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$C-$commutativity, power, $W-$core inverse and SEP matrices

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Abstract. This paper presents a series of characterizations for the SEP matrices, initially uncovering their fundamental properties with the aid of the $C-$commutativity, the Moore-Penrose inverse matrix of the power and the representation form of the $W-$core inverses.

1. Introduction

In this paper, we use $\mathbb{C}^{n\times n}$ to denote the set of all $n \times n$ matrices over $\mathbb{C}$. The matrix $B \in \mathbb{C}^{n\times n}$ is called the Moore-Penrose inverse matrix of $A \in \mathbb{C}^{n\times n}$[2], if B satisfies the following Penrose equations:

$$A = ABA, \quad B = BAB, \quad (AB)^H = AB \quad \text{and} \quad (BA)^H = BA,$$

where A^H denotes the conjugate transpose of A. The Moore-Penrose inverse matrix always exists and is unique[1, 3], which is usually written by $A^\dagger$.

B is said to be the group inverse matrix of A, if B satisfies

$$A = ABA, \quad B = BAB \quad \text{and} \quad AB = BA.$$

The group inverse matrix is uniquely determined, if it exists[4], and we denote it by $A^\#$. For convenience, we write $G_n(\mathbb{C})$ to denote the set of all group invertible matrices in $\mathbb{C}^{n\times n}$.

Let $A \in G_n(\mathbb{C})$. We call that A is an *EP* matrix if $A^\# = A^\dagger$. It is known that A is *EP* if and only if $AA^\dagger = A^\dagger A$ [1]. For the *EP* matrix, we can also refer to [9]. And A is called a partial isometry matrix if $AA^HA = A$. According to [11], A is a partial isometry matrix if and only if $A^\dagger = A^H$. In recent years, many authors studied the partial isometry. For examples, in [8], several results are presented in Hilbert spaces for a new class of operators termed the semi-generalized partial isometry, which encompass two important classes in operator theory: partial isometry and nilpotent operators. In [18], it is shown that if $A \in G_n(\mathbb{C})$, then A is a partial isometry matrix if and only if the equation

$$A^\dagger XA = A^H XA$$

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is solvable for some X in $\chi_A = \{A,\ A^{\#},\ A^{\dagger},\ A^H,\ (A^{\#})^H,\ (A^{\dagger})^H\}$.

Recall that a matrix $A \in G_n(\mathbb{C})$ is said to be *SEP* if $A^{\#} = A^{\dagger} = A^H$[22]. Clearly, A is SEP if and only if A is EP and is a partial isometry matrix. We denote the set of all $n \times n$ SEP matrices over $\mathbb{C}$ by $G_n^{SEP}(\mathbb{C})$. In [17], it is shown that $A \in G_n(\mathbb{C})$ is a SEP matrix if and only if the general solution of the equation

$$(A^{\#})^H X A^{\#} = A^{\dagger}$$

is given by

$$X = A^{\dagger} + U = AA^{\dagger}UAA^{\dagger},\ \text{where } U \in \mathbb{C}^{n\times n}.$$

In [19], it is proved that $A \in G_n(\mathbb{C})$ is a SEP matrix if and only if the equation

$$A^H XA = AA^{\#}XA^{\dagger}A$$

has at least one solution in χ_A.

Let $A,\ B$ and $C \in \mathbb{C}^{n\times n}$. Then A and B are called to be $C-$commutative if $CA = BC$. Especially, if A and B are $A-$commutative, then A is called left $B-$idempotent[6]. If A and B are $B-$commutative, then B is called right $A-$idempotent[6]. Using one-sided $X-$idempotent matrices, many nice properties of SEP matrices are discussed in [10].

Let A and $W \in \mathbb{C}^{n\times n}$. A is called to be $W-$core invertible if there is a matrix $X \in \mathbb{C}^{n\times n}$ such that

$$X = AWX^2,\ \ A = XAWA,\ \ (AWX)^H = AWX.$$

In this case, X is called the W-core inverse of A, which is unique if it exists and is always denoted by $A_W^{\textcircled{\#}}$[23].

