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Marcinkiewicz integral operators associated with the Dunkl setting and their commutators on some spaces

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Abstract. In this paper, we introduce a Marcinkiewicz integral operator $\mathcal{M}_d$ whose kernels satisfy the size and the smoothness conditions associated with the Dunkl metric d. In terms of sharp maximal estimates for $\mathcal{M}_d$ and its commutators $\mathcal{M}_{d,b}$ generated by $\mathcal{M}_d$ with $b \in \mathrm{BMO}_d(\mathbb{R}^N)$, we show that the $\mathcal{M}_d$ and the commutator $\mathcal{M}_{d,b}$ are bounded on spaces $L^p(\mathbb{R}^N)$, Morrey spaces $L_d^{p,q}(\mathbb{R}^N)$ related to the Dunkl metric d and generalized Morrey spaces $\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)$ associated with the Dunkl metric d, respectively.

1. Introduction

In 1938, to give an analogue of certain Littlewood-Paley g-functions, Marcinkiewicz [1] introduced a Marcinkiewicz integral operator, and proved that the Marcinkiewicz integral is bounded on $L^p([0, 2\pi])$, where $1 < p < \infty$. In 1958, Stein [2] present that the classical Marcinkiewicz integral operator and proved that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^N)$ for all $1 < p \le 2$ whenever $\Omega \in Lip_\alpha(\mathbb{R}^N)$, $0 < \alpha \le 1$. In 2002 [3], Al-Salman etc proved that μ_Ω is bounded in L_p if $1 < p < \infty$ and $\Omega \in L(logL)^{\frac{1}{2}}(\mathbb{S}^{n-1})$. In 2019, Hu and Qu [4] prove that if $\Omega \in L^q(S^{n-1})$ for some $q \in (1, \infty]$, then for $p \in (q', \infty)$ and $\omega \in A_p(\mathbb{R}^n)$, the following bounds hold: $\|\mu_\Omega(f)\|_{L^p(\mathbb{R}^n,\omega)} \le C\|\Omega\|_{L^q(S^{n-1})}[\omega]^{max\frac{1}{2},\frac{1}{p-q'}+max1,\frac{q'}{p-q'}}\|f\|_{L^p(\mathbb{R}^n,\omega)}$, where the constant $C > 0$ is independent of f, ω, Ω. In 2015, Lu and Tao [5] establish the boundedness of $\mathcal{M}_b$ from $L^p(\mu)$ to $L^{q,\infty}$ for $1 \le p \le n/\beta$ with $1/q = 1/p - \beta/n$, from $L^p(\mu)$ to $\mathrm{RBMO}(\mu)$ for $p = n/\beta$, and from $L^p(\mu)$ to $\mathrm{Lip}(\beta - n/p)(\mu)$ for $n/\beta \le p < \infty$. In 2023, Wu[14] establishes some bounds for the commutator of the Marcinkiewicz integral on the Hardy space H_1 and its weighted space, here the commutator is considered in Lip_β. More researches about such operators and commutators can be seen[6–8, 16–18, 20].

Based on the information provided in the search results, what is done in this paper to extend this result to the context to of Dunkl theory[15], where a similar operator a similar operator is already defined. Since then, remarkable progress has been achieved in its development, for example, in 2020, Guliyev et al.[22] discusses the boundedness of the maximal commutator $M_{b,\upsilon}$ and the commutator of the maximal operator $[b, M_\upsilon]$ on Orlicz spaces $L_\Phi(R, dm\upsilon)$, where b belongs to the $BMO(R, dm\upsilon)$ space. In 2024, Han et al.[9] show that the Lipschitz spaces Λ^β connects to the Triebel-Lizorkin spaces $\dot{F}^{\alpha,q}_{p,D}$, where $1 < p < \infty$, $1 \le q \le \infty$,

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associated with the Dunkl Laplacian Δ_D in R^N and to the commutators of the Dunkl Riesz transform and the fractional Dunkl Laplacian $\Delta_D^{-2/\alpha}$, $0 < \alpha < N$, which is represented via the functional calculus of the Dunkl heat semigroup $e^{-t\Delta_D}$. More researches see [21, 23–26].

We now recall some notations about the theory of Dunkl introduced in [9, 15]. Consider the Euclidean spaces $\mathbb{R}^N$ endowed with the standard inner product $\langle x, y\rangle = \sum_{i=0}^N x_i y_i$. If the reflection σ_α with respect to the hyperplane orthogonal to α, α is a nonzero vector α in $\mathbb{R}^N$, is given by

$$\sigma_\alpha(x) = x - 2\frac{\langle x, \alpha\rangle}{\|\alpha\|^2}\alpha.$$

A finite set $R \subset \mathbb{R}^N \setminus \{0\}$ is called a root system if $\sigma_\alpha(R) = R$ for all $\alpha \in R$. Let R be a root system in $\mathbb{R}^N$ normalized so that $\langle \alpha, \alpha\rangle = 2$ for $\alpha \in R$ and G the finite reflection group generated by the reflection $\sigma_\alpha(\alpha \in R)$, where $\sigma_\alpha = x - \langle x, \alpha\rangle\alpha$ for $x \in \mathbb{R}^N$. Corresponding to this reflection group, we denote by $\mathcal{O}(x)$ the G-orbit of a point $x \in \mathbb{R}^N$. There is a natural metric between two G-orbit $\mathcal{O}(x)$ and $\mathcal{O}(y)$, given by

$$d(x, y) := \min_{\sigma \in G} \|x - \sigma(y)\|.$$

It is clear that $d(x, y) \le \|x - y\|$ and it is possible that for certain $x, y \in \mathbb{R}^N$, $d(x, y) = 0$ while $\|x - y\| > 0$.

For a multiplicity function κ defined on R (invariant under G), let

$$\mathrm{d}\omega(x) = \prod_{\alpha \in R} |\langle \alpha, x\rangle|^{\kappa(\alpha)}\mathrm{d}x$$

be the associated measure in $\mathbb{R}^N$ (see[11]). Here and in what follows, $\mathrm{d}\omega(x)$ represents the Lebesgue measure in $\mathbb{R}^N$.

Let $B(x, r) := \{y \in \mathbb{R}^N : \|x - y\| < r\}$ stand for the ball with center $x \in \mathbb{R}^N$ and radius $r > 0$. We denote by $\mathbf{N} = N + \sum_{\alpha \in R} \kappa(\alpha)$ the homogeneous dimension. The measure $\mathrm{d}\omega(x)$ satisfies

$$\omega(B(tx, tr)) = t^{\mathbf{N}}\omega(B(x, r))$$

and that there is a positive constant C such that

$$\omega(B(x, 2r)) = C\omega(B(x, r)) < \infty$$

for all $x \in \mathbb{R}^N$, $t, r > 0$. Furthermore, we have the following estimate(see[19])

$$C^{-1}\Big(\frac{r_2}{r_1}\Big)^N \le \frac{\omega(B(x, r_2))}{\omega(B(x, r_1))} \le C\Big(\frac{r_2}{r_1}\Big)^{\mathbf{N}} \quad \text{for} \quad 0 < r_1 < r_2.$$

Obviously, we have the following results are evident from the definition of G-orbits

$$\mathcal{O}(B(x, r)) = \bigcup_{\sigma \in G} B(\sigma(x), r) = \{y \in \mathbb{R}^N : d(x, y) < r\},$$

and

$$\omega(B(x, r)) \le \omega(\mathcal{O}(B(x, r))) \le |G|\omega(B(x, r)). \tag{1}$$

Motivated by the above results, in this article, we mainly establish the boundedness of Marcinkiewicz integral operators $\mathcal{M}_d$ associated with the Dunkl metric d and their commutator $\mathcal{M}_{d,b}$ formed by $b \in \mathrm{BMO}_d(\mathbb{R}^N)$ and the $\mathcal{M}_d$ on Lebesgue spaces $L^p(\mathbb{R}^N)$, Morrey spaces $L_d^{p,q}(\mathbb{R}^N)$ and generalized Morrey spaces $\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)$, respectively.

Before stating the organization of this paper, we need to recall some necessary notions. The following definition of the space $\mathrm{BMO}_d(\mathbb{R}^N)$ associated with a Dunkl metric d is from [9].

Definition 1.1. *A real-valued function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be in the space $\text{BMO}_d(\mathbb{R}^N)$ associated with a Dunkl metric d if*

$$\|f\|_d = \sup_{B\subset\mathbb{R}^N} \frac{1}{\omega\big(O(B)\big)} \int_{O(B)} |f(x) - f_{O(B)}| \mathrm{d}\omega(x), \tag{2}$$

where $f_{O(B)}$ represents the mean value of functions f over balls B, that is,

$$f_{O(B)} = \frac{1}{\omega\big(O(B)\big)} \int_{O(B)} f(y) \mathrm{d}\omega(y).$$

We now state the definition of a Marcinkiewicz integral operator $\mathcal{M}_d$ associated with the Dunkl metric d is as follows.

