Filomat 39:33 (2025), 11797–11809
https://doi.org/10.2298/FIL2533797Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The left (b, c)-core inverse in rings and its applications

Yicheng Zhang[a,b], Li Wang[c], Huihui Zhu[d,*]

[a]*School of Management, University of Science and Technology of China, Hefei 230026, China*
[b]*School of Business, Anhui Xinhua University, Hefei 230088, China*
[c]*School of Computer and Information Technology, Anhui Vocational and Technical College, Hefei 230011, China*
[d]*School of Mathematics, Hefei University of Technology, Hefei 230009, China*

Abstract. Let R be a $*$-ring and let $a, b, c \in R$. The paper aims to introduce and investigate the left (b, c)-core inverses of a. The element $a \in R$ is left (b, c)-core invertible if there exists some $x \in R$ such that $caxc = c$, $xcab = b$ and $cax = (cax)^*$. Such an x is called a left (b, c)-core inverse of a. Several criteria for the left (b, c)-core inverse are given. Among of these, it is proved that a is left (b, c)-core invertible if and only if a is left (b, c)-invertible and c (or ca) is $\{1, 3\}$-invertible, under certain condition. Moreover, the connection between left (b, c)-core inverses and left (b, c)-inverses is established. Finally, $y = xcax$ is the (b, c)-core inverse of a if and only if the descending chain $caR \supseteq (ca)^2yR \supseteq \cdots \supseteq (ca)^{n+1}y^nR \supseteq \cdots$ stabilizes. Finally, the application of this type of generalized inverses is given to carbon emission.

1. Introduction

Generalized inverse covers a wide range of mathematical areas, such as matrix theory, operator theory, C^*-algebras, semigroups. It appears in numerous applications that include areas such as linear estimation, Markov chains, graphics, cryptography, coding theory and robotics.

In the last two decades, theory of new generalized inverses in such algebraic structures such as rings, semigroups, the settings of matrices undergoes a considerable development. The most celebrated two articles were Mary's [11] and Drazin's [8]. The development for the two new types of generalized inverses and their related extended generalized inverses was also kindled by a prior progression of the theory of matrix generalized inverses, see [6, 23]. Particularly, it is often stimulated by a wide range of their applications, among which solvability of equations plays a distinguished role; see e.g., [2] and [4] for a review of applications of the Moore–Penrose inverse in physics.

A list of properties about left versions of several types of generalized inverses are dual to that of right versions in rings, for instance, left (π-) regular and right (π-) regular elements, left inverses along an element and right inverses along an element, left (b, c)-inverses and right (b, c)-inverses. Furthermore, $a \in R$ is left

2020 *Mathematics Subject Classification*. Primary 15A09; Secondary 16W10.
Keywords. (b, c)-core inverses left (b, c)-inverses Moore–Penrose inverses carbon emission.
Received: 10 July 2025; Revised: 29 August 2025; Accepted: 30 August 2025
Communicated by Dragana Cvetković-Ilić
This research is supported by Anhui Xinhua University research team project (No. kytd202514). The authors would like to thank prof. Dragana Cvetkovic-Ilic for her useful suggestions and remarks during the preparation of this paper.
* Corresponding author: Huihui Zhu
Email addresses: zyc2019@mail.ustc.edu.cn (Yicheng Zhang), andy@uta.edu.cn (Li Wang), hhzhu@hfut.edu.cn (Huihui Zhu)
ORCID iDs: https://orcid.org/0000-0002-6676-7955 (Yicheng Zhang), https://orcid.org/0000-0002-0588-930X (Huihui Zhu)

$*$-regular if and only if it is right $*$-regular if and only if it is Moore–Penrose invertible, see [19, Theorem 2.16].

The connection between carbon dioxide and climate change is evident. The imbalance between the carbon emissions and its equivalents, and their absorption leads to a continual increase of their atmospheric concentration. In 2014, Vaninsky [15] used the Moore-Penrose inverse to give a generalized Divisia index approach.

In this paper, we aim to define the left (b,c)-core inverses in rings, and investigate its several properties. An application of the left (b,c)-core inverses is given.

2. Relevant concepts and notation

Throughout this paper, R denotes a unital ring with an involution $* : R \to R$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$ for all $a, b \in R$. An element $a \in R$ is Moore–Penrose invertible [13] if there exists an $x \in R$ such that (1) $axa = a$, (2) $xax = x$, (3) $(ax)^* = ax$, (4) $(xa)^* = xa$. Such an element x is called a Moore–Penrose inverse of a. It is unique if it exists, and is denoted by $a^\dagger$. If $a, x \in R$ satisfy the equations $\{i_1, ..., i_n\} \subseteq \{1,2,3,4\}$, then x is called a $\{i_1, ..., i_n\}$-inverse of a and is denoted by $a^{(i_1,...,i_n)}$. By $R^{(i_1,...,i_n)}$ we denote the set of all $\{i_1, ..., i_n\}$-invertible elements in R. Usually, $R^{(1,2,3,4)} = R^\dagger$ denotes the set of all Moore–Penrose invertible elements in R.

Recall that an element $a \in R$ is regular in the sense of von Neumann if $a \in aRa$, left regular if $a \in Ra^2$, right regular if $a \in a^2R$. An element $a \in R$ is called strongly regular if it is both left and right regular. Following [1], a is said to be π-regular, left π-regular, right π-regular and strongly π-regular if a^n is respectively regular, left regular, right regular and strongly regular for some positive integers n.

For any $a, b \in R$, following [10], Green's preorders and relations are defined by

(1) $a \leq_{\mathcal{L}} b$ if and only if $Ra \subseteq Rb$ if and only if $a = xb$ for some $x \in R$.

(2) $a \leq_{\mathcal{R}} b$ if and only if $aR \subseteq bR$ if and only if $a = by$ for some $y \in R$.

(3) $a \leq_{\mathcal{H}} b$ if and only if $a \leq_{\mathcal{L}} b$ and $a \leq_{\mathcal{R}} b$.

(4) $a\mathcal{L}b$ if and only if $Ra = Rb$ if and only if $a = xb, b = ya$ for some $x, y \in R$.

(5) $a\mathcal{R}b$ if and only if $aR = bR$ if and only if $a = bx, b = ay$ for some $x, y \in R$.

(6) $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$.

Based on Green's relations, Mary in [11] introduced the inverse along an element in a semigroup. We present this definition in an unital ring R. For a given $a, d \in R$, a is invertible along d if there is some $b \in R$ such that $bad = d = dab$ and $b \leq_{\mathcal{H}} d$. Such an element b is called the inverse of a along d. It is unique if it exists, and is denoted by $a^{\|d}$. By $R^{\|d}$ we denote the set of all invertible elements along d in R. It is known from [11] that $a \in R$ is group invertible if and only if it is invertible along a if and only if the inverse of 1 along a exists. In this case, $a^\# = a^{\|a}$ and $aa^\# = 1^{\|a}$. Afterward, left and right inverses along an element are defined in [19].

For any given $a, b, c \in R$, a is said to be (b,c)-invertible [8] if there exists $y \in R$ such that $y \in bRy \cap yRc$, $yab = b$ and $cay = c$. Such y is called the (b,c)-inverse of a. It is unique when it exists, and is denoted by $a^{(b,c)}$. It was shown in [8] that $a \in R$ is (b,c)-invertible if and only if $b \in Rcab$ and $c \in cabR$. In terms of this equivalence, one-sided versions of (b,c)-inverses were introduced and investigated in [9]. An element a is left (b,c)-invertible if $b \in Rcab$, or equivalently if there exists some $x \in Rc$ such that $xab = b$. Such an x is called a left (b,c)-inverse of a and denoted $a_l^{(b,c)}$. By $R_l^{(b,c)}$ we denote the set of all left (b,c)-invertible elements in R. Dually, the right (b,c)-inverse was defined. An element a is right (b,c)-invertible if $c \in cabR$, or equivalently if there exists some $x \in bR$ satisfying $cax = c$. In this case, any such x is a right (b,c)-inverse of a and we denote it by $a_r^{(b,c)}$. $R_r^{(b,c)}$ stands for the set of all right (b,c)-invertible elements in R. For any $a, d \in R$, it should be mentioned that a is (resp., left) (d,d)-invertible if and only if a is (resp., left) invertible along d, in which case, the (resp., left) (d,d)-inverse of a coincides with the (resp., left) inverse of a along d.

