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Mean sensitivity and mean equicontinuity in semiflows defined on uniform spaces

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Abstract. The notions of mean sensitivity and mean equicontinuity have been introduced and studied in semiflows defined on uniform spaces. Both these notions are studied on the arbitrary product of semiflows, on the hyperspatial semiflows and on the factor semiflows. A characterization for mean sensitivity with the help of G_δ sets is proved. For a certain class of minimal semiflows it is shown that the semiflow is either mean equicontinuous or mean sensitive. Examples and counterexamples are provided throughout the paper wherever necessary.

1. Introduction

Dynamical systems is one of the main branches of mathematics with its theory focusing on the long term behaviour of many daily life phenomena that are evolving over time according to a fixed set of rules. It has applications in almost all the fields of sciences like population growth and decline, motion of celestial bodies, cell division, weather prediction and so on. Chaos theory is one of the central topics of research in dynamical systems. Its mathematical concept was introduced by Li and Yorke [14] in 1975, and later on chaos theory was extensively studied by many researchers by introducing other definitions of chaos [1, 20] with Devaney chaos [3] most famous among them. The notion of sensitivity [1] characterizes the unpredictability of a chaotic phenomenon, hence it is a part of almost all the definitions of chaos. It basically means that we can separate any point from some nearby point by a fixed distance after a certain stage of time. Once we know a system is sensitive, it becomes essential to measure its strength. For this, Moothathu [17] introduced and studied three stronger notions of sensitivity, namely, cofinite sensitivity, multi-sensitivity and syndetic sensitivity. There was a motivation to generalize the context of topological semigroups so as the classical case of dynamical system emerges as a particular case. This idea led to the study of semiflows. Since then, there has been a significant amount of research in semiflows investigating about stronger notions of sensitivity for metric spaces and uniform spaces [2, 13, 15, 16, 19].

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Mean sensitivity, which is a stronger notion of sensitivity, was introduced and studied by Li et al. [11] in discrete systems defined on compact metric spaces. They proved a characterization for mean sensitivity with the help of G_δ sets. In [7], using entropy, shadowing, minimality, and many other topological notions the authors presented some sufficient conditions for a dynamical system to be mean sensitive. Recently, the authors [23] studied mean sensitivity in discrete systems defined on uniform spaces and proved that the finite product of systems is mean sensitive if and only if at least one of the factor systems is mean sensitive. Mean sensitivity is being studied via group actions where the acting semigroups are amenable semigroups [6, 9, 25]. In 2022, Li et al. [12] studied about mean n-sensitivity and constructed a system which is mean n-sensitive but not mean (n+1)-sensitive. In recent times, the notion of mean sensitivity is being studied for linear operators [26, 27]. It is noteworthy that the famous Auslander-Yorke dichotomy [1] has been studied in almost all of the work mentioned above. This inspired us to answer a natural question about studying mean sensitivity in semiflows where the phase space is a uniform space.

Equicontinuity is a notion which represents predictability in a dynamical system and is useful in describing ergodic theoretic properties of a system. Fomin [4] introduced a weaker notion of equicontinuity, namely, mean-L-stability while studying dynamical systems with discrete spectrum. The author proved that in minimal systems mean-L-stability implies unique ergodicity. Li et al. [11] introduced mean equicontinuity in compact dynamical systems, and proved that a compact dynamical system is mean equicontinuous if and only if it is mean-L-stable. Afterwards, many authors studied stronger and weaker notions of mean equicontinuity [6, 12, 18]. Wu et al. [23] introduced and studied mean equicontinuity in dynamical systems defined on uniform spaces. They proved the Auslander-Yorke dichotomy result as well. In the past few years, mean equicontinuity and its variants are widely being studied in group actions via amenable semigroups [5, 8, 24]. All these interesting works motivated us to make an effort to work upon mean equicontinuity in semiflows defined on uniform spaces.

The paper is structured in the following manner. In Section 2, we recall some standard definitions and concepts used in the upcoming sections. In Section 3, we introduce and study the notion of mean sensitivity for semiflows defined on uniform spaces where the acting topological semigroup is either $\mathbb{R}^+$ or $\mathbb{Z}^+$. Since uniform spaces enable us to study the arbitrary product of systems, we study the said notion on the arbitrary product of semiflows, and provide an example illustrating the result obtained. We prove a characterization for mean sensitivity involving G_δ subsets of the space and the product space. In Section 4, we introduce and study a weaker notion of equicontinuity, namely, mean equicontinuity for semiflows defined on uniform spaces where the acting topological semigroup is either $\mathbb{R}^+$ or $\mathbb{Z}^+$. We provide an example of a mean equicontinuous semiflow which is not equicontinuous. We prove that the arbitrary product of semiflows is mean equicontinuous if and only if each factor semiflow is mean equicontinuous. We also study the factor semiflow and the hyperspatial semiflow along with examples discussing these notions. We end the section by proving the Auslander-Yorke dichotomy result for special type of topological semigroups.

2. Preliminaries

Throughout the paper, we shall denote set of real numbers, non-negative real numbers, rational numbers, non-negative integers and the unit circle by $\mathbb{R}, \mathbb{R}^+, \mathbb{Q}, \mathbb{Z}^+$ and $\mathbb{S}^1$ respectively. Let X be a non-empty set and θ a subset of $X \times X$, then we denote $\theta^{-1} = \{(y, x) \in X \times X : (x, y) \in \theta\}$. θ is said to be symmetric if $\theta = \theta^{-1}$. For any two subsets θ and θ' of $X \times X$, their composition is defined as $\theta \circ \theta' = \{(x, y) \in X \times X : (x, z) \in \theta'$ and $(z, y) \in \theta$ for some $z \in X\}$. We will denote the diagonal of $X \times X$ by $\triangle$. A non-empty collection $\mathscr{U}$ of subsets of $X \times X$ is said to form a uniformity for X if $\mathscr{U}$ satisfies the following conditions:

(a) each member of $\mathscr{U}$ contains $\triangle$;

(b) if $\theta \in \mathscr{U}$, then $\theta^{-1} \in \mathscr{U}$;

(c) if $\theta \in \mathscr{U}$, then $\theta' \circ \theta' \subseteq \theta$ for some $\theta' \in \mathscr{U}$;

(d) if θ and θ' are members of $\mathscr{U}$, then $\theta \cap \theta' \in \mathscr{U}$; and

(e) if $\theta \in \mathscr{U}$ and $\theta \subseteq \theta' \subseteq X \times X$, then $\theta' \in \mathscr{U}$.

