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Gradient almost Schouten solitons and contact geometry

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Abstract. The goal of this present object is to deliberate an almost Schouten soliton and gradient almost Schouten soliton on contact geometry. Here, we classify some results on K-contact manifolds and (κ, ϖ)-contact manifolds admitting gradient almost Schouten soliton.

1. Introduction and Motivations

In mathematics, contact geometry is the study of a geometric structure on smooth manifolds given by a hyperplane distribution in the tangent bundle satisfying a condition called 'complete non-integrability'. When a manifold is endowed with a geometric structure, we have more opportunities to explore its geometric properties. In 1958, Boothby and Wang [1] examined an odd-dimensional differentiable manifold with the help of almost contact and contact structure and studied its features from a topological posterior. Contact manifolds are the odd dimensional counterparts of symplectic manifolds. In a similar vein, Sasakian manifolds are the odd-dimensional counterparts of Kähler manifolds, and are defined as odd-dimensional Riemannian manifolds (M^{2n+1}, g) whose Riemannian cone $(R_+ \times M, dr^2 \oplus r^2 g)$ endows a Kähler structure. It has broad applications in geometric optics, geometric quantization, control theory, thermodynamics, integrable systems and to the classical mechanics. Contact geometry has enlarged from the mathematical formalism of classical mechanics. The concept of Ricci flow, which is an evolution equation for metrics defined the connected almost contact metric manifolds whose automorphism groups have maximal dimensions.
The study of any geometric flow has an important aspect in the study of their associated solitons, which generates self-similar solutions to the flow and often arise as singularity models. Inspired from the notion of Ricci soliton it is interesting to consider special solutions of the flow (named as Shouten flow [2, 8]) which is called as Schouten soliton, which is a generalization of Ricci-Bourguignon soliton, that is defined by

Definition 1.1. *A Riemannian manifold (M^n, g), $n \geq 3$, is said to be an almost Schouten soliton if there is a smooth vector field V and a smooth function γ on M^n satisfying*

$$(\mathfrak{I}_V g)(W_1, W_2) + 2S(W_1, W_2) = 2(\gamma - \frac{1}{2(n-1)} r) g(W_1, W_2), \tag{1}$$

where S is the Ricci tensor of g, r is the scalar curvature and $\mathfrak{I}_V g$ is the Lie derivative of g in the V direction.

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Also, if γ is constant, then it is called a Schouten soliton.

Also it is said to be shrinking, steady or expanding according as $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$ respectively.

Definition 1.2. *When $V = \nabla f$, for some smooth function $f : M \to \mathbb{R}$, then (M, g) is called a gradient Schouten soliton and then* (1) *reduces to*

$$\nabla^2 f + S = [\gamma - \frac{1}{2(n-1)} r]g, \tag{2}$$

where $\nabla^2 f$ is the Hessian of f. If γ is a smooth function, then (2) *is called gradient almost Schouten soliton.*

It is worthy to mention that Sharma [43] first initiated the study of Ricci solitons in contact geometry. However, Ghosh [24] is the first to consider 3-dimensional Kenmotsu metric as a Ricci soliton. Recently, Venkatesha et al. [46, 47] also examined Ricci solitons on perfect fluid space time and demonstarted the nature of ρ-Einstein soliton on almost Kenmotsu manifold. Very recently, some authors in (see [10–12, 20, 22, 23, 28, 38–41]), have studied Ricci soliton and Einstein soliton and their generalizations on contact geometry. Moreover, some of the latest studies related with soliton geometry can be seen in [9, 13–19, 29, 30, 42, 44, 50, 51]. On contact metric manifolds, Ricci solitons and gradient Ricci solitons have been studied by several authors (see [7, 26, 27, 43]). In [36], Perelman investigated that every compact Ricci soliton is gradient. Next, generalizations of this result for Ricci almost soliton with constant scalar curvature has been studied by Barros et al. [3]. Further, Sharma [43] studied the K-contact manifold as a gradient Ricci soliton and a Ricci soliton with the potential vector field V point-wise collinear with ξ and the manifold becomes Einstein. After some years, Ghosh [25] proved that *if a K-contact manifold admits a gradient Ricci almost soliton, then the scalar curvature is constant. Moreover, if M is compact, then it is Einstein, Sasakian and isometric to a unit sphere".* Very recently, Patra [35] proved the result for completeness instead of compactness. Very recently, Catino [8] et al. has been investigated gradient Schouten solitons and the compact gradient Schouten solitons. They have proved the triviality of every compact gradient Schouten solitons and examine that a complete gradient steady Schouten soliton is trivial and it becomes Ricci flat. Pina and Menezes [37] has been demonstrated a complete gradient Schouten solitons.Based on the above facts and discussions in the research of contact geometry, a natural question arises.

