Filomat 39:33 (2025), 11811–11824
https://doi.org/10.2298/FIL2533811O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sufficient conditions for the existence of an isolated solution to a nonlocal boundary value problem for a nonlinear third-order pseudoparabolic equation

Nurgul Orumbayeva[a], **Alua Manat**[a], **Tenggesh Tokmagambetova**[a,*]

[a]*Institute of Applied Mathematics, Buketov Karaganda National Research University, Karagandy, Kazakhstan*

Abstract. In this paper, a third-order nonlinear pseudo-parabolic equation with nonlocal boundary conditions is studied. The original boundary value problem is reduced to a multicharacteristic nonlinear boundary value problem with functional parameters. Subsequently, the resulting problem is transformed into a system of integro-differential equations, which enables the proposal of an iterative algorithm for finding the solution. Sufficient conditions for the existence, uniqueness, and convergence of the solution are established. It is proved that the sequence of approximations generated by this algorithm converges to an isolated solution of the problem.

1. Introduction

Linear and nonlinear partial differential equations with nonlocal boundary conditions have been studied by numerous authors [1–3, 5–7, 11, 12, 14, 20, 22]. Such equations arise in various applications, including heat conduction in materials with memory, fluid filtration in porous media, and other processes involving nonlocal interactions. In this paper, we consider a boundary value problem for a third-order nonlinear pseudo-parabolic equation with nonlocal boundary conditions. The main goal is to establish sufficient conditions for the existence of an isolated solution and to propose an efficient algorithm for its computation.

To this end, the original problem is reduced to a multicharacteristic boundary value problem, to which the method of parameterization is applied. The parameterization method was introduced in [8] for solving a two-point boundary value problem for an ordinary differential equation. In earlier works [4, 13, 15–19], this method was extended to address periodic boundary value problems for linear and nonlinear partial differential equations of second and third orders.

2020 *Mathematics Subject Classification*. Primary 35K70; Secondary 35K35, 65N15.
Keywords. partial differential equations, nonlocal boundary value problem, third order equations, algorithms, convergence conditions.
Received: 27 February 2025; Accepted: 19 September 2025
Communicated by Marko Nedeljkov
Research supported by this research has been/was/is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488729).
* Corresponding author: Tenggesh Tokmagambetova
Email addresses: orumbayevanurgul@gmail.com (Nurgul Orumbayeva), aluamanat5@gmail.com (Alua Manat), tenggesh.tokmagambetova@gmail.com (Tenggesh Tokmagambetova)
ORCID iDs: https://orcid.org/0000-0003-1714-6850 (Nurgul Orumbayeva), https://orcid.org/0009-0008-5829-2786 (Alua Manat), https://orcid.org/0000-0003-1984-8485 (Tenggesh Tokmagambetova)

In the present work, the resulting multicharacteristic boundary value problem is transformed into a system of integro-differential equations, allowing us to apply an iterative algorithm for constructing an approximate solution.

The relevance of this research lies in the need to develop mathematical methods for analyzing complex nonlinear systems under nonlocal conditions. The results of this study can be applied to a wide range of problems arising in mathematical physics and engineering applications.

2. Statement of the boundary value problem

In this paper, on the domain $\Omega = [0, \omega] \times [0, T]$ we consider the following boundary value problem

$$\frac{\partial^3 u(x,t)}{\partial x^2 \partial t} = f\Big(x, t, u(x,t), \frac{\partial u(x,t)}{\partial t}, \frac{\partial^2 u(x,t)}{\partial x^2}\Big), \quad (x,t) \in \Omega, \quad u \in R, \tag{1}$$

$$u(0,t) = \varphi(t), \quad t \in [0,T], \tag{2}$$

$$\frac{\partial u(0,t)}{\partial x} = \psi(t), \quad t \in [0,T], \tag{3}$$

$$b_1(x)\frac{\partial^2 u(x,0)}{\partial x^2} + b_2(x)\frac{\partial^2 u(x,T)}{\partial x^2} + b_3(x)\frac{\partial u(x,0)}{\partial t} + b_4(x)\frac{\partial u(x,T)}{\partial t} + \\ + b_5(x)u(x,0) + b_6(x)u(x,T) = \theta(x), \quad x \in [0, X], \tag{4}$$

where $f : \Omega \times R \times R \times R \to R$ is a continuous function. The functions $\theta(x)$, $b_j(x)$, $j = \overline{1,6}$ are continuous on $[0, X]$, and the functions $\varphi(t)$, $\psi(t)$ are continuous on $[0, T]$.

A function $u(x,t) \in C(\Omega, R)$, that possesses the partial derivatives $\frac{\partial u(x,t)}{\partial x} \in C(\Omega, R)$, $\frac{\partial u(x,t)}{\partial t} \in C(\Omega, R)$, $\frac{\partial^2 u(x,t)}{\partial x^2} \in C(\Omega, R)$, $\frac{\partial^2 u(x,t)}{\partial x \partial t} \in C(\Omega, R)$, $\frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \in C(\Omega, R)$ is called a solution of problem (1)–(4) if it satisfies system (1) for all $(x,t) \in \Omega$, and the boundary conditions (2)–(4).

In order to find a solution, we introduce the function $w(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$, then we can write

$$u(x,t) = \varphi(t) + \psi(t)x + \int_0^x \int_0^\xi w(\xi_1, t) d\xi_1 d\xi, \quad \frac{\partial u(x,t)}{\partial t} = \varphi'(t) + \psi'(t)x + \int_0^x \int_0^\xi \frac{\partial w(\xi_1, t)}{\partial t} d\xi_1 d\xi$$

with this substitution, problem (1)–(4) can be rewritten as

$$\frac{\partial w(x,t)}{\partial t} = f\Big(x, t, \varphi(t) + \psi(t)x + \int_0^x \int_0^\xi w(\xi_1,t) d\xi_1 d\xi, \varphi'(t) + \psi'(t)x + \int_0^x \int_0^\xi \frac{\partial w(\xi_1,t)}{\partial t} d\xi_1 d\xi, w(x,t)\Big), \tag{5}$$

$$b_1(x)w(x,0) + b_2(x)w(x,T) + b_3(x)\int_0^x \frac{\partial w(\xi,0)}{\partial t} d\xi + b_4(x)\int_0^x \frac{\partial w(\xi,T)}{\partial t} d\xi +$$

$$+ b_5(x)\int_0^x \int_0^\xi w(\xi_1, 0) d\xi_1 d\xi + b_6(x)\int_0^x \int_0^\xi w(\xi_1, T) d\xi_1 d\xi = \widetilde{\theta}(x), \quad x \in [0, X], \tag{6}$$

where $\widetilde{\theta}(x) = \theta(x) + [b_3(x) + b_4(x)]\psi'(t) + [b_5(x) + b_6(x)][\varphi(t) + \psi(t)x]$.

The problems (1)–(4) and (5), (6) are equivalent in the following sense: if $u(x,t)$ is a solution of problem (1)–(4), then the function $w(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$ is a solution of the nonlocal boundary value problem for the nonlinear equation (5), (6). Conversely, if $w(x,t)$ is a solution of problem (5), (6), then the function

$u(x,t) = \varphi(t) + \psi(t)x + \int\limits_0^x\int\limits_0^\xi w(\xi_1,t)d\xi_1 d\xi$ is a solution of the third-order nonlocal boundary value problem (1)–(4).

At step $h > 0 : Nh = T$ we will perform the partition $[0,T) = \bigcup\limits_{r=1}^{N}[(r-1)h, rh)$, $N = 1,2,\dots$. In this case, the domain Ω is divided into N parts. By $w_r(x,t)$ we denote, respectively, the restriction of the functions $w(x,t)$ to $\Omega_r = [0,\omega]\times[(r-1)h, rh)$, $r = \overline{1,N}$. Then problem (5), (6) will be equivalent to the boundary value problem

$$\frac{\partial w_r(x,t)}{\partial t} = f\Big(x,t,\varphi(t)+\psi(t)x+\int\limits_0^x\int\limits_0^\xi w_r(\xi_1,t)d\xi_1 d\xi, \varphi'(t)+\psi'(t)x+\int\limits_0^x\int\limits_0^\xi \frac{\partial w_r(\xi_1,t)}{\partial t}d\xi_1 d\xi, w_r(x,t)\Big), \quad (7)$$

$$b_1(x)w_1(x,0) + b_2(x)\lim_{t\to T-0} w_N(x,t) + b_3(x)\int\limits_0^x \frac{\partial w_1(\xi,0)}{\partial t}d\xi + b_4(x)\lim_{t\to T-0}\int\limits_0^x \frac{\partial w_N(\xi,t)}{\partial t}d\xi +$$

$$+b_5(x)\int\limits_0^x\int\limits_0^\xi w_1(\xi_1,0)d\xi_1 d\xi + b_6(x)\int\limits_0^x\int\limits_0^\xi \lim_{t\to T-0} w_N(\xi_1,t)d\xi_1 d\xi = \widetilde{\theta}(x), \quad x\in[0,X], \quad (8)$$

$$\lim_{t\to sh-0} w_s(x,t) = w_{s+1}(x,sh), \quad x\in[0,X], \quad s=\overline{1,N-1}, \quad (9)$$

where (9) is the condition of continuity of the solution in the internal lines of the partition. The solution to problem (7)–(9) is the system of functions $w^*(x,[t]) = (w_1^*(x,t), w_2^*(x,t), ..., w_N^*(x,t))' \in C(\Omega_r, R^N)$, where each function $w_r^*(x,t)$, $r=\overline{1,N}$ has partial derivatives with respect to t that are continuous and bounded on Ω_r for all $(x,t)\in\Omega_r$, $r=\overline{1,N}$. Moreover, the equalities (8), (9) hold for $w_1(x,0)$, $\lim\limits_{t\to T-0} w_N(x,t)$, $\lim\limits_{t\to sh-0} w_s(x,t)$, $w_{s+1}(x,sh)$.