The collection $\mathscr{U}$ then induces a topology on X and $(X, \mathscr{U})$ is called a uniform space [22]. Throughout the paper, $(X, \mathscr{U})$ is a uniform space. The members of $\mathscr{U}$ are called entourages. A subfamily $\mathscr{B}$ of a uniformity $\mathscr{U}$ is a base for $\mathscr{U}$ if and only if each member of $\mathscr{U}$ contains a member of $\mathscr{B}$ [10]. It has been proved in [10] that collection of all the closed symmetric entourages as well as all the open symmetric entourages form a basis for the uniformity $\mathscr{U}$. It is well known that a metric space (X, d) becomes a uniform space where the uniformity is generated by all sets of the form $\theta_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$, $\epsilon > 0$ as basis and this uniformity is known as the usual uniformity on X. Int(G) will denote the interior of the subset G of X. For $G \subseteq X$ and $\theta \in \mathscr{U}$ denote $B(G, \theta) = \{y \in X : (x, y) \in \theta$ for some x in $G\}$. If $x \in X$, then $B(x, \theta) = B(\{x\}, \theta)$. It can be easily proved that if θ is open, then $B(x, \theta)$ is open. We say G is a G_δ subset of X if G can be written as a countable intersection of open subsets of X.

Throughout, T is either $\mathbb{R}^+$ or $\mathbb{Z}^+$ with any topology and operation making T a topological semigroup. For $A \subseteq \mathbb{Z}^+$(resp. $\mathbb{R}^+$), we define the upper density of A as

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{1}{n} |A \cap \{0, 1, \dots, n-1\}| \left(\text{resp. } \limsup_{i\to\infty} \frac{1}{i} \mu(A \cap [0, i])\right),$$

where $|A|$ and $\mu(A)$ denotes the cardinality of A and the Lebesgue outer measure of A respectively. Lower density of A, $\underline{d}(A)$, is also defined in the similar manner by replacing $\limsup\limits_{n\to\infty}$ with $\liminf\limits_{n\to\infty}$ in the above definition. We say A has density $d(A)$ if $\overline{d}(A) = d(A) = \underline{d}(A)$.

A semiflow is a triplet (T, X, Φ) where $\Phi : T \times X \to X$ is a continuous map such that $\Phi((t_1 t_2), x) = \Phi(t_1, \Phi(t_2, x))$ for any $t_1, t_2 \in T$ and for any $x \in X$. If in addition T has an identity e, then $\Phi(e, x) = x$ for any $x \in X$. We shall denote $\Phi(t, x)$ by tx and the semiflow (T, X, Φ) by (T, X). For $x \in X$, its orbit is the set $O_T(x) = \{tx : t \in T\}$. A semiflow is minimal if $O_T(x)$ is dense in X for every $x \in X$. A semiflow (T, X) is said to be sensitive if there exists an entourage θ such that for any non-empty open subset G of X there exist $x, y \in G$ and a $t \in T$ such that $(tx, ty) \notin \theta$. A notion opposite to sensitivity is equicontinuity, (T, X) is equicontinuous if for any entourage θ there exists an entourage ϕ such that if $(x, y) \in \phi$, then $(tx, ty) \in \theta$ for every $t \in T$. A semiflow (T, Y) is said to be a factor of (T, X) if there exists a continuous onto map $f : X \to Y$, called factor map, such that $f(tx) = tf(x)$ for any $t \in T$ and any $x \in X$. If the map f is a homeomorphism, then (T, X) and (T, Y) are said to be topologically conjugate.

If $\{(X_\alpha, \mathscr{U}_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of uniform spaces, where $\mathcal{A}$ is an index set, then the product uniformity on $X^* = \prod_{\alpha\in\mathcal{A}} X_\alpha$ is generated by basis consisting of all sets of the form $E_{\theta_{\alpha_1}, \theta_{\alpha_2}, \dots, \theta_{\alpha_n}} = \{(x, y) \in X^* \times X^* : (x_{\alpha_i}, y_{\alpha_i}) \in \theta_{\alpha_i}\}$, $\theta_{\alpha_i} \in \mathscr{U}_{\alpha_i}$, $i \in \{1, 2, \dots, n\}$ and $n \in \mathbb{Z}^+$ [10]. It is important to note that the topology generated by the product uniformity on X^* coincides with the product topology on it [10]. Let $\{(T, X_\alpha)\}_{\alpha\in\mathcal{A}}$ be a collection of semiflows, then the product semiflow is the semiflow (T, X^*) where the action is defined by $(t, x) \to (tx_\alpha)_{\alpha\in\mathcal{A}}$ for any $t \in T$ and for any $x = (x_\alpha)_{\alpha\in\mathcal{A}} \in X^*$. For $\alpha \in \mathcal{A}$, the map $p_\alpha : X^* \to X_\alpha$, $p_\alpha(x) = x_\alpha$ is called the α^{th} projection. A semiflow (T, X) induces a hyperspatial semiflow $(T, K(X))$ where $K(X)$ consists of all the non-empty compact subsets of X. A uniformity on $K(X)$ is generated by all sets of the form $K^\theta = \{(F, F') \in K(X) \times K(X) : F \subseteq B(F', \theta)$ and $F' \subseteq B(F, \theta)\}$, $\theta \in \mathscr{U}$ as basis [21] and the action is given by $(t, F) \to tF$ for any $t \in T$ and any $F \in K(X)$.

3. Mean sensitivity

In this section, we work upon mean sensitivity for semiflows defined on uniform spaces. We study the said notion on the arbitrary product of semiflows and on the factor semiflows along with some examples. We conclude this section with a characterization for mean sensitivity involving G_δ sets.

The notion of mean sensitivity was introduced by Li et al. [11] in discrete systems defined on metric spaces and was further studied by Wu et al. [23] in discrete systems defined on uniform spaces. These works motivated us to introduce and study the notion of mean sensitivity for semiflows defined on uniform spaces.

Definition 3.1. A semiflow (T, X) is said to be mean sensitive if there exist a $\delta > 0$ and an entourage θ such that for any non-empty open subset G of X there exist x, y in G such that $\overline{d}(\{t \in T : (tx, ty) \notin \theta\}) \geq \delta$. The corresponding θ is called a mean sensitive entourage or a sensitive entourage.

It is important to note that there will always exist a basic entourage that is a sensitive entourage because for $\theta' \subseteq \theta$, $\delta \leq \overline{d}(\{t \in T : (tx, ty) \notin \theta\}) \leq \overline{d}(\{t \in T : (tx, ty) \notin \theta'\})$.

Example 3.2. Let $T = (\mathbb{Z}^+, \cdot)$ where the operation $\cdot$ on T is defined as $s \cdot t = \max\{s, t\}$. Let $X = \cup_{n=0}^{\infty} L_n \subseteq \mathbb{R}^2$ where L_n is the line segment $[(0, n) - (1, 0))$ for $n \in \mathbb{Z}^+ \cup \{0\}$ and X has the subspace uniformity of the plane. Then any element (x, y) of X can be uniquely represented as x^n where $y = n(1 - x)$. We define the action of T on X by $(t, x^n) \to x^{\max\{t,n\}}$. With this action (T, X) becomes a semiflow. Let $0 < \delta < 1$ and θ_δ the corresponding entourage of X. Let G be any non-empty open subset of X, and x^n and y^m with $x \neq y$ two distinct elements of G. Then we can find a very large t_0 in T such that $t_0 x^n$ and $t_0 y^m$ are separated by at least a distance of δ. Therefore, $(tx^n, ty^m) \notin \theta_\delta$ for every $t \geq t_0$ and thus $\overline{d}(\{t \in T : (tx^n, ty^m) \notin \theta_\delta\}) = 1 > \delta$. Hence, (T, X) becomes mean sensitive.