Is a complete K-contact metric as a gradient almost Schouten soliton Einstein?

Here, we will answer this question affirmatively using some different techniques in the next sections.

The structure of this paper is the following. In section 2, we study some basic definitions and related conclusions on K-contact manifold and (κ, ϖ)-contact metric manifolds. In section 3, we prove our main results.

2. A brief review of contact metric manifolds

A $(2n + 1)$-dimensional smooth Riemannian manifold (M, g) is said to be an almost contact metric manifold if it admits a $(1, 1)$ tensor field ϕ, a characteristic vector field ξ, a global 1-form η and an indefinite metric g on M satisfying the following relations.

$$\phi^2(W_1) = -W_1 + \eta(W_1)\xi, \quad \eta(W_1) = g(W_1, \xi) \tag{3}$$

for all vector field W_1 on M. A Riemannian metric g is said to be an associated (or compatible) metric if it satisfies

$$g(\phi W_1, \phi W_2) = g(W_1, W_2) - \eta(W_1)\eta(W_2) \tag{4}$$

for all vector fields W_1, W_2 on M. An almost contact manifold $M^{2n+1}(\phi, \xi, \eta)$ together with a compatible metric g is known as almost contact metric manifold (see Blair [4])

On the product $M^{2n+1} \times \mathbb{R}$ of an almost contact metric manifold M^{2n+1} and $\mathbb{R}$, there exists an almost complex structure J defined by

$$J(W_1, f\frac{d}{dt}) = (\phi W_1 - f\xi, \eta(W_1)\frac{d}{dt}),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is C^∞-function on $M^{2n+1} \times \mathbb{R}$. If J is integrable, then almost contact metric structure on M^{2n+1} is said to be normal.
Given a contact metric manifold M we define a symmetric (1,1)-tensor field h and self adjoint operator l by $h = \frac{1}{2}\mathfrak{I}_\xi\phi$ and $l = R(., \xi)\xi$. Then, $h\phi = -\phi h$, $Trh = Tr\phi h = 0$, $h\xi = 0$. Also from [4], we have

$$\nabla_{W_1}\xi = -\phi W_1 - \phi h W_1, \tag{5}$$

$$g(Q\xi, \xi) = Trl = 2n - |h|^2. \tag{6}$$

A normal contact metric manifold is called a Sasakian manifold. Sasakian manifolds are K-contact and 3-dimensional K-contact manifolds are Sasakian. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_{W_1}\phi)W_2 = g(W_1, W_2)\xi - \eta(W_2)W_1 \tag{7}$$

for all $W_1, W_2 \in \chi(M)$. The vector field ξ is a killing vector with respect to g if and only if $h = 0$. A contact metric manifold M for which ξ is killing (equivalently $h = 0$ or $Trl = 2n$)is said to be a K-contact metric manifold. On a K-contact manifold the following formulas are known [4]