The early studies on the SEP matrix can be founded in [16], and the formal name appears in [22]. In the recent years, many authors explored the properties of SEP matrices and SEP elements in a ring with involution such as [12–15, 20, 21]. In these papers, the authors used the representation of SEP matrices to explore various classes of matrices. Also, they used the form of the solutions of the related constructed equations to characterize the SEP matrices or SEP elements in a ring with involution.

Inspired by these references, this paper is organised as follows, in Section 2, in terms of properties of $C-$commutativity, we give some characterizations of SEP matrices. In Section 3, we explore the properties of SEP matrices by discussing the power of product of some generalized inverse matrices. In Section 4, using the representation form of $W-$core inverse matrices, we explore the $W-$core invertibility of SEP matrices.

2. The $C-$commutativity of SEP matrices

We begin with the following lemma which appears in [5]. Lemma 2.2 follows from [10, Lemma 2.1], which gives some characterizations of SEP matrices.

Lemma 2.1. *Let $A \in G_n(\mathbb{C})$. Then*
(1) $(A^{\#})^{\dagger} = A^{\dagger}A^3A^{\dagger}$;
(2) $(A^{\dagger})^{\#} = (AA^{\#})^H A (AA^{\#})^H$;
(3) $((A^{\dagger})^{\#})^{\dagger} = AA^{\dagger}A^{\dagger}A^{\dagger}A$;
(4) $((A^{\#})^{\dagger})^{\#} = (AA^{\#})^H A^{\#} (AA^{\#})^H$.

Lemma 2.2. *Let $A \in G_n(\mathbb{C})$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if one of the following conditions holds:*
(1) $(A^{\#})^2A^{\dagger} = A^HA^{\dagger}A^{\#}$;
(2) $A^{\#}A^{\dagger}A^{\dagger} = A^HA^{\dagger}A^{\#}$;
(3) $(A^{\dagger})^3 = A^HA^{\dagger}A^{\#}$;
(4) $(A^{\#})^2A^{\dagger} = A^H(A^{\#})^2$;
(5) $A^{\#}A^{\dagger}A^{\dagger} = A^HA^{\#}A^{\dagger}$.

Combining the $C-$commutativity and Lemma 2.2, one has the following result.

Lemma 2.3. *Let $A \in G_n(\mathbb{C})$. Then the following conditions are equivalent:*
(1) $A \in G_n^{SEP}(\mathbb{C})$;
(2) $A^{\#}A^{\dagger}$ and $A^H A^{\dagger}$ are $A^{\#}-$commutative;
(3) $A^{\dagger}A^{\dagger}$ and $A^H A^{\dagger}$ are $A^{\#}-$commutative;
(4) $A^{\dagger}$ and A^H are $(A^{\#})^2-$commutative;
(5) $A^{\dagger}$ and A^H are $A^{\#}A^{\dagger}-$commutative.

Lemma 2.3 induces us to give the following theorem.

Theorem 2.1. *Let $A \in G_n(\mathbb{C})$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if $A^{\dagger}A^{\#}$ and $A^H A^{\dagger}$ are $X-$commutative for some*

$$X \in w_A = \{A,\ A^{\#},\ A^{\dagger},\ A^H,\ AA^{\#},\ (A^{\#})^{\dagger},\ (A^{\dagger})^{\#}\}.$$

Proof. Necessity. Suppose that $A \in G_n^{SEP}(\mathbb{C})$. Then

$$X = A^{\#} = A^{\dagger} = A^H = (A^{\#})^{\dagger} = (A^{\dagger})^{\#}$$

meets the requirements.