The problems (5), (6) and (7)–(9) are equivalent. Their equivalence is established in the same way as the equivalence of problems (1)–(4) and (5), (6).

In problems (7)–(9), we introduce the notation $\lambda_r(x) = w_r(x,(r-1)h)$, and perform the substitution $\widetilde{w}_r(x,t) = w_r(x,t) - \lambda_r(x)$, $r=\overline{1,N}$. Hence, we derive an equivalent boundary value problem with unknown functions $\lambda_r(x)$:

$$\frac{\partial \widetilde{w}_r(x,t)}{\partial t} = f\Big(x,t,\varphi(t)+\psi(t)x+\int\limits_0^x\int\limits_0^\xi \widetilde{w}_r(\xi_1,t)d\xi_1 d\xi + \int\limits_0^x\int\limits_0^\xi \lambda_r(\xi_1)d\xi_1 d\xi,$$

$$\varphi'(t)+\psi'(t)x+\int\limits_0^x\int\limits_0^\xi \frac{\partial \widetilde{w}_r(\xi_1,t)}{\partial t}d\xi_1 d\xi, \widetilde{w}_r(x,t)+\lambda_r(x)\Big), \quad (x,t)\in\Omega_r, \quad (10)$$

$$\widetilde{w}_r(x,(r-1)h) = 0, \quad x\in[0,X], \quad (11)$$

$$b_1(x)\lambda_1(x) + b_2(x)\lambda_N(x) + b_2(x)\lim_{t\to T-0}\widetilde{w}_N(x,t)+$$

$$+b_3(x)\int\limits_0^x \frac{\partial \widetilde{w}_1(\xi,0)}{\partial t}d\xi + b_4(x)\lim_{t\to T-0}\int\limits_0^x \frac{\partial \widetilde{w}_N(\xi,t)}{\partial t}d\xi + b_5(x)\int\limits_0^x\int\limits_0^\xi \lambda_1(\xi_1)d\xi_1 d\xi+$$

$$+b_6(x)\int\limits_0^x\int\limits_0^\xi \lim_{t\to T-0}\widetilde{w}_N(\xi_1,t)d\xi_1 d\xi + b_6(x)\int\limits_0^x\int\limits_0^\xi \lambda_N(\xi_1)d\xi_1 d\xi = \widetilde{\theta}(x), \quad x\in[0,X], \quad (12)$$

$$\lambda_s(x) + \lim_{t\to sh-0} \widetilde{w}_s(x,t) = \lambda_{s+1}(x), \quad x \in [0,X], \quad s = \overline{1,N-1}. \tag{13}$$

Here, the function $\widetilde{w}_r^*(x,t)$, which is continuous and continuously differentiable in t on Ω_r, satisfies the integro-differential equation (10) under the condition $\lambda_r(x) = \lambda_r^*(x)$, $r = \overline{1,N}$, and the initial condition (11). Furthermore, for the values $\lambda_r^*(x), \lambda_r^*(x) + \lim\limits_{t\to rh-0} \widetilde{w}_r^*(x,t)$, $r = \overline{1,N}$, the equalities (12), (13) hold.

If the system of functions $\{\lambda^*(x), \widetilde{w}^*(x,[t])\}$ is a solution of (10)–(13), then the function $w^*(x,t)$, defined by $w^*(x,t) = \lambda_r^*(x) + \widetilde{w}_r^*(x,t)$, $(x,t) \in \Omega_r$, $r = \overline{1,N}$, $w^*(x,T) = \lambda_N^*(x) + \lim\limits_{t\to Nh-0} \widetilde{w}_N^*(x,t)$ is a solution of the nonlinear boundary value problem for the integro-differential equation (5), (6).

We denote by $C(\Omega_r, R^N)$ (respectively, $C([0,X], R^N)$) the space of functions $\widetilde{w}_r : \Omega_r \to R^N$ ($\lambda_r : [0,X] \to R^N$) that are continuous on $\Omega_r, r = \overline{1,N}$, and equip these spaces with the norms $\|\widetilde{w}\|_1 = \max\limits_{r=\overline{1,N}} \sup\limits_{(x,t)\in\Omega_r} \|\widetilde{w}_r(x,t)\|$, $\|\lambda\|_2 = \max\limits_{r=\overline{1,N}} \sup\limits_{x\in[0,X]} \|\lambda_r(x)\|$.

For fixed $\lambda_r(x)$, problems (10), (11) form a one-parameter family of Cauchy problems for integro-differential equations, where $x \in [0,X]$ and they are equivalent to the following nonlinear integral equation:

$$\widetilde{w}_r(x,t) = \int_{(r-1)h}^{t} f\Big(x,\tau,\varphi(\tau) + \psi(\tau)x + \int_0^x\int_0^\xi \widetilde{w}_r(\xi_1,\tau)d\xi_1 d\xi + \int_0^x\int_0^\xi \lambda_r(\xi_1)d\xi_1 d\xi,$$

$$\varphi'(\tau) + \psi'(\tau)x + \int_0^x\int_0^\xi \frac{\partial \widetilde{w}_r(\xi_1,\tau)}{\partial t} d\xi_1 d\xi, \widetilde{w}_r(x,\tau) + \lambda_r(x)\Big)d\tau. \tag{14}$$

By taking the limit as $t \to rh-0$ in (14) and, in (12), (13) substituting the corresponding right-hand sides for the unknown functions $\lambda_r(x)$, $r = \overline{1,N}$ in place of $\lim\limits_{t\to rh-0} \widetilde{w}_r(x,t)$, $\lim\limits_{t\to T-0} \frac{\partial \widetilde{w}_N(x,t)}{\partial t}$ $r = \overline{1,N}$, and then multiplying both sides of equation (12) by $h > 0$ we obtain the following system of nonlinear equations:

$$hb_1(x)\lambda_1(x) + hb_2(x)\lambda_N(x) + hb_2(x)\int_{(N-1)h}^{Nh} f\Big(x,\tau,\varphi(\tau) + \psi(\tau)x + \int_0^x\int_0^\xi \widetilde{w}_N(\xi_1,\tau)d\xi_1 d\xi + \int_0^x\int_0^\xi \lambda_N(\xi_1)d\xi_1 d\xi,$$

$$\varphi'(\tau) + \psi'(\tau)x + \int_0^x\int_0^\xi \frac{\partial \widetilde{w}_N(\xi_1,\tau)}{\partial t} d\xi_1 d\xi, \widetilde{w}_N(x,\tau) + \lambda_N(x)\Big)d\tau +$$

$$+hb_3(x)\int_0^x \frac{\partial \widetilde{w}_1(\xi,0)}{\partial t} d\xi + hb_4(x)\lim_{t\to T-0}\int_0^x \frac{\partial \widetilde{w}_N(\xi,t)}{\partial t} d\xi + hb_5(x)\int_0^x\int_0^\xi \lambda_1(\xi_1)d\xi_1 d\xi +$$

$$+hb_6(x)\int_0^x\int_0^\xi \lim_{t\to T-0} \widetilde{w}_N(\xi_1,t)d\xi_1 d\xi + hb_6(x)\int_0^x\int_0^\xi \lambda_N(\xi_1)d\xi_1 d\xi = h\widetilde{\theta}(x), \quad x \in [0,X],$$

$$\lambda_s(x) + \int_{(s-1)h}^{sh} f\Big(x,\tau,\varphi(\tau) + \psi(\tau)x + \int_0^x\int_0^\xi \widetilde{w}_s(\xi_1,\tau)d\xi_1 d\xi + \int_0^x\int_0^\xi \lambda_s(\xi_1)d\xi_1 d\xi,$$

$$\varphi'(\tau) + \psi'(\tau)x + \int_0^x\int_0^\xi \frac{\partial \widetilde{w}_s(\xi_1,\tau)}{\partial t} d\xi_1 d\xi, \widetilde{w}_s(x,\tau) + \lambda_s(x)\Big)d\tau - \lambda_{s+1}(x) = 0,$$

$x \in [0, X]$, $s = \overline{1, N-1}$, which can be written in the form

$$Q_h\Big(x, \lambda^{(0)}(x), \widetilde{w}^{(0)}(x, [\cdot])\Big) = 0. \tag{15}$$

In order to find the system of functions $\{\lambda_r(x), \widetilde{w}_r(x,t)\}$, $r = \overline{1, N}$, we obtain a closed system consisting of equations (15), (14), which is defined by the function f and the partition step $h > 0$.

