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Elementary transformation and its applications for hybrid number matrices

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Abstract. In this paper, we define the elementary transformations of hybrid number matrices and establish their upper triangularization process based on these transformations. We introduce a new determinant by means of a real representation matrix and present the sufficient conditions for the existence of LU decompositions, together with a counterexample showing that hybrid number matrices do not necessarily admit an LU decomposition. Furthermore, we investigate the inverse of the hybrid number matrix and provide both the existence condition and a concrete computation method.

1. Introduction

In 2018, Özdemir [11] introduced a new non-commutative number system called hybrid numbers, which can be regarded as a generalization of the complex, hyperbolic, and dual number systems. Since hybrid numbers combine these three well-known systems, they have numerous applications in fields such as number theory, linear algebra, kinematics, geometry, and physics ([2], [4], [8], [7], [10], [15], [16]).

For hybrid number matrices, previous studies have conducted in-depth analyses of their algebraic and geometric properties and explored their potential applications. Altınkaya et al. [1, 3] provided a systematic theoretical foundation; Çakır (2025) investigated the consimilarity of hybrid number matrices and related matrix equations [6]; Öztürk and collaborators [12–14] examined inversion transformations, similarity, and elliptical rotations in hybrid number systems. These studies establish a solid theoretical background for understanding the structure and transformation behavior of hybrid number matrices.

However, as far as we know, no study has addressed the elementary transformations of hybrid number matrices. Building on the existing work, this paper defines the elementary transformations of hybrid number matrices and establishes a corresponding triangularization procedure. A new determinant is introduced based on the real representation matrix, and the sufficient conditions for the existence of LU decompositions are analyzed (with a counterexample showing that hybrid number matrices do not necessarily admit an LU decomposition). Furthermore, the inverses of hybrid number matrices are studied, including their existence conditions and computation methods. Compared with previous research, this

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study not only extends the theoretical framework of hybrid number matrices but also provides practical computational tools, thereby offering theoretical support and methods for solving related applied problems.

The organization of this paper is as follows. In Sect.2, some basic knowledge of hybrid numbers and hybrid number matrices are given. In Sect.3, some basic conclusions of the elementary transformation are explored, and a procedure for the triangularization of hybrid number matrices is given. In Sect.4, the determinants and inverse matrices of hybrid number matrices are studied and the computation of their inverse is given. The conclusion of the paper in Sect.5.

2. Some Basic Knowledge

In this section, we review some basic knowledge of hybrid numbers and hybrid number matrices.

In [5], the set of hybrid numbers can be represented as

$$K = \left\{ p_1 + p_2\mathbf{i} + p_3\boldsymbol{\varepsilon} + p_4\mathbf{h} \mid \mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \boldsymbol{\varepsilon}, p_1, p_2, p_3, p_4 \in \mathbf{R} \right\}.$$

The multiplication table of the units $\mathbf{i}$, $\boldsymbol{\varepsilon}$ and $\mathbf{h}$ and the product of two hybrid numbers is done with the help of this table.

Table 1: Multiplication Table of Hybrid Number Units

$\cdot$	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\boldsymbol{\varepsilon}+\mathbf{i}$
$\boldsymbol{\varepsilon}$	$1+\mathbf{h}$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$-\boldsymbol{\varepsilon}-\mathbf{i}$	$\boldsymbol{\varepsilon}$	1

For any hybrid number $p = p_1 + p_2\mathbf{i} + p_3\boldsymbol{\varepsilon} + p_4\mathbf{h}$, its conjugation is defined as

$$\bar{p} = p_1 - p_2\mathbf{i} - p_3\boldsymbol{\varepsilon} - p_4\mathbf{h}.$$

The real number

$$C_{(p)} = p\bar{p} = p_1^2 + (p_2 - p_3)^2 - p_3^2 - p_4^2.$$

The norm of a hybrid number is defined as

$$\|p\| = \sqrt{|C(p)|} = \sqrt{\left|p_1^2 + (p_2 - p_3)^2 - p_3^2 - p_4^2\right|}.$$

The inverse of the hybrid number is

$$p^{-1} = \frac{\bar{p}}{C_{(p)}}, \quad C_{(p)} \neq 0.$$

Any hybrid number p can be written as the sum of two complex numbers: $p = x_1 + x_2\mathbf{h}$ or $p = x_1 + \mathbf{h}\bar{x}_2$, where $x_1 = p_1 + (p_2 - p_3)\,\mathbf{i} \in \mathbb{C}$, $x_2 = p_4 + p_3\mathbf{i} \in \mathbb{C}$. We call x_1 the complex part, x_2 the imaginary part, and denote them as $\mathbf{comp}(p)$ and $\mathbf{imag}(p)$, respectively.

For subsequent needs, we state some results.

