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Separation and sober spaces in the category of quasi-proximity spaces

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Abstract. The main objective of this paper is to characterize sober spaces, the separation properties $\overline{T_0}$, T_0',T_0, T_1, Pre$\overline{T_2}$, PreT_2', $\overline{T_2}$ and T_2' in general in the category of quasi-proximity spaces. Moreover, we introduce two notions of closure operators in the category of quasi-proximity spaces which satisfy (weak) hereditariness, productivity, idempotency and we characterize each of T_i, $i = 0, 1, 2$, quasi-proximity spaces by using these closure operators as well as show how these subcategories are related.

1. Introduction

Proximity structure was introduced by Efremovich in 1951 [19] , He characterized the proximity relation "A is close to B" as a binary relation on subsets of a set X. A study on "Separation of Sets" was worked by Wallace ([36],[37]) in 1941. This study can be considered as a primitive form of the definition of quasi-proximity (semi-quasi). Most of the early studies were made by Smirnov ([35],[34]). All our preliminary information about quasi-proximity spaces and much more can be found in this [30]. Some researchers such as Leader [26], Lodato[27] and Pervin[31] have worked with weaker axioms than Efremovich's proximity axioms.

The sober spaces were introduced by Dieudonne and Grothendieck in [16]. Baran and Abughalwa gave various forms of sober objects in a topological category and investigate relationships among these various forms in [11].

The main objective of this paper is to characterize sober spaces, the separation properties $\overline{T_0}$, T_0',T_0 ,T_1, Pre$\overline{T_2}$, PreT_2', $\overline{T_2}$ and T_2' in general in the category of quasi-proximity spaces. Moreover, we investigated the relationships between them. Finally, we compare our results in some topological categories.

2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

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Definition 2.1. [30] A quasi-proximity is a pair (X, δ), where X is a set and δ is a binary relation on the powerset of X such that,

(Q1) $A\delta B$ implies $A, B \neq \emptyset$;
(Q2) $A \cap B \neq \emptyset$ implies $A\delta B$;
(Q3) $C\delta(A \cup B)$ if and only if $C\delta A$ or $C\delta B$;
(Q4) $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$;
(Q5) $A\bar{\delta}B$ implies there is an $C \subseteq X$ such that $A\bar{\delta}C$ and $(X - C)\bar{\delta}B$;

where $A\bar{\delta}B$ means it is not true that $A\delta B$.

If δ satisfies the symmetry condition $A\delta B$ if and only if $B\delta A$, then it is called (Efremovich) proximity on X. The $(Q5)$ axiom is called *strong axiom*, and it plays an important role in the theory of (quasi-)proximity spaces.

A function $f : (X, \delta) \to (Y, \delta')$ between two (quasi-) proximity spaces is called a *(quasi-) proximity map* if and only if $f(A)\delta' f(B)$ whenever $A\delta B$. It can easily be shown that f is a (quasi-)proximity map if and only if $f^{-1}(C)\bar{\delta}f^{-1}(D)$ whenever $C\bar{\delta}'D$.

In a (quasi-)proximity space (X, δ), we write $A \ll B$ if and only if $A\ \bar{\delta}\ (X - B)$. The relation $\ll$ is called p-neighborhood relation or the strong inclusion. When $A \ll B$, we say that B is a p-neighborhood of A or A is strongly contained in B ([30],[21]).

We denote the category of quasi-proximity spaces and quasi-proximity map by **QProx**. **QProx** is a topological category over **Set** ([15] p.31).

Definition 2.2. Let X be a non-empty set, for each $i \in I$, (X_i, δ_i) be a (quasi-)proximity space and $f_i : X \to (X_i, \delta_i)$ be a source in **Set**. Define a binary relation $\mathfrak{B}$ on $P(X)$ as follows: for $A, B \in P(X)$, $A\mathfrak{B}B$ if and only if $f_i(A)\delta_i f_i(B)$, for all $i \in I$. $\mathfrak{B}$ is a (quasi-)proximity-base on X (Theorem 3.8, [33]). The initial (quasi-)proximity structure δ on X generated by the (quasi-)proximity base $\mathfrak{B}$ is given by for $A, B \in P(X)$, $A\delta B$ if and only if for any finite covers $\{A_i : 1 \leq i \leq n\}$ and $\{B_j : 1 \leq j \leq m\}$ of A and B respectively, then there exists a pair (i, j) such that $(A_i, B_j) \in \mathfrak{B}$ [29] p.38 and [33].

Definition 2.3. Let (X, δ) be a quasi-proximity space, Y be a non-empty set and f a function from a quasi-proximity space (X, δ) onto a set Y. The quotient quasi-proximity δ^* on Y is defined as follows for every $A, B \subset Y$: $A\ \delta^*\ B$ if and only if, for each binary rational s in $[0, 1]$, there exist some $C_s \subset Y$ such that $C_0 = A$, $C_1 = B$ and if $s < t$, then $f^{-1}(C_s)\ \delta\ f^{-1}(C_t)$ [21] or [39] p.276.

Definition 2.4. We write $\triangle$ for the diagonal in X^2, where $X \in$ **Qprox**. For $X \in$ **Qprox** we define the wedge $X^2 V_{\triangle} X^2$, as the final structure, with respect to the map $X^2 \coprod X^2 \longrightarrow X^2 V_{\triangle} X^2$, that is the identification of the two copies of X^2 along the the diagonal $\triangle$. An epi sink $\{i_1, i_2 : (X^2, \delta) \longrightarrow (X^2 V_{\triangle} X^2, \delta')\}$, where i_1, i_2 are the canonical injections, in **Qprox** is a final lift if and only if the following statement holds. For each pair A, B in different component of $X^2 V_{\triangle} X^2$, $A\delta' B$ if and only if there are sets C, D and U in X^2 such that $C\delta U$ and $U\delta D$ with $i_k^{-1}(A) = C$ and $i_j^{-1}(B) = D$ for $k, j = 1, 2$ and $k \neq j$. If A and B are in the same component of wedge, then $A\delta' B$ if and only if there are sets C, D in X^2 such that $C\delta D$ and $i_k^{-1}(A) = C$ and $i_k^{-1}(B) = D$ for some $k = 1, 2$. Specially, if $i_k(E) = A$ and $i_k(F) = B$, then $(i_k(E), i_k(F)) \in \delta'$ if and only if $(i_k^{-1}(i_k(E)), i_k^{-1}(i_k(F))) = (E, F) \in \delta$. This is a special case Definition 2.3.

Definition 2.5. Let X be a non-empty set. The discrete (quasi-)proximity structure δ on X is given by for $A, B \subset X$, $A\delta B$ if and only if $A \cap B \neq \emptyset$, and the indiscrete (quasi-)proximity structure δ on X is given by for $A, B \subset X$, $A\delta B$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$ [30] p.9.

Definition 2.6. ([2]) Let B be a set so that $B^2 = B \times B$ and $B^3 = B \times B \times B$.

1) The principal axis map: A: $B^2V_\vartriangle B^2 \longrightarrow B^3$,

$$A((x,y)_i) = \begin{cases} (x,y,x), & \text{i=1} \\ (x,x,y), & \text{i=2} \end{cases}$$

2) The skewed axis map: $S : B^2V_\vartriangle B^2 \longrightarrow B^3$,

$$S((x,y)_i) = \begin{cases} (x,y,y), & \text{i=1} \\ (x,x,y), & \text{i=2} \end{cases}$$

3) The fold axis map: $\triangledown : B^2V_\vartriangle B^2 \longrightarrow B^2$, $\triangledown((x,y)_i) = (x,y)$, $i = 1,2$.

Definition 2.7. Let $U : \mathcal{E} \to$ **Set** be a topological functor, X an object in $\mathcal{E}$ and $U(X) = B$.

1. X is $\overline{T_0}$ if and only if the initial lift of the $\{A : B^2V_\vartriangle B^2 \longrightarrow U(X^3) = B^3$ and $\triangledown : B^2V_\vartriangle B^2 \longrightarrow UD(B^2) = B^2\}$ is discrete, where D is the discrete functor which is a left adjoint to U [2].

2. X is T'_0 if and only if the initial lift of the U-source $\{id : B^2V_\vartriangle B^2 \longrightarrow U(X^2V_\vartriangle X^2) = B^2V_\vartriangle B^2$ and $\triangledown : B^2V_\vartriangle B^2 \longrightarrow UD(B^2) = B^2\}$ is discrete, where $X^2V_\vartriangle X^2$ is the wedge in $\mathcal{E}$, i.e., the final lift of the U-sink $\{i_1, i_2 : U(X^2) = B^2 \to B^2V_\vartriangle B^2\}$ where i_1, i_2 denote the canonical injections [2].

3. X is T_1 if and only if the initial lift of the U-source $\{S : B^2V_\vartriangle B^2 \longrightarrow U(X^3) = B^3$ and $\triangledown : B^2V_\vartriangle B^2 \longrightarrow UD(B^2) = B^2\}$ is discrete [2].

4. X is T_0 if and only if X must not contain an indiscrete subspace with at least two elements [38], [28].

3. T_0 and T_1 quasi-proximity spaces in general and relationships

Theorem 3.1. *Let (X,δ) be a quasi-proximity space. (X,δ) is T_0 if and only if for each $x \neq y$ in X, $(\{x\},\{y\}) \notin \delta$ or $(\{y\},\{x\}) \notin \delta$.*

Proof. Let (X,δ) be T_0 space. We must show that for each $x \neq y$ in X, $(\{x\},\{y\}) \notin \delta$ or $(\{y\},\{x\}) \notin \delta$. Suppose that for some $x \neq y$ in X, $(\{x\},\{y\}) \in \delta$ and $(\{y\},\{x\}) \in \delta$. We consider that $A = \{x,y\}$. (A,δ_A) is subspace of (X,δ) with δ_A is subquasi-proximity structure generated by $i : A \longrightarrow X$ inclusion map on A. Since $(i\times i)(\{x\},\{y\}) = (\{x\},\{y\}) \in \delta$ $(\{x,y\},\{x,y\}) = A\times A \in \delta_A$ or $(i\times i)(\{y\},\{x\}) = (\{y\},\{x\}) \in \delta$ $(\{y,x\},\{y,x\}) = A\times A \in \delta_A$ by condition (Q4) of definition 2.1. Consequently this is a contradiction since δ_A is indiscrete quasi-proximity structure on $A\times A$. For this reason for each $x \neq y$ in X, $(\{x\},\{y\}) \notin \delta$ or $(\{y\},\{x\}) \notin \delta$.

