb{C}^{n\times q}$, $\operatorname{ind}(M) = k$ and $M \in \mathbb{C}^{n\times n}$.

The uniquely specified solution of the Frobenius-norm constrained minimization is derived via m-WGI [13]:

$$\min \|M^{m+1} X - M^m B\|_F \quad \text{subject to} \quad \mathcal{R}(X) \subseteq \mathcal{R}(M^k), \tag{6}$$

then $X = M^{\overline{W}_m} B$ is the unique solution to equation (6), where $\operatorname{ind}(M) = k$, $m \in \mathbb{N}$, $B \in \mathbb{C}^{n\times q}$ and $M \in \mathbb{C}^{n\times n}$.

Taking inspiration from preceding studies on optimization problem tractability, we concentrate on the most inclusive minimization formulation. This investigation strives to both generalize and incorporate existing findings, particularly regarding solutions to the Frobenius-norm constrained optimization:

$$\min \|M^{m+1} X - M^{2m} (M^m)^{\dagger} B\|_F \quad \text{subject to} \quad \mathcal{R}(X) \subseteq \mathcal{R}(M^k), \tag{7}$$

where $\operatorname{ind}(M) = k$, $m \in \mathbb{N}$, $B \in \mathbb{C}^{n\times q}$ and $M \in \mathbb{C}^{n\times n}$. Here is a detailed explanation of our results.

(1) Novel expressions and descriptions of the m-weak core inverse are developed using Moore-Penrose inverses and specific projection operators;

(2) Perturbation and continuity for the m-weak core inverse are investigated;

(3) We establish that the solution to problem (7) exists and is unique, with this solution being given by the m-weak core inverse.

The remaining sections are organized as follows. In Section 2, necessary definitions and lemmas are presented. In Section 3, various expressions for calculating the m-weak core inverse are given. Section 4 contains perturbation representations and continuity results for the m-weak core inverse. Section 5 focuses on the solvability of the minimization problem (7) and its specific cases. Numerical examples are given in Section 6. Section 7 provides some concluding remarks.

2. Preliminaries

Lemma 2.1. *[23] Let $M \in \mathbb{C}^{n\times n}$ and there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$. If* $\operatorname{rank}\left(M^k\right) = t$*, then*

$$M = U \begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix} U^*, \tag{8}$$

where $M_1 \in \mathbb{C}^{t\times t}$ is nonsingular and upper-triangular and $M_3 \in \mathbb{C}^{(n-t)\times(n-t)}$ is nilpotent of index k. Further, it follows [8], for any $m \in \mathbb{N}$:

$$M^{\oplus} = U \begin{bmatrix} M_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{9}$$

$$M^{\overline{W}_m} = U \begin{bmatrix} M_1^{-1} & M_1^{-(m+1)} \sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i} \\ 0 & 0 \end{bmatrix} U^* \tag{10}$$

and

$$M^{\textcircled{\#}_m} = U \begin{bmatrix} M_1^{-1} & M_1^{-(m+1)} \sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i} P_{M_3^m} \\ 0 & 0 \end{bmatrix} U^* \tag{11}$$

$$= U \begin{bmatrix} M_1^{-1} & M_1^{-(m+1)} \sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i} M_3^m \left(M_3^m\right)^{\dagger} \\ 0 & 0 \end{bmatrix} U^*. \tag{12}$$

The expression below represents an orthogonal projector:

$$P_{M^l} = M^l \left(M^l\right)^{\dagger} = U \begin{bmatrix} I_t & 0 \\ 0 & P_{M_3^l} \end{bmatrix} U^*, \quad l \in \mathbb{N}. \tag{13}$$

Lemma 2.2. *[14] Let $M, G, H, U \in \mathbb{C}^{n\times n}$ and $Z = \begin{bmatrix} M & MG \\ HM & U \end{bmatrix} \in \mathbb{C}^{2n\times 2n}$, the following rank equality is valid:*

$$\operatorname{rank}(M) + \operatorname{rank}(U - HMG) = \operatorname{rank}(Z). \tag{14}$$

Lemma 2.3. *[12] Let $M \in \mathbb{C}^{n\times n}$ and let a matrix Y satisfy:*

$$\mathcal{N}(Y^*) = \mathcal{R}\left(M^k\right) \quad \left(\text{or equivalently} \quad \mathcal{R}(Y) = \mathcal{N}\left(\left(M^k\right)^*\right)\right). \tag{15}$$

Then $M^{k+1} + YY^*$ is nonsingular and

$$M^{\oplus} = M^k (M^{k+1} + YY^*)^{-1}. \tag{16}$$

Lemma 2.4. *[7] Let $M \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(M) = k$ and $m \in \mathbb{N}$. Then the following statements hold:*

$$M^{\textcircled{\#}_m} = M^{(2)}_{R(M^k), N\left((M^k)^* M^m P_{M^m}\right)} = M^{(2)}_{R(M^k), N\left((M^k)^* M^{2m} (M^m)^{\dagger}\right)}.$$

3. Representations and characterizations of the m-weak core inverse

This section provides various representations of m-WCI.
If matrix M is invertible, $U = M^{-1}$ is the unique matrix such that

$$\operatorname{rank}\left(\begin{bmatrix} M & I \\ I & U \end{bmatrix}\right) = \operatorname{rank}(M).$$

It is interesting to consider analogous result for the m-weak core inverse.