We choose a step size $h > 0 : Nh = T(N = 1, 2, \ldots)$, and define the vector function

$$\lambda^{(0)}(x) = (\lambda_1^{(0)}(x), \lambda_2^{(0)}(x), \ldots, \lambda_N^{(0)}(x))' \in C([0, \omega], R^N).$$

Suppose that problem (10)–(13), where $\lambda_r(x) = \lambda_r^{(0)}(x)$, $r = \overline{1, N}$, admits a solution $\widetilde{w}_r^{(0)}(x, t) \in \widetilde{C}(\Omega_r, R)$, $r = \overline{1, N}$. We denote by $G_0(f, x, h)$, the set of all such $\lambda^{(0)}(x) \in C([0, X], R^N)$. The corresponding system of solutions to problems (10)–(13) for a given $\lambda^{(0)}(x)$ is denoted by $\widetilde{w}^{(0)}(x, [t]) = (\widetilde{w}_1^{(0)}(x, t), \widetilde{w}_2^{(0)}(x, t), \ldots, \widetilde{w}_N^{(0)}(x, t))'$. Taking $\lambda^{(0)}(x) \in G_0(f, x, h)$, the function $\widetilde{w}^{(0)}(x, [t])$, and the numbers $\phi_1 > 0, \phi_2 > 0$, we construct the following sets:

$$S(\lambda^{(0)}(x), \phi_1) = \Big\{(\lambda_1(x), \lambda_2(x), \ldots, \lambda_N(x))' \in C([0, X], R^N) : \|\lambda_r(x) - \lambda_r^{(0)}(x)\| < \phi_1,\ r = \overline{1, N}\Big\},$$

$$S(\widetilde{w}^{(0)}(x, [t]), \phi_1\phi_2) = \{(\widetilde{w}_1(x, t), \widetilde{w}_2(x, t), \ldots, \widetilde{w}_N(x, t))',\ \widetilde{w}_r(x, t) \in C(\Omega_r, R^N) :$$

$$\|\widetilde{w}_r(x, t) - \widetilde{w}_r^{(0)}(x, t)\| < \phi_1\phi_2,\ (x, t) \in \Omega_r,\ r = \overline{1, N}\},$$

$$G^0(\phi_1(x), \phi_2) = \{(x, t, w) : (x, t) \in \Omega,\ \|w - \lambda_r^{(0)}(x) - \widetilde{w}_r^{(0)}(x, t)\| < \phi_1(1 + \phi_2),\ (x, t) \in \Omega_r,\ r = \overline{1, N},$$

$$\|w - \lambda_N^{(0)}(x) - \lim_{t \to T-0} \widetilde{w}_N^{(0)}(x, t)\| < \phi_1(1 + \phi_2),\ t = T\}.$$

We denote by $U(f, L_1, L_2, L_3, x, h)$ the set of all tuples $\Big(\lambda^{(0)}(x), \widetilde{w}^{(0)}(x, [t]), u^{(0)}(x, [t]), \frac{\partial u^{(0)}(x,[t])}{\partial t}, \phi_1, \phi_2\Big)$ for which the function $f(x, t, u, u_t, w)$ in $G^0(\phi_1, \phi_2)$ has continuous partial derivatives $f'_w(x, t, u, u_t, w)$, $f'_u(x, t, u, u_t, w)$, $f'_{u_t}(x, t, u, u_t, w)$ and $\|f'_w(x, t, u, u_t, w)\| \le L_1$, $\|f'_{u_t}(x, t, u, u_t, w)\| \le L_2$, $\|f'_u(x, t, u, u_t, w)\| \le L_3$, where L_1, L_2, L_3 – *const*.

From the system $\{\lambda_r(x), \widetilde{w}_r(x, t)\}$, $r = \overline{1, N}$, we form the pair $\{\lambda(x), \widetilde{w}(x, [t])\}$, where

$$\lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_N(x))', \quad \widetilde{w}(x, [t]) = (\widetilde{w}_1(x, t), \widetilde{w}_2(x, t), \ldots, \widetilde{w}_N(x, t))'.$$

As the initial approximation for problem (10)–(13), we choose the function $\widetilde{w}^{(0)}(x, [t])$ and then construct successive approximations according to the following algorithm:

Step 1. From equations (10), (15), where $\widetilde{w}_r(x, t) = \widetilde{w}_r^{(0)}(x, t)$, we determine the functions $\frac{\partial \widetilde{w}_r^{(1)}(x,t)}{\partial t}$, $\lambda_r^{(1)}(x)$, $r = \overline{1, N}$. Using integral equation (14), we determine $\widetilde{w}_r^{(1)}(x, t)$.

Step 2. From equations (10), (15), where $\widetilde{w}_r(x, t) = \widetilde{w}_r^{(1)}(x, t)$, we determine the functions $\frac{\partial \widetilde{w}_r^{(2)}(x,t)}{\partial t}$, $\lambda_r^{(2)}(x)$, $r = \overline{1, N}$. Using integral equation(14), we determine $\widetilde{w}_r^{(2)}(x, t)$.

Continuing this process, at the k-th step we obtain the collection of triples $\Big\{\frac{\partial \widetilde{w}_r^{(k)}(x,t)}{\partial t}, \lambda_r^{(k)}(x), \widetilde{w}_r^{(k)}(x, t)\Big\}$. Sufficient conditions for the feasibility and convergence of the proposed algorithm, as well as for the existence of a solution to the multicharacteristic boundary value problem with functional parameters (10)–(13), are established as follows:

Theorem 2.1. *Suppose there exists a step size* $h > 0 : Nh = T$, $(N = 1, 2, \ldots)$, *numbers* $\phi_1 > 0$, $\phi_2 > 0$, $(\lambda^{(0)}(x), \widetilde{w}^{(0)}(x, [t]), \phi_1, \phi_2) \in U(f, L_1, L_2, L_3, x, h)$, *for which the Jacobian matrix* $\dfrac{\partial Q_h\Big(x, \lambda(x), \widetilde{w}(x, [\cdot])\Big)}{\partial \lambda}$ *is invertible for all* $(x, \lambda(x), \widetilde{w}(x, [\cdot]))$, *where* $x \in [0, X]$, $\lambda(x) \in S(\lambda^{(0)}(x), \phi_1)$, $\widetilde{w}(x, [t]) \in S(\widetilde{w}^{(0)}(x, [t]), \phi_1\phi_2)$, *and the following inequalities are satisfied:*

1) $\Big\|\Big[\dfrac{\partial Q_h\big(x,\lambda(x),\widetilde{w}(x,[\cdot])\big)}{\partial\lambda}\Big]^{-1}\Big\| \le \vartheta(h)$,

2) $\kappa(h) = h\mu\Big(1 + L_2\int\limits_0^\omega\int\limits_0^x e^{L_2\frac{\xi^2}{2}}d\xi dx\Big) < 1$,

3) $\dfrac{[h\vartheta(h)]^2}{1-\kappa(h)}\max\limits_{x\in[0,X]}\|p(x)\|\max\limits_{x\in[0,X]}\|p_0(x)\|\max\limits_{x\in[0,X]}\Big\|Q_h\big(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot])\big)\Big\| + \vartheta(h)\max\limits_{x\in[0,X]}\|Q_h(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(\overline{x},[\cdot]))\| < \phi_1$,

4) $\dfrac{h}{1-\kappa(h)}\max\limits_{x\in[0,X]}\|p_0(x)\| < \phi_2$,

where

$$\mu = e^{L_2\frac{\omega^2}{2!}}\Big(L_3\frac{\omega^2}{2} + L_1\Big)\Big(1 + h\vartheta(h)\max_{x\in[0,X]}\|p(x)\|\Big),\quad p_0(x) = \Big(L_3\frac{x^2}{2} + L_2\int\limits_0^x\int\limits_0^\xi e^{L_2\frac{\xi_1{}^2}{2}}\Big(L_3\frac{\xi_1^2}{2} + L_1\Big)d\xi_1 d\xi + L_1\Big),$$

$$p(x) = \Big(p_1(x) + \widetilde{p}_2(x) + p_3(x)\frac{x^2}{2!}\Big)e^{h\vartheta(h)\widetilde{p}_2(x)},\quad p_1(x) = L_1\max\{h\|b_2(x)\|,1\},$$

$$p_2(x) = L_3\max\{h\|b_2(x)\|,1\} + h\|b_3(x)\| + h\|b_4(x)\|,\quad p_3(x) = L_2\max\{h\|b_2(x)\|,1\} + h\|b_5(x)\| + h\|b_6(x)\|,$$