Conversely for each $x \neq y$ in X, suppose that be $(\{x\},\{y\}) \notin \delta$ or $(\{y\},\{x\}) \notin \delta$. We consider that $A = \{x,y\} \subset X$ for $x \neq y$. Note that (A,δ_A) is not indiscrete subspace of (X,δ). Therefore (X,δ) is T_0 by Definition 2.7. □

Theorem 3.2. *Let (X,δ) be a quasi-proximity space. (X,δ) is $\overline{T_0}$ if and only if for each $x \neq y$ in X, $(\{x\},\{y\}) \notin \delta$ or $(\{y\},\{x\}) \notin \delta$.*

Proof. Let (X,δ) is $\overline{T_0}$, i.e., by Definition 2.5, Definition 2.2 and Definition 2.7 for any pair U,V on $X^2V_\vartriangle X^2$. $\pi_{11}U\delta\pi_{11}V$, $\pi_{21}U\delta\pi_{21}V$, $\pi_{12}U\delta\pi_{12}V$ and $\triangledown U\delta^2_d\triangledown V$ if and only if $U\cap V \neq \emptyset$ (δ^2_d is the discrete quasi-proximity structure on X^2). We must show that the condition holds. Assume that $(\{x\},\{y\}) \in \delta$ and $(\{y\},\{x\}) \in \delta$ such that there is a pair $x,y \in \delta$. Then by Definition 2.2, Definition 2.5 and Definition 2.7 for $(U,V) \in \delta'$ (δ' is a

quasi-proximity structure on $X^2 V_\vartriangle X^2$) with $U = (x, y)_1$ and $V = (x, y)_2$,
$\pi_{11} U \delta \pi_{11} V = \pi_1 A\{(x, y)_1\} \delta \pi_1 A\{(x, y)_2\} = \pi_1\{(x, y, x)\} \delta \pi_1\{(x, x, y)\} = \{x\}\delta\{x\}$

$$\Rightarrow (\{x\}, \{x\}) \in \delta,$$

$\pi_{21} U \delta \pi_{21} V = \pi_2 A\{(x, y)_1\} \delta \pi_2 A\{(x, y)_2\} = \pi_2\{(x, y, x)\} \delta \pi_2\{(x, x, y)\} = \{y\}\delta\{x\}$

$$\Rightarrow (\{y\}, \{x\}) \in \delta,$$

$\pi_{12} U \delta \pi_{12} V = \pi_3 A\{(x, y)_1\} \delta \pi_3 A\{(x, y)_2\} = \pi_3\{(x, y, x)\} \delta \pi_3\{(x, x, y)\} = \{x\}\delta\{y\}$

$$\Rightarrow (\{x\}, \{y\}) \in \delta,$$

where $\pi_i : X^3 \longrightarrow X, i = 1, 2, 3$ are projection maps and $\nabla\{(x, y)_1\}\delta^2_d \nabla\{(x, y)_2\} = \{(x, y)\}\delta^2_d \nabla\{(x, y)\}$, i.e., $(\{(x, y)\}, \{(x, y)\}) \in \delta^2_d$. Similar results are obtained for $\pi_{11} V \delta \pi_{11} U$, $\pi_{21} V \delta \pi_{21} U$, $\pi_{12} V \delta \pi_{12} U$ and $\nabla V \delta^2_d \nabla U$ (δ^2_d is the discrete quasi-proximity structure on X^2). But $U \cap V = \emptyset$. This is a contradiction to the fact that (X, δ) is $\overline{T_0}$. Therefore if $(\{x\}, \{y\}) \in \delta$ and $(\{y\}, \{x\}) \in \delta$ then $x = y$.
Conversely, assume that for each $x \neq y$, $(\{x\}, \{y\}) \notin \delta$ or $(\{y\}, \{x\}) \notin \delta$. We need to show that (X, δ) is $\overline{T_0}$, i.e., by Definition 2.2, Definition 2.5 and Definition 2.7, we must show that the quasi-proximity structure δ' on $X^2 V_\vartriangle X^2$ induced by $A : X^2 V_\vartriangle X^2 \longrightarrow U((X^3, \delta^3)) = X^3$ and $\nabla : X^2 V_\vartriangle X^2 \longrightarrow U((X^2, \delta^2_d)) = X^2$ is discrete, where δ^3 and δ^2_d are the product quasi-proximity structure on X^3 and the dicrete quasi-proximity on X^2, respectively. Let (U, V) be any set in δ', i.e., $\pi_i A(U) \delta \pi_i A(V) (i = 1, 2, 3)$ and $\nabla U \delta^2_d \nabla V$. Since δ^2_d is discrete quasi-proximity structure and $\nabla U \delta^2_d \nabla V$, then $\nabla U \cap \nabla V \neq \emptyset$. From here there is $(x, y) \in \nabla U \cap \nabla V$. So there are $t \in U$ and $z \in V$ such that $\nabla t = (x, y) = \nabla z$. If $x = y$ then $t = (x, y)_i = z, (i = 1, 2)$ and $(x, y)_i \in U \cap V$.

If $x \neq y$ then $t = (x, y)_i$, $z = (x, y)_j$ $(i, j = 1, 2)$. We must show that $U \cap V \neq \emptyset$, i.e., U and V are in the first or in the second or in both component of $X^2 V_\vartriangle X^2$.

If U subset of the first component of $X^2 V_\vartriangle X^2$ and V subset of the second component of $X^2 V_\vartriangle X^2$, then $\{(x, y)_1\} \subseteq U$ and $\{(x, y)_2\} \subseteq V$. From here,

$$\pi_{12} U \delta \pi_{12} V = \pi_3 A\{(x, y)_1\} \delta \pi_3 A\{(x, y)_2\} = \pi_3 A\{(x, y, x)\} \delta\ \pi_3 A\{(x, x, y)\} = \{x\}\delta\{y\}$$

$$\Rightarrow (\{x\}, \{y\}) \in \delta$$

and,

$$\pi_{21} U \delta \pi_{21} V = \pi_2 A\{(x, y)_1\} \delta \pi_2 A\{(x, y)_2\} = \pi_2\{(x, y, x)\} \delta \pi_2\{(x, x, y)\} = \{y\}\delta\{x\}$$

$$\Rightarrow (\{y\}, \{x\}) \in \delta.$$

It follows that $(\{x\}, \{y\}) \in \delta$ and $(\{y\}, \{x\}) \in \delta$. Since $(\{x\}, \{y\}) \notin \delta$ or $(\{y\}, \{x\}) \notin \delta$,(by assumption),

$$(\{(x, y)_1\}, \{(x, y)_2\}) \notin \delta'$$

by condition (Q3) of Definition 2.1.

The case U subset of the second component of $X^2 V_\vartriangle X^2$ and V subset of the first component of $X^2 V_\vartriangle X^2$ can be obtained similarly. From here U and V can not be different component of $X^2 V_\vartriangle X^2$.

If U and V are in both component of $X^2 V_\vartriangle X^2$ then $U \supseteq (\{(x, y)_1\}, \{(x, y)_2\})$ and $V \supseteq (\{(x, y)_1\}, \{(x, y)_2\})$. From here $U \cap V \neq \emptyset$.

If U subset of the first component of $X^2 V_\vartriangle X^2$ and V subset of both component of $X^2 V_\vartriangle X^2$, then $\{(x, y)_1\} \subseteq U$ and $V \supseteq (\{(x, y)_1\}, \{(x, y)_2\})$. From here $U \cap V \neq \emptyset$.

If U subset of both component of $X^2 V_\vartriangle X^2$ and then $U \supseteq (\{(x, y)_1\}, \{(x, y)_2\})$ and V subset of the second component of $X^2 V_\vartriangle X^2$, then $U \supseteq (\{(x, y)_1\}, \{(x, y)_2\})$ and $\{(x, y)_2\} \subseteq V$. From here $U \cap V \neq \emptyset$.

If U and V are in first component of $X^2 V_\vartriangle X^2$, then $U \supseteq (\{(x, y)_1\}$ and $V \supseteq (\{(x, y)_1\})$. From here $U \cap V \neq \emptyset$. Similarly if U and V are in the second component of $X^2 V_\vartriangle X^2$, then $U \supseteq (\{(x, y)_2\}$ and $V \supseteq (\{(x, y)_2\})$. From here $U \cap V \neq \emptyset$.

If $(\{(x,y)_i\},\{(x,y)_i\}) \in \delta', (i=1,2)$, then

$$\pi_1 A\{(x,y)_1\}\delta\ \pi_1 A\{(x,y)_1\} = \{x\}\delta\{x\} \Rightarrow (\{x\},\{x\}) \in \delta,$$

$$\pi_2 A\{(x,y)_1\}\delta\ \pi_2 A\{(x,y)_1\} = \{y\}\delta\{y\} \Rightarrow (\{y\},\{y\}) \in \delta,$$

$$\pi_3 A\{(x,y)_1\}\delta\ \pi_3 A\{(x,y)_1\} = \{x\}\delta\{x\} \Rightarrow (\{x\},\{x\}) \in \delta,$$

and

$$\pi_1 A\{(x,y)_2\}\delta\pi_1 A\{(x,y)_2\} = \{x\}\delta\{x\} \Rightarrow (\{x\},\{x\}) \in \delta,$$

$$\pi_2 A\{(x,y)_2\}\delta\ \pi_2 A\{(x,y)_2\} = \{x\}\delta\{x\} \Rightarrow (\{x\},\{x\}) \in \delta,$$

$$\pi_3 A\{(x,y)_2\}\delta\ \pi_3 A\{(x,y)_2\} = \{y\}\delta\{y\} \Rightarrow (\{y\},\{y\}) \in \delta.$$

We must have $(U,V) \supseteq (\{(x,y)_i\},\{(x,y)_i\}), (i=1,2)$, i.e., $U \cap V \neq \emptyset$. Consequently, by Definition 2.2, Definition 2.5 and Definition 2.7, (X,δ) is $\overline{T_0}$. □

Theorem 3.3. *All the quasi-proximity spaces are T_0'.*

Proof. Let (X,δ) is T_0', i.e., by Definition 2.2, Definition 2.4, Definition 2.5 and Definition 2.7, we must show that for any $(i_k(E), i_k(F)) \in \delta'$ (δ' is a quasi-proximity structure on $X^2 V_\vartriangle X^2$), if $i_k(E,F) = (i_k(E), i_k(F)) \in \delta'(k=1,2)$ for some $(E,F) \in \delta^2$ ($E,F \subset X^2$ and δ^2 is the product quasi-proximity structure on X^2) and $(\nabla(i_k(E)), \nabla(i_k(F)) \in \delta^2_d$ (δ^2_d is the dicrete quasi-proximity structure on X^2), then we will show that $k=1,2$, i.e., $i_k(E) \cap i_k(F) \neq \emptyset$.

By reason of δ^2_d is the discrete quasi-proximity structure and $\nabla(i_k(E))\delta^2_d \nabla\ (i_k(F))$, $\nabla(i_k(E)) \cap \nabla(i_k(F)) \neq \emptyset$. From here, there are $(x,y) \in \nabla(i_k(E)) \cap \nabla(i_k(F))$. For this reason, there are $t \in i_k(E)$ and $z \in i_k(F)$ such that $\nabla t = (x,y) = \nabla z$. If $x=y$, then $t=(x,y)_k, z=(x,y)_n (k,n=1,2)$. We must show that $i_k(E) \cap i_k(F) \neq \emptyset$, i.e., $i_k(E)$ and $i_k(F)$ are in the first or in the second or in both component of $X^2 V_\vartriangle X^2$.