Definition 1.2. *A real-value measurable function $K(\cdot,\cdot) \in L^1_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N \setminus \{(x,x) : x \in \mathbb{R}^N\})$ is said to a Calderón-Zygmund kernel associated with Dunkl metric d if there is a constant C such that*

(i) *for all $x, y \in \mathbb{R}^N$ with $x \neq y$,*

$$|K(x,y)| \leq C \frac{d(x,y)}{\omega(B(x, d(x,y)))}, \tag{3}$$

(ii) *for all $x, x', y \in \mathbb{R}^N$ with $\|x - x'\| \leq \frac{d(x,y)}{2}$,*

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C \frac{\|x - x'\|}{\|x - y\|} \frac{d(x,y)}{\omega(B(x, d(x,y)))}. \tag{4}$$

Let $L^\infty_b(\mathbb{R}^N)$ be the space of all $L^\infty(\mathbb{R}^N)$ functions with bounded support. A sublinear operator $\mathcal{M}_d$ associated with the Dunkl metric d is called a Marcinkiewicz integral operator with kernels K satisfying (3) and (4) if, for all $f \in L^\infty_b(\mathbb{R}^N)$ and $x \in \big(\mathbb{R}^N \setminus \text{supp}(f)\big)$,

$$\mathcal{M}_d(f)(x) = \bigg(\int_0^\infty \bigg| \int_{d(x,y)\leq t} K(x,y) f(y) \mathrm{d}\omega(y) \bigg|^2 \frac{\mathrm{d}t}{t^3} \bigg)^{\frac{1}{2}}. \tag{5}$$

Given $b \in \text{BMO}_d(\mathbb{R}^N)$, the commutator $\mathcal{M}_{d,b}$ generated by b and the $\mathcal{M}_d$ is defined by

$$\mathcal{M}_{d,b}(f)(x) = \bigg(\int_0^\infty \bigg| \int_{d(x,y)\leq t} K(x,y)\big(b(x) - b(y)\big) f(y) \mathrm{d}\omega(y) \bigg|^2 \frac{\mathrm{d}t}{t^3} \bigg)^{\frac{1}{2}}. \tag{6}$$

It is now position to state the organization of this paper as follows. In section 2, in terms of sharp maximal estimates for the Marcinkiewicz integral operator $\mathcal{M}_d$ associated with the Dunkl metric d and its commutator $\mathcal{M}_{d,b}$ formed by $\mathcal{M}_d$ with $b \in \text{BMO}_d(\mathbb{R}^N)$, the authors prove that the $\mathcal{M}_d$ and the commutator $\mathcal{M}_{d,b}$ are bounded on spaces $L^p(\mathbb{R}^N)$. By using the boundedness of the $\mathcal{M}_d$ and the commutator $\mathcal{M}_{d,b}$ on spaces $L^p(\mathbb{R}^N)$, the authors show that the $\mathcal{M}_d$ and the commutator $\mathcal{M}_{d,b}$ are bounded from Morrey spaces $L^{p,q}_d(\mathbb{R}^N)$ associated with the Dunk metric d into itself, where $1 < q \leq p < \infty$. Under assumption that the Lebesgue measurable function $\varphi(\cdot,\cdot)$ defined on $\mathbb{R}^N \times (0,\infty)$ satisfies certain conditions, the authors show that the the $\mathcal{M}_d$ and the commutator $\mathcal{M}_{d,b}$ are bounded on generalized Morrey spaces $\mathcal{L}^{\varphi,p}_d(\mathbb{R}^N)$ associated with the Dunk metric d.

Throughout this paper, we always denote by C or $C_i (i = 1, 2)$ a positive constant being independent of the main parameters, but they may be different from line to line. Given any $p \in [1,\infty]$, we denote by p' as its conjugate index, i.e., $p' = p/(p - 1)$. Moreover, for any measurable subset $E \subset \mathbb{R}^N$, we let χ_E denote its characteristic function.

2. Boundedness for $\mathcal{M}_d$ and $\mathcal{M}_{d,b}$ on space $L^p(\mathbb{R}^N)$

In this section, the authors mainly investigate the boundedness of the Marcinkiewicz integral operator $\mathcal{M}_d$ associated with the Dunkl metric d and its commutator $\mathcal{M}_{d,b}$ which is formed by $b \in \mathrm{BMO}_d(\mathbb{R}^N)$ and the $\mathcal{M}_d$ on Lebesgue spaces $L^p(\mathbb{R}^N)$ for $1 < p < \infty$.

It is now position to state the main results of this section as follows.

Theorem 2.1. *Let kernels $K(x, y)$ satisfy conditions* (3) *and* (4), *and $1 < p < \infty$. Suppose that the M_d is bounded on spaces $L^2(\mathbb{R}^N)$. Then there exists some positive constant C such that, for all $f \in L^p(\mathbb{R}^N)$,*

$$\|\mathcal{M}_d(f)\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^p(\mathbb{R}^N)}.$$

Theorem 2.2. *Let kernels $K(x, y)$ satisfy conditions* (3) *and* (4), *$b \in \mathrm{BMO}_d(\mathbb{R}^N)$, and $1 < p < \infty$. Suppose that the M_d is bounded on spaces $L^2(\mathbb{R}^N)$. Then there exists some positive constant C such that, for all $f \in L^p(\mathbb{R}^N)$,*

$$\|\mathcal{M}_{d,b}(f)\|_{L^p(\mathbb{R}^N)} \leq C\|b\|_d\|f\|_{L^p(\mathbb{R}^N)}.$$

To prove the main theorems, we need to recall the following lemmas, see [9, 10], respectively.

Lemma 2.3. *Let b be measurable function on $\mathbb{R}^N$ and $1 < p < \infty$.*

(1) *Then there exists a constant $C > 0$ such that, for any ball B,*

$$\|b\|_d \approx \sup_{B \subset \mathbb{R}^N} \left(\frac{1}{\omega(\mathcal{O}(B))} \int_{\mathcal{O}(B)} |b(x) - b_{\mathcal{O}(B)}|^p \mathrm{d}\omega(x) \right)^{\frac{1}{p}}. \tag{7}$$

(2) *Then there exists a constant $C > 0$ such that, for any ball B and $j \in \mathbb{N}$,*

$$|b_{\mathcal{O}(B)} - b_{\mathcal{O}(2^{j+1}B)}| \leq C(j+1)\|b\|_d. \tag{8}$$

Lemma 2.4. *Let $0 < p < q < \infty$. Then there exists a positive constant $C = C_{p,q}$ such that*

$$|\omega(B)|^{-\frac{1}{p}}\|f\|_{L^p(B)} \leq C|\omega(B)|^{-\frac{1}{q}}\|f\|_{L^{q,\infty}(B)}$$

for all measurable functions f.

We now recall some definitions of maximal operators introduced in [9]. The Hardy-Littlewood maximal function $M(f)(x)$ is defined as

$$Mf(x) = \sup_{B \ni x} \frac{1}{\omega(B)} \int_B |f(y)| \mathrm{d}\omega(y), \tag{9}$$

the definition of the Hardy-Littlewood maximal operator M_d associated with the Dunkl metric d is defined by

$$M_d f(x) = \sup_{\mathfrak{B} \ni x} \frac{1}{\omega(\mathfrak{B})} \int_{\mathfrak{B}} |f(y)| \mathrm{d}\omega(y) \tag{10}$$

for any $x \in \mathbb{R}^N$ and $r > 0$, where $\mathfrak{B} := \{z \in \mathbb{R}^N : d(x, z) < r\}$. And the sharp maximal function $M^\sharp$ is defined as

$$M^\sharp f(x) = \sup_{x \in B} \int_B |f(y) - f_B| \mathrm{d}\omega(y),$$

Also, we need to establish the following lemma about the $\mathcal{M}_d$.

Lemma 2.5. *Let kernels K satisfy conditions* (3) *and* (4), *and $1 < s < \infty$. Suppose that the $\mathcal{M}_d$ is bounded from spaces $L^s(\mathbb{R}^N)$ into spaces $L^{s,\infty}(\mathbb{R}^N)$. Then there exists a positive constant C such that, for all $f \in L_b^\infty(\mathbb{R}^N)$,*

$$M^\sharp(\mathcal{M}_d(f))(x) \leq CM_d^s(f)(x).$$

Proof. Let $B = B(x, r)$ be a ball centered at x and its radius $r > 0$. Represent functions f as

$$f = f_1 + f_2 = f\chi_{O(2B)} + f\chi_{\mathbb{R}^N\setminus(O(2B))}. \tag{11}$$

Then, we write

$$\begin{aligned}
&\frac{1}{\omega(B)}\int_B \left|\mathcal{M}_d(f)(y) - \left(\mathcal{M}_d(f)\right)_B\right| d\omega(y)\\
&\leq \frac{1}{\omega(B)}\int_B \left|\mathcal{M}_d(f_1)(y) + \mathcal{M}_d(f_2)(y) - \left(\mathcal{M}_d(f_2)\right)_B - \left(\mathcal{M}_d(f)\right)_B + \left(\mathcal{M}_d(f_2)\right)_B\right| d\omega(y)\\
&\leq \frac{1}{\omega(B)}\int_B |\mathcal{M}_d(f_1)(y)| d\omega(y) + \frac{1}{\omega(B)}\int_B \left|\mathcal{M}_d(f_2)(y) - \left(\mathcal{M}_d(f_2)\right)_B\right| d\omega(y)\\
&\quad + \frac{1}{\omega(B)}\int_B \left|\left(\mathcal{M}_d(f)\right)_B - \left(\mathcal{M}_d(f_2)\right)_B\right)| d\omega(y)\\
&\leq \frac{1}{\omega(B)}\int_B |\mathcal{M}_d(f_1)(y)| d\omega(y) + \frac{1}{\omega(B)}\int_B \left|\mathcal{M}_d(f_2)(y) - \left(\mathcal{M}_d(f_2)\right)_B\right| d\omega(y)\\
&\quad + \frac{1}{\omega(B)}\int_B \left|\left(\mathcal{M}_d(f_1)\right)_B - \mathcal{M}_d(f_2)\right)_B - \left(\mathcal{M}_d(f_2)\right)_B\right)| d\omega(y)\\
&\leq \frac{2}{\omega(B)}\int_B |\mathcal{M}_d(f_1)(y)| d\omega(y) + \frac{1}{\omega(B)}\int_B |\mathcal{M}_d(f_2)(y) - \frac{1}{\omega(B)}\int_B \mathcal{M}_d(f_2)(\mathrm{w}) d\omega(\mathrm{w})| d\omega(y)\\
&\leq \frac{2}{\omega(B)}\int_B |\mathcal{M}_d(f_1)(y)| d\omega(y) + \frac{1}{\left[\omega(B)\right]^2}\int_B\int_B |\mathcal{M}_d(f_2)(y) - \mathcal{M}_d(f_2)(\mathrm{w})| d\omega(\mathrm{w}) d\omega(y)\\
&= \mathrm{D}_1 + \mathrm{D}_2,
\end{aligned}$$

