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On the linear algebraic properties of (r, n)-min and (r, n)-max matrices

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Abstract. This article introduces the (r, n)-min and (r, n)-max matrices, which generalize the concepts of the r-min, r-max, min, and max matrices. Subsequently, our focus is directed towards the discussion of the algebraic structure of the (r, n)-min and (r, n)-max matrices, their reciprocal matrices, and distinct representations. These efforts enable us to elucidate several mathematical properties, including determinants, inverses, and certain norms.

1. Introduction

Various problems in the field of applied mathematics involve special types of matrices. Depending on the specific form of these matrices it is essential to examine their particular algebraic properties, such as determinants, inverses, norms, and other related notions. Let m be a positive integer with $m \geq 1$.

$$M_{min} = \begin{pmatrix} 1 & 1 & 1 & \cdots 1 \\ 1 & 2 & 2 & \cdots 2 \\ 1 & 2 & 3 & \cdots 3 \\ \vdots & \vdots & \vdots & \ddots \vdots \\ 1 & 2 & 3 & \cdots m \end{pmatrix}$$

and

$$M_{max} = \begin{pmatrix} 1 & 2 & 3 & \cdots m \\ 2 & 2 & 3 & \cdots m \\ 3 & 3 & 3 & \cdots m \\ \vdots & \vdots & \vdots & \ddots \vdots \\ m & m & m & \cdots m \end{pmatrix}$$

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are called the min and max matrices, respectively. The min matrix was initially presented in the second volume of the seminal collection of problems by G. Polya and Szegő [14]. Let $T = \{x_1, x_2, ..., x_m\}$ be a finite multiset of real numbers, where $x_1 < x_2 < \cdots < x_m$. After this, more recently, the following matrices were defined

$$T_{min} = \begin{pmatrix} x_1 & x_1 & x_1 & \cdots x_1 \\ x_1 & x_2 & x_2 & \cdots x_2 \\ x_1 & x_2 & x_3 & \cdots x_3 \\ \vdots & \vdots & \vdots & \ddots \vdots \\ x_1 & x_2 & x_3 & \cdots x_m \end{pmatrix} \tag{1}$$

and

$$T_{max} = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots x_m \\ x_2 & x_2 & x_3 & \cdots x_m \\ x_3 & x_3 & x_3 & \cdots x_m \\ \vdots & \vdots & \vdots & \ddots \vdots \\ x_m & x_m & x_m & \cdots x_m \end{pmatrix}. \tag{2}$$

The matrices defined in (1) and (2) have been extensively studied in the literature (see [1, 4, 6, 10–13, 15]). On the other hand, one of the special matrices is the r-circulant matrices which have the following form

$$C_r(a_o, \dots, a_{m-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots a_{m-1} \\ ra_{m-1} & a_0 & a_1 & \cdots a_{m-2} \\ ra_{m-2} & ra_{m-1} & a_0 & \cdots a_{m-3} \\ \vdots & \vdots & \vdots & \ddots \vdots \\ ra_1 & ra_2 & ra_3 & \cdots a_0 \end{pmatrix}, \tag{3}$$

where $a_i \in \mathbb{C}$ for all $i = 0, 1, \dots, m-1$. Motivated by the form of the matrices (1), (2) and (3), Kızılateş and Terzioğlu in [8] defined the so-called r-min and r-max matrices as follows:

$$P_{r,m} = \begin{pmatrix} x_1 & x_1 & x_1 & \cdots x_1 \\ rx_1 & x_2 & x_2 & \cdots x_2 \\ rx_1 & rx_2 & x_3 & \cdots x_3 \\ \vdots & \vdots & \vdots & \ddots \vdots \\ rx_1 & rx_2 & rx_3 & \cdots x_m \end{pmatrix} \tag{4}$$

and

$$Q_{r,m} = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots x_m \\ rx_2 & x_2 & x_3 & \cdots x_m \\ rx_3 & rx_3 & x_3 & \cdots x_m \\ \vdots & \vdots & \vdots & \ddots \vdots \\ rx_m & rx_m & rx_m & \cdots x_m \end{pmatrix}, \tag{5}$$

where r is a complex non null parameter. When $r = 1$, they are called the min and max matrices. Moreover, the authors obtained the determinants, inverses, norms, and factorizations of these matrices and their reciprocal counterparts. Another version of the min and max matrix was studied by Fonseca et al. [5]. The authors define geometric min and geometric max matrices and discuss some important linear algebra properties of these matrices. After this, Li and Yuan [9] defined the general forms of geometric min and geometric max matrices. In [2] Anđelić et al. examined the principal characteristics of the r-min and r-max matrices, which incorporate harmonic higher-order Gauss Fibonacci numbers as entries. Specifically, they focused on the inverse, determinant, permanent, and norms of these matrices. Additionally, they established several recurrence relations for characteristic polynomials. In [3] Andrade et al. defined an (r,n)-circulant matrix as follows:

$$A_{(r,n)}(a_o,\dots,a_{m-1}) = \begin{pmatrix} a_0 & na_1 & na_2 & \cdots na_{m-1} \\ ra_{m-1} & a_0 & a_1 & \cdots a_{m-2} \\ ra_{m-2} & rna_{m-1} & a_0 & \cdots a_{m-3} \\ \vdots & \vdots & \vdots & \ddots \vdots \\ ra_1 & rna_2 & rna_3 & \cdots a_0 \end{pmatrix}, \tag{6}$$

where r, n are nonzero complex numbers. If we take $r = n = 1$ in (6), we get the circulant matrices. If we take $n = 1$ in (6), we obtain the r-circulant matrices. They then obtained the eigenvalue and inverse of this type of matrix.

In this paper, motivated by the structure of the matrix (6), our aim is to define and study the (r,n)-min and (r,n)-max matrices, which are general forms of the r-min and r-max matrices given in (4) and (5). Moreover, we provide the definition of reciprocal matrices corresponding to these matrices. Then, we provide important lemmas and their proofs, which we will obtain in matrix norms, which is one of our main results. We then analyze the important linear algebraic properties of these four special types of matrices.

2. (r,n)-min and (r,n)-max matrices

In this part of our paper, we define the (r,n)-min and (r,n)-max matrices and provide some important properties of these newly established matrices such as determinant, inverse, and norms.