If $i_k(E)$ subset of the first component of $X^2 V_\vartriangle X^2$ and $i_k(F)$ subset of the second component of $X^2 V_\vartriangle X^2$, then $\{(x,y)_1\} \subseteq i_k(E)$ and $\{(x,y)_2\} \subseteq i_k(F)$. But, if $(i_k(E), i_k(F)) \supseteq (\{(x,y)_1\},\{(x,y)_2\}) \in \delta'$ for some $(E,F) \in \delta^2$ and $k=1$(resp. $k=2$), then $(\{(x,y)_1\},\{(x,y)_2\}) \in (i_1(E), i_1(F))$ which shows that $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $X^2 V_\vartriangle X^2$, a contradiction because of $x \neq y$.

Similarly, if $i_k(E)$ subset of the second component of $X^2 V_\vartriangle X^2$ and $i_k(F)$ subset of the first component of $X^2 V_\vartriangle X^2$, then $\{(x,y)_2\} \subseteq i_k(E)$ and $\{(x,y)_1\} \subseteq i_k(F)$. But, if $(i_k(E), i_k(F)) \supseteq (\{(x,y)_2\},\{(x,y)_1\}) \in \delta'$ for some $(E,F) \in \delta^2$ and $k=1$(resp. $k=2$), then $(\{(x,y)_2\},\{(x,y)_1\}) \in (i_1(E), i_1(F))$ which shows that $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $X^2 V_\vartriangle X^2$, a contradiction because of $x \neq y$. For this reason $i_k(E)$ and $i_k(F)$ are not be able to in different component of $X^2 V_\vartriangle X^2$.

If $i_k(E)$ and $i_k(F)$ are in both component of $X^2 V_\vartriangle X^2$, then

$$i_k(E) \supseteq (\{(x,y)_1\},\{(x,y)_2\}) \text{ and } i_k(F) \supseteq (\{(x,y)_1\},\{(x,y)_2\}).$$

From here $i_k(E) \cap i_k(F) \neq \emptyset$.

If $i_k(E)$ subset of the first component of $X^2 V_\vartriangle X^2$ and $i_k(F)$ subset of both component of $X^2 V_\vartriangle X^2$, then $\{(x,y)_1\} \subseteq i_k(E)$ and $i_k(F) \supseteq (\{(x,y)_1\},\{(x,y)_2\})$. From here $i_k(E) \cap i_k(F) \neq \emptyset$.

If $i_k(E)$ subset of both component of $X^2 V_\vartriangle X^2$ and $i_k(F)$ subset of the second component of $X^2 V_\vartriangle X^2$, then $(\{(x,y)_1\},\{(x,y)_2\}) \subseteq i_k(E)$ and $\{(x,y)_2\} \subseteq i_k(F)$. From here $i_k(E) \cap i_k(F) \neq \emptyset$.

If $i_k(E)$ and $i_k(F)$ are in the first component of $X^2 V_\vartriangle X^2$, then $i_k(E) \supseteq \{(x,y)_1\}$ and $i_k(F) \supseteq \{(x,y)_1\}$. From here $i_k(E) \cap i_k(F) \neq \emptyset$. Similarly, if $i_k(E)$ and $i_k(F)$ are in the second component of $X^2 V_\vartriangle X^2$, then $i_k(E) \cap i_k(F) \neq \emptyset$.

We must have $(i_k(E), i_k(F)) \supseteq (\{(x,y)_i\},\{(x,y)_i\})$, $(i=1,2)$, i.e., $i_k(E) \cap i_k(F) \neq \emptyset$. At the same time, for $(i_k(E), i_k(F)) \supseteq (\{(y,x)_i\},\{(y,x)_i\})$ $(i=1,2)$, similar results are obtained. Consequently by Definition 2.2, Definition 2.4, Definition 2.5 and Definition 2.7, (X,δ) is T_0'. □

Theorem 3.4. *Let (X,δ) be a quasi-proximity space. (X,δ) is T_1 if and only if for each $x \neq y$ in X, $(\{x\},\{y\}) \notin \delta$ or $(\{y\},\{x\}) \notin \delta$.*

Proof. Assume that, (X, δ) is T_1, i.e., Definition 2.2, Definition 2.5 and Definition 2.7, for any sets U, V on $X^2 V_\vartriangle X^2$. $\pi_{11} U \delta \pi_{11} V$, $\pi_{21} U \delta \pi_{21} V$, $\pi_{22} U \delta \pi_{22} V$ and $\triangledown U \delta^2_d \triangledown V$ if and only if $U \cap V \neq \emptyset$ (δ^2_d is the discrete quasi-proximity structure on X^2).

We should show that the condition holds. Assume that for some $x, y \in X$, $(\{x\}, \{y\}) \in \delta$ and $(\{y\}, \{x\}) \in \delta$ with $x \neq y$. Then, by Definition 2.2, Definition 2.5 and Definition 2.7, for $(U, V) \in \delta'$ (δ' is a quasi-proximity structure on $X^2 V_\vartriangle X^2$) with $U = \{(x, y)_1\}$ and $V = \{(x, y)_2\}$,

$$\pi_{11} U \delta \pi_{11} V = \pi_1 S\{(x, y)_1\} \delta \pi_1 S\{(x, y)_2\} = \pi_1\{(x, y, y)\} \delta \pi_1 \{(x, x, y)\} = \{x\} \delta \{x\},$$

i.e., $(\{x\}, \{x\}) \in \delta$,

$$\pi_{21} U \delta\ \pi_{21} V = \pi_2 S\{(x, y)_1\} \delta\ \pi_2 S\{(x, y)_2\} = \pi_2\{(x, y, y)\} \delta\ \pi_2\{(x, x, y)\} = \{y\} \delta \{x\},$$

i.e., $(\{y\}, \{x\}) \in \delta$,

$$\pi_{22} U \delta\ \pi_{22} V = \pi_3 S\{(x, y)_1\} \delta\ \pi_3 S\{(x, y)_2\} = \pi_3\{(x, y, y)\} \delta \pi_3 \{(x, x, y)\} = \{y\} \delta \{y\},$$

i.e., $(\{y\}, \{y\}) \in \delta$ where $\pi_i : X^3 \longrightarrow X, i = 1, 2, 3$ are projection maps and $\triangledown\{(x, y)_1\} \delta^2_d \triangledown \{(x, y)_2\} = \{(x, y)\} \delta^2_d \triangledown \{(x, y)\}$, i.e., $(\{(x, y)\}, \{(x, y)\}) \in \delta^2_d$ (δ^2_d is the discrete quasi-proximity structure on X^2).

For $\pi_{11} V \delta \pi_{11} U, \pi_{21} V \delta\ \pi_{21} U, \pi_{22} V \delta\ \pi_{22} U$ and $\triangledown V \delta^2_d \triangledown U$;

$$\pi_{11} V \delta \pi_{11} U = \pi_1 S\{(x, y)_2\} \delta \pi_1 S\{(x, y)_1\} = \pi_1\{(x, x, y)\} \delta \pi_1 \{(x, y, y)\} = \{x\} \delta \{x\},$$

i.e., $(\{x\}, \{x\}) \in \delta$,

$$\pi_{21} V \delta\ \pi_{21} U = \pi_2 S\{(x, y)_2\} \delta\ \pi_2 S\{(x, y)_1\} = \pi_2\{(x, x, y)\} \delta\ \pi_2\{(x, y, y)\} = \{x\} \delta \{y\},$$

i.e., $(\{x\}, \{y\}) \in \delta$,

$$\pi_{22} V \delta\ \pi_{22} U = \pi_3 S\{(x, y)_2\} \delta\ \pi_3 S\{(x, y)_1\} = \pi_3\{(x, x, y)\} \delta \pi_3 \{(x, y, y)\} = \{y\} \delta \{y\},$$

i.e., $(\{y\}, \{y\}) \in \delta$ where $\pi_i : X^3 \longrightarrow X, i = 1, 2, 3$ are projection maps and $\triangledown\{(x, y)_2\} \delta^2_d \triangledown \{(x, y)_1\} = \{(x, y)\} \delta^2_d \triangledown \{(x, y)\}$, i.e., $(\{(x, y)\}, \{(x, y)\}) \in \delta^2_d$.

But $U \cap V = \emptyset$. This is a contradiction to the fact that (X, δ) is T_1. So that if $(\{x\}, \{y\}) \in \delta$ and $(\{y\}, \{x\}) \in \delta$ then $x = y$.

Conversely, assume that for each $x \neq y$, $(\{x\}, \{y\}) \notin \delta$ or $(\{y\}, \{x\}) \notin \delta$. We need to show that (X, δ) is T_1, i.e., by Definition 2.5, Definition 2.2 and Definition 2.7, we must show that the quasi-proximity structure δ' on $X^2 V_\vartriangle X^2$ induced by $S : X^2 V_\vartriangle X^2 \longrightarrow U((X^3, \delta^3)) = X^3$ and $\triangledown : X^2 V_\vartriangle X^2 \longrightarrow U((X^2, \delta^2_d)) = X^2$ is discrete, where δ^3 and δ^2_d are the product quasi-proximity structure on X^3 and the dicrete quasi-proximity on X^2, respectively. Let (U, V) be any set in δ', i.e., $\pi_i S(U) \delta \pi_i S(V) (i = 1, 2, 3)$ and $\triangledown U \delta^2_d \triangledown V$.

Since δ^2_d is discrete quasi-proximity structure and $\triangledown U \delta^2_d \triangledown V$, then $\triangledown U \cap \triangledown V \neq \emptyset$. From here there is $(x, y) \in \triangledown U \cap \triangledown V$. So there are $t \in U$ and $z \in V$ such that $\triangledown t = (x, y) = \triangledown z$. If $x = y$ then $t = (x, y)_i = z, (i = 1, 2)$ and $(x, y)_i \in U \cap V$.

If $x \neq y$ then $t = (x, y)_i$, $z = (x, y)_j$ $(i, j = 1, 2)$. We must show that $U \cap V \neq \emptyset$, i.e., U and V are in the first or in the second or in both component of $X^2 V_\vartriangle X^2$.

If U subset of the first component of $X^2 V_\vartriangle X^2$ and V subset of the second component of $X^2 V_\vartriangle X^2$, then $\{(x, y)_1\} \subseteq U$ and $\{(x, y)_2\} \subseteq V$. From here

$$\pi_{21} U \delta\ \pi_{21} V = \pi_2 S\{(x, y)_1\} \delta \pi_2 S\{(x, y)_2\} = \pi_2 S\{(x, y, y)\} \delta \pi_2 S\{(x, x, y)\} = \{y\} \delta \{x\},$$

$$\pi_{21} V \delta\ \pi_{21} U = \pi_2 S\{(x, y)_2\} \delta\ \pi_2 S\{(x, y)_1\} = \pi_2\{(x, x, y)\} \delta\ \pi_2\{(x, y, y)\} = \{x\} \delta \{y\},$$

i.e., $(\{x\}, \{y\}) \in \delta$ and $(\{y\}, \{x\}) \in \delta$.