Sufficiency. If there exists $X_0 \in w_A$ such that $A^{\dagger}A^{\#}$ and $A^H A^{\dagger}$ are X_0-commutative, then

$$X_0 A^{\dagger}A^{\#} = A^H A^{\dagger} X_0.$$

(1) If $X_0 \in \{A,\ A^{\#},\ AA^{\#}\}$, then $AA^{\dagger}X_0 = X_0$. It follows that

$$A^H A^{\dagger} X_0 = X_0 A^{\dagger} A^{\#} = AA^{\dagger} X_0 A^{\dagger} A^{\#} = AA^{\dagger} A^H A^{\dagger} X_0.$$

Noting that $X_0 X_0^{\#} = AA^{\#}$, one obtains

$$A^H A^{\dagger} AA^{\#} = A^H A^{\dagger} X_0 X_0^{\#} = AA^{\dagger} A^H A^{\dagger} X_0 X_0^{\#} = AA^{\dagger} A^H A^{\dagger} AA^{\#}.$$

Multiplying the equality on the right by $A(A^{\#})^H A^{\dagger}$, one gets

$$A^{\dagger} = AA^{\dagger}A^{\dagger}.$$

Hence A is EP by [15, Theorem 1.2.1].

(2) If $X_0 \in \{A^{\dagger},\ A^H,\ (A^{\#})^{\dagger},\ (A^{\dagger})^{\#}\}$, then $X_0^{\dagger}X_0 = AA^{\dagger}$ and $X_0 AA^{\dagger} = X_0$. Hence

$$\begin{aligned} A^{\dagger}A^{\dagger}A^{\#} &= A^{\dagger}AA^{\dagger}A^{\dagger}A^{\#} = A^{\dagger}X_0^{\dagger}X_0 A^{\dagger}A^{\#} = A^{\dagger}X_0^{\dagger}A^H A^{\dagger}X_0 \\ &= (A^{\dagger}X_0^{\dagger}A^H A^{\dagger}X_0)AA^{\dagger} = A^{\dagger}A^{\dagger}A^{\#}AA^{\dagger}. \end{aligned}$$

By [21, Corollary 2.10], one has $A^{\dagger}A^{\#} = A^{\dagger}A^{\#}AA^{\dagger}$, which gives

$$A^{\#} = AA^{\dagger}A^{\#} = AA^{\dagger}A^{\#}AA^{\dagger} = A^{\#}AA^{\dagger}.$$

Therefore, by [15, Theorem 1.2.1], A is EP.

In any case, one obtains A is EP, so $w_A = \{A,\ A^{\#},\ A^H,\ AA^{\#}\}$.

(a) If $X_0 = A$, then $AA^{\dagger}A^{\#} = A^H A^{\dagger} A$. Since A is EP,

$$A^{\#} = AA^{\dagger}A^{\#} = A^H A^{\dagger} A = A^H.$$

Hence $A \in G_n^{SEP}(\mathbb{C})$.

(b) If $X_0 = A^{\#}$, then $A \in G_n^{SEP}(\mathbb{C})$ by Lemma 2.3 because $A^{\#} = A^{\dagger}$.

(c) If $X_0 = A^H$, then $A^H A^{\dagger} A^{\#} = A^H A^{\dagger} A^H$, it follows that

$$A^{\dagger}A^{\#} = (A^{\#})^H A^H A^{\dagger} A^{\#} = (A^{\#})^H A^H A^{\dagger} A^H = A^{\dagger} A^H.$$

Hence $A \in G_n^{SEP}(\mathbb{C})$ by [15, Theorem 1.5.3].

(d) If $X_0 = AA^{\#}$, then $AA^{\#}A^{\dagger}A^{\#} = A^H A^{\dagger} AA^{\#}$, one gets

$$A^{\#}A^{\#} = A^H A^{\#}$$

and hence, $A \in G_n^{SEP}(\mathbb{C})$. □

Corollary 2.2. *Let $A \in G_n(\mathbb{C})$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if $A^\dagger A^\#$ and $A^H A^\dagger$ are $((A^\#)^\dagger)^\#-$commutative.*

Proof. By Lemma 2.1, we can see $((A^\#)^\dagger)^\# = (AA^\#)^H A^\# (AA^\#)^H$.