Theorem 2.1. *[5] For any $p \in K$, the following properties are satisfied.*

1. $\mathbf{h}x_1 = \bar{x}_1\mathbf{h}$ for $x_1 \in \mathbb{C}$
2. $\overline{\mathbf{h}\,(x_1 + x_2\mathbf{h})} = \bar{x}_2 + \bar{x}_1\mathbf{h}$ for $x_1, x_2 \in \mathbb{C}$
3. $\overline{p\mathbf{h}} = -\mathbf{h}\bar{p}$
4. $\mathbf{h}\boldsymbol{\varepsilon} = -\boldsymbol{\varepsilon}\mathbf{h}$
5. $\overline{\mathbf{ih}} = -\mathbf{hi}$
6. $\overline{x_1\mathbf{h}} = -x_1\mathbf{h}$ for $x_1 \in \mathbb{C}$

Through simple calculations, we get the following results.

Lemma 2.2. *For* $p = x_1 + x_2\mathbf{h}$, *then* $\bar{p} = \bar{x}_1 - x_2\mathbf{h}$.

Corollary 2.3. *Let* $p = x_1 + x_2\mathbf{h} \in K$, $x_1, x_2 \in \mathbb{C}$. *Then* $\|p\|^2 = p\bar{p} = \left| |x_1|^2 - |x_2|^2 \right|$.

To establish the validity of this corollary, we provide a formal proof.

Proof.

$$\begin{aligned}
\|p\|^2 &= |p\bar{p}| \\
&= |(x_1 + x_2\mathbf{h})(\bar{x}_1 - x_2\mathbf{h})| \\
&= |x_1\bar{x}_1 - x_1x_2\mathbf{h} + x_2\mathbf{h}\bar{x}_1 - x_2\mathbf{h}x_2\mathbf{h}| \\
&= \left| |x_1|^2 - x_1x_2\mathbf{h} + x_2x_1\mathbf{h} - |x_2|^2 \right| \quad \text{(from Theorem 2.1, item 1)} \\
&= ||x_1|^2 - |x_2|^2|
\end{aligned}$$

□

The matrix representation of hybrid numbers is particularly important to facilitate multiplication of hybrid numbers. In [5], by defining an isomorphism between 2×2 real matrices, operations between hybrid numbers will then be easy to perform and some of their properties easy to prove, the representation of hybrid numbers with 2×2 real matrices are given

$$\aleph_p = \begin{bmatrix} p_1 + p_3 & p_2 - p_3 + p_4 \\ p_3 - p_2 + p_4 & p_1 - p_3 \end{bmatrix}.$$

for $p = p_1 + p_2\mathbf{i} + p_3\boldsymbol{\varepsilon} + p_4\mathbf{h} \in K$.

Theorem 2.4. *[5] Let* $p, q \in K$. *Then*

1. $\aleph_p = \aleph_q \Longleftrightarrow p = q$,
2. $\aleph_{p+q} = \aleph_p + \aleph_q$,
3. $\aleph_{pq} = \aleph_p \aleph_q$.

The set of $m \times n$ matrices with hybrid number entries, which is denoted by $\mathrm{M}_{m\times n}(K)$. If $m = n$, then the set of hybrid matrices is denoted $\mathrm{M}_n(K)$.

In [5], the complex representation of any hybrid number and the complex representation matrix of any hybrid number matrix are given.

For $p = p_1 + p_2\mathbf{i} + p_3\boldsymbol{\varepsilon} + p_4\mathbf{h}$, its complex representation is

$$\chi_p = \begin{bmatrix} p_1 + (p_2 - p_3)\mathbf{i} & p_4 + p_3\mathbf{i} \\ p_4 - p_3\mathbf{i} & p_1 - (p_2 - p_3)\mathbf{i} \end{bmatrix}.$$

and similarly, we have $\left|\det \chi_p\right| = \left|\det \aleph_p\right|$.

For $A = C_1 + C_2\mathbf{h} \in \mathrm{M}_n(K)$, we call the $2n \times 2n$ complex matrix

$$\chi_A \equiv \begin{bmatrix} C_1 & C_2 \\ \bar{C}_2 & \bar{C}_1 \end{bmatrix} \tag{1}$$

uniquely determined by A, the complex representation matrix of the hybrid number matrix A.

In the following, the basic conclusions for complex representation matrices of hybrid number matrices are given.

Theorem 2.5. *[5] Let* $A, B \in M_n(K)$, *then the following properties are satisfied:*

(1) $\chi_{A+B} = \chi_A + \chi_B$,
(2) $\chi_{AB} = \chi_A \chi_B$,
(3) $\chi_{A^{-1}} = \chi_A^{-1}$ if A^{-1} exists,
(4) $\chi_{A^*} \neq (\chi_A)^*$ in general.