Since $(\{y\}, \{x\}) \notin \delta$ or $(\{x\}, \{y\}) \notin \delta$ (by assumption), $(\{(x, y)_1\}, \{(x, y)_2\}) \notin \delta'$ by condition (Q3) of Definition 2.1.

The case U subset of the second component of $X^2V_\triangle X^2$ and V subset of the first component of $X^2V_\triangle X^2$ can be obtained similarly. From here U and V can not be different component of $X^2V_\triangle X^2$.

If U and V are in both component of $X^2V_\triangle X^2$ then $U \supseteq (\{(x, y)_1\}, \{(x, y)_2\})$ and $V \supseteq (\{(x, y)_1\}, \{(x, y)_2\})$. From here $U \cap V \neq \emptyset$.

If U subset of the first component of $X^2V_\triangle X^2$ and V subset of both component of $X^2V_\triangle X^2$, then $U \supseteq \{(x, y)_1\}$ and $V \supseteq (\{(x, y)_1\}, \{(x, y)_2\})$. From here $U \cap V \neq \emptyset$.

If U subset of both component of $X^2V_\triangle X^2$ and then $U \supseteq (\{(x, y)_1\}, \{(x, y)_2\})$ and V subset of the second component of $X^2V_\triangle X^2$, then $U \supseteq (\{(x, y)_1\}, \{(x, y)_2\})$ and $V \supseteq \{(x, y)_2\}$. From here $U \cap V \neq \emptyset$.

If U and V are in the first component of $X^2V_\triangle X^2$, then $U \supseteq (\{(x, y)_1\}$ and $V \supseteq (\{(x, y)_1\})$. From here $U \cap V \neq \emptyset$. Similarly if U and V are in the second component of $X^2V_\triangle X^2$, then $U \supseteq (\{(x, y)_2\}$ and $V \supseteq (\{(x, y)_2\})$. From here $U \cap V \neq \emptyset$.

If $(\{(x, y)_i\}, \{(x, y)_i\}) \in \delta', (i = 1, 2)$, then

$$\pi_1 S\{(x, y)_1\}\delta\ \pi_1 S\{(x, y)_1\} = \{x\}\delta\{x\} \Rightarrow (\{x\}, \{x\}) \in \delta,$$

$$\pi_2 S\{(x, y)_1\}\delta\ \pi_2 S\{(x, y)_1\} = \{y\}\delta\{y\} \Rightarrow (\{y\}, \{y\}) \in \delta,$$

$$\pi_3 S\{(x, y)_1\}\delta\ \pi_3 S\{(x, y)_1\} = \{y\}\delta\{y\} \Rightarrow (\{y\}, \{y\}) \in \delta,$$

and

$$\pi_1 S\{(x, y)_2\}\delta\ \pi_1 S\{(x, y)_2\} = \{x\}\delta\{x\} \Rightarrow (\{x\}, \{x\}) \in \delta,$$

$$\pi_2 S\{(x, y)_2\}\delta\ \pi_2 S\{(x, y)_2\} = \{x\}\delta\{x\} \Rightarrow (\{x\}, \{x\}) \in \delta,$$

$$\pi_3 S\{(x, y)_2\}\delta\ \pi_3 S\{(x, y)_2\} = \{y\}\delta\{y\} \Rightarrow (\{y\}, \{y\}) \in \delta.$$

We must have $(U, V) \supseteq (\{(x, y)_i\}, \{(x, y)_i\}), (i = 1, 2)$, i.e., $U \cap V \neq \emptyset$. Similar results are obtained for $\pi_i S(V)\delta\pi_i S(U)(i = 1, 2, 3)$ and $\triangledown V \delta_d^2 \triangledown U$. Consequently, by Definition 2.2, Definition 2.5 and Definition 2.7, (X, δ) is T_1. □

Remark 3.5. Let (X, δ) be a quasi-proximity space. The following expressions are equivalent by Theorems 3.1, 3.2 and 3.4.

1. (X, δ) is T_0,
2. (X, δ) is $\overline{T_0}$,
3. (X, δ) is T_1,
4. For each $x, y \in X$ with $x \neq y$ $(\{x\}, \{y\}) \notin \delta$ or $(\{y\}, \{x\}) \notin \delta$.

4. Pre-Hausdorff and Hausdorff quasi-proximity spaces

In this section, the characterizations of separation axioms Pre$\overline{T_2}$, PreT_2', $\overline{T_2}$ and T_2' in **QProx** which is the category of quasi-proximity spaces are given.

Definition 4.1. ([2], [7]) Let $U : \mathcal{E} \to$ **Set** be a topological functor, X be an object in $\mathcal{E}$ and $U(X) = B$.

1. X is Pre$\overline{T_2}$ if and only if the initial lift of the U-source $\{A : B^2V_\triangle B^2 \longrightarrow U(X^3) = B^3$ and $\{S : B^2V_\triangle B^2 \longrightarrow U(X^3) = B^3$ agree.
2. X is PreT_2' if and only if the initial lift of the U-source $\{S : B^2V_\triangle B^2 \longrightarrow U(X^3) = B^3$ and the final lift of the U-sink $\{i_1, i_2 : U(X^2) = B^2 \to B^2V_\triangle B^2\}$ agree.

3. X is $\overline{T_2}$ if and only if the X is $\overline{T_0}$ and Pre$\overline{T_2}$.
4. X is T_2' if and only if the X is T_0' and PreT_2'.

Theorem 4.2. *Let (X, δ) be a quasi-proximity space.(X, δ) is Pre$\overline{T_2}$ if and only if the δ quasi-proximity structure must satisfy the symmetry condition, i.e. for every $A, B \subseteq X$, $A\delta B$ if and only if $B\delta A$.*

Proof. Let (X, δ) quasi-proximity space is Pre$\overline{T_2}$. Assume that the symmetry condition is not hold. Since (X, δ) is Pre$\overline{T_2}$, by Definition 2.2 and Definition 2.7 , for any pair U and V in the $X^2 V_\triangle X^2$ wedge product, $\pi_1 A(U)\ \delta\ \pi_1 A(V)$, $\pi_2 A(U)\ \delta\ \pi_2 A(V)$ and $\pi_3 A(U)\ \delta\ \pi_3 A(V)$ if and only if $\pi_1 S(U)\ \delta\ \pi_1 S(V)$, $\pi_2 S(U)\ \delta\ \pi_2 S(V)$ and $\pi_3 S(U)\ \delta\ \pi_3 S(V)$, respectively.

We consider different possibilities for U and V, i.e., $U \supseteq \{(x, y)_1\}, \{(x, y)_2\}$ or $(\{(x, y)_1, (x, y)_2\})$ and $V \supseteq \{(x, y)_1\}, \{(x, y)_2\}$ or $(\{(x, y)_1, (x, y)_2\})$ for some $x, y \in X$. By the condition (Q3) of Definition 2.1. That's enough to write "equality" in place of "superset" for the possibilities above.

If $U = \{(x, y)_1\}$ and $V = \{(x, y)_1\}$, then $\pi_1 A(U)\ \delta\ \pi_1 A(V) = \{x\}\delta\{x\} = \pi_1 S(U)\ \delta\ \pi_1 S(V)$, $\pi_2 A(U)\ \delta\ \pi_2 A(V) = \{y\}\delta\{y\} = \pi_2 S(U)\ \delta\ \pi_2 S(V)$ and $\pi_3 A(U)\ \delta\ \pi_3 A(V) = \{x\}\delta\{x\}$ if and only if $\pi_3 S(U)\ \delta\ \pi_3 S(V) = \{y\}\delta\{y\}$.

If $U = \{(x, y)_1\}$ and $V = \{(x, y)_2\}$, then $\pi_1 A(U)\ \delta\ \pi_1 A(V) = \{x\}\delta\{x\} = \pi_1 S(U)\ \delta\ \pi_1 S(V)$, $\pi_2 A(U)\ \delta\ \pi_2 A(V) = \{y\}\delta\{x\} = \pi_2 S(U)\ \delta\ \pi_2 S(V)$. Note that $\pi_3 A(U)\ \delta\ \pi_3 A(V) = \{x\}\delta\{y\}$ if and only if $\pi_3 S(U)\ \delta\ \pi_3 S(V) = \{y\}\delta\{y\}$,

If $U = \{(x, y)_1\}$ and $V = (\{(x, y)_1, (x, y)_2\})$, then $\pi_1 A(U)\ \delta\ \pi_1 A(V) = \{x\}\delta\{x\} = \pi_1 S(U)\ \delta\ \pi_1 S(V)$, $\pi_2 A(U)\ \delta\ \pi_2 A(V) = \{y\}\delta\{x\} = \pi_2 S(U)\ \delta\ \pi_2 S(V)$, clearly, $\pi_3 A(U)\ \delta\ \pi_3 A(V) = \{x\}\delta\{x\}$ if and only if $\pi_3 S(U)\ \delta\ \pi_3 S(V) = \{y\}\delta\{x\}$.

Similarly, if $U = \{(x, y)_2\}$ or $(\{(x, y)_1, (x, y)_2\})$, $V = \{(x, y)_1\}, \{(x, y)_2\}$ or $(\{(x, y)_1, (x, y)_2\})$ then we have $\pi_1 A(U)\ \delta\ \pi_1 A(V)$, $\pi_2 A(U)\ \delta\ \pi_2 A(V)$ and $\pi_3 A(U)\ \delta\ \pi_3 A(V)$ if and only if $\pi_1 S(U)\ \delta\ \pi_1 S(V)$, $\pi_2 S(U)\ \delta\ \pi_2 S(V)$ and $\pi_3 S(U)\ \delta\ \pi_3 S(V)$, respectively.

Now, we investigate for $\pi_1 A(V)\ \delta\ \pi_1 A(U)$, $\pi_2 A(V)\ \delta\ \pi_2 A(U)$ and $\pi_3 A(V)\ \delta\ \pi_3 A(U)$ if and only if $\pi_1 S(V)\ \delta\ \pi_1 S(U)$, $\pi_2 S(V)\ \delta\ \pi_2 S(U)$ and $\pi_3 S(V)\ \delta\ \pi_3 S(U)$, respectively.

Similarly, we consider different possibilities for U and V, i.e., $U \supseteq \{(x, y)_1\}, \{(x, y)_2\}$ or $(\{(x, y)_1, (x, y)_2\})$ and $V \supseteq \{(x, y)_1\}, \{(x, y)_2\}$ or $(\{(x, y)_1, (x, y)_2\})$ for some $x, y \in X$. By the condition (Q3) of Definition 2.1. That's enough to write "equality" in place of "superset" for the possibilities above.