$$\nabla_{W_1}\xi = -\phi W_1, \tag{8}$$

$$Q\xi = 2n\xi, \tag{9}$$

$$R(W_1, \xi)\xi = W_1 - \eta(W_1)\xi, \tag{10}$$

where ∇ is the operator of covarient differentiation of g, S is the Ricci tensor of type (0,2) such that $S(W_1, W_2) = g(QW_1, W_2)$, where Q is Ricci operator and R is the Riemann curvature tensor of g.
The notion (κ, ϖ)-nullity distribution on a contact metric manifold M was introduced by Blair et al. [5], which is defined for any $p \in M$ and $\kappa, \varpi \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k, \varpi) &= \{W_3 \in T_p(M) : R(W_1, W_2)W_3 = \kappa[g(W_2, V_3)W_1 - g(W_1, W_3)W_2] \\ &+ \varpi[g(W_2, W_3)hW_1 - g(W_1, W_3)hW_2]\} \end{aligned} \tag{11}$$

for any vector fields W_1, W_2 on $T_p(M)$, where $T_p(M)$ denotes the tangent space on M at any point $p \in M$ and R is the Riemannian tensor. In particular, if $\varpi = 0$, then the notion of (κ, ϖ)-nullity distribution reduces to the notion of k-nullity distribution, introduced by Tanno [45]. If $\kappa = 1$, the structure is Sasakian. In a (κ, ϖ)-contact metric manifold the following relations hold [5, 34]

$$h^2 = (\kappa - 1)^2, \kappa \leq 1, \tag{12}$$

$$R(W_1, W_2)\xi = \kappa[\eta(W_2)W_1 - \eta(W_1)W_2] + \varpi[\eta(W_2)hW_1 - \eta(W_1)hW_2], \tag{13}$$

$$\begin{aligned} S(W_1, W_2) &= [2(n-1) - n\varpi]g(W_1, W_2) + [2(n-1) + \varpi]g(hW_1, W_2) \\ &+ [2(1-n) + n(2k + \varpi)]\eta(W_1)\eta(W_2), \end{aligned} \tag{14}$$

$$r = 2n(2n - 2 + \kappa - n\varpi). \tag{15}$$

Here, r is the scalar curvature of the manifold.

3. Main Results

Theorem 3.1. *If a K-contact manifold $M^{(2n+1)}$ endows a gradient almost Schouten soliton, then it is Einstein with constant scalar curvature $r = 2n(2n+1)$. Further, if M is complete, then it is compact Sasakian and isometric to a unit sphere S^{2n+1}.*

Proof. We can write the equation (2) as follows

$$\nabla_{W_1} Df + QW_1 = [\gamma - \frac{1}{4n}r]W_1. \tag{16}$$

Now, taking the covarient differentiation of the identity (16) along with arbitrary vector field W_2, we obtain

$$\begin{aligned} \nabla_{W_2}\nabla_{W_1} Df &= (W_2\gamma)W_1 - \frac{1}{4n}(W_2 r)W_1 + [\gamma - \frac{1}{4n}r](W_2, W_1) \\ &- (\nabla_{W_2}Q)W_1 - Q(\nabla_{W_2}W_1). \end{aligned} \tag{17}$$

Again we displace W_1 and W_2 into the foregoing equation to yield

$$\begin{aligned} \nabla_{W_1}\nabla_{W_2} Df &= (W_1\gamma)W_2 - \frac{1}{4n}(W_1 r)W_2 + [\gamma - \frac{1}{4n}r](W_1, W_2) \\ &- (\nabla_{W_1}Q)W_2 - Q(\nabla_{W_1}W_2). \end{aligned} \tag{18}$$

Also, we get

$$\nabla_{[W_1,W_2]} Df + Q[W_1, W_2] = [\gamma - \frac{1}{4n}r][W_1, W_2]. \tag{19}$$