Theorem 3.1. *Let $M \in \mathbb{C}^{n\times n}$ be in the form (8) with* $\mathrm{ind}(M) = k$ *and* $t = \mathrm{rank}\left(M^k\right)$*. Then there exist the unique matrix P satisfying*

$$P^2 = P,\ \ \left(M^k\right)^* M^{2m} \left(M^m\right)^\dagger P = 0,\ \ PM^k = 0,\ \ \mathrm{rank}\,(P) = n - t; \tag{17}$$

the unique matrix Q satisfying

$$Q^2 = Q,\ \ \left(M^k\right)^* M^{2m} \left(M^m\right)^\dagger Q = 0,\ \ QM^k = 0,\ \ \mathrm{rank}\,(Q) = n - t; \tag{18}$$

the unique matrix R satisfying

$$\mathrm{rank}\,(M) = \mathrm{rank}\left(\begin{bmatrix} M & I-P \\ I-Q & R \end{bmatrix}\right), \tag{19}$$

the matrix R is the m-weak core inverse $M^{\textcircled{\#}_m}$ of M. Furthermore, we have

$$P = I - MM^{\textcircled{\#}_m},\ \ Q = I - M^{\textcircled{\#}_m}M.$$

Proof. Assume that matrix M is decomposed as shown in equation (8). By using (12), we get that

$$P = I - MM^{\textcircled{\#}_m} = U\begin{bmatrix} 0 & -M_1^{-m}\sum_{j=0}^{m-1} M_1^j M_2 M_3^{m-1-j} P_{M_3^m} \\ 0 & I \end{bmatrix} U^*,$$

which implies (17) for $P = I - MM^{\textcircled{\#}_m}$.
To demonstrate that (17) has a unique solution, suppose (17) for the matrices P and $P_1 = U\begin{bmatrix} C & D \\ E & F\end{bmatrix}U^*$, where $C \in \mathbb{C}^{t\times t}$. From $P_1M^k = 0$ and

$$M^k = U\begin{bmatrix} M_1^k & \sum_{i=0}^{k-1} M_1^{k-1-i} M_2 M_3^i \\ 0 & 0 \end{bmatrix} U^*,$$

we have $C = 0$ and $E = 0$. Now, $P_1^2 = P_1$ and $\mathrm{rank}\,(P_1) = n - t$ imply $D = DF$, $F = F^2$, F is invertible and $F = I$. From

$$\begin{aligned} 0 &= \left(M^k\right)^* M^{2m} \left(M^m\right)^\dagger P_1 \\ &= U\begin{bmatrix} \left(M_1^k\right)^* & 0 \\ \left(\sum_{i=0}^{k-1} M_1^{k-1-i}M_2M_3^i\right)^* & 0 \end{bmatrix}\begin{bmatrix} M_1^m & \sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i} \\ 0 & M_3^m \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & P_{M_3^m}\end{bmatrix}\begin{bmatrix} 0 & D \\ 0 & I\end{bmatrix}U^* \\ &= U\begin{bmatrix} \left(M_1^k\right)^* M_1^m & \left(M_1^k\right)^* \sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i} P_{M_3^m} \\ \left(\sum_{i=0}^{k-1} M_1^{k-1-i}M_2M_3^i\right)^* M_1^m & \left(\sum_{i=0}^{k-1} M_1^{k-1-i}M_2M_3^i\right)^* \sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i} P_{M_3^m} \end{bmatrix}\begin{bmatrix} 0 & D \\ 0 & I\end{bmatrix}U^* \\ &= U\begin{bmatrix} 0 & \left(M_1^k\right)^*\left(M_1^m D + \sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i} P_{M_3^m}\right) \\ 0 & \left(\sum_{i=0}^{k-1} M_1^{k-1-i}M_2M_3^i\right)^*\left(M_1^m D + \sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i} P_{M_3^m}\right) \end{bmatrix}U^*, \end{aligned}$$

we deduce that $M_1^m D + \sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i} P_{M_3^m} = 0$, that is, $D = -M_1^{-m}\sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i} P_{M_3^m}$. So, $P_1 = P$.
Similarly, we demonstrate that it is satisfied for a unique $Q = I - M^{\textcircled{\#}_m}M$.
Under the settings $P = I - MM^{\textcircled{\#}_m}$ and $Q = I - M^{\textcircled{\#}_m}M$, it can be verified

$$\begin{bmatrix} M & MM^{\textcircled{\#}_m} \\ M^{\textcircled{\#}_m}M & R \end{bmatrix} = \begin{bmatrix} M & I-P \\ I-Q & R \end{bmatrix}.$$

By (19) and Lemma 2.2, notice that $R = M^{\textcircled{\#}_m}MM^{\textcircled{\#}_m} = M^{\textcircled{\#}_m}$. □

The study also examines the correlation between an invertible bordered matrix and m-WCI.

Theorem 3.2. *Let $M \in \mathbb{C}^{n\times n}$ with $\text{ind}(M) = k$. Assume that two full column rank matrices G and H^* which satisfy $\mathcal{N}\left(\left(M^k\right)^* M^{2m}(M^m)^\dagger\right) = \mathcal{R}(G)$ and $\mathcal{R}\left(M^k\right) = \mathcal{N}(H)$. Then*

$$X = \begin{bmatrix} M & G \\ H & 0 \end{bmatrix}$$

is invertible and

$$X^{-1} = \begin{bmatrix} M^{\text{\textcircled{\#}}_m} & \left(I - M^{\text{\textcircled{\#}}_m}M\right)H^\dagger \\ G^\dagger\left(I - MM^{\text{\textcircled{\#}}_m}\right) & -G^\dagger\left(M - MM^{\text{\textcircled{\#}}_m}M\right)H^\dagger \end{bmatrix}. \tag{20}$$

Proof. By using Lemma 2.4, we have $M^{\text{\textcircled{\#}}_m} = M^{(2)}_{\mathcal{R}(M^k),\mathcal{N}\left(\left(M^k\right)^* M^{2m}(M^m)^\dagger\right)}$. For

$$\mathcal{R}\left(I - MM^{\text{\textcircled{\#}}_m}\right) = \mathcal{N}\left(M^{\text{\textcircled{\#}}_m}\right) = \mathcal{N}\left(\left(M^k\right)^* M^{2m}(M^m)^\dagger\right) = \mathcal{R}(G) = \mathcal{R}\left(GG^\dagger\right) = \mathcal{N}\left(I - GG^\dagger\right).$$