If $U = \{(x, y)_1\}$ and $V = \{(x, y)_1\}$, then $\pi_1 A(V)\ \delta\ \pi_1 A(U) = \{x\}\delta\{x\} = \pi_1 S(V)\ \delta\ \pi_1 S(U)$, $\pi_2 A(V)\ \delta\ \pi_2 A(U) = \{y\}\delta\{y\} = \pi_2 S(V)\ \delta\ \pi_2 S(U)$ and $\pi_3 A(V)\ \delta\ \pi_3 A(U) = \{x\}\delta\{x\}$ if and only if $\pi_3 S(V)\ \delta\ \pi_3 S(U) = \{y\}\delta\{y\}$.

If $U = \{(x, y)_1\}$ and $V = \{(x, y)_2\}$, then $\pi_1 A(V)\ \delta\ \pi_1 A(U) = \{x\}\delta\{x\} = \pi_1 S(V)\ \delta\ \pi_1 S(U)$, $\pi_2 A(V)\ \delta\ \pi_2 A(U) = \{x\}\delta\{y\} = \pi_2 S(V)\ \delta\ \pi_2 S(U)$. Note that $\pi_3 A(V)\ \delta\ \pi_3 A(U) = \{y\}\delta\{x\}$ if and only if $\pi_3 S(V)\ \delta\ \pi_3 S(U) = \{y\}\delta\{y\}$.

If $U = \{(x, y)_1\}$ and $V = (\{(x, y)_1, (x, y)_2\})$, then $\pi_1 A(V)\ \delta\ \pi_1 A(U) = \{x\}\delta\{x\} = \pi_1 S(V)\ \delta\ \pi_1 S(U)$, $\pi_2 A(V)\ \delta\ \pi_2 A(U) = \{x\}\delta\{y\} = \pi_2 S(V)\ \delta\ \pi_2 S(U)$, clearly, $\pi_3 A(V)\ \delta\ \pi_3 A(U) = \{x\}\delta\{x\}$ if and only if $\pi_3 S(V)\ \delta\ \pi_3 S(U) = \{x\}\delta\{y\}$.

Similarly, if $U = \{(x, y)_2\}$ or $(\{(x, y)_1, (x, y)_2\})$, $V = \{(x, y)_1\}, \{(x, y)_2\}$ or $(\{(x, y)_1, (x, y)_2\})$ then we have $\pi_1 A(V)\ \delta\ \pi_1 A(U)$, $\pi_2 A(V)\ \delta\ \pi_2 A(U)$ and $\pi_3 A(V)\ \delta\ \pi_3 A(U)$ if and only if $\pi_1 S(V)\ \delta\ \pi_1 S(U)$, $\pi_2 S(V)\ \delta\ \pi_2 S(U)$ and $\pi_3 S(V)\ \delta\ \pi_3 S(U)$, respectively. Consequently, since the symmetry condition is hold, there is a contradiction.

Now, assume that (X, δ) is hold the symmetry condition. In this situation, (X, δ) becomes proximity space. Here from, it is seen that (X, δ) is Pre$\overline{T_2}$ by Theorem 3.7 in [24] .

□

Theorem 4.3. *Let (X, δ) be a quasi-proximity space. (X, δ) is PreT_2' if and only if for each $x \neq y$, $(\{x\}, \{y\}) \notin \delta$ and $(\{y\}, \{x\}) \notin \delta$.*

Proof. Assume that (X, δ) is PreT_2', i.e., by Definition 2.2, Definition 2.4 and Definition 2.7, for any sets U, V on $X^2 V_\triangle X^2$ wedge product,

(a) $\pi_{11}U\,\delta\,\pi_{11}V$, $\pi_{21}U\,\delta\,\pi_{21}V$ and $\pi_{22}U\,\delta\,\pi_{22}V$ if and only if

(b) there is a pair $(a,b),(c,d) \in X^2$ such that $\{(a,b)\}\ \delta^2\ \{(c,d)\}$ and $i_k\{(a,b)\} = U$ and $i_k\{(c,d)\} = V$ for some $k = 1$ or $k = 2$, where δ^2 is the product quasi-proximity structure on X^2.

For each pair U,V in the different component of $X^2V_\vartriangle X^2$ wedge product, $U\delta' V$ if and only if there are sets C,D and A in X^2 such that $C\ \delta^2\ A$ and $A\ \delta^2\ D$ with $i_k^{-1}(U) = C$ and $i_k^{-1}(V) = D$ for $k,j = 1,2$ and $k \neq j$. If U and V are in the same component of wedge, then $U\ \delta'\ V$ if and only if there are sets C,D in X^2 such that $C\ \delta'\ D$ and $i_k^{-1}(U) = C$ and $i_k^{-1}(V) = D$ for some $k,j = 1,2$. Specially, if $i_k(E) = U$ and $i_k(F) = V$, then $(i_k(E), i_k(F)) \in \delta'$ if and only if $(i_k^{-1}(i_k(E)), (i_k^{-1}(i_k(F))) = (E,F) \in \delta'$. This is a special case of Definition 2.3. We will show that the condition holds.

Assume that for some $x,y \in X$, $(\{x\},\{y\}) \in \delta$ or $(\{y\},\{x\}) \in \delta$ with $x \neq y$. $(U,V) \in \delta'$ (δ' is a quasi-proximity structure on $X^2V_\vartriangle X^2$) with $U \supseteq \{(x,y)_1\}$ and $V \supseteq \{(x,y)_2\}$,

$$\pi_{11}U\delta\,\pi_{11}V \supseteq \pi_1S\{(x,y)_1\}\delta\pi_1S\{(x,y)_2\} = \pi_1\{(x,y,y)\}\delta\pi_1\{(x,x,y)\} = \{x\}\delta\{x\},$$

i.e., $(\{x\},\{x\}) \in \delta$,

$$\pi_{21}U\delta\,\pi_{21}V \supseteq \pi_2S\{(x,y)_1\}\delta\pi_2S\{(x,y)_2\} = \pi_2\{(x,y,y)\}\delta\pi_2\{(x,x,y)\} = \{y\}\delta\{x\},$$

i.e., $(\{y\},\{x\}) \in \delta$,

$$\pi_{22}U\delta\,\pi_{22}V \supseteq \pi_3S\{(x,y)_1\}\delta\pi_3S\{(x,y)_2\} = \pi_3\{(x,y,y)\}\delta\ \pi_3\{(x,x,y)\} = \{y\}\delta\{y\},$$

i.e., $(\{y\},\{y\}) \in \delta$ where $\pi_i : X^3 \longrightarrow X, i = 1,2,3$ are projection maps.

There are sets E,F and A in X^2 such that $E\ \delta^2\ A$ and $A\ \delta^2\ F$ with $i_k^{-1}(U) = E$ and $i_j^{-1}(V) = F$ for $k,j = 1,2$ and $k \neq j$. $i_k(i_k^{-1}(U)) = i_k(E) \subseteq U$ and $i_j(i_j^{-1}(V)) = i_j(F) \subseteq V$.

If $i_k(E)$ subset of the first component of $X^2V_\vartriangle X^2$ and $i_k(F)$ subset of the second component of $X^2V_\vartriangle X^2$, then $\{(x,y)_1\} \subseteq i_k(E)$ and $\{(x,y)_2\} \subseteq i_k(F)$. But, if $(i_k(E), i_k(F)) \supseteq (\{(x,y)_1\}, \{(x,y)_2\}) \in \delta'$ for some $(E,F) \in \delta^2$ and $k = 1$(resp. $k = 2$), then $(\{(x,y)_1\}, \{(x,y)_2\}) \in (i_1(E), i_1(F))$ which shows that $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $X^2V_\vartriangle X^2$, a contradiction because of $x \neq y$.

Similarly, if $i_k(E)$ subset of the second component of $X^2V_\vartriangle X^2$ and $i_k(F)$ subset of the first component of $X^2V_\vartriangle X^2$, then $\{(x,y)_2\} \subseteq i_k(E)$ and $\{(x,y)_1\} \subseteq i_k(F)$. But, if $(i_k(E), i_k(F)) \supseteq (\{(x,y)_2\}, \{(x,y)_1\}) \in \delta'$ for some $(E,F) \in \delta^2$ and $k = 1$(resp. $k = 2$), then $(\{(x,y)_2\}, \{(x,y)_1\}) \in (i_1(E), i_1(F))$ which shows that $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $X^2V_\vartriangle X^2$, a contradiction because of $x \neq y$. For this reason $i_k(E)$ and $i_k(F)$ are not be able to in different component of $X^2V_\vartriangle X^2$ (In the case of $V\delta' U$ it can be obtained similarly).

Conversely, assume that for each $x \neq y$ $(\{x\},\{y\}) \notin \delta$ and $(\{y\},\{x\}) \notin \delta$. We will show that (X,δ) is $\mathrm{Pre}T_2'$, i.e., by Definition 2.2, Definition 2.4 and Definition 2.7, (a) and (b) above are equivalent . Now we will show that (a) implies (b). Let (U,V) be any set in δ', i.e., $\pi_iS(U)\ \delta\ \pi_iS(V) \quad (i = 1,2,3)$.

If $i_k(E)$ subset of the first component of $X^2V_\vartriangle X^2$ and $i_k(F)$ subset of the second component of $X^2V_\vartriangle X^2$, then $\{(x,y)_1\} \subseteq i_k(E)$ and $\{(x,y)_2\} \subseteq i_k(F)$.

$$\pi_2S\{(x,y)_1\}\ \delta\ \pi_2S\{(x,y)_2\} = \pi_2\{(x,y,y)\}\delta\pi_2\{(x,x,y)\} = \{y\}\delta\{x\}$$

i.e., $(\{y\},\{x\}) \in \delta$. Since $(\{y\},\{x\}) \notin \delta$ (by assumption), $((\{(x,y)_1\}, \{(x,y)_2\}) \in \delta')$ by the condition (Q3) of Definition 2.1.

The case $i_k(E) \subseteq U$ subset of the second component of $X^2V_\vartriangle X^2$ and $i_k(F) \subseteq V$ subset of the first component of $X^2V_\vartriangle X^2$ can be obtained similarly. From here $i_k(E)$ and $i_k(F)$ can not be in different component of $X^2V_\vartriangle X^2$.

If $i_k(E) \subseteq U$ and $i_k(F) \subseteq V$ are in both component of $X^2V_\vartriangle X^2$, then $U \supseteq i_k(E) \supseteq \{(x,y)_1, (x,y)_2\}$ and $V \supseteq i_k(F) \supseteq \{(x,y)_1, (x,y)_2\}$.

If $i_k(E) \subseteq U$ subset of the first component of $X^2V_\vartriangle X^2$ and $i_k(F) \subseteq V$ subset of both component of $X^2V_\vartriangle X^2$, then $U \supseteq i_k(E) \supseteq \{(x,y)_1\}$ and $V \supseteq i_k(F) \supseteq \{(x,y)_1, (x,y)_2\}$.