Let $A = A_1 + A_2\mathbf{i} + A_3\boldsymbol{\varepsilon} + A_4\mathbf{h}$ and $A_1, A_2, A_3, A_4 \in \mathrm{M}_n(R)$. Then, since the hybrid number matrix A can be written as

$$A = A_1 + (A_2 - A_3)\mathbf{i} + (A_4 + A_3\mathbf{i})\mathbf{h},$$

we can write the $2n \times 2n$ real representation matrix of the hybrid number matrix

$$\aleph_A = \begin{bmatrix} A_1 + A_3 & A_2 - A_3 + A_4 \\ A_3 - A_2 + A_4 & A_1 - A_3 \end{bmatrix}. \tag{2}$$

3. Elementary Transformation and Triangulation

In this section, we discuss the elementary transformation for the hybrid number matrix, which is basically the same as that for the complex matrix.

The following three elementary row operations applied to the hybrid number matrix yield an upper triangular matrix.
(a) Interchanges: Interchanging the i-th row and the j-th row. The corresponding elementary matrix is denoted as $P(i, j)$.
(b) Scaling: Multiplying the i-th row by the hybrid number λ with $|\lambda| \neq 0$ from the left. The corresponding elementary matrix is denoted as $P(\lambda * i)$.
(c) Replacement: Multiplying the j-th row by the hybrid number λ from the left, then adding to the i-th row. The corresponding elementary matrix is denoted as $P(i, \lambda * j)$.

Remark 3.1. *Let $A = (a_{ij}) \in M_n(K)$, where K is a (generally non-commutative) hybrid number algebra, and denote the k-th row of A by r_k. For any elementary matrix $E = (e_{ik}) \in M_n(K)$, the matrix product satisfies*

$$(EA)_{ij} = \sum_{k=1}^{n} e_{ik} a_{kj}, \quad j = 1, \ldots, n.$$

Collecting all columns into a row vector gives

$$(EA)_{i,*} = \sum_{k=1}^{n} e_{ik} r_k,$$

which shows that the i-th row of EA is a left linear combination of the rows of A. Therefore, in the hybrid number setting, row operations can still be implemented through left multiplication by elementary matrices.

The following result is important for triangulating a hybrid number matrix.

Lemma 3.2. *For nonzero hybrid numbers $a = x_1 + x_2\mathbf{h}$, $b = y_1 + y_2\mathbf{h}$ with $x_1, x_2, y_1, y_2 \in \mathbb{C}$ and $\|a\| = \|b\| = 0$. If $\frac{y_1}{x_1} \neq \frac{y_2}{x_2}$, then either $\|a + b\| \neq 0$ or $\|a + \mathbf{i}b\| \neq 0$ holds.*

Proof. Since $\|a\| = \|b\| = 0$, it follows from Corollary 2.3 that

$$|x_1|^2 = |x_2|^2, \quad |y_1|^2 = |y_2|^2, \quad x_i \neq 0,\ y_i \neq 0,\ i = 1, 2.$$

Let $t_1 = \frac{y_1}{x_1} = c + d\mathbf{i}$, $t_2 = \frac{y_2}{x_2} = e + f\mathbf{i}$, then, by the properties of complex numbers, we obtain $\bar{y}_1 = \bar{x}_1\bar{t}_1$, $\bar{y}_2 = \bar{x}_2\bar{t}_2$. If the real parts of t_1 and t_2 are different,

$$\begin{aligned}
\|a+b\|^2 &= \left|\,\left|x_1+y_1\right|^2 - \left|x_2+y_2\right|^2\,\right| \\
&= \left|\,(x_1+y_1)(\bar{x}_1+\bar{y}_1) - (x_2+y_2)(\bar{x}_2+\bar{y}_2)\,\right| \\
&= |x_1\bar{x}_1 + x_1\bar{y}_1 + y_1\bar{x}_1 + y_1\bar{y}_1 - x_2\bar{x}_2 - x_2\bar{y}_2 - y_2\bar{x}_2 - y_2\bar{y}_2| \\
&= \left|\,|x_1|^2 + x_1\bar{y}_1 + y_1\bar{x}_1 + \left|y_1\right|^2 - |x_2|^2 - x_2\bar{y}_2 - y_2\bar{x}_2 - \left|y_2\right|^2\,\right| \\
&= \left|x_1\bar{y}_1 + y_1\bar{x}_1 - x_2\bar{y}_2 - y_2\bar{x}_2\right| \\
&= |x_1\bar{x}_1\bar{t}_1 + t_1x_1\bar{x}_1 - x_2\bar{x}_2\bar{t}_2 - t_2x_2\bar{x}_2| \\
&= \left|\bar{t}_1\,|x_1|^2 + t_1\,|x_1|^2 - \bar{t}_2\,|x_2|^2 - t_2\,|x_2|^2\right| \\
&= \left|\bar{t}_1\,|x_2|^2 + t_1\,|x_2|^2 - \bar{t}_2\,|x_2|^2 - t_2\,|x_2|^2\right| \\
&= |(\bar{t}_1+t_1) - (\bar{t}_2+t_2)|\,|x_2|^2 \\
&= \left|(c - d\mathbf{i} + c + d\mathbf{i}) - (e - f\mathbf{i} + e + f\mathbf{i})\right|\,|x_2|^2 \\
&= |2c - 2e|\,|x_2|^2 \neq 0
\end{aligned}$$