Necessity. If $A \in G_n^{SEP}(\mathbb{C})$, then $((A^\#)^\dagger)^\# = A^\#$. By Theorem 2.1, $A^\dagger A^\#$ and $A^H A^\dagger$ are $((A^\#)^\dagger)^\#-$commutative.

Sufficiency. From the assumption, one gets

$$(AA^\#)^H A^\# (AA^\#)^H A^\dagger A^\# = A^H A^\dagger (AA^\#)^H A^\# (AA^\#)^H,$$

i.e.,

$$(AA^\#)^H A^\# A^\dagger A^\# = A^H A^\dagger A^\# (AA^\#)^H.$$

Multiplying the last equality on the right by $AA^\dagger$, one yields

$$(AA^\#)^H A^\# A^\dagger A^\# = (AA^\#)^H A^\# A^\dagger A^\# AA^\dagger.$$

It follows that

$$\begin{aligned} A^\# A^\dagger A^\# &= AA^\dagger A^\# A^\dagger A^\# = AA^\dagger (AA^\#)^H A^\# A^\dagger A^\# = AA^\dagger (AA^\#)^H A^\# A^\dagger A^\# AA^\dagger \\ &= A^\# A^\dagger A^\# AA^\dagger \end{aligned}$$

and

$$A = A^3 A^\dagger A^\# = A^4 A^\# A^\dagger A^\# = A^4 A^\# A^\dagger A^\# AA^\dagger = A^2 A^\dagger.$$

Hence A is EP, which implies $((A^\#)^\dagger)^\# = A^\#$. By Theorem 2.1, $A \in G_n^{SEP}(\mathbb{C})$. □

The following lemma is trivial, we omit the proof.

Lemma 2.4. *Let $A,\ B \in G_n(\mathbb{C})$ and $C \in \mathbb{C}^{n\times n}$. If A and B are $C-$commutative, then $A^\#$ and $B^\#$ are $C-$commutative.*

Noting $(A^H A^\dagger)^\# = (AA^\#)^H A (A^\#)^H$ and $(A^\dagger A^\#)^\# = A^\dagger A^3$, it follows from Corollary 2.2 and Lemma 2.4 that

Corollary 2.3. *Let $A \in G_n(\mathbb{C})$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if $A^\dagger A^3$ and $(AA^\#)^H A (A^\#)^H$ are $((A^\#)^\dagger)^\#-$commutative.*

We can get the following theorem.

Theorem 2.4. *Let $A \in G_n(\mathbb{C})$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if $A^\dagger A^3$ and $(AA^\#)^H A (A^\#)^H$ are $A^\dagger-$commutative.*

Proof. Necessity. If $A \in G_n^{SEP}(\mathbb{C})$, then $((A^\#)^\dagger)^\# = A^+$. By Corollary 2.3, we are done.

Sufficiency. From the assumption, one gets

$$A^\dagger A^\dagger A^3 = (AA^\#)^H A (A^\#)^H A^\dagger.$$

This gives

$$\begin{aligned} (A^\#)^H A^\dagger &= A^\dagger A (A^\#)^H A^\dagger = A^\dagger (AA^\#)^H A (A^\#)^H A^\dagger = A^\dagger A^\dagger A^\dagger A^3 \\ &= (A^\dagger A^\dagger A^\dagger A^3) A^\dagger A = (A^\#)^H A^\dagger A^\dagger A \end{aligned}$$

and

$$A^\dagger = A^H (A^\#)^H A^\dagger = A^H (A^\#)^H A^\dagger A^\dagger A = A^\dagger A^\dagger A.$$

Hence A is EP, which leads to

$$A = A^\dagger A^\dagger A^3 = (AA^\#)^H A (A^\#)^H A^\dagger = A (A^\#)^H A^\dagger$$

and

$$AA^\dagger = A^\dagger A = A^\dagger A (A^\#)^H A^\dagger = (A^\#)^H A^\dagger.$$

It follows that

$$A^H = A^H AA^\dagger = A^H (A^\#)^H A^\dagger = A^\dagger.$$

Thus $A \in G_n^{SEP}(\mathbb{C})$. □

3. Using the power to characterize SEP matrices

Let $\mathbb{Z}^+$ denote the set of positive integers. The proof of the following lemma is routine, we omit it.