From $1 < s < \infty$, the $(L^s(\mathbb{R}^N), L^{s,\infty}(\mathbb{R}^N))$-boundedness of the maximal operator $\mathcal{M}_d$ and (8), it then follows that

$$\begin{aligned}
\mathrm{D}_1 &\leq C\frac{1}{\omega(B)}\|\mathcal{M}_d(f_1)\|_{L(B)}\\
&\leq C\frac{1}{[\omega(B)]^{\frac{1}{s}}}\|\mathcal{M}_d(f_1)\|_{L^{s,\infty}(\mathbb{R}^N)} \leq C\frac{1}{[\omega(B)]^{\frac{1}{s}}}\|f_1\|_{L^s(\mathbb{R}^N)}\\
&\leq C\left[\frac{\omega(O(2B)}{\omega(B)}\right]^{\frac{1}{s}}\left(\frac{1}{\omega(O(2B)}\int_{O(2B)} |f(y)|^s d\omega(y)\right)^{\frac{1}{s}}\\
&\leq CM_d^s(f)(x),
\end{aligned}$$

To estimate D_2, we first consider $|\mathcal{M}_d(f_2)(y) - \mathcal{M}_d(f_2)(\mathrm{w})|$ with $\mathrm{w}, y \in B$. By applying (3), (5) and the Hölder's inequality, we have

$$\begin{aligned}
&|\mathcal{M}_d(f_2)(y) - \mathcal{M}_d(f_2)(\mathrm{w})|\\
&\leq \int_{\mathbb{R}^n} |K(y,z) - K(z,\mathrm{w})||f_2(z)|\left(\int_{\max\{d(z,y)\leq t, d(z,y_0)\leq t\}} \frac{dt}{t^3}\right)^{\frac{1}{2}} d\omega(z)\\
&\leq C\int_{\mathbb{R}^n\setminus O(2B)} \frac{\|y-y_0\|}{\|z-y\|}\frac{1}{\omega(B(z,d(z,\mathrm{w})))}|f(z)| d\omega(z)
\end{aligned}$$

$$
\begin{aligned}
&\le Cr \int_{\mathbb{R}^n \setminus O(2B)} \frac{1}{d(z, \mathrm{w})} \frac{1}{\omega(B(\mathrm{w}, d(z, \mathrm{w})))} |f(z)| \mathrm{d}\omega(z) \\
&\le Cr \sum_{i=0}^{\infty} \int_{(2r)\times 2^i \le d(z,\mathrm{w}) \le (2r)\times 2^{i+1}} \frac{1}{d(z, \mathrm{w})} \frac{1}{\omega(B(\mathrm{w}, d(z, \mathrm{w})))} |f(z)| \mathrm{d}\omega(z) \\
&\le Cr \sum_{i=0}^{\infty} \frac{1}{2r \times 2^i} \frac{1}{\omega(B(\mathrm{w}, 2r \times 2^i))} \int_{d(z,\mathrm{w}) \le (2r)\times 2^{i+1}} |f(z)| \mathrm{d}\omega(z) \\
&\le C \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{[\omega(B(x, 2^{i+1} \times (2r)))]^{1-\frac{1}{s}}}{\omega(B(\mathrm{w}, 2r \times 2^i))} \bigg(\int_{d(z,\mathrm{w}) \le 2^{i+1}\times(2r)} |f(z)|^s \mathrm{d}\omega(z) \bigg)^{\frac{1}{s}} \\
&\le C M_d^s(f)(x),
\end{aligned}
$$

therefore, we have $\mathrm{D}_2 \le CM_d^s(f)(x)$. Hence, the proof of Lemma 1.9 is finished. □

Proof of theorem 2.1**.** By Lemma 2.5 and the $(L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))$-boundedness of the maximal operator M_d^s(see [12]), we obtain

$$
\|\mathcal{M}_d(f)\|_{L^p(\mathbb{R}^N)} \le C\|M_d^\sharp(\mathcal{M}_d(f))\|_{L^p(\mathbb{R}^N)} \le C\|M_d^s(f)\|_{L^p(\mathbb{R}^N)} \le C\|f\|_{L^p(\mathbb{R}^N)}.
$$

Hence, the proof of Theorem 2.1 is completed. □

Proof of theorem 2.2**.** Let $B = B(x_0, r)$ be the open ball centered at $x_0 \in \mathbb{R}^n$ and its radius $r > 0$. And decompose functions f using the same form in (11), that is,

$$
f = f_1 + f_2,
$$

where $f_1 = f\chi_{O(2B)}$ and $f_2 = f\chi_{\mathbb{R}^N \setminus (O(2B))}$.

We first claim that, for all $s \in (1, \infty)$, $f \in L^p(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$,

$$
M^\sharp\Big(\mathcal{M}_{d,b}(f)\Big)(x) \le C\|b\|_d \Big(M^s(\mathcal{M}_d(f))(x) + M_d^s(f)(x)\Big). \tag{12}
$$

Once (12) is proved, takeing $1 < s < p < \infty$, by applying Theorem 2.1 and the $(L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))$-boundedness of operators M^s and M_d^s, we conclude that

$$
\begin{aligned}
\|\mathcal{M}_{d,b}(f)\|_{L^p(\mathbb{R}^N)} &\le C\|M^\sharp\Big(\mathcal{M}_{d,b}(f)\Big)\|_{L^p(\mathbb{R}^N)} \\
&\le C\|b\|_d \Big(\|M^s(\mathcal{M}_d(f))\|_{L^p(\mathbb{R}^N)} + \|M_d^s(f)\|_{L^p(\mathbb{R}^N)}\Big) \\
&\le C\|b\|_d \Big(\|\mathcal{M}_d(f)\|_{L^p(\mathbb{R}^N)} + \|f\|_{L^p(\mathbb{R}^N)}\Big) \\
&\le C\|b\|_d \|f\|_{L^p(\mathbb{R}^N)},
\end{aligned}
$$

which is our desired result.

To show (12), by applying the definition of sharp maximal operator $M^\sharp$, for all x and B with $B \ni x$,

$$
\frac{1}{\omega(B)} \int_B |\mathcal{M}_{d,b}(f)(y) - h_B| \mathrm{d}\omega(y) \le C\|b\|_d \Big(M^s(\mathcal{M}_d(f))(x) + M_d^s(f)(x)\Big), \tag{13}
$$

where $h_B = (\mathcal{M}_d((b(\cdot) - b_{O(B)})f_2))_B$.

To prove (13), write

$$
\begin{aligned}
&\frac{1}{\omega(B)}\int_B |\mathcal{M}_{d,b}(f)(y)-h_B|\mathrm{d}\omega(y)\\
&\leq \frac{1}{\omega(B)}\frac{1}{\omega(B)}\int_B\int_B |\mathcal{M}_{d,b}(f)(y)-\mathcal{M}_d((b(\cdot)-b_{O(B)})f_2)(z)|\mathrm{d}\omega(y)\mathrm{d}\omega(z)\\
&\leq \frac{1}{\omega(B)}\frac{1}{\omega(B)}\int_B\int_B |b(y)-b_{O(B)}||\mathcal{M}_d(f)(y)|\mathrm{d}\omega(y)\mathrm{d}\omega(z)\\
&\quad+\frac{1}{\omega(B)}\frac{1}{\omega(B)}\int_B\int_B |\mathcal{M}_d([b(y)-b_{O(B)}](f_1))(y)|\mathrm{d}\omega(y)\mathrm{d}\omega(z)\\
&\quad\quad+\frac{1}{\omega(B)}\frac{1}{\omega(B)}\int_B\int_B |\mathcal{M}_d((b(y)-b_{O(B)})f_2))(y)-\mathcal{M}_d((b(\cdot)-b_{O(B)})f_2)(z)|\mathrm{d}\omega(y)\mathrm{d}\omega(z)\\
&=\mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3.
\end{aligned}
$$

By applying the Hölder's inequality and (7), we have

$$
\begin{aligned}
\mathrm{E}_1&=\frac{1}{\omega(B)}\frac{1}{\omega(B)}\int_B\int_B |b(y)-b_{O(B)}||\mathcal{M}_d(f)(y)|\mathrm{d}\omega(y)\mathrm{d}\omega(z)\\
&\leq\Big(\frac{1}{\omega(B)}\int_B |b(y)-b_{O(B)}|^{s}\mathrm{d}\omega(y)\Big)^{\frac{1}{s'}}\Big(\frac{1}{\omega(B)}\int_B |\mathcal{M}_d(f)(y)|^{s}\mathrm{d}\omega(y)\Big)^{\frac{1}{s}}\\
&\leq C\Big(\frac{1}{\omega(O(B))}\int_{O(B)} |b(y)-b_{O(B)}|^{s}\mathrm{d}\omega(y)\Big)^{\frac{1}{s'}}\Big(\frac{1}{\omega(B)}\int_B |\mathcal{M}_d(f)(y)|^{s}\mathrm{d}\omega(y)\Big)^{\frac{1}{s}}\\
&\leq C\|b\|_d M^{s}(\mathcal{M}_d(f))(x),
\end{aligned}
$$

where we use the following fact

$$\omega(B)\leq\omega(O(B))\leq|G|\omega(B).$$

Choosing $p=\sqrt{s}$ with $s>1$, by the $(L^p(\mathbb{R}^N),L^p(\mathbb{R}^N))$-boundedness of the $\mathcal{M}_d$, (7) and the definition of M^s_d, we deduce