Definition 2.1. *Let $P_{(r,n),m} = (p_{ij})_{i,j=1}^{m}$ and $Q_{(r,n),m} = (q_{ij})_{i,j=1}^{m}$ be matrices of order m-by-m defined by*

$$p_{ij} = \begin{cases} nx_{\min\{i,j\}} & ; \quad j > i = 1 \\ x_j & ; \quad i = j \\ x_{\min\{i,j\}} & ; \quad j \geq i \neq 1 \\ rx_{\min\{i,j\}} & ; \quad 1 = j < i \\ rnx_{\min\{i,j\}} & ; \quad 1 \neq j < i \end{cases}$$

and

$$q_{ij} = \begin{cases} nx_{\max\{i,j\}} & ; \quad j > i = 1 \\ x_j & ; \quad i = j \\ x_{\max\{i,j\}} & ; \quad j \geq i \neq 1 \\ rx_{\max\{i,j\}} & ; \quad 1 = j < i \\ rnx_{\max\{i,j\}} & ; \quad 1 \neq j < i, \end{cases}$$

where $r, n \in \mathbb{C}$ are non zero, and x_i are real numbers such that $x_1 < x_2 < \cdots < x_m$.

Note that the matrices $P_{(r,n),m}$ and $Q_{(r,n),m}$ are represented as follows:

$$P_{(r,n),m} = \begin{pmatrix} x_1 & nx_1 & nx_1 & \cdots nx_1 \\ rx_1 & x_2 & x_2 & \cdots x_2 \\ rx_1 & rnx_2 & x_3 & \cdots x_3 \\ \vdots & \vdots & \vdots & \ddots \vdots \\ rx_1 & rnx_2 & rnx_3 & \cdots x_m \end{pmatrix} \tag{7}$$

and

$$Q_{(r,n),m} = \begin{pmatrix} x_1 & nx_2 & nx_3 & \cdots nx_m \\ rx_2 & x_2 & x_3 & \cdots x_m \\ rx_3 & rnx_3 & x_3 & \cdots x_m \\ \vdots & \vdots & \vdots & \ddots \vdots \\ rx_m & rnx_m & rnx_m & \cdots x_m \end{pmatrix}. \tag{8}$$

The reciprocal matrices of $P_{(r,n),m}$ and $Q_{(r,n),m}$ are

$$P^{\circ -1}_{(r,n),m} = \begin{pmatrix} \frac{1}{x_1} & \frac{1}{nx_1} & \frac{1}{nx_1} & \cdots & \frac{1}{nx_1} \\ \frac{1}{rx_1} & \frac{1}{x_2} & \frac{1}{x_2} & \cdots & \frac{1}{x_2} \\ \frac{1}{rx_1} & \frac{1}{rnx_2} & \frac{1}{x_3} & \cdots & \frac{1}{x_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{rx_1} & \frac{1}{rnx_2} & \frac{1}{rnx_3} & \cdots & \frac{1}{rnx_m} \end{pmatrix} \tag{9}$$

and

$$Q^{\circ -1}_{(r,n),m} = \begin{pmatrix} \frac{1}{x_1} & \frac{1}{nx_2} & \frac{1}{nx_3} & \cdots & \frac{1}{nx_m} \\ \frac{1}{rx_2} & \frac{1}{x_2} & \frac{1}{x_3} & \cdots & \frac{1}{x_m} \\ \frac{1}{rx_3} & \frac{1}{rnx_3} & \frac{1}{x_3} & \cdots & \frac{1}{x_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{rx_m} & \frac{1}{rnx_m} & \frac{1}{rnx_m} & \cdots & \frac{1}{rnx_m} \end{pmatrix}. \tag{10}$$

Before starting one of the main results, we require some previous lemmas. Let $m \geq 1$. Let R_1, R_2, R_3 be the following sets (see Figure 1)

$$\begin{aligned} R_1 =& \Big\{(n,r) \in \mathbb{C}^2 \backslash \{(0,0)\} : |r|^2 \leq (m-1)|n|^2 - (m-2) \text{ and } |r|^2 < \frac{(m-1)|n|^2}{(m-2)|n|^2+1}\Big\}, \\ R_2 =& \Big\{(n,r) \in \mathbb{C}^2 \backslash \{(0,0)\} : |rn| < 1 \text{ and } |r|^2 > (m-1)|n|^2 - (m-2)\Big\}, \\ R_3 =& \Big\{(n,r) \in \mathbb{C}^2 \backslash \{(0,0)\} : |rn| \geq 1 \text{ and } |r|^2 \geq \frac{(m-1)|n|^2}{(m-2)|n|^2+1}\Big\}. \end{aligned}$$

For A and B matrices, the Hadamard product is denoted by $A \circ B = (a_{ij}b_{ij})$ and for any $C \in M_m(\mathbb{C})$, we define

$$r_1(C) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^{m} |c_{ij}|^2}; \quad c_1(C) = \max_{1 \leq j \leq m} \sqrt{\sum_{i=1}^{m} |c_{ij}|^2} \textbf{ and}$$

and

$$S_i(C) = \sum_{j=1}^{m} |c_{ij}|^2; \quad 1 \leq i \leq m.$$

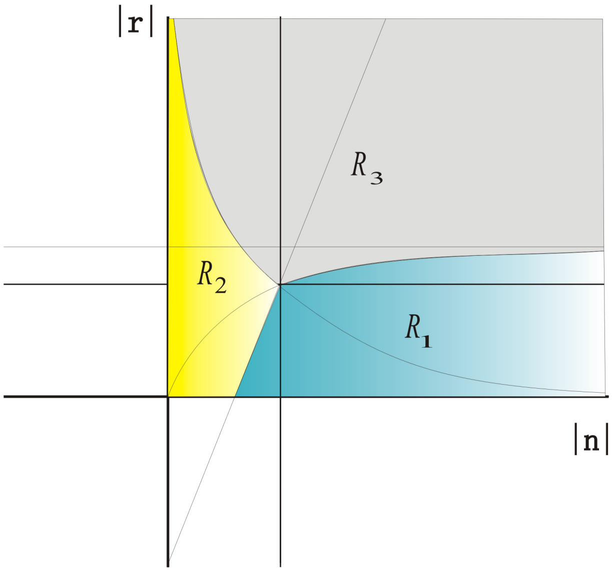

Figure 1: Regions R_i for the norm $\|\cdot\|_2$

Lemma 2.2. *Let A be the following m-by-m matrix:*

$$A = \begin{pmatrix} 1 & n & n & \cdots & n & n \\ r & 1 & 1 & \cdots & 1 & 1 \\ r & rn & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & rn & rn & \cdots & rn & 1 \end{pmatrix}.$$

Then, we have

1) *if* $(n,r) \in R_1$ *then* $r_1(A) = \sqrt{1 + (m-1)|n|^2}$,
2) *if* $(n,r) \in R_2$ *then* $r_1(A) = \sqrt{|r|^2 + (m-1)}$,
3) *if* $(n,r) \in R_3$ *then* $r_1(A) = \sqrt{(m-2)|rn|^2 + |r|^2 + 1}$.