In 2010, Baksalary and Trenkler [3] presented the concept of the core inverse of a complex matrix and established its several properties. In 2014, Rakić et al.[14] generalized the core inverse and the dual core inverse in the ring with involution. For any $a \in R$, we say that a is core invertible if there is some $x \in R$ such that $axa = a$, $xR = aR$ and $Rx = Ra^*$, or equivalently, $axa = a$, $xax = x$, $ax^2 = x$, $xa^2 = a$ and $ax = (ax)^*$. Such an

x is called the core inverse of a. It is unique and is denoted by $a^{\#}$. As usual, by $R^{\#}$ we denote the set of all core invertible elements in R.

Let $a, w \in R$. The element a is called w-core invertible if there is some $x \in R$ satisfying $awx^2 = x$, $xawa = a$ and $(awx)^* = awx$ (see [21]). Such $x \in R$ is called the w-core inverse of a. It is unique if it exists, and is denoted by $a_w^{\#}$. We denote by $R_w^{\#}$ the set of all w-core invertible elements in R. Right and left w-core inverses were introduced in [20]. An element $x \in R$ is called a left w-core inverse of a if $awxa = a$, $xawa = a$, and $awx = (awx)^*$. Such x we denote by $a_{l,w}^{\#}$. More details for right w-core inverses can be found in [22].

In [23], the (b, c)-core inverse was introduced in a $*$-monoid. Let S be a $*$-monoid and $b, c \in S$. An element $a \in S$ is called (b, c)-core invertible if there exists some $x \in S$ such that $caxc = c$, $xS = bS$ and $Sx = Sc^*$. Such x is called a (b, c)-core inverse of a. It is unique if it exists and is denoted by $a_{(b,c)}^{\#}$. The important characterization for the (b, c)-core inverse is that a is (b, c)-core invertible if and only if a is (b, c)-invertible and c is $\{1, 3\}$-invertible, in which case, $a_{(b,c)}^{\#} = a^{(b,c)}c^{(1,3)}$. It should be noted that the w-core inverse is a special case of the (b, c)-core inverse, precisely, $a_w^{\#} = w_{(a,a)}^{\#}$. Subsequently, the right (b, c)-core inverse is defined in $*$-rings [7]. For given elements a, b, c of any $*$-ring R, a is called right (b, c)-core invertible if there exists some $x \in bR$ such that $caxc = c$ and $(cax)^* = cax$. In addition, it is shown that a is right (b, c)-core invertible if and only if $c \in cabR \cap Rc^*c$ if and only if a is right (b, c)-invertible and c is $\{1,3\}$-invertible.

It is of interest to introduce the left version of (b, c)-core inverses. It should be mentioned that the left (b, c)-core inverses are not completely dual to the right version of (b, c)-core inverses defined in [7] (see, e.g., Example 4.4 and Theorem 4.5). By choosing b and c appropriately in Definition 3.1 below, we could obtain equivalent alternative notions of other left generalized inverses. So, it is meaningful to introduce left (b, c)-core inverses in R.

We remind the reader that neither dimensional analysis nor special decomposition in Hilbert spaces and C^*-algebras can be used in rings. The results in this paper are proved by a purely ring theoretical method.

The paper is organized as follows. In Section 2, for any $a, b, c \in R$, the left (b, c)-core inverse of a is defined and several criteria are given in a $*$-ring R. We characterize the left (b, c)-core inverse by left invertible elements under certain conditions. In Section 3, we get the set consisting of all left (b, c)-inverses of a in R, provided that cab is regular. Theorem 4.6 shows the equivalence between the fact that a has a unique right (b, c)-core inverse and the fact that a has the (b, c)-core inverse. However, this does not hold for the case of left (b, c)-core inverses and (b, c)-core inverses in general. We construct two examples to indicate this fact. In Section 4, several properties on left (b, c)-core inverses are given. It is proved that $x \in R$ is a left (b, c)-core inverse of a if and only if $xcax$ is a left (b, c^*)-inverse of ca and $caxc = c$. Finally, we show that $y = xcax$ is the (b, c)-core inverse of a if and only if the descending chain $caR \supseteq (ca)^2yR \supseteq \cdots \supseteq (ca)^{n+1}y^nR \supseteq \cdots$ stabilizes. Finally, the application of left (b, c)-core inverses is given to establish the model of carbon emission.

3. Left (b, c)-core inverses

In this section, we introduce the left (b, c)-core inverse and present its several characterizations.

Definition 3.1. *Let $a, b, c \in R$. The element a is called left (b, c)-core invertible if there exists some $x \in R$ such that $caxc = c$, $xcab = b$ and $cax = (cax)^*$. Such x is called a left (b, c)-core inverse of a.*

For any $a \in R$, the left (b, c)-core inverse is not unique (see Example 4.4). By $a_{l,(b,c)}^{\#}$ we denote a left (b, c)-core inverse of a. $R_{l,(b,c)}^{\#}$ stands for the set of all left (b, c)-invertible elements in R.

Next, we give two auxiliary lemmas.

Lemma 3.2. [24, Lemma 2.2] *Let $a \in R$. Then*

(i) *$a \in R^{(1,3)}$ if and only if $a \in Ra^*a$. In particular, if $a = xa^*a$ for some $x \in R$, then x^* is $\{1, 3\}$-inverse of a.*

(ii) *$a \in R^{(1,4)}$ if and only if $a \in aa^*R$. In particular, if $a = aa^*y$ for some $y \in R$, then y^* is $\{1, 4\}$-inverse of a.*

Lemma 3.3. *Let $a, b, c \in R$. The following are equivalent*:

(i) *$c \in R(ca)^*c$.*

(ii) *There exist some* $x \in R$ *such that* $caxc = c$ *and* $cax = (cax)^*$.
(iii) $c \in caRc \cap Rc^*c$.
(iv) $c \in caR \cap Rc^*c$.
(v) $c \in caR$ *and* $ca \in R(ca)^*ca$.

Proof. Implications (ii) ⇒ (iii) ⇒ (iv) are evident.

(i) ⇒ (ii) Since $c \in R(ca)^*c \subseteq Rc^*c$, it follows from Lemma 3.2 that $c \in R^{(1,3)}$ and $as^* \in c\{1,3\}$, where $s \in R$ satisfies $c = s(ca)^*c \in R(ca)^*c$. So $cas^*c = c$ and $cas^* = (cas^*)^*$.

(iv) ⇒ (v) Given (iv), then $ca \in Rc^*ca \subseteq R(caR)^*ca \subseteq R(ca)^*ca$.

(v) ⇒ (i) As $c \in caR$, then there is some $t \in R$ such that $c = cat \in R(ca)^*cat = R(ca)^*c$. □

Theorem 3.4. *Let* $a, b, c \in R$. *The following conditions are equivalent*:
(i) $a \in R^{\circledast}_{l,(b,c)}$.
(ii) $b \in Rcab$ *and* $c \in R(ca)^*c$.
(iii) $b \in Rcab$ *and* $c \in caRc \cap Rc^*c$.
(iv) $b \in Rcab$ *and* $c \in caR \cap R(ca)^*c$.
(v) *There exists some* $x \in R$ *such that* $(cax)^n c = c$, $(xca)^n b = b$ *and* $((cax)^n)^* = (cax)^n$ *for any positive integer* n.
(vi) *There exists some* $x \in R$ *such that* $(cax)^n c = c$, $(xca)^n b = b$ *and* $((cax)^n)^* = (cax)^n$ *for some positive integer* n.
If one of the conditions $(i) - (vi)$ *is satisfied, then* $x(cax)^{n-1}$ *is a left* (b, c)*-core inverse of* a.

Proof. (i) ⇒ (ii) Since $a \in R^{\circledast}_{l,(b,c)}$, there exists $x \in R$ such that $caxc = c$, $xcab = b$ and $cax = (cax)^*$. Thus, $b = xcab \in Rcab$ and $c = caxc = (cax)^*c = x^*(ca)^*c \in R(ca)^*c$.

(ii) ⇒ (i) Since $b \in Rcab$, it follows that $b = zcab$ for some $z \in R$. By Lemma 3.3, we have that $c \in R(ca)^*c$ implies $c = catc$ and $(cat)^* = cat$ for some $t \in R$. Let $x = t + z - tcaz$. Then $cax = caz + cat - catcaz = caz + cat - caz = cat = (cax)^*$, $caxc = catc = c$ and $xcab = tcab + zcab - tcazcab = tcab + b - tcab = b$.