$$
\begin{aligned}
\mathrm{E}_2&=\frac{1}{\omega(B)}\frac{1}{\omega(B)}\int_B\int_B |\mathcal{M}_d([b(y)-b_{O(B)}](f_1))(y)|\mathrm{d}\omega(y)\mathrm{d}\omega(z)\\
&=\frac{1}{\omega(B)}\int_B |\mathcal{M}_d((b(y)-b_{O(B)})f_1)(y)|\mathrm{d}\omega(y)\\
&\leq\Big(\frac{1}{\omega(B)}\int_B |\mathcal{M}_d((b(y)-b_{O(B)})f_1)(y)|^{p}\mathrm{d}\omega(y)\Big)^{\frac{1}{p}}\\
&\leq C\Big(\frac{1}{\omega(B)}\int_{O(2B)} |b(y)-b_{O(B)}|^{p}|f(y)|^{p}\mathrm{d}\omega(y)\Big)^{\frac{1}{p}}\\
&\leq C\frac{1}{[\omega(B)]^{\frac{1}{p}}}\Big(\int_{O(2B)} |b(y)-b_{O(B)}|^{pp'}\mathrm{d}\omega(y)\Big)^{\frac{1}{pp'}}\Big(\int_{O(2B)} |f(y)|^{pp}\mathrm{d}\omega(y)\Big)^{\frac{1}{pp}}\\
&\leq C\Big(\frac{1}{\omega(B)}\int_{O(2B)} |b(y)-b_{O(B)}|^{pp'}\mathrm{d}\omega(y)\Big)^{\frac{1}{pp'}}\Big(\frac{1}{\omega(B)}\int_{O(2B)} |f(y)|^{s}\mathrm{d}\omega(y)\Big)^{\frac{1}{s}}\\
&\leq C\|b\|_d M^{s}_d(f)(x).
\end{aligned}
$$

To estimate E_3, we first consider the form $|\mathcal{M}_d((b(y)-b_{O(B)})f_2))(y)-\mathcal{M}_d((b(\cdot)-b_{O(B)})f_2)(z)|$ with $y,z\in B$.

By using (5), write

$$
\begin{aligned}
&|\mathcal{M}_d((b(y)-b_{O(B)})f_2))(y)-\mathcal{M}_d((b(\cdot)-b_{O(B)})f_2)(z)|\\
&\quad\le\Big(\int_0^\infty\Big|\int_{d(y,\mathrm{w})\le t}K(y,\mathrm{w})(b(\mathrm{w})-b_{O(B)})f_2(\mathrm{w})\mathrm{d}\omega(\mathrm{w})\\
&\qquad\qquad-\int_{d(z,\mathrm{w})\le t}K(z,\mathrm{w})(b(\mathrm{w})-b_{O(B)})f_2(z)\mathrm{d}\omega(\mathrm{w})\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}\\
&\quad\le\Big(\int_0^\infty\Big[\int_{d(y,\mathrm{w})\le t\le d(z,\mathrm{w})}|K(y,\mathrm{w})||b(\mathrm{w})-b_{O(B)}||f_2(\mathrm{w})|\mathrm{d}\omega(\mathrm{w})\Big]^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}\\
&\qquad+\Big(\int_0^\infty\Big[\int_{d(z,\mathrm{w})\le t\le d(y,\mathrm{w})}|K(z,\mathrm{w})||b(\mathrm{w})-b_{O(B)}||f_2(\mathrm{w})|\mathrm{d}\omega(\mathrm{w})\Big]^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}\\
&\qquad\quad+\Big(\int_0^\infty\Big[\int_{\substack{d(z,\mathrm{w})\le t\\ d(y,\mathrm{w})\le t}}|K(y,\mathrm{w})-K(z,\mathrm{w})||b(\mathrm{w})-b_{O(B)}||f_2(\mathrm{w})|\mathrm{d}\omega(\mathrm{w})\Big]^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}\\
&\quad=\mathrm{E}_3^1+\mathrm{E}_3^2+\mathrm{E}_3^3.
\end{aligned}
$$

By applying (3), the Hölder's inequality, (7) and the M_d^s, we get

$$
\begin{aligned}
\mathrm{E}_3^1&\le C\int_{\mathbb{R}^N\setminus O(2B)}\frac{d(y,\mathrm{w})}{\omega(B(y,d(y,\mathrm{w})))}|b(z)-b_{O(B)}||f(\mathrm{w})|\Big(\int_{d(y,\mathrm{w})}^{d(z,\mathrm{w})}\frac{\mathrm{d}t}{t^3}\Big)^{\frac12}\mathrm{d}\omega(\mathrm{w})\\
&\le C\int_{\mathbb{R}^N\setminus O(2B)}\frac{d(y,\mathrm{w})}{\omega(B(y,d(y,\mathrm{w})))}|b(z)-b_{O(B)}||f(\mathrm{w})|\Big(\frac{d(x,y)}{d(y,\mathrm{w})^3}\Big)^{\frac12}\mathrm{d}\omega(\mathrm{w})\\
&\le Cr^{\frac12}\sum_{i=0}^\infty\int_{O(2^{i+1}B)\setminus O(2^{i+2}B)}\frac{1}{d(x,\mathrm{w})^{\frac12}}\frac{1}{\omega(x,d(x,\mathrm{w}))}|b(z)-b_{O(B)}||f(\mathrm{w})|\mathrm{d}\omega(\mathrm{w})\\
&\le Cr^{\frac12}\sum_{i=0}^\infty\int_{2^{i+1}r\le d(x,\mathrm{w})\le 2^{i+2}r}\frac{1}{d(x,\mathrm{w})^{\frac12}}\frac{1}{\omega(x,d(x,\mathrm{w}))}|b(z)-b_{O(B)}||f(\mathrm{w})|\mathrm{d}\omega(\mathrm{w})\\
&\le C\sum_{i=0}^\infty\frac{1}{(2^i)^{\frac12}}\int_{d(x,\mathrm{w})\le 2^{i+2}r}\frac{1}{\omega(x,2^{i+1}r)}|b(\mathrm{w})-b_{O(B)}||f(\mathrm{w})|\mathrm{d}\omega(\mathrm{w})\\
&\le C\sum_{i=0}^\infty\frac{1}{(2^i)^{\frac12}}\Big\{\int_{d(x,z)\le 2^{i+2}r}\frac{1}{\omega(x,2^{i+1}r)}|b(\mathrm{w})-b_{O(2^{i+2}B)}||f(z)|\mathrm{d}\omega(\mathrm{w})\\
&\qquad+\int_{d(x,\mathrm{w})\le 2^{i+2}r}\frac{1}{\omega(x,2^{i+1}r)}|b_{O(2^{i+2}B)}-b_{O(B)}||f(\mathrm{w})|\mathrm{d}\omega(\mathrm{w})\Big\}\\
&\le C\sum_{i=0}^\infty\frac{1}{(2^i)^{\frac12}}\Big\{\int_{d(x,\mathrm{w})\le 2^{i+2}r}\frac{1}{\omega(x,2^{i+1}r)}|b(\mathrm{w})-b_{O(2^{i+2}B)}||f(z)|\mathrm{d}\omega(\mathrm{w})\\
&\qquad+(i+1)\|b\|_d\frac{1}{\omega(x,2^{i+1}r)}\int_{d(x,\mathrm{w})\le 2^{i+2}r}|f(\mathrm{w})|\mathrm{d}\omega(\mathrm{w})\Big\}\\
&\le C\sum_{i=0}^\infty\frac{1}{(2^i)^{\frac12}}\Big\{\Big(\frac{1}{\omega(x,2^{i+1}r)}\int_{d(x,\mathrm{w})\le 2^{i+2}r}|b(\mathrm{w})-b_{O(2^{i+2}B)}|^{s'}\mathrm{d}\omega(\mathrm{w})\Big)^{\frac{1}{s'}}\\
&\qquad\times\Big(\frac{1}{\omega(x_0,2^{i+1}r)}\int_{d(x,\mathrm{w})\le 2^{i+2}r}|f(\mathrm{w})|^s\mathrm{d}\omega(\mathrm{w})\Big)^{\frac1s}
\end{aligned}
$$

$$
\begin{aligned}
&+ (i+1)\|b\|_d \frac{1}{\omega(x, 2^{i+2}r)} \int_{d(x,\mathrm{w})\leq 2^{i+2}r} |f(\mathrm{w})| d\omega(\mathrm{w}) \Big\} \\
&\leq C\|b\|_d \sum_{i=0}^{\infty} \frac{i+1}{(2^i)^{\frac{1}{2}}} \Big(\frac{1}{\omega(x, 2^{i+1}r)} \int_{d(x,\mathrm{w})\leq 2^{i+2}r} |f(\mathrm{w})|^s d\omega(\mathrm{w}) \Big)^{\frac{1}{s}} \\
&\leq C\|b\|_d M_d^s(f)(x).
\end{aligned}
$$