If $i_k(E) \subseteq U$ subset of both component of $X^2V_{\triangle}X^2$ and $i_k(F) \subseteq V$ subset of the second component of $X^2V_{\triangle}X^2$, then $U \supseteq i_k(E) \supseteq \{(x,y)_1,(x,y)_2\}$ and $V \supseteq i_k(F) \supseteq \{(x,y)_2\}$.

If $i_k(E) \subseteq U$ and $i_k(F) \subseteq V$ are in the first component of $X^2V_{\triangle}X^2$, then $U \supseteq i_k(E) \supseteq \{(x,y)_1\}$ and $V \supseteq i_k(F) \supseteq \{(x,y)_1\}$. Similarly, $i_k(E) \subseteq U$ and $i_k(F) \subseteq V$ are in the second component of $X^2V_{\triangle}X^2$, then $U \supseteq i_k(E) \supseteq \{(x,y)_2\}$ and $V \supseteq i_k(F) \supseteq \{(x,y)_2\}$.

If $\{(x,y)_i,(x,y)_i\} \in \delta'$, $(i = 1,2)$, then

$$\pi_1S\{(x,y)_1\}\delta\ \pi_1S\{(x,y)_1\} = \pi_1\{(x,y,y)\}\delta\ \pi_1\{(x,y,y)\} = \{x\}\delta\{x\}$$
$$\Rightarrow (\{x\},\{x\}) \in \delta,$$
$$\pi_2S\{(x,y)_1\}\delta\ \pi_2S\{(x,y)_1\} = \pi_2\{(x,y,y)\}\delta\ \pi_2\{(x,y,y)\} = \{y\}\delta\{y\}$$
$$(\{y\},\{y\}) \in \delta,$$
$$\pi_3S\{(x,y)_1\}\delta\ \pi_3S\{(x,y)_1\} = \pi_3\{(x,y,y)\}\delta\ \pi_3\{(x,y,y)\} = \{y\}\delta\{y\}$$
$$\Rightarrow (\{y\},\{y\}) \in \delta,$$
$$\pi_1S\{(x,y)_2\}\delta\ \pi_1S\{(x,y)_2\} = \pi_1\{(x,x,y)\}\delta\ \pi_1\{(x,x,y)\} = \{x\}\delta\{x\}$$
$$\Rightarrow (\{x\},\{x\}) \in \delta,$$
$$\pi_2S\{(x,y)_2\}\delta\ \pi_2S\{(x,y)_2\} = \pi_2\{(x,x,y)\}\delta\ \pi_2\{(x,x,y)\} = \{x\}\delta\{x\}$$
$$\Rightarrow (\{x\},\{x\}) \in \delta,$$
$$\pi_3S\{(x,y)_2\}\delta\ \pi_3S\{(x,y)_2\} = \pi_3\{(x,x,y)\}\delta\ \pi_3\{(x,x,y)\} = \{y\}\delta\{y\}$$
$$\Rightarrow (\{y\},\{y\}) \in \delta.$$

It follows that $(i_k(E),i_k(F)) \supseteq \{(x,y)_i,(x,y)_i\}$ $(i = 1,2)$, i.e., $i_k(E) \subseteq U$ and $i_k(F) \subseteq V$ are in the first or in the second or in both component of $X^2V_{\triangle}X^2$. So there are a pair $(a,b),(c,d) \in X^2$ such that $\{(a,b)\}\ \delta^2\ \{(c,d)\}$ and $i_k\{(a,b)\} = U$ and $i_k\{(c,d)\} = V$ for some $k = 1$ or $k = 2$. Similar results are obtained for $(i_k(E),i_k(F)) \supseteq \{(y,x)_i,(y,x)_i\}$ $(i = 1,2)$ This shows that (a) implies (b).

Now we show that (b) implies (a). Assume that (b) holds. We must show that for any U,V on $X^2V_{\triangle}X^2$, $\pi_{11}U\ \delta\ \pi_{11}V$, $\pi_{21}U\ \delta\ \pi_{21}V$ and $\pi_{22}U\ \delta\ \pi_{22}V$. There are a pair $(a,b),(c,d) \in X^2$ such that $\{(a,b)\}\ \delta^2\ \{(c,d)\}$ and $i_k\{(a,b)\} = U$ and $i_k\{(c,d)\} = V$ for some $k = 1$ or $k = 2$. By using the similar argument as above, we must have $(i_k\{(a,b)\},i_k\{(c,d)\} \supseteq \{(x,y)_i,(x,y)_i\})$, $(i = 1,2)$.

For $i = 1$, if $(\{(x,y)_1,(x,y)_1\}) \in \delta'$, then

$$\pi_1S\{(x,y)_1\}\delta\ \pi_1S\{(x,y)_1\} = \pi_1\{(x,y,y)\}\delta\ \pi_1\{(x,y,y)\} = \{x\}\delta\{x\}$$
$$\Rightarrow (\{x\},\{x\}) \in \delta,$$
$$\pi_2S\{(x,y)_1\}\delta\ \pi_2S\{(x,y)_1\} = \pi_2\{(x,y,y)\}\delta\ \pi_2\{(x,y,y)\} = \{y\}\delta\{y\}$$
$$\Rightarrow (\{y\},\{y\}) \in \delta,$$
$$\pi_3S\{(x,y)_1\}\delta\ \pi_3S\{(x,y)_1\} = \pi_3\{(x,y,y)\}\delta\ \pi_3\{(x,y,y)\} = \{y\}\delta\{y\}$$
$$\Rightarrow (\{y\},\{y\}) \in \delta.$$

For $i = 2$, if $(\{(x,y)_2,(x,y)_2\}) \in \delta'$, then

$$\pi_1S\{(x,y)_2\}\delta\ \pi_1S\{(x,y)_2\} = \pi_1\{(x,x,y)\}\delta\ \pi_1\{(x,x,y)\} = \{x\}\delta\{x\}$$
$$\Rightarrow (\{x\},\{x\}) \in \delta,$$
$$\pi_2S\{(x,y)_2\}\delta\ \pi_2S\{(x,y)_2\} = \pi_2\{(x,x,y)\}\delta\ \pi_2\{(x,x,y)\} = \{x\}\delta\{x\}$$
$$\Rightarrow (\{x\},\{x\}) \in \delta,$$
$$\pi_3S\{(x,y)_2\}\delta\ \pi_3S\{(x,y)_2\} = \pi_3\{(x,x,y)\}\delta\ \pi_3\{(x,x,y)\} = \{y\}\delta\{y\}$$
$$\Rightarrow (\{y\},\{y\}) \in \delta.$$

From here, $\pi_iS(U)\ \delta\ \pi_iS(V)$ $(i = 1,2,3)$. This shows that (b) implies (a). For $\pi_iS(V)\ \delta\ \pi_iS(U)$, the conditions are hold. Hence (X,δ) is PreT_2'. □

Remark 4.4. If a quasi-proximity space (X,δ) is PreT_2' then (X,δ) is Pre$\overline{T_2}$. However the converse is not true generally. For example, let $X=\{a,b\}$,

$$\delta=\{(X,X),(\{a\},\{a\}),(\{b\},\{b\}),(X,\{a\}),(X,\{b\}),(\{a\},\{b\}),(\{b\},\{a\})\}$$

Then (X,δ) is Pre$\overline{T_2}$ but (X,δ) is not PreT_2' since $(\{a\},\{b\})\in\delta$ but $a\neq b$.

Theorem 4.5. *A quasi-proximity space (X,δ) is $\overline{T_2}$ if and only if for each $x\neq y$, $(\{x\},\{y\})\notin\delta$ and $(\{y\},\{x\})\notin\delta$.*

Proof. It follows from Definition 4.1, Theorem 3.2 and Theorem 4.2. □

Theorem 4.6. *A quasi-proximity space (X,δ) is T_2' if and only if for each $x\neq y$, $(\{x\},\{y\})\notin\delta$ and $(\{y\},\{x\})\notin\delta$.*

Proof. It follows from Definition 4.1, Theorem 3.3 and Theorem 4.3. □

Remark 4.7. Let (X,δ) be a quasi-proximity space. The following expressions are equivalent by Theorems 4.3, 4.5 and 4.6.

1. (X,δ) is PreT_2',
2. (X,δ) is T_2',
3. (X,δ) is $\overline{T_2}$,
4. For each $x,y\in X$ with $x\neq y$ $(\{x\},\{y\})\notin\delta$ and $(\{y\},\{x\})\notin\delta$.

5. Closed subobjects

Let B be set and $p\in B$. The infinite wedge product $\vee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty=B\times B\times\ldots$ be the countable cartesian product of B. Define $A_p^\infty:\vee_p^\infty B\to B^\infty$ by $A_p^\infty(x_i)=(p,p,\ldots,p,x,p,\ldots)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p,p,\ldots,p,x,p,\ldots)$ (infinite principal p-axis map), and $\nabla_p^\infty:\vee_p^\infty B\to B$ by $\nabla_p^\infty(x_i)=x$ for all $i\in I$ (infinite fold map), [3].

Definition 5.1. ([3]) Let $U:\mathcal{E}\to$ **Set** be a topological functor, X an object in $\mathcal{E}$ with $U(X)=B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi U-sink $q:U(X)=B\to B/F=(B\backslash F)\cup\{*\}$, where q is the epi map that is the identity on $B\backslash F$ and identifying F with a point $\{*\}$.

Let p be a point in B.

1. p is closed if and only if the initial lift of the U-source $\{A_p^\infty:\vee_p^\infty B\to U(X^\infty)=B^\infty$ and $\nabla_p^\infty:\vee_p^\infty B\to UD(B)=B\}$ is discrete.
2. $F\subset X$ is closed if and only if $\{*\}$, the image of F, is closed in X/F or $F=\emptyset$.
3. $F\subset X$ is strongly closed if and only if X/F is T_1 at $\{*\}$ or $F=\emptyset$.
4. If $B=F=\emptyset$, then we define F to be both closed and strongly closed.

Theorem 5.2. ([25]) *Let (X,δ) be an object in* ***QProx****, $p\in X$ and $\emptyset\neq F\subset X$.*

1. *$\{p\}$ is closed in X if and only if for any $B\subset X$, if $\{p\}\delta B$ or $B\delta\{p\}$, then $p\in B$.*
2. *The following expressions are equivalent.*
   (a) *F is closed.*
   (b) *F is strongly closed.*
   (c) *$x\in F$ whenever $\{x\}\delta F$ or $F\delta\{x\}$ for any $x\in X$.*

3. *F is (strongly) open if and only if $x \in F^c$ whenever $\{x\}\delta F^c$ or $F^c\delta\{x\}$ for all $x \in X$.*

Definition 5.3. ([30] p. 106) Let (X, δ) be a quasi-proximity space and $A \subset X$. Define $\bar{A} = \{x \mid x\delta A \text{ or} A\delta\, x\}$ and if $\bar{A} = A$, then A is said to be closed.