$\widetilde{p}_2(x) = p_2(x)\int\limits_0^x\int\limits_0^\xi e^{\frac{L_2\xi_1^2}{2}}\Big(L_3\frac{\xi_1^2}{2!} + L_1\Big)d\xi_1 d\xi$. *Then, the sequence defined by the algorithm* $\big\{\lambda^{(k)}(x),\widetilde{w}^{(k)}(x,[t])\big\}$, $k = 1,2,...$, *is contained in* $S(\lambda^{(0)}(x),\phi_1)\times S(\widetilde{w}^{(0)}(x,[t]),\phi_1\phi_2)$, *converges to the solution of the problem (10)–(13)* $\big(\lambda^*(x),\widetilde{w}^*(x,[t])\big)$, *and the following estimates hold:*

$$a)\ \|\lambda^* - \lambda^{(k+1)}\|_2 \le [h\vartheta(h)]^2\max_{x\in[0,X]}\|p_0(x)\|\max_{x\in[0,X]}\|p(x)\|\frac{[\kappa(h)]^k}{1-\kappa(h)}\max_{x\in[0,X]}\Big\|Q_h\big(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot])\big)\Big\|,$$

$$b)\ \|\widetilde{w}^* - \widetilde{w}^{(k+1)}\|_1 \le \frac{[\kappa(h)]^{k+1}}{1-\kappa(h)}h\vartheta(h)\max_{x\in[0,X]}\|p_0(x)\|\max_{x\in[0,X]}\Big\|Q_h\big(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot])\big)\Big\|.$$

Moreover, any solution $\big(\lambda(x),\widetilde{w}(x,[t])\big)$ *of problem (10)–(13) in* $S(\lambda^{(0)}(x),\phi_1)\times S\big(\widetilde{w}^{(0)}(x,[t]),\phi_1\phi_2\big)$ *is isolated.*

Proof. The functions $\frac{\partial\widetilde{w}_r^{(1)}(x,t)}{\partial t}$, $\frac{\partial\widetilde{w}_r^{(0)}(x,t)}{\partial t}$, $r = \overline{1,N}$ are determined by the relations

$$\frac{\partial\widetilde{w}_r^{(1)}(x,t)}{\partial t} = f\Big(x,t,\varphi(t) + \psi(t)x + \int\limits_0^x\int\limits_0^\xi \widetilde{w}_r^{(0)}(\xi_1,t)d\xi_1 d\xi + \int\limits_0^x\int\limits_0^\xi \lambda_r^{(1)}(\xi_1)d\xi_1 d\xi,$$

$$\varphi'(t) + \psi'(t)x + \int\limits_0^x\int\limits_0^\xi \frac{\partial\widetilde{w}_r^{(1)}(\xi_1,t)}{\partial t}d\xi_1 d\xi, \widetilde{w}_r^{(0)}(x,t) + \lambda_r^{(1)}(x)\Big)$$

and

$$\frac{\partial\widetilde{w}_r^{(0)}(x,t)}{\partial t} = f\Big(x,t,\varphi(t) + \psi(t)x + \int\limits_0^x\int\limits_0^\xi \widetilde{w}_r^{(0)}(\xi_1,t)d\xi_1 d\xi + \int\limits_0^x\int\limits_0^\xi \lambda_r^{(0)}(\xi_1)d\xi_1 d\xi,$$

$$\varphi'(t) + \psi'(t)x + \int\limits_0^x\int\limits_0^\xi \frac{\partial\widetilde{w}_r^{(0)}(\xi_1,t)}{\partial t}d\xi_1 d\xi, \widetilde{w}_r^{(0)}(x,t) + \lambda_r^{(0)}(x)\Big),$$

the following estimate holds

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\left\|\frac{\partial\widetilde{w}_r^{(1)}(x,t)}{\partial t}-\frac{\partial\widetilde{w}_r^{(0)}(x,t)}{\partial t}\right\|\leq L_3\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\|\lambda_r^{(1)}(\xi_1)-\lambda_r^{(0)}(\xi_1)\|d\xi_1d\xi+$$

$$+L_2\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\left\|\frac{\partial\widetilde{w}_r^{(1)}(\xi_1,t)}{\partial t}-\frac{\partial\widetilde{w}_r^{(0)}(\xi_1,t)}{\partial t}\right\|d\xi_1d\xi+L_1\max_{r=\overline{1,N}}\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|.$$

Using a linear integral inequality that includes multiple integrals [10, p. 58], we obtain

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\left\|\frac{\partial\widetilde{w}_r^{(1)}(x,t)}{\partial t}-\frac{\partial\widetilde{w}_r^{(0)}(x,t)}{\partial t}\right\|\leq$$

$$\leq e^{L_2\frac{x^2}{2}}\Big(L_3\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\|\lambda_r^{(1)}(\xi_1)-\lambda_r^{(0)}(\xi_1)\|d\xi_1d\xi+L_1\max_{r=\overline{1,N}}\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|\Big). \tag{17}$$

The functions $\widetilde{w}_r^{(1)}(x,t),\widetilde{w}_r^{(0)}(x,t), r=\overline{1,N}$, are determined by the relations

$$\widetilde{w}_r^{(1)}(x,t)=\int_{(r-1)h}^t f\Big(x,\tau,\varphi(\tau)+\psi(\tau)x+\int_0^x\int_0^\xi\widetilde{w}_r^{(0)}(\xi_1,\tau)d\xi_1d\xi+\int_0^x\int_0^\xi\lambda_r^{(1)}(\xi_1)d\xi_1d\xi,$$

$$\varphi'(\tau)+\psi'(\tau)x+\int_0^x\int_0^\xi\frac{\partial\widetilde{w}_r^{(1)}(\xi_1,\tau)}{\partial t}d\xi_1d\xi,\widetilde{w}_r^{(0)}(x,\tau)+\lambda_r^{(1)}(x)\Big)d\tau,$$

$$\widetilde{w}_r^{(0)}(x,t)=\int_{(r-1)h}^t f\Big(x,\tau,\varphi(\tau)+\psi(\tau)x+\int_0^x\int_0^\xi\widetilde{w}_r^{(0)}(\xi_1,\tau)d\xi_1d\xi+\int_0^x\int_0^\xi\lambda_r^{(0)}(\xi_1)d\xi_1d\xi,$$

$$\varphi'(\tau)+\psi'(\tau)x+\int_0^x\int_0^\xi\frac{\partial\widetilde{w}_r^{(0)}(\xi_1,\tau)}{\partial t}d\xi_1d\xi,\widetilde{w}_r^{(0)}(x,\tau)+\lambda_r^{(0)}(x)\Big)d\tau,$$

the following estimate holds

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(x,t)-\widetilde{w}_r^{(0)}(x,t)\|\leq hL_3\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\|\lambda_r^{(1)}(\xi_1)-\lambda_r^{(0)}(\xi_1)\|d\xi_1d\xi+$$

$$+hL_2\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\left\|\frac{\partial\widetilde{w}_r^{(1)}(\xi_1,t)}{\partial t}-\frac{\partial\widetilde{w}_r^{(0)}(\xi_1,t)}{\partial t}\right\|d\xi_1d\xi+hL_1\max_{r=\overline{1,N}}\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|. \tag{18}$$

In view of inequality 3) of Theorem 2.1, there exists a number $\varepsilon_0>0$, satisfying the inequalities

$$\varepsilon_0\vartheta(h)<1,\quad \frac{\vartheta(h)}{1-\varepsilon_0\vartheta(h)}\Big\|Q_h\Big(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot])\Big)\Big\|<\phi_1,$$

and the Jacobian matrix $\dfrac{\partial Q_h\Big(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot])\Big)}{\partial\lambda}$ is uniformly continuous in $S(\lambda^{(0)}(x),\phi_1)$. Moreover, for $\varepsilon_0>0$ there exists $\delta_0\in(0,\frac{\phi_1}{2})$ such that $\left\|\dfrac{\partial Q_h\Big(x,\lambda(x),\widetilde{w}^{(0)}(x,[\cdot])\Big)}{\partial\lambda}-\dfrac{\partial Q_h\Big(x,\widehat{\lambda}(x),\widetilde{w}^{(0)}(x,[\cdot])\Big)}{\partial\lambda}\right\|<\varepsilon_0$, whenever for $\lambda(x),\widehat{\lambda}(x)\in S(\lambda^{(0)}(x),\phi_1)$, the inequality holds $\|\lambda(x)-\widehat{\lambda}(x)\|<\delta_0$, $x\in[0,X]$. Choosing a number

$$\alpha\geq\alpha_0=\max\Big\{1,\frac{\vartheta(h)}{\delta_0}\Big\|Q_h\Big(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot])\Big)\Big\|\Big\}$$

we construct the iterative process: $\lambda^{(1,0)}(x)=\lambda^{(0)}(x)$,

$$\lambda^{(1,m+1)}(x)=\lambda^{(1,m)}(x)-\frac{1}{\alpha}\Big[\frac{\partial Q_h\Big(x,\lambda^{(1,m)}(x),\widetilde{w}^{(0)}(x,[\cdot])\Big)}{\partial\lambda}\Big]^{-1}Q_h\Big(x,\lambda^{(1,m)}(x),\widetilde{w}^{(0)}(x,[\cdot])\Big),\quad m=0,1,2,.... \tag{19}$$

According to Theorem 1 [[9]] the iterative process (16) converges to $\lambda^{(1)}(x)$, an isolated solution of the equation $Q_h\Big(x,\lambda(x),\widetilde{w}^{(0)}(x,[\cdot])\Big)=0$ in $S(\lambda^{(0)}(x),\phi_1)$ and

$$\|\lambda^{(1)}(x)-\lambda^{(0)}(x)\|\leq\vartheta(h)\max_{x\in[0,X]}\Big\|Q_h\Big(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot])\Big)\Big\|<\phi_1. \tag{20}$$