With an argument similar that used in the estimate of E_3^1, it is easy to get

$$\mathrm{E}_3^2 \leq C\|b\|_d M_d^s(f)(x).$$

We now turn E_3^3. For any $y, z \in B$, by applying (4), the Hölder's inequality, (7) and M_d^s, we obtain

$$
\begin{aligned}
\mathrm{E}_3^3 &\leq C \int_{\mathbb{R}^N \setminus O(2B)} |K(y,\mathrm{w}) - K(z,\mathrm{w})||b(\mathrm{w}) - b_{O(B)}||f(z)| \Big(\int_{d(y,\mathrm{w})}^{\infty} \frac{dt}{t^3} \Big)^{\frac{1}{2}} d\omega(\mathrm{w}) \\
&\leq C \int_{\mathbb{R}^N \setminus O(2B)} \frac{\|y-z\|}{\|y-\mathrm{w}\|} \frac{d(y,\mathrm{w})}{\omega(B(y, d(y,\mathrm{w})))} |b(\mathrm{w}) - b_{O(B)}||f(\mathrm{w})| d\omega(\mathrm{w}) \\
&\leq Cr \sum_{i=0}^{\infty} \int_{O(2^{i+1}B)\setminus O(2^{i+2}B)} \frac{1}{d(x,\mathrm{w})} \frac{1}{\omega(B(x, d(x,\mathrm{w})))} |b(\mathrm{w}) - b_{O(B)}||f(\mathrm{w})| d\omega(\mathrm{w}) \\
&\leq Cr \sum_{i=0}^{\infty} \int_{2^{i+1}r \leq d(x,\mathrm{w}) \leq 2^{i+2}r} \frac{1}{d(x,\mathrm{w})} \frac{1}{\omega(B(x, d(x,\mathrm{w})))} |b(\mathrm{w}) - b_{O(B)}||f(\mathrm{w})| d\omega(\mathrm{w}) \\
&\leq Cr \sum_{i=0}^{\infty} \frac{1}{2^{i+1}r} \int_{d(x_0,z)\leq 2^{i+2}r} \frac{1}{\omega(B(x, 2^{i+1}r))} |b(\mathrm{w}) - b_{O(B)}||f(\mathrm{w})| d\omega(\mathrm{w}) \\
&\leq C \sum_{i=0}^{\infty} \frac{1}{2^i} \Big[\frac{1}{\omega(B(x, 2^{i+1}r))} \int_{d(x,\mathrm{w})\leq 2^{i+2}r} |b(z) - b_{O(2^{i+2}B)}||f(\mathrm{w})| d\omega(\mathrm{w}) \\
&\quad + \frac{1}{\omega(B(x, 2^{i+1}r))} |b_{O(2^{i+2}B)} - b_{O(B)}||f(\mathrm{w})| d\omega(\mathrm{w}) \Big] \\
&\leq C \sum_{i=0}^{\infty} \frac{1}{2^i} \Big[\Big(\frac{1}{\omega(B(x, 2^{i+1}r))} \int_{d(x,\mathrm{w})\leq 2^{i+2}r} |b(\mathrm{w}) - b_{O(2^{i+2}B)}|^{s'} d\omega(\mathrm{w}) \Big)^{\frac{1}{s'}} \\
&\quad \times \Big(\frac{1}{\omega(B(x, 2^{i+1}r))} \int_{d(x,\mathrm{w})\leq 2^{i+2}r} |f(z)|^s d\omega(\mathrm{w}) \Big)^{\frac{1}{s}} \\
&\quad + (i+1)\|b\|_d \Big(\frac{1}{\omega(B(x_0, 2^{i+1}r))} \int_{d(x,\mathrm{w})\leq 2^{i+2}r} |f(\mathrm{w})|^s d\omega(\mathrm{w}) \Big)^{\frac{1}{s}} \Big] \\
&\leq C\|b\|_d \sum_{i=0}^{\infty} \frac{i+1}{2^i} \Big(\frac{1}{\omega(B(x, 2^{i+1}r))} \int_{d(x,\mathrm{w})\leq 2^{i+2}r} |f(z)|^s d\omega(z) \Big)^{\frac{1}{s}} \\
&\leq C\|b\|_d M_d^s(f)(x),
\end{aligned}
$$

which, combining the estimates for E_3^1, E_3^2, E_1 and E_2, yields (12). Hence, the proof of Theorem 2.2 is finished. □

3. Boundedness for $\mathcal{M}_d$ and $\mathcal{M}_{d,b}$ on spaces $L_d^{p,q}(\mathbb{R}^N)$

In this section, we mainly establish the boundedness of the $\mathcal{M}_d$ and its commutator $\mathcal{M}_{d,b}$ on Morrey spaces $L_d^{p,q}(\mathbb{R}^N)$ associated with the Dunkl metric d, respectively. Before stating the main results of this section, we first state the definition of a Morrey space $L_d^{p,q}(\mathbb{R}^N)$ as follows.

Definition 3.1. *Let $1 < q \leq p < \infty$. Then the Morrey space $L_d^{p,q}(\mathbb{R}^N)$ associated with the Dunkl metric d is defined by*

$$L_d^{p,q}(\mathbb{R}^N) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^N) : \|f\|_{L_d^{p,q}(\mathbb{R}^N)} < \infty \right\}$$

where

$$\|f\|_{L_d^{p,q}(\mathbb{R}^N)} = \sup_B [\omega(\mathcal{O}(B))]^{\frac{1}{p}-\frac{1}{q}} \Big(\int_{\mathcal{O}(B)} |f(x)|^q \mathrm{d}\omega(x) \Big)^{\frac{1}{q}}. \tag{14}$$

It is now position to state the main theorems of this section as follows.

Theorem 3.2. *Let kernels $K(\cdot,\cdot)$ satisfy conditions* (3) *and* (4), *and $1 < q \leq p < \infty$. Suppose that the $\mathcal{M}_d$ is bounded on spaces $L^2(\mathbb{R}^N)$, and the following inequality*

$$\sum_{j=0}^{\infty} \Big[\frac{\omega(\mathcal{O}(B))}{\omega(\mathcal{O}(B(x, 2^{j+1}r)))} \Big]^{\tau} \leq C \tag{15}$$

holds for all $\tau \in (0,1)$. Then there is a positive constant C such that, for all $f \in L_d^{p,q}(\mathbb{R}^N)$,

$$\|\mathcal{M}_d(f)\|_{L_d^{p,q}(\mathbb{R}^N)} \leq C\|f\|_{L_d^{p,q}(\mathbb{R}^N)}.$$

Theorem 3.3. *Let kernels $K(\cdot,\cdot)$ satisfy conditions* (3) *and* (4), *$b \in \mathrm{BMO}_d(\mathbb{R}^N)$ and $1 < q \leq p < \infty$. Suppose that the $\mathcal{M}_d$ is bounded on spaces $L^2(\mathbb{R}^N)$, and the following inequality*

$$\sum_{j=0}^{\infty} (j+1) \Big[\frac{\omega(\mathcal{O}(B))}{\omega(\mathcal{O}(B(x, 2^{j+1}r)))} \Big]^{\tau} \leq C \tag{16}$$

holds for all $\tau \in (0,1)$. Then there is a positive constant C such that, for all $f \in L_d^{p,q}(\mathbb{R}^N)$,

$$\|\mathcal{M}_{d,b}(f)\|_{L_d^{p,q}(\mathbb{R}^N)} \leq C\|b\|_d \|f\|_{L_d^{p,q}(\mathbb{R}^N)}.$$

Proof of theorem 3.2. Let $B = B(x,r)$ be the open ball centered at $x \in \mathbb{R}^n$ and its radius $r > 0$. And decompose functions f using the same form in (11), that is,

$$f = f_1 + f_2,$$

where $f_1 = f\chi_{\mathcal{O}(2B)}$ and $f_2 = f\chi_{\mathbb{R}^N \setminus (\mathcal{O}(2B))}$. Then, by the Minkowski's inequality, write

$$\begin{aligned}
&\sup_B [\omega(\mathcal{O}(B))]^{\frac{1}{p}-\frac{1}{q}} \Big(\int_{\mathcal{O}(B)} |\mathcal{M}_d(f)(y)|^q \mathrm{d}\omega(y) \Big)^{\frac{1}{q}} \\
&\quad \leq \sup_B [\omega(\mathcal{O}(B))]^{\frac{1}{p}-\frac{1}{q}} \Big(\int_{\mathcal{O}(B)} |\mathcal{M}_d(f_1)(y)|^q \mathrm{d}\omega(y) \Big)^{\frac{1}{q}} \\
&\qquad + \sup_B [\omega(\mathcal{O}(B))]^{\frac{1}{p}-\frac{1}{q}} \Big(\int_{\mathcal{O}(B)} |\mathcal{M}_d(f_2)(y)|^q \mathrm{d}\omega(y) \Big)^{\frac{1}{q}} \\
&\quad = \mathrm{F}_1 + \mathrm{F}_2.
\end{aligned}$$

From Theorem 2.1, (14) and(15), it then follows that

$$
\begin{aligned}
\mathrm{F}_1 &\le \sup_B[\omega(O(B))]^{\frac{1}{p}-\frac{1}{q}}\Big(\int_{\mathbb{R}^N}|\mathcal{M}_d(f_1)(y)|^q\mathrm{d}\omega(y)\Big)^{\frac{1}{q}}\\
&\le C\sup_B[\omega(O(B))]^{\frac{1}{p}-\frac{1}{q}}\|f_1\|_{L^p(\mathbb{R}^N)}\\
&\le C\sup_B\Big[\frac{\omega(O(B))}{\omega(O(2B))}\Big]^{\frac{1}{p}-\frac{1}{q}}[\omega(O(2B))]^{\frac{1}{p}-\frac{1}{q}}\Big(\int_{O(2B)}|f(y)|^q\mathrm{d}\omega(y)\Big)^{\frac{1}{q}}\\
&\le C\|f\|_{L_d^{p,q}(\mathbb{R}^N)}.
\end{aligned}
$$