For a similar reason, if the imaginary parts of t_1 and t_2 are different,

$$\begin{aligned}
\|a+\mathbf{i}b\|^2 &= \left|\,\left|x_1+\mathbf{i}y_1\right|^2 - \left|x_2+\mathbf{i}y_2\right|^2\,\right| \\
&= \left|\,(x_1+\mathbf{i}y_1)(\bar{x}_1-\mathbf{i}\bar{y}_1) - (x_2+\mathbf{i}y_2)(\bar{x}_2-\mathbf{i}\bar{y}_2)\,\right| \\
&= |x_1\bar{x}_1 + y_1\bar{y}_1 - \mathbf{i}(x_1\bar{y}_1 - y_1\bar{x}_1) - (x_2\bar{x}_2 + y_2\bar{y}_2 - \mathbf{i}(x_2\bar{y}_2 - y_2\bar{x}_2))| \\
&= \left|\mathbf{i}y_1\bar{x}_1 + \mathbf{i}x_2\bar{y}_2 - \mathbf{i}x_1\bar{y}_1 - \mathbf{i}y_2\bar{x}_2\right| \\
&= |\mathbf{i}t_1x_1\bar{x}_1 + \mathbf{i}x_2\bar{x}_2\bar{t}_2 - \mathbf{i}x_1\bar{x}_1\bar{t}_1 - \mathbf{i}t_2x_2\bar{x}_2| \\
&= \left|\mathbf{i}t_1\,|x_1|^2 + \mathbf{i}\bar{t}_2\,|x_2|^2 - \mathbf{i}\bar{t}_1\,|x_1|^2 - \mathbf{i}t_2\,|x_2|^2\right| \\
&= \left|\mathbf{i}t_1\,|x_2|^2 + \mathbf{i}\bar{t}_2\,|x_2|^2 - \mathbf{i}\bar{t}_1\,|x_2|^2 - \mathbf{i}t_2\,|x_2|^2\right| \\
&= |\mathbf{i}\,[(t_1-\bar{t}_1) - (t_2-\bar{t}_2)]|\,|x_2|^2 \\
&= \left|\mathbf{i}\,[(c + d\,\mathbf{i} - c + d\,\mathbf{i}) - (e + f\,\mathbf{i} - e + f\,\mathbf{i})]\right|\,|x_2|^2 \\
&= \left|\mathbf{i}\,(2d\,\mathbf{i} - 2f\,\mathbf{i})\right|\,|x_2|^2 \neq 0
\end{aligned}$$

□

The step of triangulating a matrix by the elementary transformation is basically the same as that in the real field. We only take a vector as an example. Let $0 \neq a = \left(a^{(1)}, a^{(2)}, \cdots, a^{(n)}\right)^{\mathrm{T}}$ and $a^{(i)} \in K\,(i = 1, 2, \cdots, n)$.
Step 1. If $\|a^{(1)}\| \neq 0$, we take

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -a^{(2)}a^{(1)^{-1}} & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ -a^{(n)}a^{(1)^{-1}} & 0 & \cdots & 1 \end{pmatrix}$$

and then $La = \left(a^{(1)}, 0, \cdots, 0\right)^{\mathrm{T}}$.
Step 2. If $\|a^{(1)}\| = 0$ and $a^{(1)} \neq 0$, we have the following two situations.

(1) $\|a^{(i)}\| = 0$ for some $i > 1$, then we perform (I) on $P\,(1, i)\,a$.

(2) $\forall i$, if $\|a^{(i)}\| = 0$, we can further categorize it into the following two situations.

<a>

$$\frac{\mathbf{comp}\left(a^{(i)}\right)}{\mathbf{comp}(a^{(1)})} = \frac{\mathbf{imag}\left(a^{(i)}\right)}{\mathbf{imag}(a^{(1)})} \equiv t_i \quad (i = 1, 2, \cdots, n)$$

we take

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -t_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ -t_n & 0 & \cdots & 1 \end{pmatrix}$$

and then $La = \left(a^{(1)}, 0, \cdots, 0\right)^{\mathrm{T}}$.

<b>

$$g_1 = \frac{\mathbf{comp}\left(a^{(l)}\right)}{\mathbf{comp}(a^{(1)})} \neq \frac{\mathbf{imag}\left(a^{(l)}\right)}{\mathbf{imag}(a^{(1)})} = g_2,$$

for some l.

If real parts of g_1, g_2 are different, $\|a^{(l)} + a^{(1)}\| \neq 0$, we perform (I) on $P(1, l)\, P(l, 1 * 1)\, a$.