Consequently, $\left(I - GG^\dagger\right)\left(I - MM^{\text{\textcircled{\#}}_m}\right) = 0$, we get that

$$GG^\dagger\left(I - MM^{\text{\textcircled{\#}}_m}\right) = \left(I - MM^{\text{\textcircled{\#}}_m}\right).$$

Further, we know that $HM^{\text{\textcircled{\#}}_m} = 0$ by using $\mathcal{R}\left(M^{\text{\textcircled{\#}}_m}\right) = \mathcal{R}\left(M^k\right) = \mathcal{N}(H)$. Let Y represents the right-hand side of equation (20), we get that

$$\begin{aligned} XY &= \begin{bmatrix} MM^{\text{\textcircled{\#}}_m} + GG^\dagger\left(I - MM^{\text{\textcircled{\#}}_m}\right) & M\left(I - M^{\text{\textcircled{\#}}_m}M\right)H^\dagger - GG^\dagger\left(I - MM^{\text{\textcircled{\#}}_m}\right)MH^\dagger \\ HM^{\text{\textcircled{\#}}_m} & H\left(I - M^{\text{\textcircled{\#}}_m}M\right)H^\dagger \end{bmatrix} \\ &= \begin{bmatrix} MM^{\text{\textcircled{\#}}_m} + I - MM^{\text{\textcircled{\#}}_m} & \left(I - MM^{\text{\textcircled{\#}}_m}\right)MH^\dagger - \left(I - MM^{\text{\textcircled{\#}}_m}\right)MH^\dagger \\ 0 & HH^\dagger \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I. \end{aligned}$$

Therefore, matrix X is invertible, and its inverse matrix equals $X^{-1} = Y$. □

By making use of the expression for the m-WGI that was put forward in [13], we derived a new expression for the m-WCI.

Theorem 3.3. *Let $M \in \mathbb{C}^{n\times n}$, $m \in \mathbb{N}$, $k = ind(M)$ and a matrix Y satisfying (15), the matrix $M^k(M^k)^*M + YY^*$ is invertible and*

$$\begin{aligned} M^{\text{\textcircled{\#}}_m} &= (KM + YY^*)^{-1} K\left(M^D\right)^m M^k\left(M^k\right)^\dagger M^{2m}(M^m)^\dagger & (21) \\ &= (KM + YY^*)^{-1} KM^k\left(M^{k+m}\right)^\dagger M^{2m}(M^m)^\dagger, & (22) \end{aligned}$$

where $K = M^k(M^k)^*$.

Proof. By [13], we know that

$$\begin{aligned} M^{\overline{W}_m} &= (KM + YY^*)^{-1} K\left(M^D\right)^m M^k\left(M^k\right)^\dagger M^m \\ &= (KM + YY^*)^{-1} KM^k\left(M^{k+m}\right)^\dagger M^m, \end{aligned}$$

where $K = M^k(M^k)^*$. And $M^{\text{\textcircled{\#}}_m} = M^{\overline{W}_m}M^m(M^m)^\dagger$, therefore, it is proven. □

Theorem 3.4. *Let $M \in \mathbb{C}^{n\times n}$, $m \in \mathbb{N}$, $k = ind(M)$ and a matrix Y satisfying (15), the matrix $M^{k+1} + YY^*$ is invertible and*

$$M^{\text{\textcircled{\#}}_m} = \left(M^k + \left(M^D\right)^{m+1} M^k \left(M^k\right)^{\dagger} M^{2m} \left(M^m\right)^{\dagger} YY^*\right) H \tag{23}$$

$$= \left(M^k + M^k \left(M^{k+m+1}\right)^{\dagger} M^{2m} \left(M^m\right)^{\dagger} YY^*\right) H, \tag{24}$$

where $H = \left(M^{k+1} + YY^*\right)^{-1}$.

Proof. According to Lemma 2.3, note that $M^{k+1} + YY^*$ is invertible. Therefore,

$$\begin{aligned} M^{\text{\textcircled{\#}}_m}\left(M^{k+1} + YY^*\right) &= \left(M^{\oplus}\right)^{m+1} M^{2m} \left(M^m\right)^{\dagger} \left(M^{k+1} + YY^*\right) \\ &= \left(M^D\right)^{m+1} M^k \left(M^k\right)^{\dagger} M^{2m} \left(M^m\right)^{\dagger} \left(M^{k+1} + YY^*\right) \\ &= \left(M^D\right)^{m+1} M^{m+k+1} + \left(M^D\right)^{m+1} M^k \left(M^k\right)^{\dagger} M^{2m} \left(M^m\right)^{\dagger} YY^* \\ &= M^k + \left(M^D\right)^{m+1} M^k \left(M^k\right)^{\dagger} M^{2m} \left(M^m\right)^{\dagger} YY^*. \end{aligned}$$

□

When $m = 1$ in Theorems 3.3 and 3.4, we obtain novel representations for the WCI.

Corollary 3.5. *Let $M \in \mathbb{C}^{n\times n}$, $k = ind(M)$, $m = 1$ and a matrix Y satisfying (15), the matrix $M^k(M^k)^* M + YY^*$ is invertible and*

$$M^{\overline{W},\dagger} = \left(KM + YY^*\right)^{-1} KM^D M^k \left(M^k\right)^{\dagger} M^2 M^{\dagger} \tag{25}$$

$$= \left(KM + YY^*\right)^{-1} KM^k \left(M^{k+1}\right)^{\dagger} M^2 M^{\dagger}, \tag{26}$$

where $K = M^k(M^k)^*$.