Now we study the aforementioned notion on the arbitrary product of semiflows and on the factor semiflows. This improve Theorem 5.8 of [11] and Lemma 2 of [23].

Theorem 3.3. *Let $\{(T, X_\alpha)\}_{\alpha \in \mathcal{A}}$ be a collection of semiflows. The product semiflow (T, X^*) is mean sensitive if and only if (T, X_α) is mean sensitive for some $\alpha \in \mathcal{A}$.*

Proof. Let (T, X^*) be mean sensitive with $\delta > 0$ and a sensitive entourage $E_{\theta_{\alpha_1}, \theta_{\alpha_2}, \ldots, \theta_{\alpha_n}}$, where θ_{α_i} is an entourage of X_{α_i} for each $i \in \{1, 2, \ldots, n\}$. If for some $i \in \{2, 3, \ldots, n\}$ (T, X_{α_i}) is mean sensitive, then we are done. Otherwise assume that (T, X_{α_i}) is not mean sensitive for each $i \in \{2, 3, \ldots, n\}$. We claim that (T, X_{α_1}) is mean sensitive with $\delta/n > 0$ and a sensitive entourage θ_{α_1}. Let G_{α_1} be a non-empty open subset of X_{α_1}. For $i \in \{2, 3, \ldots, n\}$ as (T, X_{α_i}) is not mean sensitive, corresponding to $\delta/n > 0$ and entourage θ_{α_i} there exists a non-empty open subset G_{α_i} of X_{α_i} such that for any $p_i, q_i \in G_{\alpha_i}$, $\overline{d}(\{t \in T : (tp_i, tq_i) \notin \theta_{\alpha_i}\}) < \delta/n$. Let $G = \cap_{i=1}^{n} p_{\alpha_i}^{-1}(G_{\alpha_i})$, then G is a non-empty open subset of X^*. So, there exist $a, b \in G$ such that $\overline{d}(A = \{t \in T : (ta, tb) \notin E_{\theta_{\alpha_1}, \theta_{\alpha_2}, \ldots, \theta_{\alpha_n}}\}) \geq \delta$. For $i \in \{1, 2, \ldots, n\}$ let $a_{\alpha_i} = x_i$, $b_{\alpha_i} = y_i$ and $A_i = \{t \in T : (tx_i, ty_i) \notin \theta_{\alpha_i}\}$. Since for $i \in \{2, 3, \ldots, n\}$ both x_i and y_i lie in $p_{\alpha_i}(G) = G_{\alpha_i}$, $\overline{d}(A_i) < \delta/n$. By the definition of $E_{\theta_{\alpha_1}, \theta_{\alpha_2}, \ldots, \theta_{\alpha_n}}$, we have $A = \cup_{i=1}^{n} A_i$. If $\overline{d}(A_1) < \delta/n$, then $\overline{d}(A) = \overline{d}(\cup_{i=1}^{n} A_i) \leq \Sigma_{i=1}^{n} \overline{d}(A_i) < \Sigma_{i=1}^{n} (\delta/n) = \delta$, which contradicts to the fact that $\overline{d}(A) \geq \delta$. Therefore, $\overline{d}(A_1) \geq \delta/n$, and as $x_1, y_1 \in p_{\alpha_1}(G) = G_{\alpha_1}$, (T, X_{α_1}) becomes mean sensitive.

Let (T, X_α) be mean sensitive for some $\alpha \in \mathcal{A}$ with $\delta > 0$ and a sensitive entourage θ_α. We claim that (T, X^*) is mean sensitive with $\delta > 0$ and a sensitive entourage E_{θ_α}. Let G be a non-empty open subset of X^*. Then $p_\alpha(G)$ is a non-empty open subset of X_α. Hence, there exist $x, y \in p_\alpha(G)$ such that $\overline{d}(\{t \in T : (tx, ty) \notin \theta_\alpha\}) \geq \delta$. Let a and b be any two elements of G such that $a_\alpha = x$ and $b_\alpha = y$. Then one can observe that $\{t \in T : (ta, tb) \notin E_{\theta_\alpha}\} = \{t \in T : (tx, ty) \notin \theta_\alpha\}$. Therefore, $\overline{d}(\{t \in T : (ta, tb) \notin E_{\theta_\alpha}\}) \geq \delta$ and thus (T, X^*) becomes mean sensitive. □

Example 3.4. Let $T = (\mathbb{Z}^+, +)$, $X = \mathbb{R}$ with the usual uniformity and the action of T on X be defined by $(t, x) \to 2^t x$. With this action (T, X) becomes a semiflow. Let $0 < \delta < 1$ be any real number and θ_δ the corresponding entourage. Let G be any non-empty open subset of X, and x and y any two distinct elements of G. Then there exists a $t_0 \in T$ such that $2^{t_0}|x - y| > \delta$. Hence, $2^t|x - y| \geq 2^{t_0}|x - y| > \delta$ for every $t \geq t_0$. Therefore, $\overline{d}(\{t \in T : (tx, ty) \notin \theta_\delta\}) = 1 > \delta$. Thus, (T, X) becomes mean sensitive. Let $Y = \mathbb{S}^1$ with the usual uniformity, $q \in \mathbb{R} \setminus \mathbb{Q}$ be a fixed irrational number and the action of T on Y is defined by $(t, e^{ib}) \to e^{i[(b+2\pi tq) \bmod 2\pi]}$. With this action (T, Y) becomes a semiflow. Let $\gamma > 0$ and ψ any entourage of Y. Then there exists an $\eta > 0$ such that $\theta_\eta \subseteq \psi$. Let G' be any non-empty open subset of Y such that $\operatorname{diam}(G') < \eta$. Let x' and y' be any two distinct elements of G', then observe that for any t in T, $(tx', ty') \in \theta_\eta$. Therefore, $\overline{d}(\{t \in T : (tx', ty') \notin \psi\}) = 0 < \gamma$. Thus, (T, Y) is not mean sensitive. Let $\mathcal{A}$ be an uncountable index set. For a fixed $\alpha \in \mathcal{A}$, let $(T, X_\alpha) = (T, X)$ and for any $\beta \in \mathcal{A} \setminus \{\alpha\}$, let $(T, X_\beta) = (T, Y)$. Then X^* is a non-metrizable uniform space. Since (T, X_α) is mean sensitive, by Theorem 3.3 we have (T, X^*) is mean sensitive even though (T, X_β) is not mean sensitive for any $\beta \in \mathcal{A} \setminus \{\alpha\}$.

Remark 3.5. García-Ramos et al. [7] constructed an example of a discrete system where the induced hyperspatial system is mean sensitive but the system itself is not mean sensitive. This shows that mean sensitivity need not carry from the hyperspatial semiflow to the semiflow itself.