If imaginary parts of g_1, g_2 are different, $\|a^{(1)} + \mathbf{i}a^{(l)}\| \neq 0$, we perform (I) on $P(1, l)P(l, i * 1)P(1, l)a$.

Step 3. If $\|a^{(1)}\| = 0$ and $a^{(1)} = 0$, we have the following two situations.

(1) $\|a^{(i)}\| \neq 0$ for some $i > 1$, then we perform (I) on $P(1, i)\, a$.

(2) $\forall i$, if $\|a^{(i)}\| = 0$, $a^{(l)} \neq 0$ for some l, we perform (II) on $P(1, l)\, a$.

Referring to the literature [17], we can get the following conclusions.

Theorem 3.3. *For $A \in M_n(K)$, there exist permutation hybrid number matrices or unit lower triangular hybrid number matrices $P_1, P_2, \cdots, P_s$ and an upper triangular hybrid number matrix U such that*

$$P_s \cdots P_2 P_1 A = U.$$

As we know, for any $A \in \mathrm{F}^{n\times n}$(F, may be R, $\mathbb{C}$ or H), there exit a permutation matrix P, an unit lower triangular matrix L and an upper triangular matrix U such that

$$PA = LU.$$

But for hybrid number matrices, this conclusion is not necessarily true. For example, let

$$P\begin{pmatrix} 1 + \mathbf{h} & 1 \\ 1 - \mathbf{h} & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix}\begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix}.$$

If $P = I_2$, we have

$$\begin{pmatrix} 1 + \mathbf{h} & 1 \\ 1 - \mathbf{h} & 2 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ lu_1 & lu_2 + u_3 \end{pmatrix}.$$

Thus, we have $u_1 = 1+\mathbf{h}$, $lu_1 = 1-\mathbf{h}$, $l(1+\mathbf{h}) = 1-\mathbf{h}$, and then, by Theorem 2.4 and the real representation of hybrid number matrices in (2), we obtain $\aleph_l \aleph_{1+\mathbf{h}} = \aleph_{1-\mathbf{h}}$, therefore,

$$\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad l_{11}+l_{12} = 1,\ l_{21}+l_{22} = -1$$

which leads to a contradiction!

For the same reason, if $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have

$$\begin{pmatrix} 1-\mathbf{h} & 2 \\ 1-\mathbf{h} & 1 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ lu_1 & lu_2+u_3 \end{pmatrix}.$$

Thus, we have $u_1 = 1-\mathbf{h}$,$\mathrm{lu}_1 = 1+\mathbf{h}$, $l\,(1-\mathbf{h}) = 1+\mathbf{h}$ and then $\aleph_l \aleph_{1-\mathbf{h}} = \aleph_{1+\mathbf{h}}$, that is,

$$\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad l_{11}-l_{12} = 1,\ l_{12}-l_{11} = 1.$$

which also leads to a contradiction!

4. Some Applications

The determinant of a real or complex matrix has been intensively studied as an important tool. In this section, we will introduce a new determinant and prove some properties of it and another determinant that has been defined. With these properties, we obtain two important conclusions. At the same time, we introduced the definition of the inverse of the hybrid number matrix and proved its related properties, and finally presented its computation.

4.1. Determinant

For the determinant of the hybrid number matrix, let $A \in \mathrm{M}_n(K)$ and χ_A be the complex matrix of A. In [5], the q-determinant of A is defined as

$$|A|_q = |\chi_A|.$$

We can easily prove that $|A|_q \in \mathrm{R}$ and $\left||a|_q\right| = \|a\|^2$ for any hybrid number a.

To facilitate the subsequent study of LU decomposition and invertibility of hybrid number matrices, we introduce a new definition of the determinant.

Definition 4.1. *For $A \in M_n(K)$, we define a new determinant as*

$$\mathbf{mdet}_1(A) = \mathbf{det}(\aleph_A).$$

For ease of computation later, we denote $|\chi_A|$ as $\mathbf{mdet}_2(A)$. $\mathbf{mdet}_1(A)$ and $\mathbf{mdet}_2(A)$ have the following properties, which are almost the same as those of the normal determinant.

In the following, we can easily draw an important conclusion.