Corollary 3.6. *Let $M \in \mathbb{C}^{n\times n}$, $k = ind(M)$, $m = 1$ and a matrix Y satisfying (15), the matrix $M^{k+1} + YY^*$ is invertible and*

$$M^{\overline{W},\dagger} = \left(M^k + \left(M^D\right)^2 M^k \left(M^k\right)^{\dagger} M^2 M^{\dagger} YY^*\right) H \tag{27}$$

$$= \left(M^k + M^k \left(M^{k+2}\right)^{\dagger} M^2 M^{\dagger} YY^*\right) H, \tag{28}$$

where $H = \left(M^{k+1} + YY^*\right)^{-1}$.

Theorem 3.7 provides characterizations of the m-WCI using the full-rank decomposition of the matrix power M^k.

Theorem 3.7. *Let $M \in \mathbb{C}^{n\times n}$, $m \in \mathbb{N}$ and $k = ind(M)$. Assume that $M^k = QP$ is a full-rank decomposition, then $Q^* M^{m+1} Q$ is invertible and*

$$M^{\text{\textcircled{\#}}_m} = Q\left(Q^* M^{m+1} Q\right)^{-1} Q^* M^{2m} \left(M^m\right)^{\dagger} \tag{29}$$

$$= M^{(2)}_{\mathcal{R}(Q),\mathcal{N}\left(Q^* M^{2m} (M^m)^{\dagger}\right)}. \tag{30}$$

Proof. It follows by [13] that $Q^*M^{m+1}Q$ is invertible and

$$M^{\overline{W}_m} = Q\left(Q^*M^{m+1}Q\right)^{-1} Q^*M^m.$$

Thus

$$M^{\textcircled{\#}_m} = M^{\overline{W}_m}M^m\left(M^m\right)^\dagger = Q\left(Q^*M^{m+1}Q\right)^{-1} Q^*M^{2m}\left(M^m\right)^\dagger.$$

Finally, the derivation of (30) can be obtained by applying the full-rank characterization of outer inverses $M^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$, which specifies both range and null space constraints. This representation, originally established by Urquhart [22], takes the form $B(CMB)^\dagger C$. Further generalizations of this result have been subsequently developed in [20], providing extended formulations of the original expression. □

In the special case of $m = 1$, Theorem 3.7 yields the corresponding weak core inverse formulas.

Corollary 3.8. *Let $M \in \mathbb{C}^{n\times n}$, $k = ind(M)$, $m = 1$ and assume that $M^k = QP$ is a full-rank decomposition, then*

$$\begin{aligned} M^{\overline{W},\dagger} &= Q\left(Q^*M^2Q\right)^{-1} Q^*M^2M^\dagger \\ &= M^{(2)}_{\mathcal{R}(Q),\mathcal{N}(Q^*M^2M^\dagger)}. \end{aligned}$$

Since the expression for $M^{\textcircled{\#}_m}$ given in Theorem 3.7 depends only on Q from the full-rank decomposition $M^k = QP$, we present an alternative characterization of $M^{\textcircled{\#}_m}$.

Theorem 3.9. *Let $M \in \mathbb{C}^{n\times n}$, $k = ind(M)$, $m \in \mathbb{N}$ and $S \in \mathbb{C}^{n\times n}$ satisfying $\mathcal{R}(S) = \mathcal{N}\left(M^k\right)$, then*

$$M^{\textcircled{\#}_m} = S\left(S^*M^{m+1}S\right)^{-1} S^*M^{2m}\left(M^m\right)^\dagger \tag{31}$$

$$= M^{(2)}_{\mathcal{R}(S),\mathcal{N}\left(S^*M^{2m}(M^m)^\dagger\right)}. \tag{32}$$

Proof. It follows by [13], we have:

$$M^{\overline{W}_m} = S\left(S^*M^{m+1}S\right)^{-1} S^*M^m.$$

So, we get that

$$M^{\textcircled{\#}_m} = S\left(S^*M^{m+1}S\right)^{-1} S^*M^{2m}\left(M^m\right)^\dagger.$$

□

4. Perturbations and continuity of m-weak core inverse

In [14–17], the perturbation formula and its perturbation boundary of various generalized inverses of matrices have been studied under certain conditions recently. This section will discuss the perturbation formula of m-weak core inverse. Let's start with a lemma.

Lemma 4.1. *Let $M \in \mathbb{C}^{n\times n}$, $m \in \mathbb{N}$, $\mathrm{ind}(M) = k$ and $B = M + E \in \mathbb{C}^{n\times n}$. Assume that $E = MM^{\overline{W}_m}E$ and $\|M^{\overline{W}_m}E\| < 1$, then*

$$B^{\overline{W}_m} = \left(\left(I + M^{\overline{W}_m}E\right)^{-1} M^{\overline{W}_m}\right)^{m+1} MM^{\oplus}\left(M + E\right)^m. \tag{33}$$

Proof. Let M be defined as in (8) and

$$E = U\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} U^*,$$

where $E_1 \in \mathbb{C}^{r\times r}$. Applying (10), we have

$$MM^{\overline{W}_m}E = U\begin{bmatrix} E_1 + M_1^{-m}\sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i}E_3 & E_2 + M_1^{-m}\sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i}E_4 \\ 0 & 0 \end{bmatrix} U^*.$$

The equality $E = MM^{\overline{W}_m}E$ yields $E_3 = 0$ and $E_4 = 0$. Since $\| M^{\overline{W}_m}E \| < 1$, then

$$I + M^{\overline{W}_m}E = U\begin{bmatrix} I + M_1^{-1}E_1 & M_1^{-1}E_2 \\ 0 & I \end{bmatrix} U^*$$

is nonsingular. Therefore, $I + M_1^{-1}E_1 = M_1^{-1}(M_1 + E_1)$ is nonsingular implies that $M_1 + E_1$ is nonsingular. Because

$$B = M + E = U\begin{bmatrix} M_1 + E_1 & M_2 + E_2 \\ 0 & M_3 \end{bmatrix} U^*,$$

we have

$$B^{\oplus} = U\begin{bmatrix} (M_1 + E_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*$$

and

$$B^{\overline{W}_m} = U\begin{bmatrix} (M_1 + E_1)^{-1} & (M_1 + E_1)^{-(m+1)}\sum_{i=0}^{m-1}(M_1 + E_1)^i (M_2 + E_2) M_3^{m-1-i} \\ 0 & 0 \end{bmatrix} U^*.$$