Recall that a map $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ is said to be uniformly continuous if $(f \times f)^{-1}(\theta) \in \mathscr{U}$ for any $\theta \in \mathscr{V}$ and f is said to be semi-open if $\mathrm{Int}(f(G))$ is non-empty for any non-empty open subset G of X.

Proposition 3.6. *Let (T, X) and (T, Y) be semiflows and suppose there exists a uniformly continuous map $f : X \to Y$ such that f is semi-open and $f(tx) = tf(x)$ for any $t \in T$ and any $x \in X$. If (T, Y) is mean sensitive, then (T, X) is mean sensitive.*

Proof. Let (T, Y) be mean sensitive with $\delta > 0$ and a sensitive entourage θ. We claim that (T, X) is mean sensitive with $\delta > 0$ and a sensitive entourage $\phi = (f \times f)^{-1}(\theta)$. Let G be a non-empty open subset of X. As f is semi-open, $G' = \mathrm{Int}(f(G))$ is a non-empty open subset of Y. Thus, there exist $p, q \in G'$ such that $\overline{d}(\{t \in T : (tp, tq) \notin \theta\}) \geq \delta$. Since $p, q \in \mathrm{Int}(f(G))$, there exist $a, b \in G$ such that $f(a) = p$ and $f(b) = q$. Let $t \in T$ be such that $(tp, tq) \notin \theta$ and assume that $(ta, tb) \in \phi$. Then $(f \times f)(ta, tb) \in \theta$ and hence $(f(ta), f(tb)) = (tf(a), tf(b)) = (tp, tq) \in \theta$ which is a contradiction. Therefore, $(ta, tb) \notin \phi$ and thus $\{t \in T : (tp, tq) \notin \theta\} \subseteq \{t \in T : (ta, tb) \notin \phi\}$ which gives that $\overline{d}(\{t \in T : (ta, tb) \notin \phi\}) \geq \delta$. Hence, (T, X) becomes mean sensitive. □

The following result is an immediate consequence of the above proposition.

Proposition 3.7. *Let (T, X) be a semiflow where X is a compact uniform space and (T, Y) a factor of (T, X) such that the factor map is semi-open. If (T, Y) is mean sensitive, then (T, X) is mean sensitive.*

The following example justifies the necessity of compactness condition in the hypothesis of Proposition 3.7.

Example 3.8. Let $T = (\mathbb{Z}^+, +)$, $X = (0, \infty)$ and $Y = (1, \infty)$ with the usual uniformity. Consider the semiflows (T, X) and (T, Y) where the action of T on X and Y is $(t, x) \to x + \log 2^t$ and $(t, y) \to 2^t y$ respectively. By the similar techniques of Example 3.4, we have (T, Y) is mean sensitive. It is easy to see that (T, X) is not sensitive, hence it is not mean sensitive. Consider the map $f : X \to Y$, $f(x) = e^x$. Then f is a homeomorphism and $f(tx) = e^{x+\log 2^t} = e^x 2^t = 2^t e^x = tf(x)$ for any $t \in T$ and any $x \in X$. Therefore, (T, X) and (T, Y) are topologically conjugate to each other, hence (T, Y) is a factor of (T, X) via the factor map f which is semi-open. Note that here X is not compact.

The following characterization of mean sensitivity was studied by Li et al. [11] and Wu et al. [23] in discrete systems defined on metric spaces and uniform spaces respectively.

Theorem 3.9. *Let $T = \mathbb{Z}^+$ and (T, X) a semiflow. Then the following statements are equivalent:*

(a) *(T, X) is mean sensitive.*

(b) *There exist an entourage θ and an $\epsilon > 0$ such that the set $\mathscr{A}_{\theta,\epsilon} = \{(x, y) \in X \times X : \overline{d}(\{t \in T : (tx, ty) \notin \theta\}) \geq \epsilon\}$ is a dense G_δ subset of $X \times X$.*

(c) *There exist an entourage θ and an $\epsilon > 0$ such that for any x in X the set $\mathscr{A}_{\theta,\epsilon}(x) = \{y \in X : \overline{d}(\{t \in T : (tx, ty) \notin \theta\}) \geq \epsilon\}$ is a dense G_δ subset of X.*

Proof. (a)⇒(b) Let $\delta > 0$ and $\theta' \in \mathscr{U}$ corresponding to the mean sensitivity of (T, X). Let $\theta \in \mathscr{U}$ be a closed symmetric entourage such that $\theta \circ \theta \subseteq \theta'$ and $\epsilon = \delta/2$. We claim that $\mathscr{A}_{\theta,\epsilon}$ is a dense G_δ subset of $X \times X$. We have

$$\begin{aligned}\mathscr{A}_{\theta,\epsilon} =&\{(x,y)\in X\times X : \overline{d}(\{t\in T : (tx,ty)\notin\theta\})\geq\epsilon\}\\ =&\{(x,y)\in X\times X : \forall n\in\mathbb{Z}^+, \forall m\in\mathbb{Z}^+, \exists m' > m, \exists k\in\{0,1,\ldots,m'\} \text{ with}\\ & k/m' > \epsilon - 1/n, \exists (r_1,r_2,\ldots,r_k)\in(\mathbb{Z}^+)^k \text{ with } 0\leq r_1 < r_2 < \cdots < r_k \leq\\ & m'-1 \text{ such that } (r_i x, r_i y)\notin\theta \text{ for each } i\in\{1,2,\ldots,k\}\}\\ =&\bigcap_{n\in\mathbb{Z}^+}\bigcap_{m\in\mathbb{Z}^+}\bigcup_{m'>m}\bigcup_{k\in P_{m'}}\bigcup_{(r_1,r_2,\ldots,r_k)\in Q_{m',k}}\bigcap_{i=1}^{k} g_{r_i}^{-1}(X\times X\setminus\theta),\end{aligned}$$

where $P_{m'} = \{k\in\mathbb{Z}^+ : 0\leq k\leq m' \text{ and } k/m' > \epsilon - 1/n\}$, $Q_{m',k} = \{(r_1,r_2,\ldots,r_k)\in(\mathbb{Z}^+)^k : 0\leq r_1 < r_2 < \cdots < r_k \leq m'-1\}$ and $g_{r_i} : X\times X\to X\times X$, $g_{r_i}(x,y) = (r_i x, r_i y)$ is a continuous map. Hence, $\mathscr{A}_{\theta,\epsilon}$ is a G_δ subset of $X\times X$. Let $U\times V$ be a non-empty basic open subset of $X\times X$. Since (T,X) is mean sensitive with δ and θ', there exist $x,x'\in U$ such that $\overline{d}(A = \{t\in T : (tx,tx')\notin\theta'\})\geq\delta$. Let $y\in V$, $t\in A$, and assume that $(tx,ty)\in\theta$ and $(tx',ty)\in\theta$. Then $(tx,tx')\in\theta\circ\theta\subseteq\theta'$, which is a contradiction. Therefore, either $(tx,ty)\notin\theta$ or $(tx',ty)\notin\theta$ and thus $A\subseteq\{t\in T : (tx,ty)\notin\theta\}\cup\{t\in T : (tx',ty)\notin\theta\}$. Hence, either $\overline{d}(\{t\in T : (tx,ty)\notin\theta\})\geq\epsilon$ or $\overline{d}(\{t\in T : (tx',ty)\notin\theta\})\geq\epsilon$. In any case, as $(x,y),(x',y)\in U\times V$, we have $\mathscr{A}_{\theta,\epsilon}\cap(U\times V)\neq\emptyset$. Thus, $\mathscr{A}_{\theta,\epsilon}$ becomes a dense G_δ subset of $X\times X$.