Example 5.4. Let $X = \{a, b, c\}$. The following relations δ_i $(i = 1, 2, 3)$ on $P(X)$ are quasi-proximity relations.
$\delta_1 = \{(A, B) \in P^2(X) \mid A \cap B \neq \emptyset\}$
$\delta_2 = \delta_1 \cup \{(\{a\}, \{b\}), (\{a\}, \{b, c\}), (\{a, c\}, \{b\})\}$
$\delta_3 = \delta_1 \cup \{(\{a\}, \{b\}), (\{b\}, \{a\}), (\{a\}, \{c\}), (\{c\}, \{a\}), (\{b\}, \{c\}), (\{c\}, \{b\}), (\{a\}, \{b, c\}), (\{b, c\}, \{a\}),$
$(\{b\}, \{a, c\}), (\{a, c\}, \{b\}), (\{c\}, \{a, b\}), (\{a, b\}, \{c\})\}$
For (X, δ_1), all subsets of X are (strongly) closed since δ_1 is discrete.
For (X, δ_2), the family of all (strongly) closed subsets is given below:

$\{\emptyset, X, \{c\}, \{a, b\}\}$.

For (X, δ_3), the only (strongly) closed subsets are $\emptyset$ and X since δ_3 is indiscrete.

Theorem 5.5. ([25])

1. *Let (X, δ_1) and (Y, δ_2) be quasi-proximity spaces and $f : (X, \delta_1) \to (Y, \delta_2)$ be a p-map. If $K \subset Y$ is (strongly) closed, then $f^{-1}(K) \subset X$ is (strongly) closed.*
2. *If $K \subset L$ and $L \subset X$ are (strongly) closed in a quasi-proximity space (X, δ), so also is $K \subset X$.*

Let $\mathcal{E}$ be a set-based topological category and cl be a closure operator of $\mathcal{E}$.

1. $\mathcal{E}_{0cl} = \{X \in \mathcal{E} \mid x \in cl(\{y\}) \text{ and } y \in cl(\{x\}) \implies x = y \text{ with } x, y \in X\}$ [18].
2. $\mathcal{E}_{1cl} = \{X \in \mathcal{E} \mid cl(\{x\}) = \{x\}, \text{ for each } x \in X\}$ [18].
3. $\mathcal{E}_{2cl} = \{X \in \mathcal{E} \mid cl(\Delta) = \Delta, \text{ the diagonal}\}$ [18].

Definition 5.6. Let (X, δ) be a quasi-proximity space and $K \subset X$.

(i) $c(K) = \bigcap \{U \subset X \mid K \subset U \text{ and } U \text{ is closed}\}$ is called the closure of K.
(ii) $sc(K) = \bigcap \{U \subset X \mid K \subset U \text{ and } U \text{ is strongly closed}\}$ is called the strong closure of K.

It is shown that the notion of closedness forms closure operator [17] in some topological categories [6, 8, 10, 12, 13, 20, 22, 23].

Theorem 5.7. $(X, \delta) \in \mathbf{QProx_{0c}}$ *if and only if for any $x, y \in X$ with $x \neq y$, there exists a closed subset $K \subset X$ such that $x \notin K$ and $y \in K$ or a closed subset $L \subset X$ such that $x \in L$ and $y \notin L$.*

Proof. Similarly it is obtained from Theorem 3.9 in [14]. □

Theorem 5.8. $(X, \delta) \in \mathbf{QProx_{1c}}$ *if and only if $(\{x\}, \{y\}) \notin \delta$ and $(\{y\}, \{x\}) \notin \delta$ for any $x, y \in X$ with $x \neq y$.*

Proof. Suppose $(X, \delta) \in \mathbf{QProx_{1c}}$ and $x, y \in X$ with $x \neq y$. We have $c(\{x\}) = \{x\}$ for all $x \in X$, i.e., $\{x\}$ is closed. By Theorem 5.2 (1), if $(\{x\}, \{y\}) \in \delta$ for some $x, y \in X$ with $x \neq y$, then $y \in \overline{\{x\}}$. This is a contradiction since $\{x\}$ is closed. Hence, we have $(\{x\}, \{y\}) \notin \delta$ for any distinct pair $x, y \in X$. Similarly if $(\{y\}, \{x\}) \in \delta$ for some $x, y \in X$ with $x \neq y$, then $y \in \overline{\{x\}}$. This is a contradiction since $\{x\}$ is closed.Hence, we have $(\{y\}, \{x\}) \notin \delta$ for any distinct pair $x, y \in X$.

Conversely, suppose for any $x, y \in X$ with $x \neq y$, $(\{x\}, \{y\}) \notin \delta$ and $(\{y\}, \{x\}) \notin \delta$. It follows that for any $E \subset X$, if $(\{x\}, E) \in \delta$ or $(E, \{x\}) \in \delta$, then $x \in E$. By Theorem 5.2 (1), $\{x\}$ is closed, i.e., $\mathbf{c}(\{x\}) = \{x\}$, and consequently $(X, \delta) \in \mathbf{QProx_{1c}}$. □

Theorem 5.9. $(X, \delta) \in \mathbf{QProx_{2c}}$ *if and only if $(\{x\}, \{y\}) \notin \delta$ and $(\{y\}, \{x\}) \notin \delta$ for any $x, y \in X$ with $x \neq y$.*

Proof. Suppose $(X, \delta) \in \mathbf{QProx_{2c}}$ and $x, y \in X$ with $x \neq y$. Note that $(x, y) \notin \Delta$. Since Δ is (strongly) closed, by Theorem 5.2 (2), $(\{(x, y)\}, \Delta) \notin \delta^2$ where δ^2 is the product quasi-proximity relation on X^2. By Definition 2.2, for any $x, y \in X$ with $x \neq y$, $(\{x\}, \{y\}) \notin \delta$. Similarly, it is obtained for any $x, y \in X$ with $x \neq y$, $(\{y\}, \{x\}) \notin \delta$. $(\Delta, \{(x, y)\}) \notin \delta^2$, by Definition 2.2.

Conversely, assume that the condition satisfies and $(x, y) \in X^2$ with $x \neq y$. Then $(x, y) \notin \Delta$ and by assumption, we have $(\{(x, y)\}, \Delta) \notin \delta^2$ and $(\Delta, \{(x, y)\}) \notin \delta^2$. Assume that $(\{(x, y)\}, \Delta) \notin \delta^2$. We will show that if $(\{(*, *)\}, B^2) \in \delta'^2$, then $(*, *) \in B^2$ for any $B^2 \subset X^2/\Delta$, i.e., Δ is (strongly) closed. Suppose $(*, *) \notin B^2$ for some $B^2 \subset X^2/\Delta$. Since $(\{(*, *)\}, B^2) \in \delta'^2$, there is some $C^2_s \subset X^2/\Delta$ for each binary rational s in $[0, 1]$ such that $C^2_0 = \{*\} \times \{*\}$, $C^2_1 = B \times B$ and $s < t$ implies $((q^{-1} \times q^{-1})(C^2_s), (q^{-1} \times q^{-1})(C^2_t)) \in \delta^2$. It follows that $((q^{-1} \times q^{-1})(\{*\} \times \{*\}), (q^{-1} \times q^{-1})(B \times B)) = (\Delta, B^2) \in \delta^2$ by definition of q-map and Definition 2.3. Since $(\Delta, B^2) \in \delta^2$, there exists $(x, y) \in B^2$ ($x, y \in B$ and $x \neq y$) such that $(\Delta, \{(x, y)\}) \in \delta^2$ by the condition ($Q3$) of Definition 2.1. But for all $(x, y) \in B^2$, $(x, y) \notin \Delta$ since $(*, *) \notin B^2$. Since $(\{x\}, \{y\}) \notin \delta$ for any $x, y \in X$ with $x \neq y$, this is a contradiction. It is obtained $(\{y\}, \{x\}) \notin \delta$ for $(\Delta, \{(x, y)\}) \notin \delta^2$, similarly. Hence Δ is (strongly) closed, and consequently $(X, \delta) \in \mathbf{QProx_{2c}}$. □

Remark 5.10. Let (X, δ) be a quasi-proximity space. The following expressions are equivalent by Theorems 5.8 and 5.9.

1. $(X, \delta) \in \mathbf{QProx_{1c}}$,
2. $(X, \delta) \in \mathbf{QProx_{2c}}$,
3. For any $x, y \in X$ with $x \neq y$ $(\{x\}, \{y\}) \notin \delta$ and $(\{y\}, \{x\}) \notin \delta$.

Theorem 5.11. *If a quasi-proximity space $(X, \delta) \in \mathbf{QProx}_{ic}$, $i = 1, 2$, then $(X, \delta) \in \mathbf{QProx_{0c}}$.*

Proof. Suppose $(X, \delta) \in \mathbf{QProx}_{ic}(i = 1, 2)$ i.e., $(\{x\}, \{y\}) \notin \delta$ and $(\{y\}, \{x\}) \notin \delta$ for any $x, y \in X$ with $x \neq y$. By Theorem 5.2 (1), $\{x\}$ and $\{y\}$ is closed. Let $K = \{y\}$ or $L = \{x\}$. It follows that $x \notin K$ and $y \in K$ or $x \in L$ and $y \notin L$, and consequently $(X, \delta) \in \mathbf{QProx_{0c}}$. □

Theorem 5.12. *A quasi-proximity space $(X, \delta) \in \mathbf{QProx}_{ic}$ if and only if $(X, \delta) \in \mathbf{QProx}_{isc}$ for $i = 0, 1, 2$.*

Proof. It is obvious from Theorem 5.2 and Definition 5.6. □

Example 5.13. The quasi-proximity space (X, δ_1) defined in Example 5.4 is in $\mathbf{QProx}_{ik}$, $i = 0, 1, 2$ and $k =$ **c** or **sc**.

Remark 5.14. $\mathbb{T}$**QProx** is the full subcategory of **QProx** consisting of all $\mathbb{T}$ objects, where $\mathbb{T}$ is $PreT'_2$ (resp. $\overline{T}_2$, T'_2) which were defined in [2].

Theorem 5.15. *The following categories are isomorphic.*

1. $\mathbf{QProx}_{ik}$ *for* $i = 1, 2$ *and* $k =$ **c** *or* **sc**.
2. $\mathbb{T}$**QProx** *for* $PreT'_2, \overline{T}_2, T'_2$.

Proof. It follows from Theorems 5.11, 5.12 and Remarks 4.7, 5.10, 5.14. □

Remark 5.16. 1. By Remark 5.10 and Theorems 5.11, 5.12, we have

$$\mathbf{QProx_{2c}} = \mathbf{QProx_{2sc}} = \mathbf{QProx_{1c}} = \mathbf{QProx_{1sc}} \subset \mathbf{QProx_{0c}} = \mathbf{QProx_{0sc}}.$$

2. For the category **Prox**, by Remark 3.19 of [14],

$$\mathbf{Prox_{2c}} = \mathbf{Prox_{2sc}} = \mathbf{Prox_{1c}} = \mathbf{Prox_{1sc}} \subset \mathbf{Prox_{0c}} = \mathbf{Prox_{0sc}}.$$

3. For the category **Top**, by Remark 3.5 of [6],

$$\mathbf{Top}_{2cl} = \mathbf{Top}_{2scl} \subset \mathbf{Top}_{1cl} = \mathbf{Top}_{1scl} \subset \mathbf{Top}_{0cl} = \mathbf{Top}_{0scl}.$$

Definition 5.17. ([2], [4]) Let $\mathcal{E}$ be a set-based topological category and X an object in $\mathcal{E}$.