By Lemma 3.3 we have the following equivalences: (ii) ⇔ (iii) ⇔ (iv).

(i) ⇒ (v) Given (i), then there exists an $x \in R$ such that $caxc = c$, $xcab = b$ and $cax = (cax)^*$. So $cax = (caxc)ax = (cax)^2 = \cdots = (cax)^n$ and $b = xcab = xca(xcab) = (xca)^2 b = \cdots = (xca)^n b$ for any positive integer n. Therefore, $c = caxc = (cax)^n c$ and $cax = (cax)^n = ((cax)^n)^*$.

(v) ⇒ (vi) is evident.

(vi) ⇒ (i) Let $y = x(cax)^{n-1}$. Then $cay = cax(cax)^{n-1} = (cax)^n = ((cax)^n)^* = (cay)^*$, $cayc = cax(cax)^{n-1}c = (cax)^n c = c$ and $ycab = x(cax)^{n-1}cab = (xca)^{n-1}xcab = (xca)^n b = b$. □

From Theorem 3.4, it follows that if a is left (b, c)-core invertible then $c \in R(ca)^*c \subseteq Rc^*c$. It is natural to inquire the question that whether a is left (b, c)-core invertible if and only if $b \in Rcab$ and $c \in Rc^*c$. The answer to this question is negative, as we will see in the following example.

Example 3.5. Let R be the ring of all 2×2 complex matrices and let the involution $*$ be the conjugate transpose. Assume $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $c = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in R. Then $b \in Rcab$ and $c \in Rc^*c$. However, a is not left (b, c)-core invertible as there is no element $x \in R$ such that $caxc = 0 \neq c$.

Corollary 3.6. *Let* $a, b, c \in R$. *The following are equivalent*:
(i) $a \in R^{\circledast}_{l,(b,c)}$.
(ii) $a \in R_l^{(b,c)}$, $c \in R^{(1,3)}$ *and* $c \in caR$.
(iii) $a \in R_l^{(b,c)}$, $ca \in R^{(1,3)}$ *and* $c \in caR$.
If one of the conditions $(i) - (iii)$ *is satisfied, then* $a_l^{(b,c)}c^{(1,3)} + rc^{(1,3)} - rc^{(1,3)}caa_l^{(b,c)}c^{(1,3)}$ *is a left* (b, c)*-core inverse of* a *for any* $a_l^{(b,c)}$ *and* $c^{(1,3)}$, *where* $r \in R$ *satisfies* $c = car$.

Proof. (i) ⇔ (iii) ⇔ (iv) follow from Theorem 3.4.

Let us show that $x = a_l^{(b,c)}c^{(1,3)} + rc^{(1,3)} - rc^{(1,3)}caa_l^{(b,c)}c^{(1,3)}$ is a left (b, c)-core inverse of a. Indeed,

(1) $cax = caa_l^{(b,c)}c^{(1,3)} + carc^{(1,3)} - carc^{(1,3)}caa_l^{(b,c)}c^{(1,3)} = caa_l^{(b,c)}c^{(1,3)} + cc^{(1,3)} - caa_l^{(b,c)}c^{(1,3)} = cc^{(1,3)}$, where $r \in R$ satisfies $c = car$.

(2) $caxc = cc^{(1,3)}c = c$.

(3) Note that $a_l^{(b,c)} \in Rc$ and $a_l^{(b,c)}ab = b$. Then $xcab = a_l^{(b,c)}c^{(1,3)}cab + rc^{(1,3)}cab - rc^{(1,3)}caa_l^{(b,c)}c^{(1,3)}cab = a_l^{(b,c)}ab + rc^{(1,3)}cab - rc^{(1,3)}caa_l^{(b,c)}ab = b + rc^{(1,3)}cab - rc^{(1,3)}cab = b$. □

Remark 3.7. Let $r \in R$ be such that $c = car$. Then $ca(rc^{(1,3)}) = cc^{(1,3)} = (ca(rc^{(1,3)}))^*$ and $ca(rc^{(1,3)})ca = ca$. Hence $rc^{(1,3)} \in (ca)\{1,3\}$. Further, $a(ca)^{(1,3)} \in c\{1,3\}$. A left (b,c)-core inverse of a can be given as $a_l^{(b,c)}a(ca)^{(1,3)} + (ca)^{(1,3)} - (ca)^{(1,3)}caa_l^{(b,c)}a(ca)^{(1,3)}$.

We know from [20] that $a \in R$ is left w-core invertible if there exists some $x \in R$ such that $xawa = a$, $awxa = a$ and $awx = (awx)^*$. It was proved that $a \in R$ is left w-core invertible if and only if w is left (a,a)-core invertible. In this case, $w_{l,(a,a)}^{\oplus} = a_{l,w}^{\oplus}$.

Applying Corollary 3.6, we have the following result.

Corollary 3.8. [20, Theorem 2.7] *Let $a, w \in R$. The following are equivalent*:

(i) $a \in R_{l,w}^{\oplus}$.

(ii) $w \in R_l^{\|a}$, $a \in R^{(1,3)}$ *and* $a \in awR$.

(iii) $w \in R_l^{\|a}$, $aw \in R^{(1,3)}$ *and* $a \in awR$.

If one of the conditions (i) − (iii) is satisfied, then $w_l^{\|a}a^{(1,3)} + ra^{(1,3)} - ra^{(1,3)}aww_l^{\|a}a^{(1,3)}$ *is a left w-core inverse of a for any* $w_l^{\|a}$ *and* $a^{(1,3)}$, *where* $r \in R$ *satisfies* $a = awr$.

For any $a, b, c \in R$, it follows from [23] that $a \in R_{(b,c)}^{\oplus}$ if and only if there exists some $x \in R$ such that $caxc = c$, $xR = bR$ and $Rx \subseteq Rc^*$. Especially, $a \in R$ is (a,a)-core invertible if and only if a is core invertible. In this case, $a^{\oplus} = 1_{(a,a)}^{\oplus} = aa_{(a,a)}^{\oplus}$.

Given any $a, b \in R$, the right preorder $a \leq^{\circ} b$ is defined as: if $bs = bt$ then $as = at$ for all $s, t \in R$. Dually, the left preorder $a^{\circ} \leq b$ is defined as: if $sb = tb$ then $sa = ta$ for all $s, t \in R$. Obviously, $Ra \subseteq Rb$ implies $a \leq^{\circ} b$, and $aR \subseteq bR$ implies $a^{\circ} \leq b$. The following result characterizes left (b,c)-core invertible elements in terms of the preorder.

Recall that an element $p \in R$ is an idempotent if $p = p^2$, in addition, if $p = p^*$, then p is a projection.

Theorem 3.9. *Let $a, b, c \in R$. The following conditions are equivalent*:

(i) $a \in R_{l,(b,c)}^{\oplus}$.

(ii) *There exists $y \in R$ such that $cayc = c$, $b = ycab$, $(cay)^* = cay$ and $ycay = y$.*

(iii) *There exists $y \in R$ such that $cayc = c$, $y^*R = cR$ and $bR \subseteq yR$.*

(iv) *There exists $y \in R$ such that $cayc = c$, $y^*R \subseteq cR$ and $bR \subseteq yR$.*

(v) *There exists $y \in R$ such that $cayc = c$, $y^{*\circ} \leq c$ and $b^{\circ} \leq y$.*

(vi) *There exists $y \in R$ such that $ycay = y$, $y^*R = cR$ and $bR \subseteq yR$.*

(vii) *There exists $y \in R$ such that $ycay = y$, $cR \subseteq y^*R$ and $bR \subseteq yR$.*

(viii) *There exists $y \in R$ such that $ycay = y$, $c^{\circ} \leq y^*$ and $b^{\circ} \leq y$.*

(ix) *There exist an unique projection $p \in R$ and an idempotent $e \in R$ such that $cR = pR = caR$, $Rca = Re$ and $bR \subseteq eR$.*

(x) *There exist a projection $p \in R$ and an idempotent $e \in R$ such that $cR = pR = caR$, $Rca = Re$ and $bR \subseteq eR$. If one of the conditions (i) − (x) is satisfied, then $e(ca)^{(1)}p$ is a left (b,c)-core inverse of a, where $(ca)^{(1)} \in (ca)\{1\}$.*

Proof. (iii) ⇒ (iv) ⇒ (v), (vi) ⇒ (vii) ⇒ (viii) and (ix) ⇒ (x) are clear.