Proof. First we note that

$$S_1(A) = 1 + (m-1)|n|^2,$$
$$S_i(A) = |r|^2 + (m-(i-1)) + (i-2)|rn|^2; \quad 2 \le i \le m.$$

Moreover, we get

$$\begin{aligned} S_{i+1}(A) &= |r|^2 + (m-i) + (i-1)|rn|^2 \\ &= |r|^2 + (m-(i-1)-1) + ((i-2)+1)|rn|^2 \\ &= |r|^2 + (m-(i-1)) + (i-2)|rn|^2 + |rn|^2 - 1 \\ &= S_i(A) + |rn|^2 - 1; \quad 2 \le j \le m. \end{aligned}$$

For item 1) we suppose that $(n,r) \in R_1$. Then, we have

$$|r|^2 \le (m-1)|n|^2 - (m-2) \text{ and } |r|^2 < \frac{(m-1)|n|^2}{(m-2)|n|^2+1}.$$

Also, we note that $|r|^2 \leq (m-1)|n|^2 - (m-2)$ if and only if $S_2(A) \leq S_1(A)$ and $|r|^2 < \frac{(m-1)|n|^2}{(m-2)|n|^2+1}$, if and only if $S_m(A) < S_1(A)$.

If $|rn| \geq 1$, then

$$S_2(A) \leq S_3(A) \leq \cdots \leq S_m(A) < S_1(A).$$

If $|rn| \leq 1$, then

$$S_m(A) \leq S_{m-1}(A) \leq \cdots \leq S_2(A) \leq S_1(A).$$

Therefore, we have

$$r_1(A) = \max_{1 \leq i \leq m} \sqrt{S_i(A)} = \sqrt{S_1(A)} = \sqrt{1 + (m-1)|n|^2}.$$

For item 2) we suppose that $(n, r) \in R_2$. Then $|rn| < 1$ and $|r|^2 > (m-1)|n|^2 - (m-2)$. Also we note that

$$|r|^2 > (m-1)|n|^2 - (m-2), \quad \textit{if and only if} \quad S_2(A) > S_1(A).$$

Thus

$$S_m(A) \leq S_{m-1}(A) \leq \cdots \leq S_2(A) \text{ and } S_1(A) < S_2(A).$$

Therefore, we have

$$r_1(A) = \max_{1 \leq i \leq m} \sqrt{S_i(A)} = \sqrt{S_2(A)} = \sqrt{|r|^2 + (m-1)}.$$

For item 3) we suppose that $(n, r) \in R_3$. Then $|rn|^2 - 1 \geq 0$ and $|r|^2 \geq \frac{(m-1)|n|^2}{(m-2)|n|^2+1}$. Moreover, we can write

$$S_{i+1}(A) \geq S_i(A); \quad 2 \leq j \leq m.$$

Also we note that

$$|r|^2 \geq \frac{(m-1)|n|^2}{(m-2)|n|^2 + 1}, \text{ if and only if } \quad S_1(A) \leq S_m(A).$$

Then

$$S_1(A) \leq S_m(A) \text{ and } S_2(A) \leq S_3(A) \leq \cdots \leq S_m(A).$$

Therefore, we have

$$r_1(A) = \max_{1 \leq i \leq m} \sqrt{S_i(A)} = \sqrt{S_m(A)} = \sqrt{(m-2)|rn|^2 + |r|^2 + 1}.$$

And so the proof is finished. □

Remark 2.3. *We note that if $n = 1$ then $R_1 = \{r \in \mathbb{C} : |r| < 1\}$, $R_2 = \emptyset$ and $R_3 = \{r \in \mathbb{C} : |r| \geq 1\}$. So, we obtain the following result in* [8] *(see Theorem 2.4). For*

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ r & 1 & 1 & \cdots & 1 & 1 \\ r & r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & r & \cdots & r & 1 \end{pmatrix}. \tag{11}$$

We have

1) *if $|r| < 1$ then $r_1(A) = \sqrt{m}$,*

2) *if $|r| \geq 1$ then $r_1(A) = \sqrt{(m-1)|rn|^2 + 1}$.*

Let m be a positive integer. We define T_1, T_2, T_3 as:

$$\begin{aligned}
T_1 =&\Big\{(n,r)\in\mathbb{C}^2\backslash\{(0,0)\}\} : |rn|\geq 1 \text{ and } |r|^2 \leq \frac{|n|^2}{(m-1)-(m-2)|n|^2}\Big\}\\
&\cup\Big\{(n,r)\in\mathbb{C}^2\backslash\{(0,0)\}\} : |n|\geq \frac{3}{2} \text{ and } |r|^2 \geq \frac{|n|^2}{(m-1)-(m-2)|n|^2}\Big\},\\
T_2 =&\Big\{(n,r)\in\mathbb{C}^2\backslash\{(0,0)\}\} : |n|\leq 1 \text{ and } |r|^2 \geq \frac{1}{m-2}|n|^2+\frac{m-2}{m-1}\Big\}\\
&\cup\Big\{(n,r)\in\mathbb{C}^2\backslash\{(0,0)\}\} : |n|\geq \frac{3}{2} \text{ and } r|^2 > \frac{|n|^2}{(m-1)-(m-2)|n|^2}\Big\},\\
T_3 =&\Big\{(n,r)\in\mathbb{C}^2\backslash\{(0,0)\}\} : |n|\leq 1 \text{ and } |r|^2 \geq \frac{1}{m-2}|n|^2+\frac{m-2}{m-1}\Big\}\Big\}.
\end{aligned}$$