(b)⇒(c) Let $\theta\in\mathscr{U}$ and $\epsilon > 0$ such that $\mathscr{A}_{\theta,\epsilon}$ is a dense G_δ subset of $X\times X$. Let $\theta'\in\mathscr{U}$ be a closed symmetric entourage such that $\theta'\circ\theta'\subseteq\theta$ and $\gamma = \epsilon/2$. Let $x\in X$ be arbitrary; we claim that $\mathscr{A}_{\theta',\gamma}(x)$ is a dense G_δ subset of X. By similar techniques of the previous part, we have $\mathscr{A}_{\theta',\gamma}(x) = \bigcap_{n\in\mathbb{Z}^+}\bigcap_{m\in\mathbb{Z}^+}\bigcup_{m'>m}\bigcup_{k\in P_{m'}}\bigcup_{(r_1,r_2,\ldots,r_k)\in Q_{m',k}}\bigcap_{i=1}^{k} g_{r_i}^{-1}(X\times X\setminus\theta')$, where $P_{m'} = \{k\in\mathbb{Z}^+ : 0\leq k\leq m' \text{ and } k/m' > \epsilon - 1/n\}$, $Q_{m',k} = \{(r_1,r_2,\ldots,r_k)\in(\mathbb{Z}^+)^k : 0\leq r_1 < r_2 < \cdots < r_k\leq m'-1\}$ and $g_{r_i} : X\to X\times X$, $g_{r_i}(y) = (r_i x, r_i y)$ is a continuous map. Hence, $\mathscr{A}_{\theta',\gamma}(x)$ is a G_δ subset of X. Let U be a non-empty open subset of X. Then $U\times U$ is a non-empty open subset of $X\times X$. Therefore, $\mathscr{A}_{\theta,\epsilon}\cap(U\times U)\neq\emptyset$. Let $y,z\in U$ be such that $\overline{d}(A = \{t\in T : (ty,tz)\notin\theta\})\geq\epsilon$. Since $\theta'\circ\theta'\subseteq\theta$, $A\subseteq\{t\in T : (tx,ty)\notin\theta'\}\cup\{t\in T : (tx,tz)\notin\theta'\}$. Therefore, either $\overline{d}(\{t\in T : (tx,ty)\notin\theta'\})\geq\gamma$ or $\overline{d}(\{t\in T : (tx,tz)\notin\theta'\})\geq\gamma$. In any case, as $y,z\in U$, we have $\mathscr{A}_{\theta',\gamma}(x)\cap U\neq\emptyset$. Thus, $\mathscr{A}_{\theta',\gamma}(x)$ becomes a dense G_δ subset of X.

(c)⇒(a) Let $\theta\in\mathscr{U}$ and $\epsilon > 0$ such that for any $x\in X$ the set $\mathscr{A}_{\theta,\epsilon}(x)$ is a dense G_δ subset of X. Let U be a non-empty open subset of X and $x\in U$. Then $\mathscr{A}_{\theta,\epsilon}(x)\cap U\neq\emptyset$. Let $y\in U$ be such that $\overline{d}(\{t\in T : (tx,ty)\notin\theta\})\geq\epsilon$. Since $x,y\in U$, therefore (T,X) becomes mean sensitive. □

4. Mean equicontinuity

In this section, we work upon the notion of mean equicontinuity in semiflows defined on uniform spaces. We provide an example of a mean equicontinuous semiflow which is not equicontinuous. We study the said notion on the arbitrary product of semiflows, on the hyperspatial semiflow and on the factor semiflows along with some examples. We end this section by proving the Auslander-Yorke dichotomy result for a certain class of topological semigroups.

The notion of mean equicontinuity has been studied in discrete systems by considering phase spaces as metric spaces [11] and uniform spaces [23]. Motivated by these works, we introduce and study mean equicontinuity for semiflows defined on uniform spaces.

Definition 4.1. A semiflow (T,X) is said to be mean equicontinuous if for any $\epsilon > 0$ and any entourage θ there exists an entourage ϕ such that for any $(x,y)\in\phi$, $\overline{d}(\{t\in T : (tx,ty)\notin\theta\}) < \epsilon$.

It is noteworthy that in this definition we can work with basic entourages, because for $\theta'\subseteq\theta$, $\overline{d}(\{t\in T : (tx,ty)\notin\theta\})\leq\overline{d}(\{t\in T : (tx,ty)\notin\theta'\}) < \epsilon$.

From Definition 4.1, it is clear that an equicontinuous semiflow implies a mean equicontinuous semiflow. The following example is inspired from [6] and it shows that the set of equicontinuous semiflows is properly contained in the set of mean equicontinuous semiflows.

Example 4.2. Let $T = (\mathbb{Z}^+, +)$, $X = \{(x_i)_{i\geq 0} : x_i = 0 \text{ or } 1\}$ with the uniformity generated by the metric d where $d(x, y) = \Sigma_{i\geq 0}|x_i - y_i|/2^i$, and the action of T on X is defined by $(t, x) \to y$ where $y = (x_t, x_{t+1}, \ldots)$. Let Y be the subspace of X consisting of all those sequences having 0 everywhere except at finitely many places. Since $tY \subseteq Y$ for any $t \in T$, we consider the semiflow (T, Y). Let $0 < \epsilon < 1$ and consider the entourage θ_ϵ of Y. Let ϕ be any entourage of Y and $(y, y') \in \phi$ such that $y \neq y'$. Then there exists a $t \in T$ such that $(y)_t \neq (y')_t$. Therefore, $(ty, ty') \notin \theta_\epsilon$ and hence (T, Y) is not equicontinuous. Let $\gamma > 0$ and θ any entourage of Y. Let $\psi = \theta$ and $(y, y') \in \psi$. Since $y, y' \in Y$, they both can have 1 at most at finite number of places. Therefore, there exists a $t_0 \in T$ such that $(y)_t = 0 = (y')_t$, for every $t \geq t_0$. Thus, $(ty, ty') \in \theta$ for every $t \geq t_0$ and hence $\overline{d}(\{t \in T : (ty, ty') \notin \theta\}) = 0 < \gamma$. Therefore, (T, Y) is mean equicontinuous.

Now we study mean equicontinuity on the product semiflows, on the hyperspatial semiflows and on the factor semiflows along with examples illustrating these results.