To estimates F_2, we first consider the form $|\mathcal{M}_d(f_2)(y)|$ with $y\in O(B)$. By using (2), (14), the Hölder's inequality and (15), we have

$$
\begin{aligned}
|\mathcal{M}_d f_2(y)| &\le \int_{\mathbb{R}^N\setminus O(2B)}|K(y,z)||f(z)|\Big(\int_{d(y,z)}^{\infty}\frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}}\mathrm{d}\omega(z)\\
&\le C\int_{\mathbb{R}^N\setminus O(2B)}\frac{1}{\omega(B(x,d(y,z)))}|f(z)|\mathrm{d}\omega(z)\\
&\le C\sum_{j=0}^{\infty}\int_{O(2^{j+2}B)\setminus O(2^{j+1}B)}\frac{1}{\omega(B(x,d(y,z)))}|f(z)|\mathrm{d}\omega(z)\\
&\le C\sum_{j=0}^{\infty}\frac{1}{\omega(B(x,2^{j+1}r))}\int_{O(2^{j+2}B)}|f(z)|\mathrm{d}\omega(z)\\
&\le C\sum_{j=0}^{\infty}\frac{1}{\omega(B(x,2^{j+1}r))}\Big(\int_{O(B(x,2^{j+2}r))}|f(z)|^q\mathrm{d}\omega(z)\Big)^{\frac{1}{q}}\Big[\omega(O(B(x,2^{j+2}r)))\Big]^{\frac{1}{q'}}\\
&\le C\|f\|_{L_d^{p,q}(\mathbb{R}^N)}\Big\{\sum_{j=0}^{\infty}\frac{[\omega(O(B(x,2^{j+2}r)))]^{1-\frac{1}{p}}}{\omega(B(x,2^{j+2}r))}\Big\},
\end{aligned}
$$

further, by (14) and (15), we deduce

$$
\begin{aligned}
\mathrm{F}_2 &\le C\|f\|_{L_d^{p,q}(\mathbb{R}^N)}\sup_B[\omega(O(B))]^{\frac{1}{p}}\Big\{\sum_{j=0}^{\infty}\frac{[\omega(O(B(x,2^{j+2}r)))]^{1-\frac{1}{p}}}{\omega(B(x,2^{j+2}r))}\Big\}\\
&\le C\|f\|_{L_d^{p,q}(\mathbb{R}^N)}\sup_B\Big[\sum_{j=0}^{\infty}\frac{\omega(O(2B))}{\omega(O(B(x,2^{j+2}r)))}\Big]^{\frac{1}{p}}\\
&\le C\|f\|_{L_d^{p,q}(\mathbb{R}^N)},
\end{aligned}
$$

which, together with the estimate for F_1, yields the desired result. Hence, the proof of Theorem 3.2 is finished. □

Proof of theorem 3.3. Let $B=B(x,r)$ be the open ball centered at $x\in\mathbb{R}^n$ and its radius $r>0$. And decompose functions f using the same form in (11), namely,

$$f=f_1+f_2,$$

where $f_1=f\chi_{O(2B)}$ and $f_2=f\chi_{\mathbb{R}^N\setminus(O(2B))}$. Then, using the Minkowski's inequality, write

$$\|\mathcal{M}_d(f)\|_{L_d^{p,q}(\mathbb{R}^N)}\le\|\mathcal{M}_d(f_1)\|_{L_d^{p,q}(\mathbb{R}^N)}+\|\mathcal{M}_d(f_2)\|_{L_d^{p,q}(\mathbb{R}^N)}=\mathrm{G}_1+\mathrm{G}_2.$$

From (14), Theorem 2.2 and (16), it follows that

$$\begin{aligned}
G_1 &= \sup_B [\omega(O(B))]^{\frac{1}{p}-\frac{1}{q}} \Big(\int_{\mathbb{R}^N} |\mathcal{M}_{d,b} f_1(y)|^q \mathrm{d}\omega(y) \Big)^{\frac{1}{q}} \\
&\le C\|b\|_d \sup_B [\omega(O(B))]^{\frac{1}{p}-\frac{1}{q}} \|f_1\|_{L^q(\mathbb{R}^N)} \\
&\le C\|b\|_d \sup_B \Big[\frac{\omega(O(B))}{\omega(O(2B))} \Big]^{\frac{1}{p}-\frac{1}{q}} [\omega(O(2B))]^{\frac{1}{p}-\frac{1}{q}} \Big(\int_{O(2B)} |f(y)|^q \mathrm{d}\omega(y) \Big)^{\frac{1}{q}} \\
&\le C\|b\|_d \|f\|_{L_d^{p,q}(\mathbb{R}^N)}.
\end{aligned}$$

To estimate G_4, we first consider the equation $|\mathcal{M}_{d,b}(f_2)(y)|$ with $y \in B$. By applying (2), (5), the Hölder's inequality, (7) and (8), we get

$$\begin{aligned}
|\mathcal{M}_{d,b}(f_2)(y)| &\le \int_{\mathbb{R}^N \setminus O(2B)} |K(y,z)||b(y)-b(z)||f(z)| \Big(\int_{d(y,z)}^{\infty} \frac{dt}{t^3} \Big)^{\frac{1}{2}} \mathrm{d}\omega(z) \\
&\le C \int_{\mathbb{R}^N \setminus O(2B)} \frac{|b(y)-b(z)|}{\omega(B(z,d(y,z)))} |f(z)| \mathrm{d}\omega(z) \\
&\le C|b(y)-b_{O(2B)}| \int_{\mathbb{R}^N \setminus O(2B)} \frac{1}{\omega(B(z,d(y,z)))} |f(z)| \mathrm{d}\omega(z) \\
&\quad + C \int_{\mathbb{R}^N \setminus O(2B)} \frac{|b(z)-b_{O(2B)}|}{\omega(B(z,d(y,z)))} |f(z)| \mathrm{d}\omega(z) \\
&\le C|b(y)-b_{O(2B)}| \sum_{i=1}^{\infty} \int_{O(2^{i+2}B) \setminus O(2^{i+1}B)} \frac{1}{\omega(B(x,d(y,z)))} |f(z)| \mathrm{d}\omega(z) \\
&\quad + \sum_{i=1}^{\infty} \int_{O(2^{i+2}B) \setminus O(2^{i+1}B)} \frac{|b(z)-b_{O(2B)}|}{\omega(B(x,d(y,z)))} |f(z)| \mathrm{d}\omega(z) \\
&\le C|b(y)-b_{O(2B)}| \sum_{i=1}^{\infty} \frac{1}{\omega(O(2^{i+1}B))} \int_{O(2^{i+2}B)} |f(z)| \mathrm{d}\omega(z) \\
&\quad + \sum_{i=1}^{\infty} \frac{1}{\omega(O(2^{i+1}B))} \int_{O(2^{i+2}B)} |b(z)-b_{O(2B)}||f(z)| \mathrm{d}\omega(z) \\
&\le C|b(y)-b_{O(2B)}| \sum_{i=1}^{\infty} \frac{1}{\omega(O(2^{i+1}B))} \int_{O(2^{i+2}B)} |f(z)| \mathrm{d}\omega(z) \\
&\quad + \sum_{i=1}^{\infty} \frac{|b_{O(2B)}-b_{O(2^{i+2}B)}|}{\omega(O(2^{i+1}B))} \int_{O(2^{i+2}B)} |f(z)| \mathrm{d}\omega(z) \\
&\qquad + \sum_{i=1}^{\infty} \frac{1}{\omega(O(2^{i+1}B))} \int_{O(2^{i+2}B)} |b(z)-b_{O(2^{i+2}B)}||f(z)| \mathrm{d}\omega(z) \\
&\le C|b(y)-b_{O(2B)}| \sum_{i=1}^{\infty} \frac{1}{[\omega(O(2^{i+1}B))]^{\frac{1}{q}}} \Big(\int_{O(2^{i+2}B)} |f(z)|^q \mathrm{d}\omega(z) \Big)^{\frac{1}{q}} \\
&\quad + C\|b\|_d \sum_{i=1}^{\infty} i \frac{1}{[\omega(O(2^{i+1}B))]^{\frac{1}{q}}} \Big(\int_{O(2^{i+2}B)} |f(z)|^q \mathrm{d}\omega(z) \Big)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+ C\sum_{i=1}^{\infty}\frac{1}{\omega(O(2^{i+1}B))}\Big(\int_{O(2^{i+2}B)}|b(z)-b_{O(2^{i+2}B)}|^{q'}\mathrm{d}\omega(z)\Big)^{\frac{1}{q'}}\\
&\quad\times\Big(\int_{O(2^{i+2}B)}|f(z)|^{q}\mathrm{d}\omega(z)\Big)^{q}\\
&\leq C\|f\|_{L_d^{p,q}(\mathbb{R}^N)}|b(y)-b_{O(2B)}|\sum_{i=1}^{\infty}\frac{1}{[\omega(O(2^{i+1}B))]^{\frac{1}{p}}}\\
&\quad + C\|b\|_d\|f\|_{L_d^{p,q}(\mathbb{R}^N)}\sum_{i=1}^{\infty}(i+1)\frac{1}{[\omega(O(2^{i+1}B))]^{\frac{1}{p}}}\\
&\leq C\|f\|_{L_d^{p,q}(\mathbb{R}^N)}\Big\{|b(y)-b_{O(2B)}|\sum_{i=1}^{\infty}\frac{1}{[\omega(O(2^{i+2}B))]^{\frac{1}{p}}}+\|b\|_d\sum_{i=1}^{\infty}\frac{i+1}{[\omega(O(2^{i+2}B))]^{\frac{1}{p}}}\Big\},
\end{aligned}
$$

by this, (8) and (16), we deduce

$$
\begin{aligned}
\mathrm{G}_2 &\leq C\|f\|_{L_d^{p,q}(\mathbb{R}^N)}\sup_B[\omega(O(B))]^{\frac{1}{p}-\frac{1}{q}}\Big\{\sum_{i=1}^{\infty}\frac{1}{\big[\omega(O(2^{i+2}B))\big]^{\frac{1}{p}}}\Big(\int_{O(B)}|b(y)-b_{O(2B)}|^{q}\mathrm{d}\omega(y)\Big)^{\frac{1}{q}}\\
&\quad+\sum_{i=1}^{\infty}\big[\omega(O(B))\big]^{\frac{1}{q}}\frac{(i+1)\|b\|_d}{\big[\omega(O(2^{i+2}B))\big]^{\frac{1}{p}}}\Big\}\\
&\leq C\|b\|_d\|f\|_{L_d^{p,q}(\mathbb{R}^N)}\sup_B\Big\{\sum_{i=0}^{\infty}(i+1)\frac{\big[\omega(O(B))\big]^{\frac{1}{p}}}{\big[\omega(O(B(x,2^{i+1}r)))\big]^{\frac{1}{p}}}\Big\}\\
&\leq C\|b\|_d\|f\|_{L_d^{p,q}(\mathbb{R}^N)}.
\end{aligned}
$$

Which, combining the estimate for G_1, yields our desired result. Therefore, the proof of Theorem 3.3 is completed. □

4. Boundedness for $\mathcal{M}_d$ and $\mathcal{M}_{d,b}$ on spaces $\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)$

In this section, we mainly investigate the boundedness of the operators $\mathcal{M}_d$ and $\mathcal{M}_{d,b}$ on generalized Morrey spaces $\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)$ associated with the Dunkl metric d, where $1 < p < \infty$ and $\varphi(x,r) : \mathbb{R}^N \times (0,\infty) \to (0,\infty)$ is a Lebesgue measurable function.