Lemma 4.2. *Let $A, C \in M_n(K)$, $B \in R^{n\times n}$, $\lambda \in K$, and $w = 1,2$. Then*

(1) $\mathbf{mdet}_w(B) = \mathbf{det}(B)^2$,

(2) $\mathbf{mdet}_w(P(i,j)) = -1$; $\quad \mathbf{mdet}_w(P(\lambda * i)) = \bar{\lambda}\lambda$; $\quad \mathbf{mdet}_w(P(j, \lambda * i)) = 1$,

(3) $\mathbf{mdet}_w(P(i,j)A) = -\mathbf{mdet}_w(A)$; $\quad \mathbf{mdet}_w(P(\lambda * i)A) = \bar{\lambda}\lambda\, \mathbf{mdet}_w(A)$;
$\mathbf{mdet}_w(P(j, \lambda * i)A) = \mathbf{mdet}_w(A)$,

(4) $\mathbf{mdet}_w(AC) = \mathbf{mdet}_w(A)\, \mathbf{mdet}_w(C)$,

(5) $\mathbf{mdet}_w(A) = \bar{a}_{11}a_{11}, \ldots, \bar{a}_{nn}a_{nn}$ *if $A = (a_{ij})$ is an upper triangular matrix.*

Proof. All properties follow from the real or complex representation of hybrid number matrices.
(1) Since B is real, its representation is block diagonal:

$$\aleph_B = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}, \quad \chi_B = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix},$$

hence

$$\mathbf{mdet}_w(B) = \det(\aleph_B) = \det(B)^2.$$

(2) Using standard determinant properties for row operations:

<a> Since

$$\mathbf{mdet}_w(P) = \det(\aleph_P) \ \text{ or } \ \det(\chi_P),$$

in the real or complex representation, $\aleph_{p(\mathrm{i,j})}$ is also a block matrix that exchanges two rows. Its determinant properties are the same as those of ordinary real or complex matrices. Since the determinant of a matrix that exchanges two rows is known to be -1, we have

$$\mathbf{mdet}_w(P(i, j)) = -1.$$

<b> In the real or complex representation, $P(\lambda * i)$ is a block upper triangular matrix, with only the i-th diagonal block multiplied on the left by λ, while the other diagonal blocks are identity matrices I. By the property that the determinant of a block upper triangular matrix equals the product of the determinants of its diagonal blocks, we have

$$\mathbf{mdet}_w(P(\lambda * i)) = \bar{\lambda}\lambda.$$

<c> $P(j, \lambda * i)$ in the real or complex representation, it remains a block upper triangular matrix with all diagonal blocks being identity matrices, hence

$$\mathbf{mdet}_w\left(P\left(j, \lambda * i\right)\right) = 1.$$

(3) By multiplicativity of determinants:

$$\mathbf{mdet}_w(PA) = \mathbf{mdet}_w(P)\mathbf{mdet}_w(A),$$

hence the formulas follow directly from (2).
(4) Since $\mathbf{mdet}_w(A) = \det(\aleph_A)$ or $\det(\chi_A)$,

$$\mathbf{mdet}_w(AC) = \det(\aleph_A\aleph_C) = \det(\aleph_A)\det(\aleph_C) = \mathbf{mdet}_w(A)\mathbf{mdet}_w(C).$$

(5) If A is upper triangular, then $\aleph_A$ or χ_A is block upper triangular. By the property of block upper triangular matrices, the determinant equals the product of the determinants of its diagonal blocks, we obtain

$$\mathbf{mdet}_w(A) = \prod_{i=1}^{n} \bar{a}_{ii}a_{ii}.$$

□

By means of theorem 3.3, we can prove that the two determinants are the same.

Theorem 4.3. *For* $A \in M_n(K)$, $\mathbf{mdet}_1(A) = \mathbf{mdet}_2(A)$.

Proof. From Theorem 3.3, we know that there exist permutation matrices or unit lower triangular hybrid number matrices $P_1P_2,\cdots,P_s$ and an upper triangular hybrid number matrix U such that $P_s\cdots P_2P_1A = U = \left(u_{ij}\right)$. From (2) and (3) of Lemma 4.2, we have

$$\mathbf{mdet}_1(A) = \mathbf{mdet}_1(U) = \mathbf{det}(\aleph_U)\,, \quad \mathbf{mdet}_2(A) = \mathbf{mdet}_2(U) = \mathbf{det}(\chi_U)\,.$$

where $\aleph_U$ and χ_U are block upper triangular matrices with diagonal blocks $\aleph_{u_{ii}}$ and $\chi_{u_{ii}}$. Based on the properties of hybrid numbers and determinants, namely $\left|\det \chi_p\right| = \left|\det \aleph_p\right|$ and the fact that the determinant of a block upper triangular matrix equals the product of the determinants of its diagonal blocks, we can obtain

$$\mathbf{mdet}_1(A) = \prod_{i=1}^{n}\mathbf{det}(\aleph_{u_{ii}}) = \prod_{i=1}^{n}\mathbf{det}(\chi_{u_{ii}}) = \mathbf{mdet}_2(A)\,.$$

□

Remark 4.4. *In Theorem 3.3, U is not unique, and the product of its diagonal elements is also not unique. However, the modulus of the product is unique.*

At the end of this section, referring to the literature [17], we give an important conclusion.