Theorem 4.3. *Let $\{(T, X_\alpha)\}_{\alpha\in\mathcal{A}}$ be a collection of semiflows. The product semiflow (T, X^*) is mean equicontinuous if and only if (T, X_α) is mean equicontinuous for every $\alpha \in \mathcal{A}$.*

Proof. Let (T, X^*) be mean equicontinuous and $\alpha \in \mathcal{A}$ fixed. We claim that (T, X_α) is mean equicontinuous. Let $\epsilon > 0$ and θ_α any entourage of X_α. Since (T, X^*) is mean equicontinuous, corresponding to the above $\epsilon > 0$ and the entourage E_{θ_α} of X^* there exists an entourage $E_{\theta_{\alpha_1},\theta_{\alpha_2},\ldots,\theta_{\alpha_n}}$ such that for any $(a, b) \in E_{\theta_{\alpha_1},\theta_{\alpha_2},\ldots,\theta_{\alpha_n}}$, $\overline{d}(\{t \in T : (ta, tb) \notin E_{\theta_\alpha}\}) < \epsilon$. If for some $i \in \{1, 2, \ldots, n\}$ θ_{α_i} is an entourage of X_α, then for any $(x, y) \in \theta_{\alpha_i}$ we can find two elements p and q of X^* such that $p_{\alpha_i} = x$, $q_{\alpha_i} = y$ and $(p, q) \in E_{\theta_{\alpha_1},\theta_{\alpha_2},\ldots,\theta_{\alpha_n}}$. Therefore, $\overline{d}(\{t \in T : (tp, tq) \notin E_{\theta_\alpha}\}) < \epsilon$ and as $\{t \in T : (tp, tq) \notin E_{\theta_\alpha}\} = \{t \in T : (tx, ty) \notin \theta_\alpha\}$, $\overline{d}(\{t \in T : (tx, ty) \notin \theta_\alpha\}) < \epsilon$ and thus (T, X_α) becomes mean equicontinuous. If for any $i \in \{1, 2, \ldots, n\}$ θ_{α_i} is not an entourage of X_α, then let ϕ_α be any entourage of X_α and $(x, y) \in \phi_\alpha$. We can find two elements p and q of X^* such that $p_\alpha = x$, $q_\alpha = y$ and $(p, q) \in E_{\theta_{\alpha_1},\theta_{\alpha_2},\ldots,\theta_{\alpha_n},\phi_\alpha} \subseteq E_{\theta_{\alpha_1},\theta_{\alpha_2},\ldots,\theta_{\alpha_n}}$. Therefore, $\overline{d}(\{t \in T : (tp, tq) \notin E_{\theta_\alpha}\}) < \epsilon$ and as $\{t \in T : (tp, tq) \notin E_{\theta_\alpha}\} = \{t \in T : (tx, ty) \notin \theta_\alpha\}$, $\overline{d}(\{t \in T : (tx, ty) \notin \theta_\alpha\}) < \epsilon$ and thus (T, X_α) becomes mean equicontinuous.

Let (T, X_α) be mean equicontinuous for every $\alpha \in \mathcal{A}$. Let $\epsilon > 0$ and $E_{\theta_{\alpha_1},\theta_{\alpha_2},\ldots,\theta_{\alpha_n}}$ any basic entourage of X^*. Since for $i \in \{1, 2, \ldots, n\}$ (T, X_{α_i}) is mean equicontinuous, corresponding to $\epsilon/n > 0$ and entourage θ_{α_i} there exists an entourage ϕ_{α_i} such that for any $(p_i, q_i) \in \phi_{\alpha_i}$, $\overline{d}(\{t \in T : (tp_i, tq_i) \notin \theta_{\alpha_i}\}) < \epsilon/n$. Consider the entourage $E_{\phi_{\alpha_1},\phi_{\alpha_2},\ldots,\phi_{\alpha_n}}$ of X^* and let $(x, y) \in E_{\phi_{\alpha_1},\phi_{\alpha_2},\ldots,\phi_{\alpha_n}}$. For each $i \in \{1, 2, \ldots, n\}$ as $(x_{\alpha_i}, y_{\alpha_i}) \in \phi_{\alpha_i}$, $\overline{d}(A_i = \{t \in T : (tx_{\alpha_i}, ty_{\alpha_i}) \notin \theta_{\alpha_i}\}) < \epsilon/n$. Since $A = \{t \in T : (tx, ty) \notin E_{\theta_{\alpha_1},\theta_{\alpha_2},\ldots,\theta_{\alpha_n}}\} = \cup_{i=1}^n A_i$, $\overline{d}(A) \leq \Sigma_{i=1}^n \overline{d}(A_i) < \Sigma_{i=1}^n \epsilon/n = \epsilon$. Hence, (T, X^*) becomes mean equicontinuous. □

Example 4.4. Let $T = (\mathbb{Z}^+, +)$, $X = \mathbb{S}^1$ with the usual uniformity, $q \in \mathbb{Q}$ be a fixed rational number and the action of T on Y be defined by $(t, e^{ib}) \to e^{i[(b+2\pi tq) \bmod 2\pi]}$. With this action (T, X) becomes a semiflow. Let θ be an entourage of X. Then there exists a $\delta > 0$ such that $\theta_\delta \subseteq \theta$. Let $\phi = \theta_\delta$ and $(x, y) \in \phi$. Then for any $t \in T$ we have $(tx, ty) \in \theta_\delta \subseteq \theta$. Therefore, (T, X) becomes equicontinuous and hence mean equicontinuous. Let $Y = \mathbb{R}$ with the usual uniformity and consider the semiflow (T, Y) where the action of T on Y is defined by $(t, y) \to 2^t y$. Let $0 < \gamma < 1$ and consider the corresponding entourage θ_γ of Y. Let ψ be an entourage of Y, then there exists an $\eta > 0$ such that $\theta_\eta \subseteq \psi$. Let x' and y' be any two distinct elements of θ_η. Then there exists a $t_0 \in T$ such that $(tx', ty') \notin \theta_\gamma$ for every $t \geq t_0$. Therefore, $\overline{d}(\{t \in T : (tx', ty') \notin \theta_\gamma\}) = 1 > \gamma$. Thus, (T, Y) is not mean equicontinuous. Let $\mathcal{A}$ be an uncountable index set. For a fixed $\alpha \in \mathcal{A}$, let $(T, X_\alpha) = (T, Y)$ and for any $\beta \in \mathcal{A} \setminus \{\alpha\}$, let $(T, X_\beta) = (T, X)$. Then X^* is a non-metrizable uniform space. Since (T, X_β) is not mean equicontinuous, by Theorem 4.3 we have (T, X^*) is not mean equicontinuous even though (T, X_β) is mean equicontinuous for every $\beta \in \mathcal{A} \setminus \{\alpha\}$. If we substitute $(T, X_\alpha) = (T, X)$, then by Theorem 4.3 (T, X^*) becomes mean equicontinuous.