Definition 4.1. *Let $1 \leq p < \infty$ and $\varphi(x,r) : \mathbb{R}^N \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. Then the generalized Morrey space $\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)$ associated with the Dunkl metric d is defined by*

$$\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N) = \Big\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^N) : \|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)} < \infty\Big\},$$

where

$$\|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N, r>0}[\varphi(x,r)]^{-1}\Big(\int_{O(B(x,r))}|f(y)|^p\mathrm{d}\omega(y)\Big)^{\frac{1}{p}}, \tag{17}$$

and the supremum is taken over all balls $B \subset \mathbb{R}^N$.

We now state the definition of a class W_ϵ as follows.

Definition 4.2. *Let $\epsilon \in (0,1)$. A Lebesgue measurable function $\varphi(x,r) : \mathbb{R}^N \times (0,\infty) \to (0,\infty)$ is said to be in the class $\mathbb{W}_\epsilon$ if*

$$\sum_{i=0}^{\infty} \left[\frac{\omega(O(B(x,r)))}{\omega(O(B(x,2^{i+1}r)))}\right]^{\epsilon} \frac{\varphi(x,2^{i+1}r)}{\varphi(x,r)} \leq C_1, \tag{18}$$

and there exists a positive constant C_2 such that

$$\varphi(x,r) \geq C_2, \tag{19}$$

for all $r \geq 1$ and $x \in \mathbb{R}^N$.

It is now position to state the main results of this section as follows.

Theorem 4.3. *Let kernels $K(x,y)$ satisfy conditions* (3) *and* (4)*, $\varphi(\cdot,\cdot)$ be a Lebesgue measurable function defined as Definition* 4.2*, and $1 < p < \infty$. Suppose that $\mathcal{M}_d$ is bounded on spaces $L^2(\mathbb{R}^N)$. Then there exists some positive constant C such that, for all $f \in \mathcal{L}_{\varphi,p}(\mathbb{R}^N)$,*

$$\|\mathcal{M}_d(f)\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)} \leq C\|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)}.$$

Theorem 4.4. *Let kernels $K(x,y)$ satisfy conditions* (3) *and* (4)*, $b \in \mathrm{BMO}_d(\mathbb{R}^N)$, $1 < p < \infty$, and the Lebesgue measurable function $\varphi(\cdot,\cdot)$ meet the following condition*

$$\sum_{i=0}^{\infty} (i+1)\left[\frac{\omega(O(B(x,r)))}{\omega(O(B(x,2^{i+1}r)))}\right]^{\epsilon} \varphi(x,2^{i+1}r) \leq C\varphi(x,r) \tag{20}$$

for $\epsilon \in (0,1)$. Suppose that $\mathcal{M}_d$ is bounded on spaces $L^2(\mathbb{R}^N)$. Then there exists some positive constant C such that, for all $f \in \mathcal{L}^{\varphi,p}(\mathbb{R}^N)$,

$$\|\mathcal{M}_{d,b}(f)\|_{L_d^{\varphi,p}(\mathbb{R}^N)} \leq C\|b\|_d\|f\|_{L_d^{\varphi,p}(\mathbb{R}^N)}.$$

To prove the above theorem, we need to establish the following lemma.

Lemma 4.5. *Let $\epsilon \in (0,1)$ and $\varphi(x,r) \in \mathbb{W}_\epsilon$. Then there exists a positive constant C_N such that, for all ball B,*

$$\frac{\varphi(x,2r)}{\varphi(x,r)} \leq C_N.$$

In the other word, $\varphi(x,r)$ is a double function respecting to the radius r.

Proof. By applying (18) and (1), we deduce that

$$\begin{aligned}
\left[\frac{\omega(O(B(x,r)))}{\omega(O(B(x,2r)))}\right]^{\epsilon} \frac{\varphi(x,2^{i+1}r)}{\varphi(x,r)} \leq C_1 \Leftrightarrow \frac{\varphi(x,2^{i+1}r)}{\varphi(x,r)} &\leq C_1\left[\frac{\omega(O(B(x,2r)))}{\omega(O(B(x,r)))}\right]^{\epsilon} \\
\Rightarrow \frac{\varphi(x,2^{i+1}r)}{\varphi(x,r)} &\leq C_1\left[\frac{|G|\omega(B(x,2r))}{\omega(B(x,r))}\right]^{\epsilon} \\
\Rightarrow \frac{\varphi(x,2^{i+1}r)}{\varphi(x,r)} &\leq C_1\left(\frac{2r}{r}\right)^{N\epsilon} \\
\Rightarrow \frac{\varphi(x,2^{i+1}r)}{\varphi(x,r)} &\leq C_N. \quad \square
\end{aligned}$$

It is now position to state the proofs of Theorem 4.3 and Theorem 4.4 as follows.

Proof of theorem 4.3. Let $B = B(x,r)$ be the open ball centered at $x \in \mathbb{R}^n$ and its radius r. And decompose functions f using the same form in (11), that is,

$$f = f_1 + f_2,$$

where $f_1 = f\chi_{O(2B)}$ and $f_2 = f\chi_{\mathbb{R}^N\setminus(O(2B))}$. Then, by the Minkowski's inequality, write

$$\|\mathcal{M}_d(f)\|_{\mathcal{L}_{\varphi,p}(\mathbb{R}^N)} \le \|\mathcal{M}_d(f_1)\|_{\mathcal{L}_{\varphi,p}(\mathbb{R}^N)} + \|\mathcal{M}_d(f_2)\|_{\mathcal{L}_{\varphi,p}(\mathbb{R}^N)} = \mathrm{H}_1 + \mathrm{H}_2.$$

From (17), Theorem 2.1 and Lemma 4.5, it follows that

$$\begin{aligned}
\mathrm{H}_1 &\le \sup_{x\in\mathbb{R}^N, r>0} [\varphi(x,r)]^{-1}\Big(\int_{\mathbb{R}^N} |\mathcal{M}_d(f_1)(y)|^p \mathrm{d}\omega(y)\Big)^{\frac{1}{p}} \\
&\le \sup_{x\in\mathbb{R}^N, r>0} [\varphi(x,r)]^{-1}\|f_1\|_{L^p(\mathbb{R}^N)} \\
&\le \sup_{x\in\mathbb{R}^N, r>0} \frac{[\varphi(x,2r)]}{[\varphi(x,r)]}[\varphi(x,2r)]^{-1}\|f\chi_{O(2B)}\|_{L^p(\mathbb{R}^N)} \\
&\le C_N\|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)}
\end{aligned}$$

To estimate H_2, we first consider $|\mathcal{M}_d(f_2)(y)|$ with $y \in B(x,r)$. By applying (2), the Hölder's inequality, (17) and (18), we get

$$\begin{aligned}
|\mathcal{M}_d(f_2)(y)| &\le \int_{\mathbb{R}^N\setminus O(2B)} |K(y,z)||f(z)|\Big(\int_{d(y,z)\le t}^{\infty} \frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}} \mathrm{d}\omega(z) \\
&\le C\int_{\mathbb{R}^N\setminus O(2B)} \frac{1}{\omega(B(z,d(y,z)))}|f(z)|\mathrm{d}\omega(z) \\
&\le C\sum_{i=0}^{\infty}\int_{O(2^{i+2}B)\setminus O(2^{i+1}B)} \frac{1}{\omega(B(x,d(y,z)))}|f(z)|\mathrm{d}\omega(z) \\
&\le C\sum_{i=0}^{\infty}\frac{1}{\omega(B(x,2^{i+1}r))}\int_{O(2^{i+2}B)} |f(z)|\mathrm{d}\omega(z) \\
&\le C\sum_{i=0}^{\infty}\frac{1}{\omega(B(x,2^{i+1}r))}\Big(\int_{O(2^{i+2}B)} |f(z)|^p\mathrm{d}\omega(z)\Big)^{\frac{1}{p}}\Big[\omega(O(B(x,2^{i+2}r)))\Big]^{\frac{1}{p'}} \\
&\le C\|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)}\Bigg\{\sum_{i=0}^{\infty}\frac{1}{\Big[\omega(O(B(x,2^{i+2}r)))\Big]^{\frac{1}{p}}}\varphi(x,2^{l+2}r)\frac{\omega(O(B(x,2^{i+2}r)))}{\omega(B(x,2^{i+1}r))}\Bigg\} \\
&\le C\|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)}\Bigg\{\sum_{i=0}^{\infty}\frac{1}{\Big[\omega(O(B(x,2^{i+2}r)))\Big]^{\frac{1}{p}}}\varphi(x,2^{i+2}r)\Bigg\},
\end{aligned}$$

further, by applying (17) and (18), we have

$$\begin{aligned}
\mathrm{H}_2 &= \sup_{x\in\mathbb{R}^N, r>0} [\varphi(x,r)]^{-1}\Big(\int_{O(B(x,r))} |\mathcal{M}_d f_2(y)|^p \mathrm{d}\omega(y)\Big)^{\frac{1}{p}} \\
&\le C\|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)} \sup_{x\in\mathbb{R}^N, r>0}\Bigg\{\sum_{i=0}^{\infty}\Big\{\frac{\omega(O(B(x,r)))}{\omega(O(B(x,2^{i+1}r)))}\Big\}^{\frac{1}{p}}\frac{\varphi(x,2^{i+1}r)}{\varphi(x,r)}\Bigg\} \\
&\le C\|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)}.
\end{aligned}$$