1) X is KT_2 if and only if X is Pre$\overline{T_2}$ and T_0'.

2) X is LT_2 if and only if X is PreT_2' and $\overline{T_0}$.

3) X is NT_2 if and only if X is Pre$\overline{T_2}$ and T_0.

Theorem 5.18. *Let (X, δ) be a quasi-proximity space. (X, δ) is KT_2 if and only if the δ quasi-proximity structure must satisfy the symmetry condition, i.e. for every $A, B \subseteq X$, $A\delta B$ if and only if $B\delta A$.*

Proof. It is obivious that by Definition 5.17,Theorem 3.3 and Theorem 4.2. □

Theorem 5.19. *Let (X, δ) be a quasi-proximity space. (X, δ) is NT_2 if and only if for each $x \neq y$ in X, $(\{x\}, \{y\}) \notin \delta$ and $(\{y\}, \{x\}) \notin \delta$.*

Proof. It is obvious that by Definition 5.17,Theorem 3.1 and Theorem 4.2. □

Theorem 5.20. *Let (X, δ) be a quasi-proximity space. (X, δ) is LT_2 if and only if for each $x \neq y$ in X, $(\{x\}, \{y\}) \notin \delta$ and $(\{y\}, \{x\}) \notin \delta$.*

Proof. It is obvious that by Definition 5.17,Theorem 3.2 and Theorem 4.3. □

Remark 5.21. 1. Let (X, δ) be a quasi-proximity space. (X, δ) is NT_2 if and only if (X, δ) is LT_2 by Theorem 5.19 and Theorem 5.20.

2. Let (X, δ) be a quasi-proximity space. If (X, δ) is NT_2 and LT_2 then (X, δ) is KT_2 . But the converse is not true in general. For example, let $X = \{a, b\}$,

$$\delta = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}), (\{a\}, X), (X, \{b\}), (\{b\}, X), (\{a\}, \{b\}), (\{b\}, \{a\})\}$$

Then (X, δ) is KT_2 but (X, δ) is not NT_2 or LT_2 since $(\{a\}, \{b\}) \in \delta$ but $a \neq b$.

6. Sober quasi-proximity spaces

In this section, we characterize sober and quasi-sober in the quasi-proximity spaces.

Definition 6.1. Let ε be a topological category over **Set** and $X \in Ob(\varepsilon)$ [32].

$$cl(Z) = \bigcap \{U \subset X : Z \subset U \quad and \quad U \quad is \quad closed\}$$

is called the closure of a subobject Z of X.

Definition 6.2. ([11], [8]) Let ε be a topological category and $X \in Ob(\varepsilon)$.

(1) X is said to be irreducible if A, B are closed subobjects of X and $X = A \cup B$, then $A = X$ or $B = X$.

(2) X is called quasi-sober if every nonempty irreducible closed subset of X is closure of a point.

(3) X is called $\overline{T_0}$ sober if X is $\overline{T_0}$ and a quasi-sober.

(4) X is called T_0' sober if X is T_0' and a quasi-sober.

(5) X is called T_0 sober if X is T_0 and a quasi-sober.

Theorem 6.3. *Every quasi-proximity space is quasi sober.*

Proof. Let (X, δ) is any quasi-proximity space. Now, we shall show that (X, δ) is quasi sober. For every $A, B \subseteq X$ there are two possible cases either $A \cap B \neq \emptyset$ or $A \cap B = \emptyset$.

Firstly, we assume that for some $A, B \subseteq X$, $A \cap B = \emptyset$ with $A\delta B$ or $B\delta A$. By Definition 2.1 for some $a \in A$ and $b \in B$ (a ≠ b) if $\{a\}\delta\{b\}$ or $\{b\}\delta\{a\}$ then $\overline{\{a\}} = \{a, b\}$, $\overline{\{b\}} = \{a, b\}$ and for every $c \notin \{a, b\}$, $\overline{\{c\}} = \{c\}$ by Definition 5.3. Therefore $\{a, b\}$, $\{c\}$ are irredicible closed subsets of X and these are generated by one point. In this case (X, δ) is quasi-sober space.

Secondly, we assume that $A \cap B \neq \emptyset$ when $A\delta B$ or $B\delta A$. We must show that (X, δ) is quasi-sober space. In this case for some $A \subseteq X$ there are irredicible closed subset of X so that there exist at least two generic point, i.e., for some $x, y \in X$ $(x \neq y)$ $\overline{\{x\}} = A$, $\overline{\{y\}} = A$. By Definition 5.3 $\{x\}\delta A$ or $A\delta\{x\}$, $\{y\}\delta A$ or $A\delta\{y\}$. Let $\{x\}\delta A$, $y \in A$ since $\overline{\{y\}} = A$. From here by Definition 2.1 $\{x\}\delta\{y\}$ but $\{x\} \cap \{y\} = \emptyset$. This is a contradiction. Therefore X has just one generic point. The proof is similar for $B\delta A$. The results are similar for other cases.
□

Theorem 6.4. *Let (X, δ) is quasi-sober space. Then, the following are equivalent.*

(1) (X, δ) is T_0 sober,

(2) (X, δ) is T_0' sober,

(3) (X, δ) is $\overline{T_0}$ sober,

(4) (X, δ) is T_0 or $\overline{T_0}$ or T_1 ,

(5) For each $x \neq y$ in X, $(\{x\}, \{y\}) \notin \delta$ or $(\{y\}, \{x\}) \notin \delta$.

Proof. It follows from Theorem 3.3, Remark 3.5 and Definition 6.2. □

Remark 6.5. Let (X, δ) is quasi-proximity space. If (X, δ) is quasi-sober space then, (X, δ) is T_0 sober, T_0' sober and $\overline{T_0}$ sober but the converse is not true in general. For example, let $X = \{a, b\}$. The following relations δ_i $(i = 1, 2)$ on $P(X)$ are quasi-proximity relations. =

$\delta_1 = \{(A, B) \in P^2(X) \mid A \cap B \neq \emptyset\}$

$\delta_2 = \delta_1 \cup \{(\{a\}, \{b\}), (\{b\}, \{a\})\}$.

Since for $a \neq b$ in X, $(\{a\}, \{b\}) \in \delta$ and $(\{b\}, \{a\}) \in \delta$, (X, δ) is not T_0 sober, T_0' sober and $\overline{T_0}$ sober by Theorem 6.4 but (X, δ) is quasi-sober by Theorem 6.3.

7. Comparative evaluation

We examined our findings in some topological categories and we infer:

(1) In ***Top***,

(i) By Theorem 2.2.11 of [2], Remark 1.3 of [1] and Remark 2.6 of [5],

$$\mathbf{Top}_{2cl} = \mathbf{Top}_{2scl} = \mathbf{LT_2Top} = \mathbf{NT_2Top} = \mathbf{KT_2Top} \subset \mathbf{Top}_{1cl} = \mathbf{Top}_{1scl} \subset \mathbf{Top}_{0cl} = \mathbf{Top}_{0scl} = \mathbf{T_0Top} = \overline{\mathbf{T_0}}\mathbf{Top} = \mathbf{T_0'Top}$$

(ii) By Remark 3.4 of [11],

$$\mathbf{T_0SobTop} = \overline{\mathbf{T_0}}\mathbf{SobTop} = \mathbf{T_0'SobTop}$$

(2) In ***QProx***,

(i) By Theorems 5.19 and 5.20, Remarks 5.16 (1) and 5.21,

$$\mathbf{QProx}_{1cl} = \mathbf{QProx}_{1scl} = \mathbf{QProx}_{2cl} = \mathbf{QProx}_{2scl} = \mathbf{LT_2QProx} = \mathbf{NT_2QProx} \subset \mathbf{KT_2QProx}$$

(ii) By Remarks 3.5 and 5.10,

$$\mathbf{QProx}_{1cl} = \mathbf{QProx}_{1scl} = \mathbf{QProx}_{2cl} = \mathbf{QProx}_{2scl} \subset \mathbf{T_1QProx} = \mathbf{T_0QProx} = \overline{\mathbf{T_0}}\mathbf{QProx}$$

(iii) By Theorems 4.2 and 5.18, Remarks 4.7 and 5.21,

$$\mathbf{LT_2QProx} = \mathbf{NT_2QProx} = \overline{\mathbf{T_2}}\mathbf{QProx} = \mathbf{T_2'QProx} = \mathbf{preT_2'QProx} \subset \mathbf{KT_2QProx} = \mathbf{pre}\overline{\mathbf{T_2}}\mathbf{QProx}$$

(iv) By Theorem 6.4,

$$\overline{\mathbf{T_0}}\mathbf{QProx} = \mathbf{T_1QProx} = \mathbf{T_0QProx} = \mathbf{T_0SobQProx} = \overline{\mathbf{T_0}}\mathbf{SobQProx} = \mathbf{T_0'SobQProx} = \mathbf{QSobQProx}$$

where **QSobQprox** is the full subcategory of **QProx** consisting of all quasi-sober quasi-proximity space.

(3) In ***Rel***,(the category of relation spaces and relation preserving functions)

(i) By Theorem 3.3 of [9],

$$\mathbf{Rel}_{1cl} = \mathbf{Rel}_{2cl} = \mathbf{Rel}_{1Q} = \mathbf{Rel}_{2Q} = \mathbf{Rel}_{1SQ} = \mathbf{Rel}_{2SQ}$$

(ii) By Theorem 4.5 of [9],

$$\mathbf{LT_2Rel} \subset \mathbf{NT_2Rel} \subset \mathbf{KT_2Rel} = \overline{\mathbf{T_2}}\mathbf{Rel} = \mathbf{pre}\overline{\mathbf{T_2}}\mathbf{Rel} \subset \mathbf{Rel}_{1Q} = \mathbf{T_1Rel} = \mathbf{T_0'Rel} = \overline{\mathbf{T_0}}\mathbf{Rel} = \mathbf{Rel}$$

(iii) By Theorem 3.3 of [9],

$$\overline{\mathbf{T_0}}\mathbf{SobRel} = \mathbf{T_0'SobRel} = \mathbf{QSobRel}$$

where **QSobRel** is the full subcategory of **Rel** consisting of all quasi-sober relation spaces.

8. Conclusion

In this work, we characterized sober spaces, the separation properties $\overline{T_0}$, T'_0,T_0 ,T_1, Pre$\overline{T_2}$, PreT'_2, $\overline{T_2}$, and T'_2 in general in the category of quasi-proximity spaces. Then, we investigated the relationships between them. Finally, we compared our results in some topological categories. For future work, the Tietze extension theorem and Urysohn's lemma can be studied within the topological category of quasi-proximity spaces.

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