Theorem 4.5. *If the hyperspatial semiflow $(T, K(X))$ is mean equicontinuous, then the semiflow (T, X) is mean equicontinuous.*

Proof. Let $(T, K(X))$ be mean equicontinuous. Let $\epsilon > 0$ and θ any entourage of X. Since $(T, K(X))$ is mean equicontinuous, corresponding to the above $\epsilon > 0$ and the entourage K^θ of $K(X)$ there exists an entourage K^ϕ, where ϕ is a symmetric entourage of X, such that for any $(F, F') \in K^\phi$, $\overline{d}(\{t \in T : (tF, tF') \notin K^\theta\}) < \epsilon$. Let x and y be any two elements of X such that $(x, y) \in \phi$. Let $F = \{x\}$ and $F' = \{y\}$, then since $(x, y) \in \phi$, $(F, F') \in K^\phi$. Therefore, $\overline{d}(\{t \in T : (tF, tF') \notin K^\theta\}) < \epsilon$. If $t \in T$ satisfies that $(tx, ty) \notin \theta$, then $tF' \nsubseteq B(tF, \theta)$ implying that $(tF, tF') \notin K^\theta$. Therefore, $\{t \in T : (tx, ty) \notin \theta\} \subseteq \{t \in T : (tF, tF') \notin K^\theta\}$ and thus $\overline{d}(\{t \in T : (tx, ty) \notin \theta\}) < \epsilon$. Hence, (T, X) becomes mean equicontinuous. □

Remark 4.6. The Theorem 1.8 of [7] is an example of a discrete system where the system is mean equicontinuous but the induced hyperspatial system is not mean equicontinuous. This shows that the converse of Theorem 4.5 need not be true.

Example 4.7. Let $T = (\mathbb{R}, \cdot)$ where $\cdot$ is the product operation, $X = \mathbb{R}$ with the usual uniformity and consider the semiflow (T, X) where the action of T on X is defined by $(t, x) \to t \cdot x$. Let $0 < \epsilon < 1$ and consider the corresponding entourage θ_ϵ of X. Let ϕ be any entourage of X, and x and y any two distinct elements of X such that $(x, y) \in \phi$. Then there exists a t_0 in T such that $(tx, ty) \notin \theta_\epsilon$ for every $t \geq t_0$. Therefore, $\overline{d}(\{t \in T : (tx, ty) \notin \theta_\epsilon\}) = 1 > \epsilon$. Hence, (T, X) is not mean equicontinuous and thus by Theorem 4.5 $(T, K(X))$ is not mean equicontinuous.

The following result generalizes Theorem 4 of [23] where the authors studied it on discrete dynamical systems.

Proposition 4.8. *Let (T, X) be a semiflow where $(X, \mathscr{U})$ is a compact Hausdorff space and (T, Y) a factor of (T, X) where $(Y, \mathscr{V})$ is Hausdorff. If (T, X) is mean equicontinuous, then (T, Y) is mean equicontinuous.*

Proof. Let $f : X \to Y$ be the factor map and assume that (T, Y) is not mean equicontinuous. Then there exist an $\epsilon > 0$ and a $\theta \in \mathscr{V}$ such that for any $\phi \in \mathscr{V}$ there exists a $(y_\phi, y'_\phi) \in \phi$ such that $\overline{d}(A_{y_\phi y'_\phi} = \{t \in T : (ty_\phi, ty'_\phi) \notin \theta\}) \geq \epsilon$. Let $\theta' \in \mathscr{V}$ be symmetric such that $\theta' \circ \theta' \subseteq \theta$. Then for any $\phi \in \mathscr{V}$, $\overline{d}(\{t \in T : (ty_\phi, ty'_\phi) \notin \theta'\}) \geq \epsilon$. As Y is a compact Hausdorff space, without loss of generality, we can assume that $(y_\delta)_{\delta \in \mathscr{V}} \to y$ and $(y'_\delta)_{\delta \in \mathscr{V}} \to y$ for some $y \in Y$. For any $\phi \in \mathscr{V}$, let $A_{y_\phi} = \{t \in T : (ty_\phi, ty) \notin \theta'\}$ and $A_{y'_\phi} = \{t \in T : (ty'_\phi, ty) \notin \theta'\}$. Since θ' is symmetric and $\theta' \circ \theta' \subseteq \theta$, for any $\phi \in \mathscr{V}$ we have $A_{y_\phi y'_\phi} \subseteq A_{y_\phi} \cup A_{y'_\phi}$. Therefore, for any $\phi \in \mathscr{V}$ either $\overline{d}(A_{y_\phi}) \geq \epsilon/2$ or $\overline{d}(A_{y'_\phi}) \geq \epsilon/2$. For $\phi \in \mathscr{V}$, let

$$z_\phi = \begin{cases} y_\phi, & \text{if } \overline{d}(A_{y_\phi}) \geq \epsilon/2, \\ y'_\phi, & \text{otherwise.} \end{cases}$$

Since $(y_\delta)_{\delta \in \mathscr{V}} \to y$ and $(y'_\delta)_{\delta \in \mathscr{V}} \to y$, $(z_\phi)_{\phi \in \mathscr{V}} \to y$. Also, $\overline{d}(\{t \in T : (tz_\phi, ty) \notin \theta'\}) \geq \epsilon/2$ for any $\phi \in \mathscr{V}$. As f is an onto map, for any $\phi \in \mathscr{V}$ there exists an $x_\phi \in X$ such that $f(x_\phi) = z_\phi$. Since X is compact, we can assume that $(x_\phi)_{\phi \in \mathscr{V}} \to x$ for some $x \in X$, and as f is continuous and Y is Hausdorff, $f(x) = y$. Since (T, X) is mean equicontinuous, corresponding to $\epsilon/2 > 0$ and the entourage $\eta = (f \times f)^{-1}(\theta')$ of X there exists an open symmetric entourage η' of X such that for any $(p, q) \in \eta'$, $\overline{d}(\{t \in T : (tp, tq) \notin \eta\}) < \epsilon/2$. Consider the open neighbourhood $B(x, \eta')$ of x, then since $(x_\phi)_{\phi \in \mathscr{V}} \to x$ there exists a $\phi \in \mathscr{V}$ such that $x_\phi \in B(x, \eta')$, that is, $(x_\phi, x) \in \eta'$. Thus, $\overline{d}(\{t \in T : (tx_\phi, tx) \notin \eta\}) < \epsilon/2$. Let $t \in T$ be such that $(tz_\phi, ty) \notin \theta'$ and assume that $(tx_\phi, tx) \in \eta$. Then $(f(tx_\phi), f(tx)) = (tf(x_\phi), tf(x)) = (tz_\phi, ty) \in \theta'$, which is a contradiction. Therefore, $\{t \in T : (tz_\phi, ty) \notin \theta'\} \subseteq \{t \in T : (tx_\phi, tx) \notin \eta\}$ and hence $\overline{d}\{t \in T : (tx_\phi, tx) \notin \eta\} \geq \epsilon/2$, which is a contradiction. Thus, (T, Y) is mean equicontinuous. □

The following example shows the importance of compactness in Proposition 4.8.

Example 4.9. Let $T = (\mathbb{Z}^+, +)$, $X = (0, \infty)$ and $Y = (1, \infty)$ with the usual uniformity. Consider the semiflows (T, X) and (T, Y) where the action of T on X and Y is $(t, x) \to x + \log 2^t$ and $(t, y) \to 2^t y$ respectively. By similar arguments as given in Example 4.4, it can be proved that (T, Y) is not mean equicontinuous. It is easy to see that (T, X) is equicontinuous, hence mean equicontinuous. Moreover, X and Y are Hausdorff spaces, X is not compact and by Example 3.8 (T, X) and (T, Y) are topologically conjugate to each other.