Therefore, by (9),

$$\begin{aligned}
&\left(\left(I + M^{\overline{W}_m}E\right)^{-1} M^{\overline{W}_m}\right)^{m+1} MM^{\oplus}(A + E)^m \\
= \; & U\begin{bmatrix} (M_1 + E_1)^{-1} & (M_1 + E_1)^{-1} M_1^{-m}\sum_{i=0}^{m-1} M_1^i M_2 M_3^{m-1-i} \\ 0 & 0 \end{bmatrix}^{m+1} U^* \\
& \times U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (M_1 + E_1)^m & \sum_{i=0}^{m-1}(M_1 + E_1)^i (M_2 + E_2) M_3^{m-1-i} \\ 0 & 0 \end{bmatrix} U^* \\
= \; & U\begin{bmatrix} (M_1 + E_1)^{-1} & (M_1 + E_1)^{-(m+1)}\sum_{i=0}^{m-1}(M_1 + E_1)^i (M_2 + E_2) M_3^{m-1-i} \\ 0 & 0 \end{bmatrix} U^* = B^{\overline{W}_m}.
\end{aligned}$$

□

Theorem 4.2. *Let $M \in \mathbb{C}^{n\times n}$, $m \in \mathbb{N}$, $\operatorname{ind}(M) = k$ and $B = M + E \in \mathbb{C}^{n\times n}$. Assume that $E = MM^{\overline{W}_m}E$ and $\|M^{\overline{W}_m}E\| < 1$, then*

$$B^{\textcircled{\#}_m} = \left(\left(I + M^{\overline{W}_m}E\right)^{-1} M^{\overline{W}_m}\right)^{m+1} MM^{\oplus}(M + E)^m P_{(M+E)^m} \tag{34}$$

$$= \left(\left(I + M^{\overline{W}_m}E\right)^{-1} M^{\overline{W}_m}\right)^{m+1} MM^{\oplus}(M + E)^{2m}\left((M + E)^m\right)^{\dagger}. \tag{35}$$

Proof. The proof follows from Lemma 4.1 and $M^{\textcircled{\#}_m} = M^{\overline{W}_m}P_{(M)^m} = M^{\overline{W}_m}M^m(M^m)^{\dagger}$. □

Lemma 4.3. *Let $M \in \mathbb{C}^{n\times n}$, $m \in \mathbb{N}$, $\operatorname{ind}(M) = k$ and $B = M + E \in \mathbb{C}^{n\times n}$. Assume that $E = M^{\overline{W}_m}ME$ and $\|M^{\overline{W}_m}E\| < 1$, then*

$$B^{\overline{W}_m} = \left(\left(I + M^{\overline{W}_m}E\right)^{-1} M^{\overline{W}_m}\right)^{m+1} MM^{\oplus}(M + E)^m. \tag{36}$$

Theorem 4.4. *Let $M \in \mathbb{C}^{n\times n}$, $m \in \mathbb{N}$, $\operatorname{ind}(M) = k$ and $B = M + E \in \mathbb{C}^{n\times n}$. Assume that $E = M^{\overline{W}_m}ME$ and $\|M^{\overline{W}_m}E\| < 1$, then*

$$\begin{aligned} B^{\circledast_m} &= \left(\left(I + M^{\overline{W}_m}E\right)^{-1} M^{\overline{W}_m}\right)^{m+1} MM^{\oplus}(M+E)^m P_{(M+E)^m} \\ &= \left(\left(I + M^{\overline{W}_m}E\right)^{-1} M^{\overline{W}_m}\right)^{m+1} MM^{\oplus}(M+E)^{2m}\left((M+E)^m\right)^{\dagger}. \end{aligned} \tag{37}$$

According to the relationship between the m-weak core inverse and the weak group inverse, we can obtain the following theorem.

Theorem 4.5. *Let $M \in \mathbb{C}^{n\times n}$, $m \in \mathbb{N}$, $\operatorname{ind}(M) = k$ and $B = M + E \in \mathbb{C}^{n\times n}$. Assume that $E = M^{\overline{W}}ME$ and $\|M^{\overline{W}}E\| < 1$, then*

$$\begin{aligned} B^{\circledast_m} &= \left(\left(\left(I + M^{\overline{W}}E\right)^{-1} M^{\overline{W}}\right)^{2} MM^{\oplus}(M+E)\right)^{m} (M+E)^{m-1} P_{(M+E)^m} \\ &= \left(\left(I + M^{\overline{W}}E\right)^{-1} M^{\overline{W}}\left(I + M^{\overline{W}}E\right)^{-1} M^{\oplus}(M+E)\right)^{m} (M+E)^{m-1} P_{(M+E)^m}. \end{aligned} \tag{38}$$

Proof. From in [18, Theorem 4.1], we know

$$B^{\overline{W}} = \left(\left(I + M^{\overline{W}}E\right)^{-1} M^{\overline{W}}\right)^{2} MM^{\oplus}(M+E) = \left(I + M^{\overline{W}}E\right)^{-1} M^{\overline{W}}\left(I + M^{\overline{W}}E\right)^{-1} M^{\oplus}(M+E).$$

Also, according to [6, Theorem 4.5],

$$M^{\circledast_m} = \left(M^{\overline{W}}\right)^{m} M^{m-1}P_{(M^m)}.$$

Thus, we get that

$$\begin{aligned} B^{\circledast_m} &= \left(\left(\left(I + M^{\overline{W}}E\right)^{-1} M^{\overline{W}}\right)^{2} MM^{\oplus}(M+E)\right)^{m} (M+E)^{m-1} P_{(M+E)^m} \\ &= \left(\left(I + M^{\overline{W}}E\right)^{-1} M^{\overline{W}}\left(I + M^{\overline{W}}E\right)^{-1} M^{\oplus}(M+E)\right)^{m} (M+E)^{m-1} P_{(M+E)^m}. \end{aligned}$$

□

The continuity of the Moore-Penrose inverse was examined in [21], while [19] addressed this property for the m-WGI. In the current section, we focus on investigating continuity characteristics of the m-WCI. An auxiliary result concerning the m-WGI is provided in Lemma 4.6.