Substituting (20) into (18), we obtain

$$\|\widetilde{w}_r^{(1)}-\widetilde{w}_r^{(0)}\|_1\leq h\max_{x\in[0,X]}\|p_0(x)\|\|\lambda^{(1)}-\lambda^{(0)}\|_2\leq h\vartheta(h)\max_{x\in[0,X]}\|p_0(x)\|\max_{x\in[0,X]}\Big\|Q_h\Big(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot])\Big)\Big\|.$$

Since the functions $\frac{\partial\widetilde{w}_r^{(2)}(x,t)}{\partial t},\frac{\partial\widetilde{w}_r^{(1)}(x,t)}{\partial t}$, $r=\overline{1,N}$ are determined by the relations

$$\frac{\partial\widetilde{w}_r^{(1)}(x,t)}{\partial t}=f\Big(x,t,\varphi(t)+\psi(t)x+\int_0^x\int_0^\xi\widetilde{w}_r^{(0)}(\xi_1,t)d\xi_1d\xi+\int_0^x\int_0^\xi\lambda_r^{(1)}(\xi_1)d\xi_1d\xi,$$

$$\varphi'(t)+\psi'(t)x+\int_0^x\int_0^\xi\frac{\partial\widetilde{w}_r^{(1)}(\xi_1,t)}{\partial t}d\xi_1d\xi,\widetilde{w}_r^{(0)}(x,t)+\lambda_r^{(1)}(x)\Big)$$

and

$$\frac{\partial\widetilde{w}_r^{(2)}(x,t)}{\partial t}=f\Big(x,t,\varphi(t)+\psi(t)x+\int_0^x\int_0^\xi\widetilde{w}_r^{(1)}(\xi_1,t)d\xi_1d\xi+\int_0^x\int_0^\xi\lambda_r^{(2)}(\xi_1)d\xi_1d\xi,$$

$$\varphi'(t)+\psi'(t)x+\int_0^x\int_0^\xi\frac{\partial\widetilde{w}_r^{(2)}(\xi_1,t)}{\partial t}d\xi_1d\xi,\widetilde{w}_r^{(1)}(x,t)+\lambda_r^{(2)}(x)\Big),$$

the following estimate holds

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{w}_r^{(2)}(x,t)}{\partial t}-\frac{\partial\widetilde{w}_r^{(1)}(x,t)}{\partial t}\Big\|\leq L_3\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(\xi_1,t)-\widetilde{w}_r^{(0)}(\xi_1,t)\|d\xi_1d\xi+$$

$$+L_3\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\|\lambda_r^{(2)}(\xi_1)-\lambda_r^{(1)}(\xi_1)\|d\xi_1d\xi+L_2\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big\|\frac{\partial\widetilde{w}_r^{(2)}(\xi_1,t)}{\partial t}-\frac{\partial\widetilde{w}_r^{(1)}(\xi_1,t)}{\partial t}\Big\|d\xi_1d\xi+$$

$$+L_1 \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{w}_r^{(1)}(x,t)-\widetilde{w}_r^{(0)}(x,t)\| + L_1 \max_{r=\overline{1,N}} \|\lambda_r^{(2)}(x)-\lambda_r^{(1)}(x)\|.$$

Using a linear integral inequality containing multiple integrals [10, p. 58], we have

$$\max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \left\| \frac{\partial \widetilde{w}_r^{(2)}(x,t)}{\partial t} - \frac{\partial \widetilde{w}_r^{(1)}(x,t)}{\partial t} \right\| \leq$$

$$\leq e^{L_2\frac{x^2}{2}}\Big(L_3 \int_0^x \int_0^\xi \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{w}_r^{(1)}(\xi_1,t)-\widetilde{w}_r^{(0)}(\xi_1,t)\| d\xi_1 d\xi + L_3 \int_0^x \int_0^\xi \max_{r=\overline{1,N}} \|\lambda_r^{(2)}(\xi_1)-\lambda_r^{(1)}(\xi_1)\| d\xi_1 d\xi +$$

$$+L_1 \max_{r=\overline{1,N}} \sup_{t\in[(r-1)h,rh)} \|\widetilde{w}_r^{(1)}(x,t)-\widetilde{w}_r^{(0)}(x,t)\| + L_1 \max_{r=\overline{1,N}} \|\lambda_r^{(2)}(x)-\lambda_r^{(1)}(x)\|\Big). \tag{21}$$

From the structure of the operator $Q_h\big(x,\lambda(x),\widetilde{w}^{(1)}(x,[\cdot])\big)$ and the equality $Q_h\big(x,\lambda^{(1)}(x),\widetilde{w}^{(0)}(x,[\cdot])\big)=0$ follows that

$$\vartheta(h)\big\|Q_h\big(x,\lambda^{(1)}(x),\widetilde{w}^{(1)}(x,[\cdot])\big)\big\| = \vartheta(h)\big\|Q_h\big(x,\lambda^{(1)}(x),\widetilde{w}^{(1)}(x,[\cdot])\big) - Q_h\big(x,\lambda^{(1)}(x),\widetilde{w}^{(0)}(x,[\cdot])\big)\big\| \leq$$

$$\leq h\vartheta(h) \max_{x\in[0,X]} \|p(x)\| \|\widetilde{w}^{(1)}-\widetilde{w}^{(0)}\|_1. \tag{22}$$

Let us take $\phi_{1,1}=\vartheta(h)\max_{x\in[0,X]}\|Q_h(x,\lambda^{(1)}(x),\widetilde{w}^{(1)}(x,[\cdot]))\|$ and show that $S(\lambda^{(1)}(\overline{x}),\phi_{1,1})\in S(\lambda^{(0)}(\overline{x}),\phi_1)$. Indeed, if

$$\|\lambda(x)-\lambda^{(1)}(x)\| < \phi_{1,1} = \vartheta(h)\max_{x\in[0,X]}\|Q_h(x,\lambda^{(1)}(x),\widetilde{w}^{(1)}(x,[\cdot]))\|,$$

then, considering inequality (22) and

$$\|\lambda^{(1)}(x)-\lambda^{(0)}(x)\| \leq \vartheta(h)\max_{x\in[0,X]}\|Q_h(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot]))\|$$

we have

$$\max_{r=\overline{1,N}}\|\lambda_r(x)-\lambda_r^{(0)}(x)\| \leq \max_{r=\overline{1,N}}\|\lambda_r(x)-\lambda_r^{(1)}(x)\| + \max_{r=\overline{1,N}}\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\| \leq$$

$$\leq \vartheta(h)h\max_{x\in[0,X]}\|p(x)\|\|\widetilde{w}^{(1)}-\widetilde{w}^{(0)}\|_1 + \vartheta(h)\max_{x\in[0,X]}\|Q_h(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot]))\| < \phi_1.$$

From the conditions of the theorem, it follows that the operator $Q_h(x,\lambda^{(1)}(x),\widetilde{w}^{(1)}(x,[\cdot]))$ in $S(\lambda^{(1)}(x),\phi_{1,1})$ satisfies all the conditions of Theorem 1 [9]. Therefore, the iterative process: $\lambda^{(2,0)}(x)=\lambda^{(1)}(x)$,

$$\lambda^{(2,m+1)}(x)=\lambda^{(2,m)}(x)-\frac{1}{\alpha}\Big[\frac{\partial Q_h\big(x,\lambda^{(2,m)}(x),\widetilde{w}^{(1)}(x,[\cdot])\big)}{\partial\lambda}\Big]^{-1} Q_h\big(x,\lambda^{(2,m)}(x),\widetilde{w}^{(1)}(x,[\cdot])\big),$$

$m=0,1,2,...$ converges to $\lambda^{(2)}(x)$ an isolated solution of the equation $Q_h\big(x,\lambda(x),\widetilde{w}^{(1)}(x,[\cdot])\big)=0$ in the set $S(\lambda^{(1)}(x),\phi_{1,1})$, and

$$\|\lambda^{(2)}(x)-\lambda^{(1)}(x)\| \leq \vartheta(h)\max_{x\in[0,X]}\big\|Q_h\big(x,\lambda^{(1)}(x),\widetilde{w}^{(1)}(x,[\cdot])\big)\big\|. \tag{23}$$

Substituting (21) and (22) into (23):

$$\|\lambda^{(2)}(x)-\lambda^{(1)}(x)\| = \max_{r=\overline{1,N}}\|\lambda_r^{(2)}(x)-\lambda_r^{(1)}(x)\| \leq$$

$$\leq h\vartheta(h)\Big(p_1(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(x,t)-\widetilde{w}_r^{(0)}(x,t)\|+$$

$$+p_2(x)\int_0^x\int_0^\xi e^{L_2\frac{\xi_1^2}{2}}\Big(L_3\int_0^{\xi_1}\int_0^{\xi_2}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(\xi_3,t)-\widetilde{w}_r^{(0)}(\xi_3,t)\|d\xi_3\xi_2+$$

$$+L_3\int_0^{\xi_1}\int_0^{\xi_2}\max_{r=\overline{1,N}}\|\lambda_r^{(2)}(\xi_3)-\lambda_r^{(1)}(\xi_3)\|d\xi_3d\xi_2+$$

$$+L_1\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(\xi_1,t)-\widetilde{w}_r^{(0)}(\xi_1,t)\|+L_1\max_{r=\overline{1,N}}\|\lambda_r^{(2)}(\xi_1)-\lambda_r^{(1)}(\xi_1)\|\Big)d\xi_1d\xi+$$

$$+p_3(x)\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(\xi_1,t)-\widetilde{w}_r^{(0)}(\xi_1,t)\|d\xi_1d\xi\Big),$$