Lemma 2.4. *Let B be the following m-by-m matrix:*

$$B=\begin{pmatrix} 1 & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n}\\ \frac{1}{r} & 1 & 1 & \cdots & 1 & 1\\ \frac{1}{r} & \frac{1}{rn} & 1 & \cdots & 1 & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \frac{1}{r} & \frac{1}{rn} & \frac{1}{rn} & \cdots & \frac{1}{rn} & 1 \end{pmatrix}.$$

Then, we have

1) *if $(n,r)\in T_1$ then $r_1(B)=\sqrt{1+\frac{m-1}{|n|^2}}$,*

2) *if $(n,r)\in T_2$ then $r_1(B)=\sqrt{(m-1)+\frac{1}{|r|^2}}$,*

3) *if $(n,r)\in T_3$ then $r_1(B)=\sqrt{1+\frac{1}{|r|^2}+\frac{m-2}{|rn|^2}}$.*

Proof. We analyze the three cases corresponding to the regions T_1,T_2, and T_3 for the specific values of $r_1(B)$. We start by calculating the necessary quantities $S_i(B)$ in a similar form to the previous lemma. We define the quantities $S_i(B)$ that describe the behavior of the elements in terms of $|r|$, $|n|$, and m:

$$S_1(B) = 1+\frac{m-1}{|n|^2},$$

and for $i\geq 2$,

$$S_i(B) = \frac{1}{|r|^2}+(m-(i-1))+(i-2)\frac{1}{|rn|^2}.$$

We establish the relationship between $S_i(B)$ and $S_{i+1}(B)$ for $i\geq 2$:

$$S_{i+1}(B) = \frac{1}{|r|^2}+(m-i)+(i-1)\frac{1}{|rn|^2}.$$

This leads to the relation:

$$S_{i+1}(B) = S_i(B)+\frac{1}{|rn|^2}-1.$$

For the case $(n,r)\in T_1$, we have:

$$|rn|\geq 1 \quad\text{and}\quad |r|^2\leq\frac{|n|^2}{(m-1)-(m-2)|n|^2}.$$

Then $S_1(B) \geq S_2(B)$, and also

$$S_m(B) < S_1(B).$$

Therefore, under these conditions, we conclude that:

$$r_1(B) = \max_{1\leq i\leq m} \sqrt{S_i(B)} = \sqrt{S_1(B)} = \sqrt{1 + \frac{m-1}{|n|^2}}.$$

For the case $(n, r) \in T_2$, we have

$$|n| \leq 1 \quad \text{and} \quad |r|^2 \geq \frac{1}{m-2}|n|^2 + \frac{m-2}{m-1}.$$

Then $S_2(B) > S_1(B)$ which implies that

$$S_m(B) \leq S_{m-1}(B) \leq \cdots \leq S_2(B),$$

and therefore:

$$r_1(B) = \max_{1\leq i\leq m} \sqrt{S_i(B)} = \sqrt{S_2(B)} = \sqrt{(m-1) + \frac{1}{|r|^2}}.$$

For the case $(n, r) \in T_3$ we have

$$|n| \leq 1 \quad \text{and} \quad |r|^2 \geq \frac{1}{m-2}|n|^2 + \frac{m-2}{m-1}.$$

This implies that $S_1(B) \leq S_m(B)$ and that:

$$S_2(B) \leq S_3(B) \leq \cdots \leq S_m(B).$$

Thus, we have

$$r_1(B) = \max_{1\leq i\leq m} \sqrt{S_i(B)} = \sqrt{S_m(B)} = \sqrt{1 + \frac{1}{|r|^2} + \frac{m-2}{|rn|^2}}.$$

So the proof is completed. □

Remark 2.5. *Note that, unlike the regions in Lemma 11, the regions T_i in Lemma 2.4 are expressed as unions of sets, since the structure of the matrix B allows for cases such as $|n| > 3/2$ to arise.*

2.1. Properties for the matrix $P_{(r,n),m}$

Theorem 2.6. *The determinant of the matrix $P_{(r,n),m}$ is*

$$\det(P_{(r,n),m}) = x_1 \prod_{i=1}^{m-1}(x_{i+1} - rnx_i).$$

Proof. Applying appropriate column operations to $P_{(r,n),m}$, we have

$$\det(P_{(r,n),m}) = \begin{vmatrix} x_1 & 0 & 0 & \cdots & 0 & 0 \\ rx_1 & x_2 - rnx_1 & 0 & \cdots & 0 & 0 \\ rx_1 & rn(x_2 - x_1) & x_3 - rnx_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rx_1 & rn(x_2 - x_1) & rn(x_3 - x_2) & \cdots & x_{m-1} - rnx_{m-2} & 0 \\ rx_1 & rn(x_2 - x_1) & rn(x_3 - x_2) & \cdots & rn(x_{m-1} - x_{m-2}) & x_m - rnx_{m-1} \end{vmatrix}$$

$$= x_1 \prod_{i=1}^{m-1}(x_{i+1} - rnx_i).$$

□

Theorem 2.7. *Let $P_{(r,n),m}$ be defined as in (7). If $x_1 \neq 0$ and $x_{i+1} \neq nrx_i$, for all $i = 1, \ldots, m-1$, then $P_{(r,n),m}$ is nonsingular and $P^{-1}_{(r,n),m}$ is the following matrix:*

$$\begin{pmatrix} P^{-1}_{(r,n),m-1} + \mu P^{-1}_{(r,n),m-1}(nE_1 + E_2)(rE_1^T + rnE_2^T)P^{-1}_{(r,n),m-1} & -\mu P^{-1}_{(r,n),m-1}(nE_1 + E_2) \\ -\mu(rE_1^T + rnE_2^T)P^{-1}_{(r,n),m-1} & \mu \end{pmatrix} \tag{12}$$

with $\mu = \frac{1}{x_m - rnx_{m-1}}$, $E_1 = (x_1, 0, \ldots, 0)^T \in \mathbb{R}^{m-1}$ and $E_2 = (0, x_2, \ldots, x_{m-1})^T$.