Lemma 3.1. *Let $A \in G_n(\mathbb{C})$ and $m \in \mathbb{Z}^+$. Then*
(1) $(((A^\#)^2A^\dagger)^m)^\dagger = (A^4A^\dagger)^m = (((A^\#)^2A^\dagger)^m)^\#$;
(2) $(((A^\#)^2A^\dagger)^m)^\dagger((A^\#)^2A^\dagger)^m = AA^\dagger$.

Theorem 3.1. *Let $A \in G_n(\mathbb{C})$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if*

$$((A^\#)^2A^\dagger)^k = (A^HA^\dagger A^\#)^k$$

for $k = m,\ n_1$, where $m,\ n_1 \in \mathbb{Z}^+$ and $(m,\ n_1) = 1$.

Proof. Necessity. Assume that $A \in G_n^{SEP}(\mathbb{C})$. Then $(A^\#)^2A^\dagger = A^HA^\dagger A^\#$ by Lemma 2.2. Hence

$$((A^\#)^2A^\dagger)^k = (A^HA^\dagger A^\#)^k$$

for any $k \in \mathbb{Z}^+$.

Sufficiency. From the hypothesis, one has

$$((A^\#)^2A^\dagger)^m = (A^HA^\dagger A^\#)^m, \tag{1}$$

$$((A^\#)^2A^\dagger)^{n_1} = (A^HA^\dagger A^\#)^{n_1}. \tag{2}$$

Multiplying (1) on the right by $A^\dagger A$, one obtains

$$((A^\#)^2A^\dagger)^m = ((A^\#)^2A^\dagger)^mA^\dagger A.$$

By Lemma 3.1, one yields

$$AA^\dagger = (((A^\#)^2A^\dagger)^m)^\dagger((A^\#)^2A^\dagger)^m = (((A^\#)^2A^\dagger)^m)^\dagger((A^\#)^2A^\dagger)^mA^\dagger A = AA^\dagger A^\dagger A.$$

Hence A is EP, and so (1) and (2) can be changed as

$$((A^\#)^3)^m = (A^HA^\#A^\#)^m \quad \text{and} \quad ((A^\#)^3)^{n_1} = (A^HA^\#A^\#)^{n_1}.$$

Since $(m,\ n_1) = 1$, there exist $s,\ t \in \mathbb{Z}$ such that $ms + n_1t = 1$. Without loss of the generality, we can set $s>0$ and $t<0$. Then

$$\begin{aligned}(A^\#)^3((A^\#)^3)^{n_1|t|} &= ((A^\#)^3)^{1-n_1t} = ((A^\#)^3)^{ms} = (((A^\#)^3)^m)^s \\ &= ((A^HA^\#A^\#)^m)^s = (A^HA^\#A^\#)^{ms} = (A^HA^\#A^\#)^{1-n_1t} \\ &= A^HA^\#A^\#(A^HA^\#A^\#)^{n_1|t|} = A^HA^\#A^\#((A^HA^\#A^\#)^{n_1})^{|t|} \\ &= A^HA^\#A^\#(((A^\#)^3)^{n_1})^{|t|} = A^HA^\#A^\#((A^\#)^3)^{n_1|t|}\end{aligned}$$

This gives

$$\begin{aligned}(A^\#)^2 &= (A^\#)^3A = (A^\#)^3((A^\#)^3)^{n_1|t|}A^{3n_1|t|+1} = A^HA^\#A^\#((A^\#)^3)^{n_1|t|}A^{3n_1|t|+1} \\ &= A^HA^\#.\end{aligned}$$

Hence $A \in G_n^{SEP}(\mathbb{C})$ by [15, Theorem 1.5.3]. □

The following corollary is an immediate result of Theorem 3.1.