Using the inequalities of Theorem 2.1, we obtain

$$\max_{r=\overline{1,N}}\|\lambda_r^{(2)}(x)-\lambda_r^{(1)}(x)\|\le$$

$$\le\exp\Big(h\vartheta(h)p_2(x)\int_0^x\int_0^\xi e^{L_2\frac{\xi_1^2}{2}}\Big(L_3\frac{\xi_1^2}{2!}+L_1\Big)d\xi_1d\xi\Big)h\vartheta(h)\Big[p_1(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(x,t)-\widetilde{w}_r^{(0)}(x,t)\|+$$

$$+p_2(x)\int_0^x\int_0^\xi e^{L_2\frac{\xi_1^2}{2}}\Big(L_3\int_0^{\xi_1}\int_0^{\xi_2}\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(\xi_3,t)-\widetilde{w}_r^{(0)}(\xi_3,t)\|d\xi_3d\xi_2+$$

$$+L_1\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(\xi_1,t)-\widetilde{w}_r^{(0)}(\xi_1,t)\|\Big)d\xi_1d\xi+$$

$$+p_3(x)\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(\xi_1,t)-\widetilde{w}_r^{(0)}(\xi_1,t)\|d\xi_1d\xi\Big].$$

Thus,

$$\|\lambda^{(2)}-\lambda^{(1)}\|_2\le h\vartheta(h)\max_{x\in[0,X]}\Big[\Big(p_1(x)+\widetilde{p}_2(x)+p_3(x)\frac{x^2}{2!}\Big)e^{h\vartheta(h)\widetilde{p}_2(x)}\Big]\|\widetilde{w}^{(1)}-\widetilde{w}^{(0)}\|_1\le$$

$$\le h\vartheta(h)\max_{x\in[0,X]}\|p(x)\|\|\widetilde{w}_r^{(1)}-\widetilde{w}_r^{(0)}\|_1. \tag{24}$$

Substituting (24) into (21), we obtain

$$\Big\|\frac{\partial\widetilde{w}_r^{(2)}}{\partial t}-\frac{\partial\widetilde{w}_r^{(1)}}{\partial t}\Big\|_1\le\mu\|\widetilde{w}_r^{(1)}-\widetilde{w}_r^{(0)}\|_1. \tag{25}$$

The functions $\widetilde{w}_r^{(2)}(x,t),\widetilde{w}_r^{(1)}(x,t),r=\overline{1,N}$ are determined by the following relations:

$$\widetilde{w}_r^{(2)}(x,t)=\int_{(r-1)h}^t f\Big(x,\tau,\varphi(\tau)+\psi(\tau)x+\int_0^x\int_0^\xi\widetilde{w}_r^{(1)}(\xi_1,\tau)d\xi_1d\xi+\int_0^x\int_0^\xi\lambda_r^{(2)}(\xi_1)d\xi_1d\xi,$$

$$\varphi'(\tau)+\psi'(\tau)x+\int_0^x\int_0^\xi\frac{\partial\widetilde{w}_r^{(2)}(\xi_1,\tau)}{\partial t}d\xi_1d\xi,\widetilde{w}_r^{(1)}(x,\tau)+\lambda_r^{(2)}(x)\Big)d\tau,$$

$$\widetilde{w}_r^{(1)}(x,t) = \int\limits_{(r-1)h}^{t} f\Big(x,\tau,\varphi(\tau)+\psi(\tau)x+\int\limits_0^x\int\limits_0^\xi \widetilde{w}_r^{(0)}(\xi_1,\tau)d\xi_1 d\xi + \int\limits_0^x\int\limits_0^\xi \lambda_r^{(1)}(\xi_1)d\xi_1 d\xi,$$

$$\varphi'(\tau)+\psi'(\tau)x+\int\limits_0^x\int\limits_0^\xi \frac{\partial \widetilde{w}_r^{(1)}(\xi_1,\tau)}{\partial t}d\xi_1 d\xi, \widetilde{w}_r^{(0)}(x,\tau)+\lambda_r^{(1)}(x)\Big)d\tau,$$

the following estimate holds:

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(2)}(x,t)-\widetilde{w}_r^{(1)}(x,t)\| \le hL_3\int\limits_0^x\int\limits_0^\xi \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(\xi_1,t)-\widetilde{w}_r^{(0)}(\xi_1,t)\|d\xi_1 d\xi +$$

$$+hL_3\int\limits_0^x\int\limits_0^\xi \max_{r=\overline{1,N}}\|\lambda_r^{(1)}(\xi_1)-\lambda_r^{(0)}(\xi_1)\|d\xi_1 d\xi + hL_2\int\limits_0^x\int\limits_0^\xi \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\left\|\frac{\partial \widetilde{w}_r^{(1)}(\xi_1,t)}{\partial t}-\frac{\partial \widetilde{w}_r^{(0)}(\xi_1,t)}{\partial t}\right\|d\xi_1 d\xi +$$

$$+hL_1\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^{(1)}(x,t)-\widetilde{w}_r^{(0)}(x,t)\| + hL_1\max_{r=\overline{1,N}}\|\lambda_r^{(1)}(x)-\lambda_r^{(0)}(x)\|. \tag{26}$$

Substituting (25) and (24) into (26), we obtain

$$\|\widetilde{w}_r^{(2)}-\widetilde{w}_r^{(1)}\|_1 \le \kappa(h)\|\widetilde{w}_r^{(1)}-\widetilde{w}_r^{(0)}\|_1.$$

An estimate holds

$$\|\widetilde{w}^{(2)}-\widetilde{w}^{(0)}\|_1 \le [1+\kappa(h)]\max_{r=\overline{1,N}}\|\widetilde{w}^{(1)}-\widetilde{w}^{(0)}\|_1 \le$$

$$\le [1+\kappa(h)]h\vartheta(h)\max_{x\in[0,X]}\|p_0(x)\|\max_{x\in[0,X]}\left\|Q_h\Big(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot])\Big)\right\| < \phi_1\phi_2.$$

Assuming that the pair $(\lambda^{(k)}(x),\widetilde{w}^{(k)}(x,[t]) \in S(\lambda^{(0)}(x),\phi_1)\times S(\widetilde{w}(x,[t]),\phi_1\phi_2)$ is defined and that the following estimates are established:

$$\|\widetilde{w}^{(k+1)}-\widetilde{w}^{(k)}\|_1 \le [\kappa(h)]^{k-1}\|\widetilde{w}^{(1)}-\widetilde{w}^{(0)}\|_1, \tag{27}$$

$$\vartheta(h)\max_{x\in[0,X]}\|Q_h(x,\lambda^{(k)}(x),\widetilde{w}^{(k)}(x,[\cdot]))\| \le h\vartheta(h)\max_{x\in[0,X]}\|p(x)\|\|\widetilde{w}^{(k)}-\widetilde{w}^{(k-1)}\|_1. \tag{28}$$

Let us determine the $(k+1)$-th approximation of the functional parameter $\lambda^{(k+1)}(x)$ from the equation $Q_h(x,\lambda(x),\widetilde{w}^{(k)}(x,[\cdot]))=0$. Using inequality (22) and the equality $Q_h(x,\lambda^{(k)}(x),\widetilde{w}^{(k-1)}(x,[\cdot]))=0$, we establish the validity of the following inequality:

$$\vartheta(h)\max_{x\in[0,X]}\|Q_h(x,\lambda^{(k)}(x),\widetilde{w}^{(k)}(x,[\cdot]))\| \le h\vartheta(h)\max_{x\in[0,X]}\|p(x)\|[\kappa(h)]^{k-1}\|\widetilde{w}^{(1)}-\widetilde{w}^{(0)}\|_1. \tag{29}$$

Let us define $\phi_k = \vartheta(h)\max\limits_{x\in[0,X]}\|Q_h(x,\lambda^{(k)}(x),\widetilde{w}^{(k)}(x,[\cdot]))\|$ and show that $S(\lambda^{(k)}(\overline{x}),\phi_k)\subset S(\lambda^{(0)}(x),\phi_1)$. Indeed, in view of inequalities (27)–(29) and condition (3)

$$\|\lambda-\lambda^{(0)}\|_2 \le h\vartheta(h)\max_{x\in[0,X]}\|p(x)\|\phi_2 + \vartheta(h)\max_{x\in[0,X]}\|Q_h(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot]))\| < \phi_1.$$