Proof. This theorem can be proved by employing the method of mathematical induction on m. $P_{(r,n),m}$ can be expressed as follows:

$$P_{(r,n),m} = \begin{pmatrix} P_{(r,n),m-1} & nE_1 + E_2 \\ rE_1^T + rnE_2^T & x_m \end{pmatrix},$$

with $E_1 = (x_1, 0, \ldots, 0)^T \in \mathbb{R}^{m-1}$ and $E_2 = (0, x_2, \ldots, x_{m-1})^T$.

If $m = 2$, from (12) we have

$$\begin{aligned} \begin{pmatrix} \frac{1}{x_1} + \frac{1}{x_2 - rnx_1}\frac{1}{x_1}rx_1 \cdot nx_1\frac{1}{x_1} & -\frac{1}{x_2 - rnx_1}\frac{1}{x_1}nx_1 \\ -\frac{1}{x_2 - rnx_1}rx_1\frac{1}{x_1} & \frac{1}{x_2 - rnx_1} \end{pmatrix} &= \begin{pmatrix} \frac{x_2}{x_1(x_2 - rnx_1)} & -\frac{n}{x_2 - rnx_1} \\ -\frac{r}{x_2 - rnx_1} & \frac{1}{x_2 - rnx_1} \end{pmatrix} \\ &= \frac{1}{x_1(x_2 - rnx_1)}\begin{pmatrix} x_2 & -nx_1 \\ -rx_1 & x_1 \end{pmatrix} \\ &= \frac{1}{det(P_{(r,n),2})}\begin{pmatrix} x_2 & -nx_1 \\ -rx_1 & x_1 \end{pmatrix} \\ &= P^{-1}_{(r,n),2}. \end{aligned}$$

We will use induction on m, we assume that the assertion is true for $m = t-1$, that is,$P^{-1}_{(r,n),t-1}P_{(r,n),t-1} = I_{(t-1)\times(t-1)}$. Namely, we have

$$P^{-1}_{(r,n),t-1}\begin{pmatrix} nx_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{t-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

We use this fact to prove that the statement for $m = t$, we have

$$P^{-1}_{(r,n),t}P_{(r,n),t} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where,

$$\begin{aligned}
A_{11} =& \left[P^{-1}_{(r,n),t-1} + \mu P^{-1}_{(r,n),t-1}(nE_1 + E_2)(rE_1^T + rnE_2^T)P^{-1}_{(r,n),t-1}\right] P_{(r,n),t-1} \\
&+ \left[-\mu P^{-1}_{(r,n),t-1}(nE_1 + E_2)\right](rE_1^T + rnE_2^T) \\
=& I_{(t-1)\times(t-1)} + \mu P^{-1}_{(r,n),t-1}(nE_1 + E_2)(rE_1^T + rnE_2^T) \\
&- \mu P^{-1}_{(r,n),t-1}(nE_1 + E_2)(rE_1^T + rnE_2^T) \\
=& I_{(t-1)\times(t-1)}, \\
A_{12} =& \left[P^{-1}_{(r,n),t-1} + \mu P^{-1}_{(r,n),t-1}(nE_1 + E_2)(rE_1^T + rnE_2^T)P^{-1}_{(r,n),t-1}\right](nE_1 + E_2) \\
&- \mu P^{-1}_{(r,n),t-1}(nE_1 + E_2)x_t \\
=& P^{-1}_{(r,n),t-1}(nE_1 + E_2) + \mu P^{-1}_{(r,n),t-1}(nE_1 + E_2)(rE_1^T + rnE_2^T)P^{-1}_{(r,n),t-1}(nE_1 + E_2) \\
&- \mu P^{-1}_{(r,n),t-1}(nE_1 + E_2)x_t \\
=& (0,\dots,0,1)^T + \mu(0,\dots,0,1)^T(rE_1^T + rnE_2^T)(0,\dots,0,1)^T - \mu(0,\dots,0,1)^T x_t \\
=& (0,\dots,0,1)^T\left[1 + \mu rnx_{t-1} - \mu x_t\right] \\
=& (0,\dots,0,1)^T\left[1 - \mu(x_t - rnx_{t-1})\right] = 0_{t-1\times 1}, \\
A_{21} =& \left[-\mu(rE_1^T + rnE_2^T)P^{-1}_{(r,n),t-1}\right]P_{(r,n),t-1} + \mu(rE_1^T + rnE_2^T) = 0_{1\times t-1}, \\
A_{22} =& \left[-\mu(rE_1^T + rnE_2^T)P^{-1}_{(r,n),t-1}\right](nE_1 + E_2) + \mu x_t \\
=& -\mu(rE_1^T + rnE_2^T)(0,\dots,0,1)^T + \mu x_t \\
=& -\mu rnx_{t-1} + \mu x_t = \mu(x_t - rnx_{t-1}) = 1.
\end{aligned}$$

Therefore, $P^{-1}_{(r,n),t}P_{(r,n),t} = I_{t\times t}$, for all positive integer t. On the other hand $P_{(r,n),t}P^{-1}_{(r,n),t} = I_{t\times t}$ can be similarly obtained. So the proof is completed. □

Theorem 2.8. *Let $P_{(r,n),m}$ be defined as in (7). The Euclidean norm of $P_{(r,n),m}$ is given by*

$$\|P_{(r,n),m}\|_E = \left[\left((m-1)(|r|^2 + |n|^2) + 1\right)x_1^2 + \sum_{l=2}^{m}\left((m-l)(|rn|^2 + 1) + 1\right)x_l^2\right]^{\frac{1}{2}}.$$

Proof. The sum of the elements $p^2_{1,1}, p^2_{1,j}$ and $p^2_{i,1}$ for $i, j = 2, \dots, m$ is given by

$$\begin{aligned}
s_1 =& \left[(m-1)|n|^2x_1^2\right] + \left[(m-1)|r|^2x_1^2)\right] + \left[x_1^2\right] \\
=& \left((m-1)(|r|^2 + |n|^2) + 1\right)x_1^2.
\end{aligned}$$

The sum of the elements $p^2_{2,2}, p^2_{2,j}$ and $p^2_{i,2}$ for $i, j = 3, \dots, m$ is given by

$$\begin{aligned}
s_2 =& \left[(m-2)x_2^2\right] + \left[(m-2)|rn|^2x_2^2\right] + \left[x_2^2\right] \\
=& \left((m-2)(|rn|^2 + 1) + 1\right)x_2^2.
\end{aligned}$$