(i) ⇒ (ii) Since $a \in R_{l,(b,c)}^{\oplus}$, there exists $x \in R$ such that $caxc = c$, $xcab = b$ and $cax = (cax)^*$. Let $y = xcax$, then $cayc = c$, $ycab = b$, $cay = (cay)^*$ and $ycay = y$.

(ii) ⇒ (iii) Assume that (*ii*) holds. Then $ca = cayca = (cay)^*ca = y^*(ca)^*ca \in y^*R$ and $y = ycay = y(cay)^* = yy^*(ca)^* \in R(ca)^*$. Hence $y^*R = caR$. Also, $bR = ycabR \subseteq yR$. Since $cayc = c$, it follows that

$cR = caycR \subseteq caR \subseteq cR$, and $cR = caR$. (v) $\Rightarrow$ (i) Since $cayc = c$ and $y^* \overset{\circ}{\leq} c$, it follows that $cayy^* = y^*$. Post-multiplying $cayy^* = y^*$ by $(ca)^*$ yields $(cay)^* = cay(cay)^* = cay$. Hence $y = y(cay)^* = ycay$. Post-multiplying again by $b \overset{\circ}{\leq} y$, we get $ycab = b$.

(iii) $\Rightarrow$ (vi) Let $y^*R \subseteq cR$. Then there exists $t \in R$ such that $y^* = ct = cayct = cayy^*$. Now $(cay)^* = y^*(ca)^* = cayy^*(ca)^* = cay(cay)^* = cay$. Therefore, $y = y(cay)^* = ycay$.

(viii) $\Rightarrow$ (ix) Since $ycay = y$, we have $y^* = (cay)^*y^*$, which together with $c \overset{\circ}{\leq} y^*$ gives that $(cay)^*c = c$. Post-multiplying $(cay)^*c = c$ by ay yields $cay = (cay)^*cay = (cay)^*$. So, $c = (cay)^*c = cayc$. Let $p = cay$ and $e = yca$. Then $p = p^2 = p^*$ and $e^2 = e$. Thus, $cR = caycR = pcR \subseteq pR \subseteq caR \subseteq cR$ which implies $pR = cR = caR$. From $y = ycay$, it follows that $b = ycab = eb$ and consequently $bR \subseteq eR$. Also, $Rca = Rcayca \subseteq Re = Ryca \subseteq Rca$, i.e., $Rca = Re$.

To prove the uniqueness, let us suppose that the projections $p_1, p_2 \in R$ both satisfy conditions. Then $p_1R = cR = caR = p_2R$, and hence $p_1 = p_2$ by [14, Lemma 2.10].

(x) $\Rightarrow$ (i) By $cR = pR = caR$, we get $c = pc$, $ca = pca$ and $p = cav$ for some $v \in R$. Hence, $ca = cavca$, i.e., ca is regular. The condition $Rca = Re$ implies $ca = cae$ and $e = tca$ for some $t \in R$. Also, $bR \subseteq eR$ gives $b = eb$. Let $x = e(ca)^{(1)}p$, where $(ca)^{(1)} \in (ca)\{1\}$. Then $cax = cae(ca)^{(1)}p = ca(ca)^{(1)}p = ca(ca)^{(1)}cau = cau = p = p^* = (cax)^*$, $caxc = pc = c$ and $xcab = e(ca)^{(1)}pcab = tca(ca)^{(1)}cab = tcab = eb = b$. □

Remark 3.10. The left (b, c)-core inverse could be defined in a semigroup S with involution $*$ satisfying $(a^*)^* = a$ and $ab^* = b^*a^*$ for all $a, b \in S$. Theorem 3.9 can be extended to a semigroup with involution.

For any $a, b, c \in R$, characterizations of the (b, c)-core inverse of a are described in terms of properties of the left (right) annihilators and ideals in [23]. It is proved that $a \in R^{\textcircled{\#}}_{(b,c)}$ if and only if $R = R(cab)^* \oplus {}^0c = Rca \oplus {}^0b$ which is further equivalent to $R = R(cab)^* + {}^0c = Rca + {}^0b$. Inspired by this characterization, we will give the characterization of left (b, c)-core invertible elements in R.

Theorem 3.11. *Let $a, b, c \in R$. The following statements are equivalent*:

(i) $a \in R^{\textcircled{\#}}_{l,(b,c)}$.

(ii) $R = R(ca)^* \oplus {}^0c = Rca + {}^0b$.

(iii) $R = R(ca)^* + {}^0c = Rca + {}^0b$.

Proof. (i) $\Rightarrow$ (ii) Since $a \in R^{\textcircled{\#}}_{l,(b,c)}$, by Theorem 3.4, we have $c \in R(ca)^*c$. Hence $c = r(ca)^*c = ra^*c^*c$ for some $r \in R$ and $1 - r(ca)^* \in {}^0c$. Now, for any $s \in R$, we have $s = s[(1 - r(ca)^*) + r(ca)^*] = s(1 - r(ca)^*) + sr(ca)^* \in {}^0c + R(ca)^*$. So $R = {}^0c + R(ca)^*$. Note also that $ar^* \in c\{1, 3\}$. For any $z \in R(ca)^* \cap {}^0c$, we have $zc = 0$ and there is some $t \in R$ such that $z = t(ca)^* = t(cc^{(1,3)}ca)^* = t(car^*ca)^* = t(ca)^*(car^*)^* = zcar^* = 0$. So, $R = R(ca)^* \oplus^0 c$.

On the other hand, $a \in R^{\textcircled{\#}}_{l,(b,c)}$ implies $b \in Rcab$. So there is some $v \in R$ such that $b = vcab$ and consequently $1 - vca \in {}^0b$. Therefore, $R = Rca + {}^0b$ since $1 = vca + (1 - vca) \in Rca + {}^0b$.

(ii) $\Rightarrow$ (iii) is evident.

(iii) $\Rightarrow$ (i) Assume that $R = R(ca)^* + {}^0c$. Then $c \in Rc \subseteq R(ca)^*c$. By $R = Rca + {}^0b$, we have $b \in Rb = Rcab$. So, by Theorem 3.4 we have $a \in R^{\textcircled{\#}}_{l,(b,c)}$. □

If $b\mathcal{L}c$, then $b \in R^{(1)}$ if and only if $c \in R^{(1)}$, which is also true for the case of {1,4}-inverses. A dual result can be given under the condition $b\mathcal{R}c$.

Lemma 3.12. *Let $b, c \in R$ be such that $b\mathcal{L}c$. Then*

(i) *$b \in R^{(1)}$ if and only if $c \in R^{(1)}$. Moreover, $b = bc^{(1)}c$ and $c = cb^{(1)}b$, for any $b^{(1)} \in b\{1\}$ and $c^{(1)} \in c\{1\}$.*

(ii) *$b \in R^{(1,4)}$ if and only if $c \in R^{(1,4)}$. Moreover, $b = bc^{(1,4)}c$ and $c = cb^{(1,4)}b$, for any $b^{(1,4)} \in b\{1, 4\}$ and $c^{(1,4)} \in c\{1, 4\}$.*

Lemma 3.13. *Let $b, c \in R$ be such that $b\mathcal{R}c$. Then*

(i) *$b \in R^{(1)}$ if and only if $c \in R^{(1)}$. Moreover, $b = cc^{(1)}b$ and $c = cb^{(1)}b$ for any $b^{(1)} \in b\{1\}$ and $c^{(1)} \in c\{1\}$.*

(ii) *$b \in R^{(1,3)}$ if and only if $c \in R^{(1,3)}$. Moreover, $b = cc^{(1,3)}b$ and $c = bb^{(1,3)}c$ for any $b^{(1,3)} \in b\{1, 3\}$ and $c^{(1,3)} \in c\{1, 3\}$.*

Lemma 3.14. [19, Theorem 2.11] *Let $a \in R$. The following conditions are equivalent*:
(i) $a \in R^{\dagger}$.
(ii) $a \in aa^*aR$.
(iii) $a \in Raa^*a$.
In this case, $a^{\dagger} = (ax)^* = (ya)^*$, where $x, y \in R$ satisfy $a = aa^*ax = yaa^*a$.

For given $a \in R$, the symbols a_l^{-1} and a_r^{-1} stand for a left inverse and a right inverse of a, respectively. By R_l^{-1} and R_r^{-1} we denote the sets of all left invertible and right invertible elements in R, respectively.