Corollary 3.2. *[10, Theorem 2.2] Let $A \in G_n(\mathbb{C})$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if*

$$((A^{\#})^2A^{\dagger})^k = (A^HA^{\dagger}A^{\#})^k$$

for $k = 2, 3$.

Using the representation form of Moore-Penrose inverse matrix of the power of $(A^{\#})^2A^{\dagger}$, we have the following theorem.

Theorem 3.3. *Let $A \in G_n(\mathbb{C})$ and $m \in \mathbb{Z}^+$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if*

$$(((A^{\#})^2A^{\dagger})^m)^{\dagger} = (A^{\#})^HA^3A^{\dagger}(A^4A^{\dagger})^{m-1}.$$

Proof. Necessity. Suppose that $A \in G_n^{SEP}(\mathbb{C})$. Then $(A^{\#})^H = A$. By Lemma 3.1,

$$(((A^{\#})^2A^{\dagger})^m)^{\dagger} = (A^4A^{\dagger})^m = A^4A^{\dagger}(A^4A^{\dagger})^{m-1} = (A^{\#})^HA^3A^{\dagger}(A^4A^{\dagger})^{m-1}.$$

Sufficiency. Using the hypothesis and Lemma 3.1, one has

$$(A^4A^{\dagger})^m = (A^{\#})^HA^3A^{\dagger}(A^4A^{\dagger})^m.$$

Multiplying the equality on the right by $((A^4A^{\dagger})^{m-1})^{\dagger}$, one gets

$$A^4A^{\dagger}AA^{\dagger} = (A^{\#})^HA^3A^{\dagger}AA^{\dagger},$$

i.e.,

$$A^4A^{\dagger} = (A^{\#})^HA^3A^{\dagger}.$$

It follows that

$$\begin{aligned} A &= A^4A^{\dagger}(A^{\#})^2 = (A^{\#})^HA^3A^{\dagger}(A^{\#})^2 = (A^{\#})^HAA^{\#} = A^{\dagger}A(A^{\#})^HAA^{\#} \\ &= A^{\dagger}A^2. \end{aligned}$$

Hence A is EP and so $A = (A^{\#})^HAA^{\#} = (A^{\dagger})^HAA^{\#} = (A^{\dagger})^H = (A^{\#})^H$. Therefore, $A \in G_n^{SEP}(\mathbb{C})$. □

It is well known that if $A \in G_n(\mathbb{C})$, then $A + E_n - AA^{\#}$ is an invertible matrix and

$$(A + E_n - AA^{\#})^{-1} = A^{\#} + E_n - AA^{\#}.$$

Hence Lemma 3.1 implies

$$[((A^{\#})^2A^{\dagger})^m + E_n - AA^{\dagger}]^{-1} = (A^4A^{\dagger})^m + E_n - AA^{\dagger}.$$

Using Theorem 3.3, we can get the following result.

Corollary 3.4. *Let $A \in G_n(\mathbb{C})$ and $m \in \mathbb{Z}^+$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if*

$$[((A^{\#})^2A^{\dagger})^m + E_n - AA^{\dagger}]^{-1} = (A^{\#})^HA^3A^{\dagger}(A^4A^{\dagger})^{m-1} + E_n - AA^{\dagger}.$$

4. Using w−core invertibility to characterize SEP matrices

By Lemma 3.1, we can get $(A^4A^{\dagger})^{\dagger} = (A^{\#})^2A^{\dagger}$ and the following lemma.