Since $\|Q_h(x,\lambda(x),\widetilde{w}^{(k)}(x,[\cdot]))\|$ in $S(\lambda^{(k)}(x),\phi_k)$ satisfies all the conditions of Theorem 1 from [D.S. Jumabayev], there exists $\lambda^{(k+1)}(x)$ a solution to the equation $\|Q_h(x,\lambda(x),\widetilde{w}^{(k)}(x,[\cdot]))\|=0$ in $S(\lambda^{(k)}(x),\phi_k)$ and the following estimate holds:

$$\|\lambda_r(x) - \lambda_r^{(k)}(x)\| \leq \vartheta(h) \max_{x\in[0,X]} \|Q_h(x, \lambda^{(k)}(x), \widetilde{w}^{(k)}(x,[\cdot]))\|.$$

We establish the following estimates:

$$\|\lambda^{(k+1)} - \lambda^{(k)}\|_2 \leq h\vartheta(h) \max_{x\in[0,\omega]} \|p(x)\|\|\widetilde{w}^{(k)} - \widetilde{w}^{(k-1)}\|_1, \tag{30}$$

$$\|\widetilde{w}^{(k+1)} - \widetilde{w}^{(k)}\|_1 \leq \kappa(h)\|\widetilde{w}^{(k)} - \widetilde{w}^{(k-1)}\|_1. \tag{31}$$

From the inequalities (30), (31), and $\kappa(h) < 1$ imply that the sequence $\{\lambda^{(k)}(x), \widetilde{w}^{(k)}(x,[t])\}$ converges as $k \to \infty$ to $\{\lambda^*(x), \widetilde{w}^*(x,[t])\}$, which is the solution to the problem (10)–(13) in $S(\lambda^{(0)}(x), \phi_1) \times S(\widetilde{w}^{(0)}(x,[t]), \phi_1\phi_2)$.

The following estimates hold:

$$\|\lambda^{(k+n)} - \lambda^{(k+1)}\|_2 \leq \|\lambda^{(k+n)} - \lambda^{(k+n-1)}\|_2 + \|\lambda^{(k+n-1)} - \lambda^{(k+n-2)}\|_2 + ... + \|\lambda^{(k+2)} - \lambda^{(k+1)}\|_2 \leq$$

$$\leq [\kappa(h)]^{k+1} \sum_{i=0}^{n-2} [\kappa(h)]^i h\vartheta(h) \max_{x\in[0,X]} \|p_0(x)\| \max_{x\in[0,X]} \left\|Q_h\Big(x, \lambda^{(0)}(x), \widetilde{w}^{(0)}(x,[\cdot])\Big)\right\|.$$

Thus, as $n \to \infty$ we obtain estimates a) and b) of Theorem 2.1.

We will show the isolation of the solution. Let us choose a number $\varepsilon < 0$ such that

$$\varepsilon\vartheta(h) < 1, \quad \kappa(h) < 1 - \varepsilon\vartheta(h).$$

Since $f'_w(x,t,u,u_t,w)$ is uniformly continuous in $G^0(\phi_1,\phi_2)$, and considering the structure of the Jacobian matrix $\frac{\partial Q_h\left(x,\lambda(x),\widetilde{w}(x,[\cdot])\right)}{\partial\lambda}$ it follows that it is uniformly continuous in $S(\lambda^*(x), \phi_1) \times S(\widetilde{w}^*(x,[t]), \phi_1\phi_2)$. Therefore, there exist numbers $\delta_1 > 0, \delta_2 > 0$ such that $\left\|\frac{\partial Q_h\left(x,\lambda(x),\widetilde{w}(x,[\cdot])\right)}{\partial\lambda} - \frac{\partial Q_h\left(x,\lambda^*(x),\widetilde{w}^*(x,[\cdot])\right)}{\partial\lambda}\right\| < \varepsilon$, for all $x \in [0,X]$ $(\lambda(x), \widetilde{w}(x,[t])) \in S(\lambda^*(x), \delta_1) \times S(\widetilde{w}^*(x,[t]), \delta_1\delta_2)$. Note that if $(\lambda^*(x), \widetilde{w}^*(x,[t]))$ is a solution to the problem (10)-(13), then $Q_h(x, \lambda^*(x), \widetilde{w}^*(x,[\cdot]) = 0$.

Let $\{\widehat{\lambda}(x), \widehat{\widetilde{w}}(x,[t])\} \in S(\lambda^*(x), \delta_1) \times S(\widetilde{w}^*(x,[t]), \delta_1\delta_2)$ be another solution to the problem (10)–(13). Since $Q_h(x, \lambda^*(x), \widetilde{w}^*(x,[\cdot])) = 0$ and $Q_h(x, \widehat{\lambda}^*(x), \widehat{\widetilde{w}}^*(x,[\cdot])) = 0$, it follows from the equalities

$$\lambda^*(x) = \lambda^*(x) - \left\|\frac{\partial Q_h\Big(x, \lambda(x), \widetilde{w}(x,[\cdot])\Big)}{\partial\lambda}\right\|^{-1} Q_h\Big(x, \lambda^*(x), \widetilde{w}^*(x,[\cdot])\Big),$$

$$\widehat{\lambda}^*(x) = \widehat{\lambda}^*(x) - \left\|\frac{\partial Q_h\Big(x, \lambda(x), \widetilde{w}(x,[\cdot])\Big)}{\partial\lambda}\right\|^{-1} Q_h\Big(x, \widehat{\lambda}^*(x), \widehat{\widetilde{w}}^*(x,[\cdot])\Big)$$

that

$$\lambda^*(x) - \widehat{\lambda}^*(x) = \lambda^*(x) - \widehat{\lambda}^*(x) - \left\|\frac{\partial Q_h\Big(x, \lambda^*(x), \widetilde{w}^*(x,[\cdot])\Big)}{\partial\lambda}\right\|^{-1} \Big[Q_h\Big(x, \lambda^*(x), \widetilde{w}^*(x,[\cdot])\Big) - Q_h\Big(x, \widehat{\lambda}^*(x), \widehat{\widetilde{w}}^*(x,[\cdot])\Big)\Big].$$

Applying the finite increment formula of Lagrange [21] to the difference $Q_h\Big(x, \lambda^*(x), \widetilde{w}^*(x,[\cdot])\Big) - Q_h\Big(x, \widehat{\lambda}^*(x), \widehat{\widetilde{w}}^*(x,[\cdot])\Big)$ we obtain that

$$\lambda^*(x) - \widehat{\lambda}^*(x) = -\Big[\frac{\partial Q_h\Big(x, \lambda^*(x), \widetilde{w}^*(x,[\cdot])\Big)}{\partial\lambda}\Big]^{-1} \times$$

$$\times \int_0^1 \Big(\frac{\partial Q_h\Big(x, \widehat{\lambda}(x) + t(\lambda^*(x) - \widehat{\lambda}(x)), \widetilde{w}^*(x,[\cdot])\Big)}{\partial\lambda} - \frac{\partial Q_h\Big(x, \lambda^*(x), \widetilde{w}^*(x,[\cdot])\Big)}{\partial\lambda}\Big) dt [\lambda^*(x) - \widehat{\lambda}^*(x)] -$$

$$-\Big[\frac{\partial Q_h\big(x,\lambda^*(x),\widetilde{w}^*(x,[\cdot])\big)}{\partial\lambda}\Big]^{-1}\Big[Q_h\big(x,\widehat{\lambda}^*(x),\widetilde{w}^*(x,[\cdot])\big)-Q_h\big(x,\widehat{\lambda}^*(x),\widehat{\widetilde{w}}^*(x,[\cdot])\big)\Big],$$

whence

$$\|\lambda^*(x)-\widehat{\lambda}^*(x)\|\le\frac{\vartheta(h)}{1-\varepsilon\vartheta(h)}\Big\|Q_h\big(x,\lambda^*(x),\widetilde{w}^*(x,[\cdot])\big)-Q_h\big(x,\widehat{\lambda}^*(x),\widehat{\widetilde{w}}^*(x,[\cdot])\big)\Big\|\le$$

$$\le h\vartheta(h)\Big(p_1(x)\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^*(x,t)-\widehat{\widetilde{w}}_r^*(x,t)\|+$$

$$+p_2(x)\int_0^x\int_0^\xi e^{L_2\frac{x^2}{2}}\Big(L_3\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^*(\xi_1,t)-\widehat{\widetilde{w}}_r^*(\xi_1,t)\|d\xi_1d\xi+$$

$$+L_1\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^*(x,t)-\widehat{\widetilde{w}}_r^*(x,t)\|\Big)d\xi_1d\xi+p_3(x)\int_0^x\int_0^\xi\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\|\widetilde{w}_r^*(\xi_1,t)-\widehat{\widetilde{w}}_r^*(\xi_1,t)\|d\xi_1d\xi\Big).$$

Since

$$\|\widetilde{w}^*-\widehat{\widetilde{w}}^*\|_1\le\kappa(h)\|\widetilde{w}^*-\widehat{\widetilde{w}}^*\|_1,$$

it follows that $\widetilde{w}_r^*(x,t)=\widehat{\widetilde{w}}_r^*(x,t)$, $\lambda_r^*(x)=\widehat{\lambda}_r^*(x)$, for all $(x,t)\in\Omega_r$, $r=\overline{1,N}$. Theorem 2.1 is proven. □