Now we have a dichotomy result for a certain class of topological semigroups and minimal semiflows on compact spaces.

Definition 4.10. Let $T = \mathbb{R}^+$ or $\mathbb{Z}^+$, then the topological semigroup $(T, .)$ is said to have the upper density preserving property or T is a udp semigroup if for any $t \in T$ and any $A \subseteq T$ with $\overline{d}(A) > 0$, there exists a $B \subseteq T$ such that $B.t \subseteq A$ and $\overline{d}(B) = \overline{d}(A)$.

Example 4.11. $(\mathbb{R}^+, +)$ and any of its proper subsemigroups; ($\mathbb{R}^+$ or $\mathbb{Z}^+, .$) where $x.y = \max\{x, y\}$; $(\mathbb{R}^+, .)$ where $x.y = x$ and any of its proper subsemigroups are udp semigroups. On the other hand $(\mathbb{Z}^+, .)$ where . is the usual multiplication is not a udp semigroup.

Theorem 4.12. *Let T be a udp semigroup, X compact and (T, X) a minimal semiflow. Then (T, X) is either mean equicontinuous or mean sensitive.*

Proof. If (T, X) is mean equicontinuous, then we are done.

Otherwise, assume that (T, X) is not mean equicontinuous. Then there exist an $\epsilon > 0$ and an entourage θ such that for any entourage ϕ there exists a $(p, q) \in \phi$ such that $\overline{d}(\{t \in T : (tp, tq) \notin \theta\}) \geq \epsilon$. Let $\delta = \epsilon/2$ and θ' a symmetric entourage such that $\theta' \circ \theta' \subseteq \theta$. Firstly, we claim that there exists a $z \in X$ such that for any entourage ψ, there exists a $y \in B(z, \psi)$ such that $\overline{d}(\{t \in T : (tz, ty) \notin \theta'\}) \geq \delta$. Assume that this does not hold, that is, for any $x \in X$ there exists an entourage ψ_x such that for any $y \in B(x, \psi_x)$, $\overline{d}(\{t \in T : (tx, ty) \notin \theta'\}) < \delta$. For $x \in X$, let ϕ_x be an open symmetric entourage such that $\phi_x \circ \phi_x \subseteq \psi_x$. Consider the open cover $\{B(x, \phi_x) : x \in X\}$ of X. As X is compact, there exist $x_1, x_2, \ldots, x_n$ in X such that $X = \cup_{i=1}^n B(x_i, \phi_{x_i})$. Let $\phi = \cap_{i=1}^n \phi_{x_i} \in \mathscr{U}$, then there exists a $(p, q) \in \phi$ such that $\overline{d}(\{t \in T : (tp, tq) \notin \theta\}) \geq \epsilon$. Now, corresponding to p, there exists a $k \in \{1, 2, \ldots, n\}$ such that $p \in B(x_k, \phi_{x_k}) \subseteq B(x_k, \psi_{x_k})$. As $(p, q) \in \phi \subseteq \phi_{x_k}$ and $(x_k, p) \in \phi_{x_k}$, $(x_k, q) \in \phi_{x_k} \circ \phi_{x_k} \subseteq \psi_{x_k}$. Therefore, $p, q \in B(x_k, \psi_{x_k})$. Since $\{t \in T : (tp, tq) \notin \theta\} \subseteq \{t \in T : (tx_k, tp) \notin \theta'\} \cup \{t \in T : (tx_k, tq) \notin \theta'\}$, either $\overline{d}(\{t \in T : (tx_k, tp) \notin \theta'\}) \geq \delta$ or $\overline{d}(\{t \in T : (tx_k, tq) \notin \theta'\}) \geq \delta$, which is a contradiction. Hence the claim. Let $z \in X$ be such that for any entourage ψ, there exists a $y \in B(z, \psi)$ such that $\overline{d}(\{t \in T : (tz, ty) \notin \theta'\}) \geq \delta$. We claim that (T, X) is mean sensitive with the above $\delta > 0$ and $\theta' \in \mathscr{U}$. Let G be a non-empty open subset of X. As (T, X) is minimal, there exists an $s \in T$ such that $sz \in G$. Therefore, there exists an entourage η such that $z \in B(z, \eta) \subseteq s^{-1}G$. By the claim, there exists a $z' \in B(z, \eta)$ such that $\overline{d}(A = \{t \in T : (tz, tz') \notin \theta'\}) \geq \delta$. Now, corresponding to s and A, there exists a $B \subseteq T$ such that $Bs \subseteq A$ and $\overline{d}(B) = \overline{d}(A)$. Thus, $\overline{d}(\{t \in T : (t(sz), t(sz')) \notin \theta'\}) \geq \overline{d}(B) \geq \delta$, where $sz, sz' \in G$. Hence, (T, X) is mean sensitive. □

Corollary 4.13. *Let T be a udp semigroup, $K(X)$ compact and $(T, K(X))$ a minimal semiflow. If (T, X) is mean sensitive, then $(T, K(X))$ is mean sensitive.*

Remark 4.14. The dichotomy result Corollary 5.5 [11] follows from Theorem 4.12 by taking the usual action of $T = (\mathbb{Z}^+, +)$ on X.

The following example shows the necessity of conditions in the hypothesis of Theorem 4.12.

Example 4.15. Consider the semiflow (T, X), where $T = (\mathbb{Z}^+, +)$, $X = \mathbb{R}$ with the usual uniformity and the action of T on X is defined by $(t, x) \to x^{2^t}$. Observe that T is a udp semigroup, X is not compact and (T, X) is not minimal. Let $0 < \epsilon < 1$ and θ_ϵ an entourage of X. Let ϕ be any entourage of X, then there exists a $\delta > 0$ such that the entourage $\theta_\delta \subseteq \phi$. We can choose very large distinct real numbers x and y such that $(x, y) \in \theta_\delta \subseteq \phi$. Then there exists a $t_0 \in T$ such that $(tx, ty) \notin \theta_\epsilon$ for every $t \geq t_0$. Therefore,

$\overline{d}(\{t \in T : (tx, ty) \notin \theta_\epsilon\}) = 1 > \epsilon$. Thus, (T, X) is not mean equicontinuous. Let $\gamma > 0$ and ψ an entourage of X. Then there exists an $\eta \in (0, 1)$ such that $\theta_\eta \subseteq \psi$. Consider the open subset $G = (0, \eta/10)$ of X. Then observe that $tG \subseteq G$ for any $t \in T$. Therefore, for any $x, y \in G$, $\{t \in T : (tx, ty) \in \theta_\eta\} = T$ and thus $\overline{d}(\{t \in T : (tx, ty) \notin \psi\}) = \overline{d}(\emptyset) = 0 < \gamma$. Hence, (T, X) is not mean sensitive.

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