And so, the sum of the elements $p^2_{l,l}, p^2_{l,j}$ and $p^2_{i,l}$ for $i, j = l+1, \dots, m$ and $l = 2, \dots, m$ is given by

$$\begin{aligned}
s_l =& \left[(m-l)x_l^2\right] + \left[(m-l)|rn|^2x_l^2\right] + \left[x_l^2\right] \\
=& \left((m-l)(|rn|^2 + 1) + 1\right)x_l^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|P_{(r,n),m}\|_E &= \left[s_1 + \sum_{l=2}^{m} s_l\right]^{\frac{1}{2}} \\ &= \left[\Big((m-1)(|r|^2 + |n|^2) + 1\Big)x_1^2 + \sum_{l=2}^{m}\Big((m-l)(|rn|^2 + 1) + 1\Big)x_l^2\right]^{\frac{1}{2}}.\end{aligned}$$

□

Remark 2.9. *It should be noticed that $n = 1$ in Theorem 2.8 the expression $\|P_{(r,n),m}\|_E$ simplifies to $\|P_{r,m}\|_E$, as appears in* [8].

Lemma 2.10 ([8])**.** *Let P_1 be the following matrix:*

$$P_1 = \begin{pmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ x_1 & x_2 & x_2 & \cdots & x_2 \\ x_1 & x_2 & x_3 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_m \end{pmatrix},$$

where $x_1 \leq x_2 \leq \cdots \leq x_m$. Then, we have

$$c_1(P_1) = \sqrt{\sum_{l=1}^{m} x_l^2}.$$

Theorem 2.11. *Let m be a positive integer and R_1, R_2, R_3 be the sets as above. Then*

1) If $(n, r) \in R_1$, we have

$$\|P_{(r,n),m}\|_2 \leq \sqrt{((m-2)|rn|^2 + |r|^2 + 1)\sum_{l=1}^{m} x_l^2}.$$

2) If $(n, r) \in R_2$, we have

$$\|P_{(r,n),m}\|_2 \leq \sqrt{(1 + (m-1)|n|^2)\sum_{l=1}^{m} x_l^2}.$$

3) If $(n, r) \in R_3$, we have

$$\|P_{(r,n),m}\|_2 \leq \sqrt{(|r|^2 + (m-1))\sum_{l=1}^{m} x_l^2}.$$

Proof. Since $P_{(r,n),m} = A \circ P_1$ and $\|A \circ P_1\| \leq r_1(A)c_1(P_1)$, using Lemma 2.2 and Lemma 2.10, we obtain our assertions. □

Remark 2.12. *If we take $n = 1$ in Theorem 2.11, we obtain $\|P_{(r,n),m}\|_2 = \|P_{r,m}\|_2$ given by [8].*

2.2. Properties for the matrix $Q_{(r,n),m}$

Theorem 2.13. *The determinant of the matrix $Q_{(r,n),m}$ is*

$$\det(Q_{(r,n),m}) = x_m \prod_{i=1}^{m-1}(x_i - rnx_{i+1}).$$

Proof. It is clear that with appropriate rows operations we have

$$\begin{aligned}\det(Q_{(r,n),m}) &= \begin{vmatrix} x_1 - rnx_2 & 0 & 0 & \cdots & 0 & 0 \\ r(x_2 - x_3) & x_2 - rnx_3 & 0 & \cdots & 0 & 0 \\ r(x_3 - x_4) & rn(x_3 - x_4) & x_3 - rnx_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r(x_{m-1} - x_m) & rn(x_{m-1} - x_m) & rn(x_{m-1} - x_m) & \cdots & x_{m-1} - rnx_m & 0 \\ rx_m & rnx_m & rnx_m & \cdots & rnx_m & x_m \end{vmatrix} \\ &= x_m \prod_{i=1}^{m-1} (x_i - rnx_{i+1}).\end{aligned}$$

□

Theorem 2.14. *Let $Q_{(r,n),m}$ be defined as in (8). If, $x_m \neq 0$ and $x_i \neq nrx_{i+1}$, for all $i = 1, \dots, m-1$, then $Q_{(r,n),m}$ is nonsingular and $Q^{-1}_{(r,n),m}$ is the following matrix:*

$$\begin{pmatrix} Q^{-1}_{(r,n),m-1} + vQ^{-1}_{(r,n),m-1}(nF_1 + F_2)(rF_1^T + rnF_2^T)Q^{-1}_{(r,n),m-1} & -vQ^{-1}_{(r,n),m-1}(nF_1 + F_2) \\ -v(rF_1^T + rnF_2^T)Q^{-1}_{(r,n),m-1} & v \end{pmatrix}, \tag{13}$$

with $F_1 = (x_m, 0, \dots, 0)^T \in \mathbb{R}^{m-1}$, $F_2 = (0, x_m, \dots, x_m)^T$ and $v = \frac{x_{m-1}}{x_m(x_{m-1} - rnx_m)}$.

Proof. The proof of this theorem can be established by utilizing the technique of mathematical induction on m, similar to the proof in Theorem 2.7, considering

$$Q_{(r,n),m} = \begin{pmatrix} Q_{(r,n),m-1} & nF_1 + F_2 \\ rF_1^T + rnF_2^T & x_m \end{pmatrix}.$$

□

Theorem 2.15. *Let $Q_{(r,n),m}$ be defined as in (7). The Euclidean norm of $Q_{(r,n),m}$ is:*

$$\|Q_{(r,n),m}\|_E = \left[x_1^2 + \sum_{\ell=2}^{m} \left((\ell - 1)(|rn|^2 + 1) - |rn|^2 + |r|^2 + |n|^2 \right) x_\ell^2 \right]^{\frac{1}{2}}.$$