Lemma 3.15. *Let $a, b \in R$. Then $\alpha = 1 + ab$ is left (right) invertible if and only if $\beta = 1 + ba$ is left (right) invertible. Moreover, $\beta_l^{-1} = 1 - b\alpha_l^{-1}a$.*

In the next theorem, we will give characterizations for the class of left (b, c)-core inverses, generalizing the results for the left w-core inverses given in [20]. First, we will give an auxiliary result on the left (b, c)-inverse.

Theorem 3.16. *Let $a, b, c, p \in R$ be such that b is regular, $b\mathcal{L}c$, $b \leq_{\mathcal{L}} pb$ and $c \leq_{\mathcal{R}} cq$. The following are equivalent*:
(i) $a \in R_l^{(pbq,cq)}$.
(ii) $u = bqap + 1 - bb^{(1)} \in R_l^{-1}$.
(iii) $v = qapb + 1 - b^{(1)}b \in R_l^{-1}$.
(iv) $w = qapb + 1 - c^{(1)}c \in R_l^{-1}$.
If any one of the conditions $(i) - (iv)$ is satisfied, then $a_l^{(b,c)} = pu_l^{-1}b$.

Proof. Since $b \in R^{(1)}$ and $b\mathcal{L}c$, it follows from Lemma 3.12 that $c \in R^{(1)}$, $b = bc^{(1)}c$ and $c = cb^{(1)}b$. Also, $b \leq_{\mathcal{L}} pb$ and $c \leq_{\mathcal{R}} cq$ imply $b = rpb$ and $c = cqt$, for some $r, t \in R$.
(i) ⇒ (ii) Since $a \in R_l^{(pbq,cq)}$, we have $pbq \in Rcqapbq$, and consequently $pbq = xcqapbq$, for some $x \in R$. By $c = cqt$ and $b\mathcal{L}c$ it follows that $b = bqt$ and $pb = pbqt = xcqapbqt = xcqapb$. Hence, we get $bqapbb^{(1)} + 1 - bb^{(1)} \in R_l^{-1}$. Let $y = bb^{(1)}rxcb^{(1)} + 1 - bb^{(1)}$. Using that $c = cb^{(1)}b$, we have

$$\begin{aligned} y(bqapbb^{(1)} + 1 - bb^{(1)}) &= (bb^{(1)}rxcb^{(1)} + 1 - bb^{(1)})(bqapbb^{(1)} + 1 - bb^{(1)}) \\ &= bb^{(1)}rxcb^{(1)}bqapbb^{(1)} + bb^{(1)}rxcb^{(1)}(1 - bb^{(1)}) \\ &\quad +(1 - bb^{(1)})bqapbb^{(1)} + (1 - bb^{(1)})(1 - bb^{(1)}) \\ &= bb^{(1)}rxcqapbb^{(1)} + 0 + 0 + 1 - bb^{(1)} = 1. \end{aligned}$$

Hence, $bqapbb^{(1)} + 1 - bb^{(1)} \in R_l^{-1}$. Lemma 3.15 again ensures that $u = bqap + 1 - bb^{(1)} \in R_l^{-1}$.
(ii) ⇒ (i) Since $ub = bqapb$, we have $b = u_l^{-1}bqapb$ and consequently $pbq = pu_l^{-1}bqapbq = pu_l^{-1}bc^{(1)}cqapbq \in Rcqapbq$. Now $pu_l^{-1}bc^{(1)}c = pu_l^{-1}b$ is a left (pbq, cq)-inverse of a.
(ii) ⇔ (iii) This follows by Lemma 3.15.
(iii) ⇔ (iv) Note that $v = qapb + 1 - b^{(1)}b = qapbc^{(1)}c + 1 - b^{(1)}bc^{(1)}c \in R_l^{-1}$ if and only if $c^{(1)}cqapb + 1 - c^{(1)}cb^{(1)}b = c^{(1)}cqapb + 1 - c^{(1)}c \in R_l^{-1}$ if and only if $qapb + 1 - c^{(1)}c \in R_l^{-1}$. □

In the case when $p = q = 1$ in Theorem 3.16, we get the following corollary:

Corollary 3.17. *Let $a, b, c \in R$ be such that b is regular and $b\mathcal{L}c$. The following conditions are equivalent*:
(i) $a \in R_l^{(b,c)}$.
(ii) $u = ba + 1 - bb^{(1)} \in R_l^{-1}$.
(iii) $v = ab + 1 - b^{(1)}b \in R_l^{-1}$.
(iv) $w = ab + 1 - c^{(1)}c \in R_l^{-1}$.
If one of the conditions $(i) - (iv)$ is satisfied, then $a_l^{(b,c)} = u_l^{-1}b$.

Taking $b = c = d$ in Corollary 3.17, we get the criterion for the left inverse along an element.

Corollary 3.18. [19, Corollary 3.3] *Let $a, d \in R$ be such that d is regular. The following conditions are equivalent:*
(i) $a \in R_l^{\|d}$.
(ii) $u = da + 1 - dd^{(1)} \in R_l^{-1}$.
(iii) $v = ad + 1 - d^{(1)}d \in R_l^{-1}$.
If one of the conditions (i) − (iii) is satisfied, then $a_l^{\|d} = u_l^{-1}d$.

Applying Corollaries 3.6 and 3.17, one has the following result.

Theorem 3.19. *Let $a, b, c \in R$ be such that $b\mathcal{H}c$. The following conditions are equivalent*:
(i) $a \in R^{\#}_{l,(b,c)}$.
(ii) $c \in caR$ *and* $u = ba + 1 - bb^{(1,3)} \in R_l^{-1}$ *for any* $b^{(1,3)} \in b\{1, 3\}$.
(iii) $c \in caR$ *and* $v = ab + 1 - b^{(1,3)}b \in R_l^{-1}$ *for any* $b^{(1,3)} \in b\{1, 3\}$.
(iv) $c \in caR$ *and* $w = ab + 1 - c^{(1,3)}c \in R_l^{-1}$ *for any* $c^{(1,3)} \in c\{1, 3\}$.

It is clear that left (b, c)-core invertible element may not be left (c, b)-core invertible. There are lots of examples to illustrate this fact. Now, we consider when an element is both left (b, c)-core invertible and left (c, b)-core invertible.

Lemma 3.20. [17, Proposition 3.2] *Let $a, b, c \in R$ be such that b, c are regular with inner inverses $b^{(1)}$, $c^{(1)}$, respectively. The following statements are equivalent*:
(i) *a is left (b, c)-invertible and left (c, b)-invertible.*
(ii) $u_1 = abac + 1 - c^{(1)}c$, $u_2 = acab + 1 - b^{(1)}b$ *are left invertible.*
(iii) $v_1 = caba + 1 - cc^{(1)}$, $v_2 = baca + 1 - bb^{(1)}$ *are left invertible.*

Proposition 3.21. *Let $a, b, c \in R$ be such that $c \in caR$ and $b \in baR$. The following statements are equivalent*:
(i) $a \in R^{\#}_{l,(b,c)} \cap R^{\#}_{l,(c,b)}$.
(ii) $u_1 = abac + 1 - c^{(1,3)}c$, $u_2 = acab + 1 - b^{(1,3)}b$ *are left invertible.*
(iii) $v_1 = caba + 1 - cc^{(1,3)}$, $v_2 = baca + 1 - bb^{(1,3)}$ *are left invertible.*

Proof. (i) ⇔ (ii) follows from Corollary 3.6 and Lemma 3.20, and (ii) ⇔ (iii) follows by Lemma 3.15. □

4. The set consisting of all left (b, c)-core invertible elements

In this section, we mainly discuss the properties of the set consisting of all left (b, c)-core invertible elements. Also, the relation between right (b, c)-core inverses and (b, c)-core inverses is given.

For any $a, w \in R$ with $a \in R^{\#}_{l,w}$, it follows that awa is regular and aw is {1,3}-invertible (see [20]). Moreover, $a(awa)^{(1)} + (aw)^{(1,3)} - (aw)^{(1,3)}awa(awa)^{(1)}$ is a left w-core inverse of a, where $(awa)^{(1)} \in (awa)\{1\}$ and $(aw)^{(1,3)} \in (aw)\{1, 3\}$. However, for the case of left (b, c)-core inverse, cab may not be regular, for instance, let $R = \mathbb{Z}$ be the ring of all integers and the involution be the identity map. Take $a = c = 1$, $b = 2$, then a is left (b, c)-core invertible with the inverse $x = 1$. However, $cab = 2$ is not regular.

For any $a, b, c \in R$ with cab regular, we give the set composed of all (b, c)-core inverses of a in the following result.