Hence, the proof of Theorem 4.3 is finished. □

Proof of Theorem 4.4. Let $B = B(x,r)$ be the open ball centered at $x \in \mathbb{R}^n$ and its radius r. And decompose functions f using the same form in (11), that is,

$$f = f_1 + f_2,$$

where $f_1 = f\chi_{O(2B)}$. Then, by the Minkowski's inequality, write

$$\|\mathcal{M}_{d,b}(f)\|_{\mathcal{L}_{\varphi,p}(\mathbb{R}^N)} \leq \|\mathcal{M}_{d,b}(f_1)\|_{\mathcal{L}_{\varphi,p}(\mathbb{R}^N)} + \|\mathcal{M}_{d,b}(f_2)\|_{\mathcal{L}_{\varphi,p}(\mathbb{R}^N)} = \mathrm{I}_1 + \mathrm{I}_2.$$

From (17), Theorem 2.2 and Lemma 4.5, it then follows that

$$\begin{aligned}
\mathrm{I}_1 &\leq \sup_{x\in\mathbb{R}^N, r>0} [\varphi(x,r)]^{-1}\Big(\int_{\mathbb{R}^N} |\mathcal{M}_{d,b}f_1(y)|^p \mathrm{d}\omega(y)\Big)^{\frac{1}{p}} \\
&\leq C\|b\|_d \sup_{x\in\mathbb{R}^N, r>0} [\varphi(x,r)]^{-1}\|f_1\|_{L^p(\mathbb{R}^N)} \\
&\leq C\|b\|_d \sup_{x\in\mathbb{R}^N, r>0} \frac{\varphi(x,2r)}{\varphi(x,r)}[\varphi(x,2r)]^{-1}\|f\chi_{O(2B)}\|_{L^p(\mathbb{R}^N)} \\
&\leq C\|b\|_d\|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)}.
\end{aligned}$$

To estimate I_2, we first consider the form $|\mathcal{M}_{d,b}(f_2)(y)|$ with $y \in B$. Since

$$|\mathcal{M}_{d,b}(f_2)(y)| \leq |b(y) - b_{O(2B)}||\mathcal{M}_d(f_2)(y)| + |\mathcal{M}_d((b(\cdot) - b_{O(2B)})f_2)(y)|,$$

write

$$\begin{aligned}
\|\mathcal{M}_{d,b}(f_2)\|_{\mathcal{L}_{\varphi,p}(\mathbb{R}^N)} &= \sup_{x\in\mathbb{R}^N, r>0} [\varphi(x,r)]^{-1}\Big(\int_{O(B(x,r))} |\mathcal{M}_{d,b}(f_2)(y)|^p \mathrm{d}\omega(y)\Big)^{\frac{1}{p}} \\
&\leq \sup_{x\in\mathbb{R}^N, r>0} [\varphi(x,r)]^{-1}\Big(\int_{O(B(x,r))} |b(y) - b_{O(2B)}|^p|\mathcal{M}_d(f_2)(y)|^p \mathrm{d}\omega(y)\Big)^{\frac{1}{p}} \\
&\quad + \sup_{x\in\mathbb{R}^N, r>0} [\varphi(x,r)]^{-1}\Big(\int_{O(B(x,r))} |\mathcal{M}_d((b(\cdot) - b_{O(2B)})f_2)(y)|^p \mathrm{d}\omega(y)\Big)^{\frac{1}{p}} \\
&= \mathrm{I}_2^1 + \mathrm{I}_2^2.
\end{aligned}$$

By applying the estimate of $|\mathcal{M}_d(f_2)(y)|$, (7), (8) and (18), we obtain

$$\begin{aligned}
\mathrm{I}_2^1 &\leq C\|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)} \sup_{x\in\mathbb{R}^N, r>0} [\varphi(x,r)]^{-1}\Big\{\sum_{i=0}^{\infty} \frac{1}{\big[\omega(O(B(x,2^{i+2}r)))\big]^{\frac{1}{p}}}\varphi(x,2^{i+2}r)\Big\} \\
&\quad \times \Big(\int_{O(B(x,r))} |b(y) - b_{O(2B)}|^p |\mathrm{d}\omega(y)\Big)^{\frac{1}{p}} \\
&\leq C\|b\|_d\|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)} \sup_{x\in\mathbb{R}^N, r>0} \Big\{\sum_{i=0}^{\infty}\Big[\frac{\omega(O(B(x,r)))}{\omega(O(B(x,2^{i+2}r)))}\Big]^{\frac{1}{p}} \frac{\varphi(x,2^{i+2}r)}{\varphi(x,r)}\Big\} \\
&\leq C\|b\|_d\|f\|_{\mathcal{L}_d^{\varphi,p}(\mathbb{R}^N)}.
\end{aligned}$$

We now turn I_2^2, we first consider $|\mathcal{M}_d((b(\cdot) - b_{O(2B)})(f_2)(y)|$ with $y \in B$. By applying (3), the Hölder's inequality, (17), we have

$$
\begin{aligned}
&|\mathcal{M}_d((b(\cdot)-b_{O(2B)})f_2)(y)|\\
&\leq \int_{\mathbb{R}^N\setminus O(2B)} |K(y,z)||b(z)-b_{O(2B)}||f(z)|\Big(\int_{d(y,z)}^{\infty}\frac{dt}{t^3}\Big)^{\frac{1}{2}}\mathrm{d}\omega(z)\\
&\leq C\sum_{i=0}^{\infty}\int_{O(2^{i+2}B)\setminus O(2^{i+1}B)}\frac{|b(z)-b_{O(2B)}|}{\omega(B(x,d(y,z)))}|f(z)|\mathrm{d}\omega(z)\\
&\leq C\sum_{i=0}^{\infty}\frac{1}{\omega(B(x,2^{i+1}r))}\int_{O(2^{i+2}B)}|b(z)-b_{O(2^{i+2}B)}||f(z)|\mathrm{d}\omega(z)\\
&\leq C\sum_{i=0}^{\infty}\frac{|b_{O(2^{i+2}B)}-b_{O(2B)}|}{\omega(B(x,2^{i+1}r))}\int_{O(2^{i+2}B)}|f(z)|\mathrm{d}\omega(z)\\
&\quad + C\sum_{i=0}^{\infty}\frac{1}{\omega(B(x,2^{i+1}r))}\int_{O(2^{i+2}B)}|b(z)-b_{O(2^{i+2}B)}||f(z)|\mathrm{d}\omega(z)\\
&\leq C\|b\|_d\sum_{i=0}^{\infty}(i+1)\frac{1}{\omega(B(x,2^{i+1}r))}\Big(\int_{O(2^{i+2}B)}|f(z)|^p\mathrm{d}\omega(z)\Big)^{\frac{1}{p}}[\omega(O(2^{i+2}B))]^{1-\frac{1}{p}}\\
&\quad + C\sum_{i=0}^{\infty}\frac{1}{\omega(B(x,2^{i+1}r))}\Big(\int_{O(2^{i+2}B)}|f(z)|^p\mathrm{d}\omega(z)\Big)^{\frac{1}{p}}\Big(\int_{O(2^{i+2}B)}|b(z)-b_{O(2^{i+2}B)}|^{p'}\mathrm{d}\omega(z)\Big)^{\frac{1}{p'}}\\
&\leq C\|b\|_d\|f\|_{L_d^{\varphi,p}(\mathbb{R}^N)}\sum_{i=0}^{\infty}(i+1)\frac{\varphi(x,2^{i+2}r)}{\omega(B(x,2^{i+1}r))}[\omega(O(2^{i+2}B))]^{1-\frac{1}{p}}\\
&\leq C\|b\|_d\|f\|_{L_d^{\varphi,p}(\mathbb{R}^N)}\sum_{i=0}^{\infty}(i+1)\frac{\varphi(x,2^{i+2}r)}{[\omega(B(x,2^{i+2}r))]^{\frac{1}{p}}},
\end{aligned}
$$

further, by (17) and (20), we deduce

$$
\begin{aligned}
\mathrm{I}_2^2 &= \sup_{x\in\mathbb{R}^N,r>0}[\varphi(x,r)]^{-1}\Big(\int_{\mathbb{R}^N}|\mathcal{M}_d((b(\cdot)-b_{O(2B)})f_2)(y)|^p\mathrm{d}\omega(y)\Big)^{\frac{1}{p}}\\
&\leq C\|b\|_d\|f\|_{L_d^{\varphi,p}(\mathbb{R}^N)}\sup_{x\in\mathbb{R}^N,r>0}\left\{\sum_{i=0}^{\infty}(i+1)\Big[\frac{\omega(B)}{\omega(B(x,2^{i+2}r))}\Big]^{\frac{1}{p}}\frac{\varphi(x,2^{i+2}r)}{\varphi(x,r)}\right\}\\
&\leq C\|b\|_d\|f\|_{L_d^{\varphi,p}(\mathbb{R}^N)},
\end{aligned}
$$

which, combining the estimates of I_2^1 and I_1, yields our desired result. Hence, we finish the proof of Theorem 4.4. □

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