Proof. The sum of the elements $q^2_{1,j}$ and $q^2_{i,1}$ is given by

$$s_1 = x_1^2 + |n|^2 \sum_{\ell=2}^{m} x_\ell^2 + |r|^2 \sum_{\ell=2}^{m} x_\ell^2 = x_1^2 + \sum_{\ell=2}^{m} (|n|^2 + |r|^2) x_\ell^2 \tag{14}$$

and

$$Q_{(r,n),m} = \begin{pmatrix} x_1 & nG^T \\ rG & Q_{rn,m} \end{pmatrix},$$

where $G = (x_2, x_3, \dots, x_m)^T$ and $Q_{rn,m}$ be a rn-max matrix defined in [8]. Then, by (14) and [8, Theorem 2.6] we have

$$\begin{aligned}\|Q_{(r,n),m}\|_E^2 &= x_1^2 + \sum_{\ell=2}^{m} (|n|^2 + |r|^2) x_\ell^2 + \sum_{\ell=2}^{m} \left((\ell - 1)(|rn|^2 + 1) - |rn|^2 \right) x_\ell^2 \\ &= x_1^2 + \sum_{\ell=2}^{m} \left((\ell - 1)(|rn|^2 + 1) - |rn|^2 + |r|^2 + |n|^2 \right) x_\ell^2\end{aligned}$$

and the result follows. □

Lemma 2.16 ([8]). *Let Q_1 be the following matrix:*

$$Q_1 = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots x_m \\ x_2 & x_2 & x_3 & \cdots x_m \\ x_3 & x_3 & x_3 & \cdots x_m \\ \vdots & \vdots & \vdots & \ddots \vdots \\ x_m & x_m & x_m & \cdots x_m \end{pmatrix}, \tag{15}$$

where $x_1 \leq x_2 \leq \cdots \leq x_m$. Then, we have

$$c_1(Q_1) = |x_m| \sqrt{m}.$$

Theorem 2.17. *Let m be a positive integer R_1, R_2, R_3 be the sets defined above. Then*

1) If $(n,r) \in R_1$, we have

$$\|Q_{(r,n),m}\|_2 \leq \sqrt{m\left((m-2)|rn|^2 + |r|^2 + 1\right) x_m^2},$$

2) If $(n,r) \in R_2$, we have

$$\|Q_{(r,n),m}\|_2 \leq \sqrt{m\left(1 + (m-1)|n|^2\right) x_m^2},$$

3) If $(n,r) \in R_3$, we have

$$\|Q_{(r,n),m}\|_2 \leq \sqrt{m\left(|r|^2 + (m-1)\right) x_m^2}.$$

Proof. Because $Q_{(r,n),m} = A \circ Q_1$ and $\|A \circ Q_1\| \leq r_1(A) c_1(Q_1)$, and using Lemma 2.2 and Lemma 2.16, we obtain our assertions. □

2.3. Properties for matrix $P^{\circ -1}_{(r,n),m}$

Theorem 2.18. *The determinant of the matrix $P^{\circ -1}_{(r,n),m}$ is*

$$\det(P^{\circ -1}_{(r,n),m}) = \frac{1}{x_1} \prod_{i=1}^{m-1} \left(\frac{1}{x_{i+1}} - \frac{1}{rnx_i}\right).$$

Proof. Applying appropriate column operations to $P^{\circ -1}_{(r,n),m}$, we have

$$\det(P^{\circ -1}_{(r,n),m}) = \det \begin{pmatrix} \frac{1}{x_1} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{rx_1} & \frac{1}{x_2} - \frac{1}{rnx_1} & 0 & \cdots & 0 & 0 \\ \frac{1}{rx_1} & \frac{1}{rn}(\frac{1}{x_2} - \frac{1}{x_1}) & \frac{1}{x_3} - \frac{1}{rnx_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{rx_1} & \frac{1}{rn}(\frac{1}{x_2} - \frac{1}{x_1}) & \frac{1}{rn}(\frac{1}{x_3} - \frac{1}{x_2}) & \cdots & \frac{1}{x_{m-1}} - \frac{1}{rnx_{m-2}} & 0 \\ \frac{1}{rx_1} & \frac{1}{rn}(\frac{1}{x_2} - \frac{1}{x_1}) & \frac{1}{rn}(\frac{1}{x_3} - \frac{1}{x_2}) & \cdots & \frac{1}{rn}(\frac{1}{x_{m-1}} - \frac{1}{x_{m-2}}) & \frac{1}{x_m} - \frac{1}{rnx_{m-1}} \end{pmatrix}$$

$$= \frac{1}{x_1} \prod_{i=1}^{m-1} \left(\frac{1}{x_{i+1}} - \frac{1}{rnx_i}\right).$$

□

Theorem 2.19. *Consider $P_{(r,n),m}$, then*

$$P^{\circ -1}_{(r,n),m} = \begin{pmatrix} P^{\circ -1}_{(r,n),m-1} & \frac{1}{n}G_1 + G_2 \\ \frac{1}{r}G_1^T + \frac{1}{rn}G_2^T & \frac{1}{x_m} \end{pmatrix},$$

where $G_1 = (\frac{1}{x_1}, 0, \dots, 0)^T$ and $G_2 = (0, \frac{1}{x_2}, \dots, \frac{1}{x_{m-1}})^T$. If $P^{\circ -1}_{(r,n),m}$ is a nonsingular matrix, then the inverse of $P^{\circ -1}_{(r,n),m}$ is

$$\begin{pmatrix} C + \mu\, C(\frac{1}{n}G_1 + +G_2)(\frac{1}{r}G_1^T + \frac{1}{rn}G_2^T)\, C & -\mu\, C(\frac{1}{n}G_1 + G_2) \\ -\mu(\frac{1}{r}G_1^T + \frac{1}{rn}G_2^T)\, C & \mu \end{pmatrix},$$

where $\mu = \frac{rnx_m x_{m-1}}{rnx_{m-1} - x_m}$ and $C = (P^{\circ -1}_{(r,n),m-1})^{-1}$.

Proof. The proof is similar to the proof of Theorem 2.14. □

Theorem 2.20. *Let $P^{\circ -1}_{(r,n),m}$ be defined as in (7). The Euclidean norm of $P^{\circ -1}_{(r,n),m}$ is:*

$$\|P^{\circ -1}_{(r,n),m}\|_E = \left[\left((m-1)\Big(\frac{1}{|r|^2} + \frac{1}{|n|^2}\Big) + 1 \right) \frac{1}{x_1^2} + \sum_{l=2}^{m} \left((m-l)\Big(\frac{1}{|rn|^2} + 1\Big) + 1 \right) \frac{1}{x_l^2} \right]^{\frac{1}{2}}.$$