Lemma 4.6. *Let $M \in \mathbb{C}^{n\times n}$, $m \in \mathbb{N}$ and $M_y \in \mathbb{C}^{n\times n}$, $y \in \mathbb{N}$ be a sequence which satisfying $M_y \to M$ as $y \to \infty$. Then*

(1) *$M_y^{\dagger} \to M^{\dagger}$ as $y \to \infty$ iff there is $y_0 \in \mathbb{N}$ satisfying $\operatorname{rank}(M_y) = \operatorname{rank}(M)$, for $y \geq y_0$;*

(2) *$M_y^{\overline{W}_m} \to M^{\overline{W}_m}$ as $y \to \infty$ iff there is $y_0 \in \mathbb{N}$ satisfying $\operatorname{rank}(M_y^l) = \operatorname{rank}(M^l)$, for $y \geq y_0$ and $l = max\{\operatorname{ind}(M), \operatorname{ind}(M_y), \operatorname{ind}(M_{y+1}), ...\}$.*

Theorem 4.7. *Let $M \in \mathbb{C}^{n\times n}$, $m \in \mathbb{N}$ and $M_y \in \mathbb{C}^{n\times n}$, $y \in \mathbb{N}$ be a sequence which satisfying $M_y \to M$ as $y \to \infty$. Then ${M_y}^{\circledast_m} \to M^{\circledast_m}$ as $y \to \infty$ iff there is $y_0 \in \mathbb{N}$ satisfying $\operatorname{rank}(M_y^l) = \operatorname{rank}(M^l)$ and $\operatorname{rank}(M_y^m) = \operatorname{rank}(M^m)$, for $y \geq y_0$ and $l = max\{\operatorname{ind}(M), \operatorname{ind}(M_y), \operatorname{ind}(M_{y+1}), ...\}$.*

Proof. By using Lemma 4.6, if $y_0 \in \mathbb{N}$ and $\text{rank}(M_y^l) = \text{rank}(M^l)$ and $\text{rank}(M_y^m) = \text{rank}(M^m)$, for $y \geq y_0$ and $l = max\{\text{ind}(M), \text{ind}(M_y), \text{ind}(M_{y+1}), ...\}$, we conclude that $M_y^{\overline{W}_m} \to M^{\overline{W}_m}$ and $(M_y^m)^\dagger \to (M^m)^\dagger$. Hence,

$$M_y{}^{\oplus_m} = M_y^{\overline{W}_m} M_y^m (M_y^m)^\dagger \to M^{\overline{W}_m} M^m (M^m)^\dagger = M^{\oplus_m}.$$

On the other hand, let $M_y{}^{\oplus_m} \to M^{\oplus_m}$. Because $M_y \to M$, we have $M_y{}^{\oplus_m} M_y \to M^{\oplus_m} M$ as $y \to \infty$, which is equivalent to $M_y^{\overline{W}_m} M_y \to M^{\overline{W}_m} M$ as $y \to \infty$. To verify that $(M_y^m)^\dagger \to (M^m)^\dagger$ as $y \to \infty$, assume that $(M_y^m)^\dagger \nrightarrow (M^m)^\dagger$ as $y \to \infty$. Therefore,

$$M_y{}^{\oplus_m} = M_y^{\overline{W}_m} M_y^m (M_y^m)^\dagger \nrightarrow M^{\overline{W}_m} M^m (M^m)^\dagger = M^{\oplus_m},$$

which is a contradiction with $M_y{}^{\oplus_m} \to M^{\oplus_m}$. Thus, $(M_y^m)^\dagger \to (M^m)^\dagger$ as $y \to \infty$, which gives, for a sufficiently large y, $\text{rank}(M_y^m) = \text{rank}(M^m)$ by Lemma 4.6.
Because $M_y{}^{\oplus_m} M_y$ and $M^{\oplus_m} M$ are projectors, according to [30], there exists $y_0 \in \mathbb{N}$ such that $\text{rank}(M_y{}^{\oplus_m} M_y) = \text{rank}(M^{\oplus_m} M)$, for $y \geq y_0$. Set $l = max\{\text{ind}(M), \text{ind}(M_y), \text{ind}(M_{y+1}), ...\}$, we have $\text{rank}(M_y^l) = \text{rank}(M_y^{ind(M_y)}) = \text{rank}(M_y{}^{\oplus_m}) = \text{rank}(M_y{}^{\oplus_m} M_y)$ and $\text{rank}(M^l) = \text{rank}(M^{ind(M)}) = \text{rank}(M^{\oplus_m}) = \text{rank}(M^{\oplus_m} M)$. So

$$\text{rank}(M_y^l) = \text{rank}(M_y{}^{\oplus_m} M_y) = \text{rank}(M^{\oplus_m} M) = \text{rank}(M^l),$$

for $y \geq y_0$. □

5. Solvability of (7) based on the m-weak core inverse

Theorem 5.1. *For any $m \in \mathbb{N}$, the optimization problem (7) possesses a unique solution in the following form*

$$X = M^{\oplus_m} B. \tag{39}$$

Proof. (*a*) Let $m < \text{ind}(M)$. The subspace constraint $\mathcal{R}(X) \subseteq \mathcal{R}(M^k)$ implies the matrix factorization $X = M^k Y$ for some $Y \in \mathbb{C}^{n \times q}$. Assume that M is given by (8) as well as

$$B = U \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad Y = U \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \quad B_1, Y_1 \in \mathbb{C}^{t \times m} .$$