We define the function $w_r^{(k)}(x,t)$, $k=0,1,2,\dots$ by the equality:

$$w_r^{(k)}(x,t)=\begin{cases}\lambda_r^{(k)}(x)+\widetilde{w}_r^{(k)}(x,t), & \text{as } (x,t)\in\Omega_r,\ r=\overline{1,N},\\ \lambda_N^{(k)}(x)+\lim\limits_{t\to T-0}\widetilde{w}_N^{(k)}(x,t), & \text{as } t=Nh,\end{cases}$$

$$u(x,t)=\varphi(t)+\psi(t)x+\int_0^x\int_0^\xi w(\xi_1,t)d\xi_1d\xi.$$

Let $S(u^{(0)}(x,t),\Phi(x))$, $\Phi(x)=\varphi(t)+\psi(t)x+\frac{x^2}{2}\phi_1(1+\phi_2)$ denote the set of piecewise continuously differentiable functions $u:\Omega\to R$ that satisfy the inequalities

$$\|u(x,t)-\varphi(t)-\psi(t)x-\int_0^x\int_0^\xi\big(\lambda^{(0)}(x)-\widetilde{w}^{(0)}(\xi_1,t)\big)d\xi_1d\xi\|<\Phi(x),$$

$$\|u(x,T)-\varphi(T)-\psi(T)x-\int_0^x\int_0^\xi\big(\lambda^{(0)}(x)-\widetilde{w}^{(0)}(\xi_1,T)\big)d\xi_1d\xi\|<\Phi(x).$$

Due to the equivalence of problems (1)–(4) and (10)–(13), it follows from Theorem 2.1 that:

Theorem 2.2. *If the conditions of Theorem 2.1 are satisfied, then the sequence of functions* $\{u^{(k)}(x,t)\}$, $k=1,2,\dots$, *is contained in* $S(u^{(0)}(x,t),\Phi(x))$, *converges to* $u^*(x,t)$ *the solution of problem (1)–(4) in* $S(u^{(0)}(x,t),\Phi(x))$ *and satisfies the inequality*

$$\|u^*(x,t)-u^{(k)}(x,t)\|\le\big(h\vartheta(h)\max_{x\in[0,X]}\|p(x)\|+\kappa(h)\big)h\vartheta(h)\max_{x\in[0,X]}\|p_0(x)\|\frac{[\kappa(h)]^k}{1-\kappa(h)}\max_{x\in[0,X]}\Big\|Q_h\big(x,\lambda^{(0)}(x),\widetilde{w}^{(0)}(x,[\cdot])\big)\Big\|,$$

$(x,t)\in\Omega$. *Moreover, any solution of problem (1)–(4) in* $S(u^{(0)}(x,t),\Phi(x))$ *is isolated.*

3. Conclusion

In this work, a third-order nonlinear pseudo-parabolic equation with nonlocal boundary conditions was investigated. The original problem was reduced to an equivalent system of integro-differential equations, which made it possible to propose an iterative algorithm for finding the solution. Sufficient conditions for the existence, uniqueness, and convergence of the solution were established. It was proven that the sequence of approximations generated by the algorithm converges to an isolated solution of the problem.

The results of this study are of significant importance for analyzing and solving similar problems involving nonlinear pseudo-parabolic equations with nonlocal conditions. The proposed algorithm can be applied to practical problems in mathematical physics, such as modeling the behavior of viscoelastic materials that exhibit both viscous and elastic properties; in acoustics, for modeling sound propagation in complex media (e.g., porous materials or turbulent atmospheres); and in geophysics, for modeling seismic wave propagation in rock formations with complex structures and properties.

References

[1] P.B. Abdimanapova, S.M. Temesheva, *On a Solution of a Nonlinear Nonlocal Boundary Value Problem for one Class of Hyperbolic Equation*, Lobachevskii J. Math. **44(7)** (2023), 2529–2541. https://doi.org/10.1134/S1995080223070028.

[2] B. Ahmad, S.K. Ntouyas, H.H. Alsulami, *Existence of Solutions or Nonlinear nth-order Differential Equations and Inclusions with Nonlocal and Integral Boundary Conditions via Fixed Point Theory*, Filomat. **28(10)** (2014), 2149–2162. https://doi.org/10.2298/FIL1410149A.

[3] A.Kh. Attaev, *Boundary control problem for a hyperbolic equation loaded along one of its characteristics*, Bull. Karaganda Univ. Math. Ser. **106(2)** (2022), 49–58. https://doi.org/10.31489/2022M2/49-58.

[4] A.T. Assanova, Z.S. Tokmurzin, *Method of functional parametrization for solving a semi-periodic initial problem for fourth-order partial differential equations*, Bull. Karaganda Univ. Math. Ser. **100(4)** (2020), 5–16. https://doi.org/10.31489/2020M4/5-16.

[5] A.T. Assanova, A.E. Imanchiyev, *A nonlocal problem with multipoint conditions for partial differential equations of higher order*, Filomat. **38(1)** (2024), 295–304. https://doi.org/10.2298/FIL2401295A.

[6] K. Belakroum, A. Ashyralyev, A. Guezane-Lakoud, *A Note on the Nonlocal Boundary Value Problem for a Third Order Partial Differential Equation*, Filomat. **32(3)** (2018), 801–808. https://doi.org/10.2298/FIL1803801B.

[7] L. Bougoffa, R.C. Rach, *Solving nonlocal boundary value problems for first- and second-order differential equations by the Adomian decomposition method*, Kybernetes. **42(4)** (2013), 641–664. https://doi.org/10.1108/K-09-2012-0045.

[8] D.S. Dzhumabayev, *Criteria for the Unique Solvability of a Linear Boundary-Value Problem for an Ordinary Differential-Equation*, USSR Comput. Math. Math. Phys. **29(1)** (1989), 34–46. https://doi.org/10.1016/0041-5553(89)90038-4.

[9] D.S. Dzhumabayev, *Iterative Processes with Damping Factors and Their Application*, Math. J. **1(1)** (2001), 20–30. [in Russian]

[10] A.N. Filatov, L.V. Sharova, *Integral Inequalities and the Theory of Nonlinear Oscillations*, Nauka, Moscow, 1976. [in Russian]

[11] T. Hartland, R. Shankar, *A Strong Maximum Principle for Nonlinear Nonlocal Diffusion Equations*, Axioms. **12(11)** (2023), 1059. https://doi.org/10.3390/axioms12111059.

[12] I. Kiguradze, T. Kiguradze, *Optimal conditions of solvability of nonlocal problems for second-order ordinary differential equations*, Nonlinear Anal. **74(3)** (2011), 757–767. https://doi.org/10.1016/j.na.2010.09.023.

[13] A.B. Keldibekova, *Solvability of a semi-periodic boundary value problem for a third order differential equation with mixed derivative*, Bull. Karaganda Univ. Math. Ser. **98(2)** (2020), 84–99. https://doi.org/10.31489/2020M2/84-99.

[14] A.Y. Lepin, A.D. Myshkis, *General nonlocal nonlinear boundary value problem for differential equation of 3rd order*, Nonlinear Anal. **28(9)** (1997), 1533–1543. https://doi.org/10.1016/S0362-546X(96)00003-X.

[15] A.M. Manat, N.T. Orumbayeva, *On one approximate solution of a nonlocal boundary value problem for the Benjamin-Bon-Mahoney equation*, Bull. Karaganda Univ. Math. Ser. **110(2)** (2023), 84–92. https://doi.org/10.31489/2023M2/84-92.

[16] N.T. Orumbayeva, G. Sabitbekov, *A boundary value problem for nonlinear differential equation with arbitrary functions*, Bull. Karaganda Univ. Math. Ser. **85(1)** (2017), 71–76. https://doi.org/10.31489/2017M1/71-76.

[17] N.T. Orumbayeva, T.D. Tokmagambetova, *On the solvability of a semiperiodic boundary value problem for a pseudohyperbolic equation*, Filomat. **37(3)** (2023), 925–933. https://doi.org/10.2298/FIL2303925O.

[18] N.T. Orumbayeva, T.D. Tokmagambetova, *On One Solution of the Boundary Value Problem for a Pseudohyperbolic Equation of Fourth Order*, Lobachevskii J. Math. **42(15)** (2021), 3705–3714. https://doi.org/10.1134/S1995080222030167.

[19] N.T. Orumbayeva, A.T. Assanova, A.B. Keldibekova, *On an Algorithm of Finding an Approximate Solution of a Periodic Problem for a Third-Order Differential Equation*, Eur. Math. J. **13(1)** (2022), 69–85. https://doi.org/10.32523/2077-9879-2022-13-1-69-85.

[20] N.H.M. Qumami, R.S. Jain, B.S. Reddy, *On Impulsive Nonlocal Nonlinear Fuzzy Integro-Differential Equations in Banach Space*, Contemp. Math. **5(4)** (2024), 4399–4413. https://doi.org/10.37256/cm.5420242578.

[21] V.A. Trenogin, *Functional Analysis*, Nauka, Moscow, 1980. [in Russian]

[22] T.K. Yuldashev, A.K. Fayziyev, *Integral Condition with Nonlinear Kernel for an Impulsive System of Differential Equations with Maxima and Redefinition Vector*, Lobachevskii J. Math. **43(8)** (2022), 2332–2340. https://doi.org/10.1134/S1995080222110312.