Proof. The proof is similar to the are given in Theorem 2.8. □

Lemma 2.21 ([8]). *Let $P_1^{\circ -1}$ be the following matrix:*

$$P_1^{\circ -1} = \begin{pmatrix} \frac{1}{x_1} & \frac{1}{x_1} & \frac{1}{x_1} & \cdots & \frac{1}{x_1} \\ \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_2} & \cdots & x_2 \\ \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} & \cdots & \frac{1}{x_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} & \cdots & \frac{1}{x_m} \end{pmatrix},$$

where where $x_1 \le x_2 \le \cdots \le x_m$. Then, we have

$$c_1(P_1^{\circ -1}) = \frac{\sqrt{m}}{|x_1|}.$$

Theorem 2.22. *Let m be a positive integer and T_1, T_2, T_3 be the sets defined above.*

1) *if $(n, r) \in T_1$, we have*

$$\|P^{\circ -1}_{(r,n),m}\|_2 \le \frac{1}{|x_1|} \sqrt{\left(1 + (m-1)\frac{1}{|n|^2}\right) m},$$

2) *if $(n, r) \in T_2$, we have*

$$\|P^{\circ -1}_{(r,n),m}\|_2 \le \frac{1}{|x_1|} \sqrt{\left(\frac{1}{|r|^2} + (m-1)\right) m},$$

3) *if $(n, r) \in T_3$, we have*

$$\|P^{\circ -1}_{(r,n),m}\|_2 \le \frac{1}{|x_1|} \sqrt{\left((m-2)\frac{1}{|rn|^2} + \frac{1}{|r|^2} + 1\right) m}.$$

Proof. Because of $P^{\circ -1}_{(r,n),m} = B \circ P_1^{\circ -1}$ and $\|B \circ P_1^{\circ -1}\| \le r_1(B) c_1(P_1^{\circ -1})$, and using Lemma 2.4 and Lemma 2.21, the proof follows. □

2.4. Properties for the matrix $Q^{\circ-1}_{(r,n),m}$

Theorem 2.23. *The determinant of the matrix $Q^{\circ-1}_{(r,n),m}$ is*

$$\det(Q^{\circ-1}_{(r,n),m}) = \frac{1}{x_m}\prod_{i=1}^{m-1}(\frac{1}{x_i} - \frac{1}{rnx_{i+1}}).$$

Proof. The proof is similar to the are given in Theorem 2.13. □

Theorem 2.24. *Consider $Q_{(r,n),m}$, then*

$$Q^{\circ-1}_{(r,n),m} = \begin{pmatrix} Q^{\circ-1}_{(r,n),m-1} & \frac{1}{n}F_1 + F_2 \\ \frac{1}{r}F_1^T + \frac{1}{rn}F_2^T & \frac{1}{x_m} \end{pmatrix},$$

where $F_1 = (\frac{1}{x_m}, 0, \ldots, 0)^T$ and $F_2 = (0, \frac{1}{x_m}, \ldots, \frac{1}{x_m})^T$. If $Q^{\circ-1}_{(r,n),m}$ is a nonsingular matrix, then the inverse of $Q^{\circ-1}_{(r,n),m}$ is

$$(Q^{\circ-1}_{(r,n),m})^{-1} = \begin{pmatrix} D + v\,D(\frac{1}{n}F_1 + F_2)(\frac{1}{r}F_1^T + \frac{1}{rn}F_2^T)\,D & -v\,D(\frac{1}{n}F_1 + F_2) \\ -v(\frac{1}{r}F_1^T + \frac{1}{rn}F_2^T)\,D & v \end{pmatrix}$$

where $v = \frac{rnx_m^2}{rnx_m - x_{m-1}}$ and $D = (Q^{\circ-1}_{(r,n),m-1})^{-1}$.

Proof. The proof is similar to the proof of Theorem 2.19. □

Theorem 2.25. *Let $Q^{\circ-1}_{(r,n),m}$ be defined as in (7). The Euclidean norm of $Q^{\circ-1}_{(r,n),m}$ is:*

$$||Q^{\circ-1}_{(r,n),m}||_E = \left[\frac{1}{x_1^2} + \sum_{\ell=2}^{m}\left(\ell(\frac{1}{|rn|^2} + 1) - \frac{1}{|rn|^2} + \frac{1}{|r|^2} + \frac{1}{|n|^2}\right)\frac{1}{x_\ell^2}\right]^{\frac{1}{2}}.$$

Proof. The proof is similar to the proof of Theorem 2.20. □

Lemma 2.26 ([8]). *Let $Q_1^{\circ-1}$ be the following matrix:*

$$Q_1^{\circ-1} = \begin{pmatrix} \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} & \cdots & \frac{1}{x_m} \\ \frac{1}{x_2} & \frac{1}{x_2} & \frac{1}{x_3} & \cdots & \frac{1}{x_m} \\ \frac{1}{x_3} & \frac{1}{x_3} & \frac{1}{x_3} & \cdots & \frac{1}{x_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_m} & \frac{1}{x_m} & \frac{1}{x_m} & \cdots & \frac{1}{x_m} \end{pmatrix}. \tag{16}$$

Then,

$$c_1(Q_1^{\circ-1}) = \sqrt{\sum_{l=1}^{m}\frac{1}{|x_l|^2}}.$$

Theorem 2.27. *Let m be a positive integer and T_1, T_2, T_3 be the sets as above. Then*

1) if $(n, r) \in T_1$ then

$$||Q^{\circ-1}_{(r,n),m}||_2 \leq \sqrt{\left(1 + (m-1)\frac{1}{|n|^2}\right)\sum_{l=1}^{m}\frac{1}{x_l^2}};$$

2) if $(n, r) \in T_2$ then

$$||Q^{\circ-1}_{(r,n),m}||_2 \leq \sqrt{\left(\frac{1}{|r|^2} + (m-1)\right)\sum_{l=1}^{m}\frac{1}{x_l^2}}.$$

3) if $(n, r) \in T_3$ *then*

$$\|Q^{\circ -1}_{(r,n),m}\|_2 \leq \sqrt{\left((m-2)\frac{1}{|rn|^2} + \frac{1}{|r|^2} + 1\right)\sum_{l=1}^{m} \frac{1}{x_l^2}};$$

Proof. As $Q^{\circ -1}_{(r,n),m} = B \circ Q^{\circ -1}_1$ and $\|B \circ Q^{\circ -1}_1\| \leq r_1(B)c_1(P^{\circ -1}_1)$. Then by Lemma 2.4 and Lemma 2.26 the proof follows. □

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