Proposition 4.1. *Let $a, b, c \in R$ be such that cab is regular and $a \in R^{\#}_{l,(b,c)}$. Then $b(cab)^{(1)} + (ca)^{(1,3)} - (ca)^{(1,3)}cab(cab)^{(1)}$ is a left (b, c)-core inverse of a, where $cab^{(1)} \in (cab)\{1\}$ and $(ca)^{(1,3)} \in (ca)\{1, 3\}$.*

Proof. Let $y = b(cab)^{(1)} + (ca)^{(1,3)} - (ca)^{(1,3)}cab(cab)^{(1)}$. Then

(1) Since $a \in R^{\#}_{l,(b,c)}$, by Corollary 3.6, we get that $c = car$, for some $r \in R$. So $cayc = cab(cab)^{(1)}c + ca(ca)^{(1,3)}c - ca(ca)^{(1,3)}cab(cab)^{(1)}c = ca(ca)^{(1,3)}c = ca(ca)^{(1,3)}car = car = c$.

(2) Since $a \in R^{\#}_{l,(b,c)}$, we have $b = scab$ for some $s \in R$. Now, $ycab = b(cab)^{(1)}cab + (ca)^{(1,3)}cab - (ca)^{(1,3)}cab(cab)^{(1)}cab = scab(cab)^{(1)}cab = scab = b$,

(3) $cay = cab(cab)^{(1)} + ca(ca)^{(1,3)} - ca(ca)^{(1,3)}cab(cab)^{(1)} = ca(ca)^{(1,3)} = (cay)^*$. □

Suppose $a \in R^{(1,3)}$. Then the set of all {1,3}-inverses of a can be expressed as $a\{1,3\} = \{a^{(1,3)} + (1 - a^{(1,3)}a)z : z \in R\}$ (see [5]).

In the following two results, we present the set consisting of all left (b,c)-core inverses of a.

Theorem 4.2. *Let $a, b, c \in R$ be such that $a \in R^{\textcircled{\#}}_{l,(b,c)}$ and cab is regular. Then $\{a^{\textcircled{\#}}_{l,(b,c)}\} = \{b(cab)^{(1)} + (ca)^{(1,3)} - (ca)^{(1,3)}cab(cab)^{(1)} + (1 - (ca)^{(1,3)}ca)v(1 - cab(cab)^{(1)}),$ where $v \in R$ is an arbitrary element$\}$.*

Proof. For any $v \in R$, by a direct calculation, we get that $b(cab)^{(1)}+(ca)^{(1,3)}-(ca)^{(1,3)}cab(cab)^{(1)}+(1-(ca)^{(1,3)}ca)v(1-cab(cab)^{(1)})$ is a left (b,c)-core inverse of a. So, we have $\{b(cab)^{(1)}+(ca)^{(1,3)}-(ca)^{(1,3)}cab(cab)^{(1)}+(1-(ca)^{(1,3)}ca)v(1-cab(cab)^{(1)}),$ where $v \in R\} \subseteq \{a^{\textcircled{\#}}_{l,(b,c)}\}$.

Conversely, let $y \in R$ be a left (b,c)-core inverse of a. As $y \in (ca)\{1,3\}$, then there exists $z \in R$ such that $y = (ca)^{(1,3)} + (1 - (ca)^{(1,3)}ca)z$, and hence

$$\begin{aligned}
&(1 - (ca)^{(1,3)}ca)z(1 - cab(cab)^{(1)}) \\
&= (1 - (ca)^{(1,3)}ca)z - (1 - (ca)^{(1,3)}ca)zcab(cab)^{(1)} \\
&= y - (ca)^{(1,3)} - (y - (ca)^{(1,3)})cab(cab)^{(1)} \\
&= y - (ca)^{(1,3)} - ycab(cab)^{(1)} + (ca)^{(1,3)}cab(cab)^{(1)} \\
&= y - (ca)^{(1,3)} - b(cab)^{(1)} + (ca)^{(1,3)}cab(cab)^{(1)}
\end{aligned}$$

So, $y = b(cab)^{(1)} + (ca)^{(1,3)} - (ca)^{(1,3)}cab(cab)^{(1)} + (1 - (ca)^{(1,3)}ca)z(1 - cab(cab)^{(1)})$ and consequently $y \in \{b(cab)^{(1)} + (ca)^{(1,3)} - (ca)^{(1,3)}cab(cab)^{(1)} + (1 - (ca)^{(1,3)}ca)v(1 - cab(cab)^{(1)}), v \in R\}$. □

Let $a, b, c \in R$ be such that cab is regular and a is left (b,c)-invertible. Then $\{a_l^{(b,c)}\} = \{b(cab)^{(1)}c + v[1 - cab(cab)^{(1)}]c, v \in R\}$. Suppose that $a, b, c, t \in R$ are such that cab regular and $b = tcab$. Then $t = tcab(cab)^{(1)} + t(1 - cab(cab)^{(1)}) \in \{b(cab)^{(1)} + v[1 - cab(cab)^{(1)}], v \in R\}$. For any $v \in R$, $(b(cab)^{(1)} + v[1 - cab(cab)^{(1)}])cab = a_l^{(b,c)}ab = b$. Therefore, $\{t | b = tcab\} = \{b(cab)^{(1)} + v[1 - cab(cab)^{(1)}] : v \in R\}$. So, Theorem 4.2 can be simplified and $\{a^{\textcircled{\#}}_{l,(b,c)}\} = \{t + (ca)^{(1,3)} - (ca)^{(1,3)}cat : t \in R, b = tcab\}$.

Proposition 4.3. *Let $a, b, c \in R$ and let cab be regular and $a \in R^{\textcircled{\#}}_{l,(b,c)}$. If $x \in R$ is a left (b,c)-core inverse of a, then $\{a^{\textcircled{\#}}_{l,(b,c)}\} = \{x + (1 - xca)r(1 - cab(cab)^{(1)}),$ where $r \in R, (cab)^{(1)} \in (cab)\{1\}\}$.*

Proof. Since x is a left (b,c)-core inverse of a, it is easy to check that $x + (1 - xca)r(1 - cab(cab)^{(1)})$ is also a left (b,c)-core inverse of a. On the other hand, since $a\{1,3\} = \{a^{(1,3)} + (1 - a^{(1,3)}a)r,$ where $r \in R\}$, for any $y \in \{a^{\textcircled{\#}}_{l,(b,c)}\}$, we have that $y \in (ca)\{1,3\}$ and consequently y can be written as $y = x + (1 - xca)r$, for some $r \in R$. So $b = ycab = xcab + (1 - xca)rcab = b + (1 - xca)rcab$ and hence $(1 - xca)rcab = 0$. Now, $y = x + (1 - xca)r(1 - cab(cab)^{(1)}) \in \{x + (1 - xca)r(1 - cab(cab)^{(1)}),$ where $r \in R, (cab)^{(1)} \in (cab)\{1\}\}$. □

The following example implies the fact that a is (b,c)-core invertible does not imply that a has a unique left (b,c)-core inverse in general. Example 4.5 below also shows that a has a unique left (b,c)-core inverse also does not imply that a is (b,c)-core invertible.

Example 4.4. Let R be the ring of all 2×2 complex matrices and let the involution $*$ be the conjugate transpose. Let $a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $b = c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the (b,c)-core inverse of a. However, any matrix of the form $\begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$ is left (b,c)-core inverse of a.

Example 4.5. Let $R = M_2(\mathbb{Z}_3)$ with transpose as involution. Take $a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $x = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$ is a unique left (b,c)-core inverse of a. But $c \notin cabR$, and hence a is not (b,c)-core invertible.

For any $a, b, c \in R$, it is known from [7, Theorem 3.14] that a is right (b, c)-core invertible if and only if ca is right (b, c^*)-invertible. Moreover, the two inverses coincide with each other. It is of interest to consider the relation between the (b, c)-core inverse of a and uniqueness of the right (b, c)-core inverse of a.