Theorem 4.5. *For* $A = \left(a_{ij}^{(1)}\right) \in M_n(K)$, *if* $\mathbf{mdet}_1\left(A^{(k)}\right) \neq 0$, $k = 1, 2, \cdots, n-1$, *there exist an unit lower triangular matrix and L and an upper triangular matrix U such that*

$$A = LU.$$

where $A^{(k)} = A\,(1:k, 1:k)$ *is the k-order leading principal submatrix of A.*

Proof. Because $\left|a_{11}^{(1)}\right|^2 = \left|\mathbf{mdet}_1\left(A^{(1)}\right)\right| \neq 0$, there exists

$$L_1 = I + \left(0, -a_{21}^{(1)}\left(a_{11}^{(1)}\right)^{-1}, \cdots, -a_{n1}^{(1)}\left(a_{11}^{(1)}\right)^{-1}\right)^{\mathrm{T}} e_1^{\mathrm{T}}.$$

such that

$$L_1A = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{n2}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} \end{pmatrix}. \tag{3}$$

Partitioninng L_1 and A into

$$L_1 = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \text{ and } A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{4}$$

where L_{11} and A_{11} are 2×2 matrices and L_{11} is unit low triangular, we get

$$L_{11}A_{11} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ 0 & a_{22}^{(2)} \end{pmatrix}.$$

Since

$$0 \neq \left|\mathbf{mdet}_1\left(A^{(2)}\right)\right| = \left|\mathbf{mdet}_1\left(A_{11}\right)\right| = \left|\mathbf{mdet}_1\left(L_{11}A_{11}\right)\right| = \left|a_{11}^{(1)}\right|^2\left|a_{22}^{(2)}\right|^2.$$

it follows that $\left|a_{22}^{(2)}\right| \neq 0$ and Gaussian elimination process can continue. Detailed information on the Gaussian elimination method is available in [9]. In the end, we can get

$$L_j = I + \left(\underbrace{0,\cdots,0,}_{j} - a_{j+1,j}^{(j)}\left(a_{jj}^{(j)}\right)^{-1}, \cdots, -a_{nj}^{(j)}\left(a_{jj}^{(j)}\right)^{-1}\right)^{\mathrm{T}} e_j^{\mathrm{T}} \quad (j = 1, 2, \cdots, n-1)$$

with

$$L_{n-1}\cdots L_2 L_1 A = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{(n)} \end{pmatrix} \equiv U.$$

Let $L = L_1^{-1}L_2^{-1}\cdots L_{n-1}^{-1}$. Then we have $A = LU$.
□

4.2. The Inverse

In this subsection, we give the definition of the inverse matrix and discuss its properties and computation method.

In [5], it is known that for $A, B \in \mathrm{M}_n(K)$, if $AB = BA = I$, then A is called an invertible matrix and B is called the inverse of A .

Theorem 4.6. *[5] For any $C \in M_n(K)$ the following properties generally hold.*

1. $(\bar{C})^{-1} \neq \overline{(C^{-1})}$,
2. $\left(C^{\mathrm{T}}\right) \neq \left(C^{-1}\right)^{\mathrm{T}}$.

In [5], the authors proved that A is invertible if and only if χ_A is invertible. Naturally, we get the following conclusions.

Theorem 4.7. *Let $A = A_1 + A_2\mathbf{i} + A_3\boldsymbol{\varepsilon} + A_4\mathbf{h} \in M_n(K)$, A is invertible if and only if the real representation matrix $\aleph_A$ of A is invertible.*

Proof. If A is invertible, then there is B such that $AB = BA = I$, which means $\aleph_{AB} = \aleph_{BA} = \aleph_I = I_{2n}$, that is, $\aleph_A$ is invertible.

On the contrary, if $\aleph_A$ is invertible, then there is $B \in \mathrm{R}^{2n\times 2n}$ such that $\aleph_A B = B\aleph_A = I_{2n}$. Partitioned B into $B = \left(B_{ij}\right)$ with $\left(B_{ij}\right) \in \mathrm{R}^{2\times 2}, i, j = 1, 2, \cdots, n$, then there is an unique set of matrices: $B_{ij}^{(1)}, B_{ij}^{(2)}, B_{ij}^{(3)}$ and $B_{ij}^{(4)}$ satisfying

$$B_{ij}^{(1)} + B_{ij}^{(3)} = B_{ij}(1,1), \quad B_{ij}^{(2)} - B_{ij}^{(3)} + B_{ij}^{(4)} = B_{ij}(1,2),$$

$$B_{ij}^{(3)} - B_{ij}^{(2)} + B_{ij}^{(4)} = B_{ij}(2,1), \quad B_{ij}^{(1)} - B_{ij}^{(3)} = B_{ij}(2,2).$$

where $B_{ij}(w,h)$, is located in the w-th row and h-th column of the chunked matrix B.