Lemma 4.1. *Let $A \in G_n(\mathbb{C})$. Then $A^{4 \textcircled{\#}}_{A^{\dagger}} = (A^{\#})^2A^{\dagger}$.*

From Lemma 4.1 and Lemma 2.2, we have

Lemma 4.2. *Let $A \in G_n(\mathbb{C})$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if $A^{4 \textcircled{\#}}_{A^\dagger} = A^H A^\dagger A^\#$.*

By [7, Lemma 3.1, Theorem 4.1 and Theorem 4.3], we have the following lemma.

Lemma 4.3. *Let $A, W, B \in \mathbb{C}^{n\times n}$ with $A^{\textcircled{\#}}_W = B$. Then*
(1) B is EP and $B^\# = B^\dagger = (AW)^2 B$;
(2) $AW \in G_n(\mathbb{C})$ and $(AW)^\# = B^2 AW$.

Theorem 4.1. *Let $A \in G_n(\mathbb{C})$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if $(A^H A^\dagger A^\#)^\dagger = A^4 A^\dagger$.*

Proof. Necessity. Assume that $A \in G_n^{SEP}(\mathbb{C})$. Then, by Lemma 2.2, $(A^\#)^2 A^\dagger = A^H A^\dagger A^\#$. From Lemma 4.2 and Lemma 4.3, one gets

$$(A^H A^\dagger A^\#)^\dagger = (A^4 A^\dagger)^2 (A^H A^\dagger A^\#) = A^7 A^\dagger (A^H A^\dagger A^\#) = A^7 A^\dagger (A^\#)^2 A^\dagger = A^4 A^\dagger.$$

Sufficiency. Since

$$(A^H A^\dagger A^\#)^\dagger = (A^\#)^\dagger (A^\dagger)^\# (A^\dagger)^H = A^\dagger A^3 A^\dagger (AA^\#)^H A (AA^\#)^H (A^\dagger)^H = A^\dagger A^3 (A^\#)^H A^\dagger A,$$

by the hypothesis, one gets

$$A^4 A^\dagger = (A^H A^\dagger A^\#)^\dagger = A^\dagger A^3 (A^\#)^H A^\dagger A = (A^\dagger A^3 (A^\#)^H A^\dagger A) A^\dagger A = A^4 A^\dagger A^\dagger A.$$

This induces $AA^\dagger = AA^\dagger A^\dagger A$. Hence A is EP, which gives

$$A^3 = A^4 A^\dagger = A^\dagger A^3 (A^\#)^H A^\dagger A = A^2 (A^\dagger)^H A^\dagger A = A^2 (A^\dagger)^H$$

and

$$(A^\dagger)^H = (A^\#)^2 A^2 (A^\dagger)^H = (A^\#)^2 A^3 = A.$$

Therefore, $A \in G_n^{SEP}(\mathbb{C})$. □

Combining Lemma 4.1 with $A^{2 \textcircled{\#}}_{A^\dagger} = AA^\# A^\dagger$ and $A^{3 \textcircled{\#}}_{A^\dagger} = A^\# A^\dagger = A(A^\#)^2 A^\dagger$, one can immediately obtain the following result.

Lemma 4.4. *Let $A \in G_n(\mathbb{C})$ and $m \in \mathbb{Z}^+$. Then*

$$A^{m \textcircled{\#}}_{A^\dagger} = A(A^\#)^{m-1} A^\dagger = (A^\#)^{m-2} A^\dagger.$$

Theorem 4.2. *Let $A \in G_n(\mathbb{C})$ and $3 \le m \in \mathbb{Z}^+$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if*

$$A^{m \textcircled{\#}}_{A^\dagger} = A^\dagger (A^\#)^{m-3} A^H.$$

Proof. Necessity. Applying $A \in G_n^{SEP}(\mathbb{C})$ and Lemma 4.4, one gets $A^\# = A^\dagger = A^H$ and

$$A^{m \textcircled{\#}}_{A^\dagger} = (A^\#)^{m-2} A^\dagger = A^\dagger (A^\#)^{m-3} A^H.$$