Theorem 4.6. *Let $a, b, c \in R$. Then a has a unique right (b, c)-core inverse if and only if a is (b, c)-core invertible.*

Proof. Suppose that a has a unique right (b, c)-core inverse. Then ca has the unique right (b, c^*)-inverse, and cab is regular. Note that $(c^{(1,3)})^*c^*cab = (cc^{(1,3)})^*cab = cc^{(1,3)}cab = cab$, and hence $(cab)\mathcal{L}(c^*cab)$. So c^*cab is regular and $\{(ca)_r^{(b,c^*)}\} = \{b(c^*cab)^{(1)}c + b[1 - (c^*cab)^{(1)}c^*cab]s, s \in R\}$ has a unique element which means that $b[1 - (c^*cab)^{(1)}c^*cab] = 0$. Therefore, $b = b(c^*cab)^{(1)}c^*cab \in Rcab$, so a is left (b, c)-invertible. Thus, $a \in R^{(b,c)}$ and $c \in R^{(1,3)}$, i.e., a is (b, c)-core invertible.

Conversely, let $x = a^{\oplus}_{(b,c)}$ and let $y \in R$ be a right (b, c)-core inverse of a. Since $x, y \in bR$, there exist some $v, w \in R$ such that $x = bv$ and $y = bw$. Now, $y = bw = xcabw = xcay = xcaxcay = x(cax)^*(cay)^* = x(caycax)^* = x(cax)^* = xcax = x$, as required. □

From Examples 4.4, 4.5 and Theorem 4.6, we conclude the fact that left (b, c)-core inverse is not completely dual to the right (b, c)-core inverse.

5. Connections with other generalized inverses

In this section, we mainly consider the relation between the left (b, c)-core inverse and other types of generalized inverses.

Proposition 5.1. *Let $a, b, c \in R$ be such that ca is left invertible. Then the following conditions are equivalent:*
(i) *a is left (b, c)-core invertible.*
(ii) *$ca \in R^{\dagger}$ and $c \in caR$.*
If (i) or (ii) is satisfied, then $(ca)^{\dagger}$ is a left (b, c)-core inverse of a.

Proof. (i) ⇒ (ii) Since ca is left invertible, we have $xca = 1$ for some $x \in R$. As a is left (b, c)-core invertible, then there exists $y \in R$ such that $cayc = c$, $ycay = y$, $ycab = b$ and $(cay)^* = cay$. Hence $c = cayc \in caR$. Pre-multiplying and post-multiplying $cayc = c$ by x and a imply $yca = xcayca = xca = 1$, and consequently $(yca)^* = yca$. So, $ca \in R^{\dagger}$ and $(ca)^{\dagger} = y$.

(ii) ⇒ (i) Let $t = (ca)^{\dagger}$. Then $catca = ca$, $tcat = t$, $(cat)^* = cat$, $(tca)^* = tca$ and $pca = 1$ for some $p \in R$. Pre-multiplying $catca = ca$ by p, we get $tca = 1$ and consequently $tcab = b$. Moreover, $catca = ca$ can be reduced to $catc = c$. So, t is a left (b, c)-core inverse of a. □

Proposition 5.2. *Let $a \in R$. The following statements are equivalent:*
(i) $a \in R^{\dagger}$.
(ii) $a \in R^{\oplus}_{l,(a^*,a^*)}$.
(iii) $a^* \in R^{\oplus}_{l,(a,a)}$.
(iv) $a^* \in R_l^{(a,a)}$.
(v) $a \in R_l^{(a^*,a^*)}$.

Proof. (i) ⇔ (ii) By Lemma 3.14, it follows that $a \in R^{\dagger}$ if and only if $a \in aa^*aR$. Also, by Theorem 3.4, $a \in R^{\oplus}_{l,(a^*,a^*)}$ if and only if $a^* \in Ra^*aa^* \cap Ra^*aa^*$ if and only if $a \in aa^*aR$.

(i) ⇔ (iii) It follows from Lemma 3.14 that $a \in R^{\dagger}$ if and only if $a \in Raa^*a$. The rest proof can be obtained by a similar way of (i) ⇔ (ii).

(i) ⇔ (iv) ⇔ (v) are clear. □

Theorem 5.3. *Let $a, b, c \in R$. Then $x \in R$ is a left (b, c)-core inverse of a if and only if $caxc = c$ and $xcax$ is a left (b, c^*)-inverse of ca.*

Proof. Suppose $x \in R$ is a left (b,c)-core inverse of a. By Theorem 3.9 (i) $\Rightarrow$ (ii), we have $caxc = c$, $xcax = x$, $xcab = b$ and $cax = (cax)^*$. Hence $b = xcab = (xcax)cab = x(cax)^*cab = x(ax)^*c^*cab \in Rc^*cab$. So, $xcax$ is a left (b,c^*)-inverse of ca.

Conversely, as $y = xcax$ be a left (b,c^*)-inverse of ca, then $b = ycab = xcaxcab = x(caxc)ab = xcab$ and $y = tc^*$ for some $t \in R$. To prove that x is a left (b,c)-core inverse of a, it suffices to show that $cax = (cax)^*$. Note that $cayc = ca(xcax)c = cax(caxc) = caxc = c$. Then $c^* = c^*(cay)^*$. Now, $y = tc^* = tc^*(cay)^* = y(cay)^*$ and $cay = cay(cay)^* = (cay)^*$, which in turn gives $y = ycay$ and $cax = (caxc)ax = ca(xcax) = cay = (cay)^* = (cax)^*$. □

Remark 5.4. Remark that the condition $caxc = c$ in Theorem 5.3 can not be removed. Let R be an infinite complex matrix ring whose rows and columns are both finite and let the involution $*$ be the conjugate transpose. Let $a = b = c = \sum_{i=1}^{\infty} e_{i+1,i}$. Then $c^*c = a^*a = 1$ and $b = a^*c^*cab \in Rc^*cab$, i.e., ca is left (b,c^*)-invertible. However, for any $x \in R$, $caxc \neq c$. In fact, if there exists some $x \in R$ such that $caxc = c^2xc = c$, then $1 = c^*c = c^*c^2xc = cxc$, which guarantees that c is invertible. This is a contradiction.

The following result gives the connection between left (b,c)-core inverse and left inverse along an element.

Theorem 5.5. *Let $a,b,c \in R$ be such that $b\mathcal{L}c$ and $c \in R^{(1,4)}$. Then $x \in R$ is a left (b,c)-core inverse of a if and only if $xcax$ is a left inverse of ca along bc^* and $c = caxc$.*

Proof. Since $c \in R^{(1,4)}$ and $b\mathcal{L}c$, we have $b = bc^{(1,4)}c$ and $c = tb$ for some $t \in R$. Suppose $x \in R$ is a left (b,c)-core inverse of a. Then $caxc = c$, $xcax = x$, $xcab = b$ and $(cax)^* = cax$. So

$$\begin{aligned} bc^* &= xcabc^* = xcaxcabc^* = xx^*a^*c^*cabc^* = xx^*a^*(cc^{(1,4)}c)^*cabc^* = xx^*a^*c^{(1,4)}cc^*cabc^* \\ &= xx^*a^*c^{(1,4)}tbc^*cabc^* \in Rbc^*cabc^*. \end{aligned}$$

Hence, $xcax$ is a left inverse of ca along bc^*.

Conversely, let $y = xcax$ be a left inverse of ca along bc^*, i.e., $ycabc^* = bc^*$ and $y \in Rbc^* \subseteq Rc^*$. Then $b = bc^*(c^{(1,4)})^* = ycabc^*(c^{(1,4)})^* = ycab = xcaxcab = xcab$. Note that $cayc = caxcaxc = caxc = c$ and $y \in Rc^*$. So $y = y(cay)^*$ and $cay = (cay)^*$. Hence $cax = caxcax = cay = (cay)^* = (cax)^*$, as required. □

A set C of submodules M satisfies that for every descending chain $T_1 \supseteq T_2 \supseteq \cdots \supseteq T_n \supseteq \cdots$ in C, then $T_n = T_{n+i}$ for some integer n $(i = 1,2,\ldots)$.

Let $a,b,c \in R$ be such that $b\mathcal{R}c$. If $x \in R$ is a left (b,c)-core inverse of a, then $xcac = c$. For any $n \in \mathbb{N}$, it follows that

$$\begin{aligned} (ca)^{n+2}x^{n+1} &= (ca)^{n+1}(x(ca)^2)x^{n+1} = (ca)^{n+1}(x^2(ca)^3)x^{n+1} \\ &= \cdots \\ &= (ca)^{n+1}(x^n(ca)^{n+1})x^{n+1} \in (ca)^{n+1}x^nR. \end{aligned}$$

Similarly, we get that $(ca)^{n+1}x^{n+2} \in (ca)^n x^{n+1}R$.