Remark that the solution condition for X in (7) is equivalent to Y is a solution to

$$\min \left\| M^{m+k+1} Y - M^{2m} (M^m)^\dagger B \right\|_F^2 .$$

Using $M^k = U \begin{bmatrix} M_1^k & \sum_{i=0}^{k-1} M_1^{k-1-i} M_2 M_3^i \\ 0 & 0 \end{bmatrix} U^*$ and $M^m = U \begin{bmatrix} M_1^m & \sum_{j=0}^{m-1} M_1^j M_2 M_3^{m-1-j} \\ 0 & M_3^m \end{bmatrix} U^*$. So we can get that

$$\begin{aligned} M^{m+k+1} Y &= U \begin{bmatrix} M_1^{m+1} & \sum_{j=0}^{m-1} M_1^{j+1} M_2 M_3^{m-1-j} + M_2 M_3^m \\ 0 & M_3^{m+1} \end{bmatrix} U^* M^k Y \\ &= U \begin{bmatrix} M_1^{m+k+1} & \sum_{i=0}^{m-1} M_1^{m+k-i} M_2 M_3^i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \\ &= U \begin{bmatrix} M_1^{m+k+1} Y_1 + \sum_{i=0}^{m-1} M_1^{m+k-i} M_2 M_3^i Y_2 \\ 0 \end{bmatrix}. \end{aligned}$$

So,

$$\begin{aligned} M^{2m}(M^m)^\dagger B &= M^m M^m (M^m)^\dagger B = U\begin{bmatrix} M_1^m & \sum_{j=0}^{m-1} M_1^j M_2 M_3^{m-1-j} \\ 0 & M_3^m \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & M_3^m\left(M_3^m\right)^\dagger \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ &= U\begin{bmatrix} M_1^m B_1 + \sum_{j=0}^{m-1} M_1^j M_2 M_3^{m-1-j} M_3^m \left(M_3^m\right)^\dagger B_2 \\ M_3^{2m}\left(M_3^m\right)^\dagger B_2 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} &\left\|M^{m+k+1}Y - M^{2m}(M^m)^\dagger B\right\|_F^2 \\ =\ &\left\|\begin{bmatrix} M_1^{m+k+1}Y_1 + \sum_{i=0}^{m-1} M_1^{m+k-i}M_2M_3^iY_2 - M_1^mB_1 - \sum_{j=1}^{m-1} M_1^jM_2M_3^{m-1-j}M_3^m\left(M_3^m\right)^\dagger B_2 \\ -M_3^{2m}\left(M_3^m\right)^\dagger B_2 \end{bmatrix}\right\|_F^2 \\ =\ &\left\|M_1^{m+k+1}Y_1 + \sum_{i=0}^{m-1} M_1^{m+k-i}M_2M_3^iY_2 - M_1^mB_1 - \sum_{j=1}^{m-1} M_1^jM_2M_3^{m-1-j}M_3^m\left(M_3^m\right)^\dagger B_2\right\|_F^2 + \left\|M_3^{2m}\left(M_3^m\right)^\dagger B_2\right\|_F^2, \end{aligned}$$

which implies

$$\min_Y \left\|M^{m+k+1}Y - M^{2m}(M^m)^\dagger B\right\|_F^2 = \left\|M_3^{2m}\left(M_3^m\right)^\dagger B_2\right\|_F^2,$$

for arbitrary $Y_2 \in \mathbb{C}^{(n-t)\times q}$ and

$$Y_1 = -M_1^{-(m+k+1)}\left(\sum_{i=0}^{m-1} M_1^{m+k-i}M_2M_3^iY_2 - M_1^mB_1 - \sum_{j=1}^{m-1} M_1^jM_2M_3^{m-1-j}M_3^m\left(M_3^m\right)^\dagger B_2\right).$$

From formula (12), it can be derived

$$M^{\textcircled{\#}_m}B = U\begin{bmatrix} M_1^{-1}B_1 + M_1^{-(m+1)}\sum_{j=0}^{m-1} M_1^jM_2M_3^{m-1-j}M_3^m\left(M_3^m\right)^\dagger B_2 \\ 0 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} X = M^kY &= U\begin{bmatrix} M_1^kY_1 + \sum_{i=0}^{k-1} M_1^{k-1-i}M_2M_3^iY_2 \\ 0 \end{bmatrix} \\ &= U\begin{bmatrix} -M_1^{-(m+1)}\left(\sum_{i=0}^{m-1} M_1^{m+k-i}M_2M_3^iY_2 - M_1^mB_1 - \sum_{j=1}^{m-1} M_1^jM_2M_3^{m-1-j}M_3^m\left(M_3^m\right)^\dagger B_2\right) + \sum_{i=0}^{k-1} M_1^{k-1-i}M_2M_3^iY_2 \\ 0 \end{bmatrix} \\ &= U\begin{bmatrix} M_1^{-1}B_1 + M_1^{-(m+1)}\sum_{j=0}^{m-1} M_1^jM_2M_3^{m-1-j}M_3^m\left(M_3^m\right)^\dagger B_2 \\ 0 \end{bmatrix} = M^{\textcircled{\#}_m}B. \end{aligned}$$

Obtained the unique solution for problem (7).
(*b*) Let $m \geq \mathrm{ind}(M)$, the result follows directly from [12], where it was established that for $k = \mathrm{ind}(M)$, the core-EP inverse solution $M^{\textcircled{\dagger}}B$ uniquely solves the minimization problem

$$\min \|MX - B\|_F \quad \text{subject to} \quad \mathcal{R}(X) \subseteq \mathcal{R}(M^k).$$

It suffices to observe $X = M^kY$, $Y \in \mathbb{C}^{n\times n}$ and $B = M^kD$, $D \in \mathbb{C}^{n\times n}$. $\square$

When $B = I_n$ in Theorem 5.1, the m-WCI emerges as the unique solution to the minimization problem.