Sufficiency. Under the condition "$A^{m \textcircled{\#}}_{A^\dagger} = A^\dagger (A^\#)^{m-3} A^H$" and Lemma 4.4, one has

$$(A^\#)^{m-2} A^\dagger = A^\dagger (A^\#)^{m-3} A^H = A^\dagger A (A^\dagger (A^\#)^{m-3} A^H) = A^\dagger A (A^\#)^{m-2} A^\dagger$$

and

$$A = (A^\#)^{m-2} A^\dagger A^m = A^\dagger A (A^\#)^{m-2} A^\dagger A^m = A^\dagger A^2.$$

Hence A is EP, which leads to

$$(A^\#)^{m-1} = (A^\#)^{m-2} A^\dagger = A^\dagger (A^\#)^{m-3} A^H = (A^\#)^{m-2} A^H$$

and

$$A = A^m (A^\#)^{m-1} = A^m (A^\#)^{m-2} A^H = A^2 A^H.$$

Hence $A \in G_n^{SEP}(\mathbb{C})$ by [15, Theorem 1.5.3]. □

Corollary 4.3. *Let $A \in G_n(\mathbb{C})$ and $3 \leq m \in \mathbb{Z}^+$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if*

$$A_{A^\dagger}^{m\text{\textcircled{\#}}} = A^H(A^\#)^{m-3}A^\dagger.$$

Proof. Necessity. It is a direct result of Theorem 4.2 because $A^H = A^\dagger$.

Sufficiency. Applying the assumption "$A_{A^\dagger}^{m\text{\textcircled{\#}}} = A^H(A^\#)^{m-3}A^\dagger$" and Lemma 4.4, one gets

$$(A^\#)^{m-2}A^\dagger = A^H(A^\#)^{m-3}A^\dagger = A^\dagger A(A^H(A^\#)^{m-3}A^\dagger) = A^\dagger A((A^\#)^{m-2}A^\dagger) = A^\dagger(A^\#)^{m-3}A^\dagger$$

and

$$A = (A^\#)^{m-2}A^\dagger A^m = A^\dagger(A^\#)^{m-3}A^\dagger A^m = A^\dagger A^2.$$

Hence A is EP, it follows that

$$(A^\#)^{m-1} = (A^\#)^{m-2}A^\dagger = A^H(A^\#)^{m-3}A^\dagger = A^H(A^\#)^{m-2}$$

and

$$A = (A^\#)^{m-1}A^m = A^HA^2.$$

Hence $A \in G_n^{SEP}(\mathbb{C})$ by [15, Theorem 1.5.3]. □

Noting that $((A^\dagger)^\dagger)^H = A^H$ and $A_{A^H}^{m\text{\textcircled{\#}}} = (A^\dagger)^H(A^\#)^{m-1}A^\dagger$, and observing Corollary 4.3, we were induced to write the following result, whose proof is left to the reader.

Proposition 4.1. *Let $A \in G_n(\mathbb{C})$ and $3 \leq m \in \mathbb{Z}^+$. Then $A \in G_n^{SEP}(\mathbb{C})$ if and only if*

$$A_{A^H}^{m\text{\textcircled{\#}}} = A^\dagger(A^\#)^{m-3}A^\dagger.$$

However, we do not know whether the following result holds.

Problem 4.1. *Let $A \in G_n(\mathbb{C})$ and $3 < m \in \mathbb{Z}^+$. Does $A \in G_n^{SEP}(\mathbb{C})$ if and only if $A_X^{m\text{\textcircled{\#}}} = (X^\dagger)^H(A^\#)^{m-3}A^\dagger$ for some $X \in \chi_A$?*

5. Conclusion

Using the C−commutativity, power and W−core inverse of matrix, we have explored many new characterizations of SEP matrices. These methods of characterizations are novel and have unique algebraic value, which also can be generalized to a ring with involution or C^*−algebra.

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