Taking $\bar{b}_{ij} = B_{ij}^{(1)} + B_{ij}^{(2)}\mathbf{i} + B_{ij}^{(3)}\boldsymbol{\varepsilon} + B_{ij}^{(4)}\mathbf{h}$ and $\bar{B} = \left(\bar{b}_{ij}\right)$, we have $\aleph_{\bar{B}} = B$ and $\aleph_A\aleph_{\bar{B}} = \aleph_{\bar{B}}\aleph_A = I_{2n}$, which means $A\bar{B} = \bar{B}A = I_n$, that is, A is invertible.
□

Corollary 4.8. *A is invertible if and only if $\mathbf{mdet}_1(A) \neq 0$.*

Corollary 4.9. *The equivalence of invertibility between a hybrid matrix and its real or complex representation.*

In the following, we give the computation method the inverse matrix.

Theorem 4.10. *If $A \in M_n(K)$ is an invertible hybrid number matrix, then there exists an invertible hybrid number matrix P such that $PA = I$.*

Proof. From Theorem 3.3, there exist permutation matrices or unit lower triangular matrices $P_1, P_2, \cdots, P_s$ and an upper triangular matrice U such that $P_s \cdots P_2 P_1 A = U = \left(u_{ij}\right)$.

It follows from $0 \neq \mathbf{mdet}_1(A) \neq \mathbf{mdet}_1(U)$ that $|u_{ii}| \neq 0$ for all i. So there exist scaling elementary matrices or unit upper triangular matrices $Q_1, Q_2, \cdots, Q_l$ such that

$$Q_l \cdots Q_2 Q_1 P_s \cdots P_2 P_1 A = Q_l \cdots Q_2 Q_1 U = I.$$

Let $P = Q_l \cdots Q_2 Q_1 P_s \cdots P_2 P_1$.Then we complete the proof.
□

With the above theorem, we can learn that the method used to compute the inverse of a hybrid number matrix is the same as the method used to compute a real or complex number matrix. Implementing three elementary row transformations on (A, E), we convert (A, E) to (E, B) and B is the inverse matrix of A.

Example 4.11. *Let* $A = \begin{pmatrix} 1+\mathbf{i} & \mathbf{i}+\varepsilon & 1-\mathbf{h} \\ \mathbf{h} & 1 & \varepsilon \\ 1 & \mathbf{h} & 1+\varepsilon \end{pmatrix} \in M_3(K)$. *Then the inverse of A is computed as follows.*

$$A = \begin{pmatrix} 1+\mathbf{i} & \mathbf{i}+\varepsilon & 1-\mathbf{h} & 1 & 0 & 0 \\ \mathbf{h} & 1 & \varepsilon & 0 & 1 & 0 \\ 1 & \mathbf{h} & 1+\varepsilon & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3-\mathbf{h}R_2,\ R_1-(\mathbf{i}+\varepsilon)R_2} \begin{pmatrix} 1 & 0 & 0 & 1 & -\mathbf{i}-\varepsilon & 0 \\ \mathbf{h} & 1 & \varepsilon & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -\mathbf{h} & 1 \end{pmatrix}$$

$$\xrightarrow{R_2-\mathbf{h}R_1} \begin{pmatrix} 1 & 0 & 0 & 1 & -\mathbf{i}-\varepsilon & 0 \\ 0 & 1 & \varepsilon & -\mathbf{h} & 1-\mathbf{i} & 0 \\ 0 & 0 & 1 & 0 & -\mathbf{h} & 1 \end{pmatrix} \xrightarrow{R_2-\varepsilon R_3} \begin{pmatrix} 1 & 0 & 0 & 1 & -\mathbf{i}-\varepsilon & 0 \\ 0 & 1 & 0 & -\mathbf{h} & 1-\mathbf{i}-\varepsilon & -\varepsilon \\ 0 & 0 & 1 & 0 & -\mathbf{h} & 1 \end{pmatrix}.$$

Step 1. During the first elementary transformation, the two corresponding elementary matrices are respectively

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\mathbf{h} & 1 \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & -(\mathbf{i}+\varepsilon) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Step 2. During the second elementary transformation, the corresponding elementary matrix is

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ -\mathbf{h} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Step 3. During the third elementary transformation, the corresponding elementary matrix is

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, we obtain

$$B = \begin{pmatrix} 1 & -\mathbf{i}-\varepsilon & 0 \\ -\mathbf{h} & 1-\mathbf{i}-\varepsilon & -\varepsilon \\ 0 & -\mathbf{h} & 1 \end{pmatrix},$$

and by verifying that

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

it follows that B is the inverse of A.

5. Conclusions

In this paper, we define the elementary transformations for hybrid number matrix and give its upper triangularization process. Then, we define a new determinant by means of a real representation matrix and summarize the sufficient conditions for the existence of LU decompositions. Additionally, through the elementary transformations, we are able to solve for the inverse matrices of hybrid number matrices, providing effective tools for related computations. Now, our study of hybrid number matrix decomposition is not deep enough and comprehensive enough, in the future, we will go on to continue the study of other decompositions of hybrid number matrices.

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