Corollary 5.2. *Let $M \in \mathbb{C}^{n\times n}$, $m \in \mathbb{N}$ and $k = \operatorname{ind}(M)$. Then the constrained minimization problem*

$$\min \left\| M^{m+1}X - M^{2m}\left(M^{m}\right)^{\dagger} \right\|_F, \quad \text{subject to} \quad \mathcal{R}(X) \subseteq \mathcal{R}\left(M^k\right)$$

possesses a unique solution in the following form

$$X = M^{\textcircled{\#}_m}.$$

The next proposition is a direct consequence of Theorem 5.1.

Corollary 5.3. *Let $M \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^n$, $m \in \mathbb{N}$ and $k = \operatorname{ind}(M)$. Then the constrained minimization problem*

$$\min \left\| M^{m+1}x - M^{2m}\left(M^{m}\right)^{\dagger} b \right\|_F, \quad \text{subject to} \quad x \in \mathcal{R}\left(M^k\right)$$

possesses a unique solution in the following form

$$x = M^{\textcircled{\#}_m} b.$$

6. Numerical examples

Example 6.1. *In this example, M is of relatively high indices and with rational entries, generated with the aim to perform numerical experiments applying exact calculation. Input matrix M is defined as*

$$M = \begin{bmatrix} 1&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&0&1\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

The powers of matrices M fulfill $\operatorname{rank}(M) = 7$, $\operatorname{rank}\left(M^2\right) = 6$, $\operatorname{rank}\left(M^3\right) = 5$, $\operatorname{rank}\left(M^4\right) = 4$, $\operatorname{rank}\left(M^5\right) = 4$, *therefore* $k = \operatorname{ind}(M) = 4$.

(*a*) *At the initial stage of this example, we calculate the core-EP inverse and m-weak core inverses according to their definitions. The core-EP inverse of M is*

$$M^{\oplus} = M^k\left(M^{k+1}\right)^{\dagger} = M^4\left(M^5\right)^{\dagger} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix},$$

The weak core inverse (or 1-weak core inverse) inverse of M is given by

$$M^{\overline{W},\dagger} = (M^{\oplus})^2 M^2 M^{\dagger} = \begin{bmatrix} 1&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix},$$

the 2-weak core inverse inverse of M is given by

$$M^{\textcircled{\#}_2} = (M^{\oplus})^3 M^4 (M^2)^{\dagger} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the 3-weak core inverse inverse of M is given by

$$M^{\textcircled{\#}_3} = (M^{\oplus})^3 M^4 (M^2)^{\dagger} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and its m-weak core inverse inverse, for $m \geq k = 4$, as

$$M^{\textcircled{\#}_m} = (M^{\oplus})^{m+1} M^m P_{M^m} = M^{\oplus}.$$

(*b*) *Consider the matrix*

$$B = \begin{bmatrix} 2 & 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

to empirically verify Theorem 5.1. The solution of the least squares minimization.
$\min\left\{\|M^2X - M^2M^{\dagger}B\|_F, \mathcal{R}(X) \subseteq \mathcal{R}(M^4)\right\}$ *is provided by*

$$M^{\overline{W},\dagger}B = \begin{bmatrix} 3 & 2 & 3 & 1 & 1 \\ 4 & 2 & 4 & 4 & 2 \\ 0 & 2 & 0 & 2 & 2 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the solution to $\min\left\{\|M^3X - M^4(M^2)^\dagger B\|_F, \mathcal{R}(X) \subseteq \mathcal{R}(M^4)\right\}$ *equals*

$$M^{\text{Ⓦ}_2}B = \begin{bmatrix} 5 & 3 & 5 & 3 & 2 \\ 4 & 2 & 4 & 4 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the solution to $\min\left\{\|M^4X - M^6(M^3)^\dagger B\|_F, \mathcal{R}(X) \subseteq \mathcal{R}(M^4)\right\}$ *equals*

$$M^{\text{Ⓦ}_3}B = \begin{bmatrix} 3 & 2 & 3 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Ultimately, for each integer $m \geq \text{ind}(M) = 4$*, the constrained matrix equation* $\|M^{m+1}X - M^{2m}(M^m)^\dagger B\|_F$*, subject to the range condition* $\mathcal{R}(X) \subseteq \mathcal{R}(M^4)$ *admits the solution*

$$M^{\text{Ⓦ}_m}B = M^{\text{Ⓦ}}B = \begin{bmatrix} 2 & 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

7. Conclusion

The m-weak core inverse concept was put forward in [6], serving as an extension of the weak core inverse concept. References [7, 29] elaborate some properties of the m-weak core inverse. In this paper, we introduce several extra representations of the m-WCI. The results are established by means of full-rank factorizations of rank-stable matrix powers M^k with $k \geq \text{ind}(M)$. Based on the perturbation analysis and continuity theory developed in [14–17, 21], we extend these concepts to the proposed inverses. Our approach is further motivated by the constrained optimization framework in [13], which was originally addressed via m-weak group inverses. By employing the m-weak core inverse, we obtain the unique solution to the Frobenius-norm minimization: $\min\left\|M^{m+1}X - M^{2m}(M^m)^\dagger B\right\|_F$, subject to $\mathcal{R}(X) \subseteq \mathcal{R}(M^k)$, where $m \in \mathbb{N}$, $B \in \mathbb{C}^{n\times q}$, $M \in \mathbb{C}^{n\times n}$ and $\text{ind}(M) = k$. Some well-known results regarding the weak core inverse and the core-EP inverse are special cases of the considered minimization problem. Notably, existing results for weak core and core-EP inverses emerge as special cases.

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