For any $a,b,c,x \in R$, we have two following descending chains in the set of right ideals,

$$\begin{gathered} caR \supseteq (ca)^2xR \supseteq \cdots \supseteq (ca)^{n+1}x^nR \supseteq \cdots, \\ xR \supseteq cax^2R \supseteq \cdots \supseteq (ca)^n x^{n+1}R \supseteq \cdots, \end{gathered}$$

provided that x is a left (b,c)-core inverse of a.

In the following result we will give a characterization for the (b,c)-core inverse in terms of the descending sequences of right principal ideals.

Theorem 5.6. *Let $a,b,c \in R$ be such that $b\mathcal{R}c$ and let $x \in R$ be a left (b,c)-core inverse of a. Let $y = xcax$. Then the following statements are equivalent*:

(i) *y is the (b,c)-core inverse of a.*

(ii) *The set $\{(ca)^{n+1}y^n, n \in \mathbb{N}\}$ is finite.*

(iii) *The descending chain $caR \supseteq (ca)^2yR \supseteq \cdots \supseteq (ca)^{n+1}y^nR \supseteq \cdots$ stabilizes.*

Proof. (i) ⇒ (ii) Since y is the (b,c)-core inverse of a, then $y \in bR$, $cayc = c$ and $b\mathcal{R}c$. Hence $y = cay^2$. By induction, we get that $(ca)^{n+1}y^n = (ca)^2y$ for any $n \in \mathbb{N}$. So, the set $\{(ca)^{n+1}y^n, n \in \mathbb{N}\}$ has at most two elements and hence is finite.

(ii) ⇒ (iii) is evident.

(iii) ⇒ (i) Since x is a left (b,c)-core inverse of a, by the proof of Theorem 3.9 (i) ⇒ (ii), $y = xcax$ is a left (b,c)-core inverse of a, so that $cayc = c$, $ycab = b$, $ycay = y$ and $cay = (cay)^*$. To show that y is the (b,c)-core inverse of a, it is sufficient to prove that $y \in bR$. Note that $b\mathcal{R}c$ and $ycab = b$. Then $ycac = c$, and we get $y = ycay = y(ycac)ay \in y^2R$, i.e., y is right regular. So, $yR = y^2R = y^3R = \cdots = y^nR = \cdots$ for any positive integer n. Thus, we have $(ca)^{n+1}y^nR = (ca)^{n+1}yR = (ca)^n(cay)R = (ca)^n(ca)R = (ca)^{n+1}R$ for any positive integer n. So the descending chain reduces to: $caR \supseteq (ca)^2R \supseteq (ca)^3R\cdots$ stabilizes. Then there exists a positive integer n such that $(ca)^nR = (ca)^{n+1}R$. In the case when $n = 1$: We have that $caR = (ca)^2R$. In the case when $n \geq 2$: Since $ycac = c \Rightarrow y(ca)^2 = ca$, it follows that $y(ca)^{n+1} = (ca)^n$ for any positive integer n. Pre-multiplying $(ca)^nR = (ca)^{n+1}R$ by y gives $(ca)^{n-1}R = (ca)^nR$. Repeating this process, we get $caR = (ca)^2R$. Therefore, $ycaR = y(ca)^2R = caR$, which implies $yca = cas$ for some $s \in R$. Thus, $y = ycay = casy \in bR$ since $b\mathcal{R}c$. □

Remark 5.7. The condition $b\mathcal{R}c$ in Theorem 5.6 is necessary. Let R be the ring of all 2×2 complex matrices and let the involution $*$ be the conjugate transpose. Assume $a = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ and $c = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $x = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{2}{3} \end{bmatrix}$ is the left (b,c)-core inverse of a. Let $y = xcax$. It is easy to check that $y = xcax = x$ is the (b,c)-core inverse of a. However, the set $\{(ca)^{n+1}y^n, n \in \mathbb{N}\} = \{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & (\frac{2}{3})^{n-1} \end{bmatrix}, n \in \mathbb{N}^*\}$ is infinite.

Similarly, we establish the other criteria for the (b,c)-core inverse of a.

Theorem 5.8. *Let $a,b,c \in R$ be such that $b\mathcal{R}c$ and let $x \in R$ be a left (b,c)-core inverse of a. Let $y = xcax$. Then the following are equivalent:*

(i) *y is the (b,c)-core inverse of a.*

(ii) *Set $\{(ca)^ny^{n+1}, n \in \mathbb{N}\}$ is finite.*

(iii) *The descending chain $yR \supseteq cay^2R \supseteq \cdots \supseteq (ca)^ny^{n+1}R \supseteq \cdots$ stabilizes.*

In [21], it was proved that for any $a,w \in R$, a is w-core invertible if and only if $a \in awR$ and aw is core invertible. We will give a similar characterization for left (b,c)-core inverses.

Proposition 5.9. *Let $a,b,c \in R$ be such that $b \in baR$. Then $a \in R^{\#}_{l,(b,c)}$ if and only if $c \in caR$ and $1 \in R^{\#}_{l,(ba,ca)}$. In this case, the left (ba,ca)-core inverse of 1 coincides with the left (b,c)-core inverse of a.*

Proof. Suppose $a \in R^{\#}_{l,(b,c)}$. Then by Theorem 3.4 (i) ⇒ (iii), we have $b \in Rcab$, $c \in R(ca)^*c$ and $c \in caR$, so that $ca \in R(ca)^*ca$ and $ba \in Rcaba$. Again, by Theorem 3.4 (ii) ⇒ (i), it follows that $1 \in R^{\#}_{l,(ba,ca)}$.

Conversely, as $1 \in R^{\#}_{l,(ba,ca)}$, then $ba \in Rcaba$ and $ca \in R(ca)^*ca$. From $b \in baR$, it follows that $ba \in Rcaba$ and hence $b \in Rcab$. Note that $c \in caR$. Then $ca \in R(ca)^*ca$ implies $c \in R(ca)^*c$. Therefore, $a \in R^{\#}_{l,(b,c)}$. □

An element $a \in R$ is called left core invertible [20] if it is left 1-core invertible. By $R^{\#}_{l}$ we denote the set of all left core invertible elements in R.

As a special case of Proposition 5.9, characterizations for the left w-core inverse and left core inverse are given.

Corollary 5.10. *Let $a,w \in R$. Then $a \in R^{\#}_{l,w}$ if and only if $a \in awR$ and $aw \in R^{\#}_{l}$. In this case, the left w-core inverse of a coincides with the left core inverse of aw.*

6. Possible applications to modeling of carbon emission

The scientific forecasting of carbon emission trends is imperative to understand carbon emission levels and achieve dual carbon goals. Carbon emission prediction models based on back propagation neural networks and multiple linear regression methods have yielded relatively favourable results. Many carbon emission models have fewer equations than unknowns

In the 1980s, Vaninsky [16] investigated the Divisia index method to interconnected indicators. Let the resulting indicator Z be a function of the factorial indicators $X_1, X_2, \cdots, X_n$ that are interconnected by the system of equations: $Z = f(X) = f(X_1, X_2, \cdots S_n)$, $\Phi_j(X_1, \cdots, X_n) = 0$. The following formula $\triangle Z[X|\Phi] = \int_L \nabla Z^T(I - \Phi_X \Phi_X^\dagger)dX$ for prediction models was proved by Vaninsky in 1984, where upper index "†" denotes the Moore-Penrose inverse, and I is the identity matrix, $\triangle Z[X|\Phi]$ are components of the factorial decomposition of the change in the resulting indicator Z in the presence of the factors' interconnections, Φ_X is a Jacobian matrix for the matrix-valued function $\Phi(X)$.

In the past several decades, the topic of carbon emissions has attracted lots of scholars. For instance, In 2014, Vaninsky [15] provided a more general theoretical framework - an introduction to the generalized Divisia index method. Modeling and forecasting of carbon emission are of great significance for building carbon emission reduction and policy formulation. it was shown that see the paper of Vaninsky [15] for details. In order to predict carbon evolution trend reasonably, this paper provides a reference for carbon emission reduction work by using our defined generalized inverses.

As shown above, the Moore-Penrose inverse is a special case of the left (b, c)-core inverse, Precisely, a is Moore-Penrose invertible if and only if a is left (a^*, a^*)-core invertible. Motivated by [15, 16], we aim to reconsider the formula $\triangle Z[X|\Phi] = \int_L \nabla Z^T(I - \Phi_X \Phi_{X_{l,(b,c)}}^{\#})dX$ by replacing the Moore-Penrose inverse with the left (b, c)-core inverse of a.

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