Now, we apply the expression of Riemannian curvature tensor $R(W_1, W_2)Df = \nabla_{W_1}\nabla_{W_2}Df - \nabla_{W_2}\nabla_{W_1}Df - \nabla_{[W_1,W_2]}Df$ to obtain

$$\begin{aligned} R(W_1, W_2)Df &= [(W_1\gamma)W_2 - (W_2\gamma)W_1] - \frac{1}{4n}[(W_1 r)W_2 - (W_2 r)W_1] \\ &- [(\nabla_{W_1}Q)W_2 - (\nabla_{W_2}Q)W_1]. \end{aligned} \tag{20}$$

Now, differentiating the identity (9) along with vector field W_2 in view of the identity (10), we get

$$(\nabla_{W_1})\xi = Q\phi W_1 - 2n\phi W_1. \tag{21}$$

Also, taking an inner product of the equation (20) with respect to ξ, we obtain

$$\begin{aligned} g(R(W_1, W_2)Df, \xi) &= [(W_1\gamma)\eta(W_2) - (W_2\gamma)\eta(W_1)] - \frac{1}{4n}[(W_1 r)\eta(W_2) \\ &- (W_2 r)\eta(W_1)] - [(\nabla_{W_1}Q)\eta(W_2) \\ &- (\nabla_{W_2}Q)\eta(W_1)]. \end{aligned} \tag{22}$$

In light of the equations (10) and (21), we replace W_2 by ξ into (22) with using the relation $g(R(W_1, W_2)Df, \xi) = -g(R(W_1, W_2)\xi, Df)$ to acquire

$$W_1(f + \gamma - \frac{1}{4n}r) = \xi(f + \gamma - \frac{1}{4n}r)\eta(W_1), \tag{23}$$

i.e.,

$$d(f + \gamma - \frac{1}{4n}r) = \xi(f + \gamma - \frac{1}{4n}r)\eta. \tag{24}$$

Now, we utilize the foregoing identity by d and with the help of Poincare lemma i.e., $d^2 = 0$ to yield

$$d\xi(f + \gamma - \frac{1}{4n}r) \wedge \eta + \xi(f + \gamma - \frac{1}{4n}r)d\eta = 0. \tag{25}$$

Also, using the relation $\eta \wedge \eta = 0$, the preceding equation reads

$$\xi(f + \gamma - \frac{1}{4n}r)d\eta \wedge \eta = 0. \tag{26}$$

As, $d\eta \neq 0$ on M, we get

$$\xi(f + \gamma - \frac{1}{4n}r) = 0. \tag{27}$$

i.e.,

$$D(f + \gamma - \frac{1}{4n}r) = 0.$$

So, we can say that $f + \gamma - \frac{1}{4n}r$ is constant on M.
Now, taking a Lie derivative of the equation (16) with respect to ξ, we get

$$\begin{aligned}\mathfrak{I}_\xi(\nabla_{W_1}Df) \quad &+ \quad [(\mathfrak{I}_\xi Q)W_1 + Q(\mathfrak{I}_\xi W_1)] = [(\mathfrak{I}_\xi\gamma) \\ &- \quad \frac{1}{4n}(\mathfrak{I}_\xi r)]W_1 + [\gamma - \frac{1}{4n}r](\mathfrak{I}_\xi W_1).\end{aligned} \tag{28}$$

As, ξ is killing on K-contact manifold, so we have

$$\begin{aligned}\mathfrak{I}_\xi(\nabla_{W_1}Df) \quad &+ \quad Q(\mathfrak{I}_\xi W_1) = [\xi\gamma - \frac{1}{4n}(\xi r)]W_1 + [\gamma \\ &- \quad \frac{1}{4n}r](\mathfrak{I}_\xi W_1).\end{aligned} \tag{29}$$

We take a Lie derivative of Df using the equations (8) and (9) to find

$$\mathfrak{I}_\xi Df = \nabla_\xi Df - \nabla_{Df}\xi = (\gamma - \frac{1}{4n}r)\xi - 2n\xi + \phi Df. \tag{30}$$

We differentiate covariantly the preceding equation along with the vector field W_2 in terms of (8) to yield

$$\nabla_{W_2}\mathfrak{I}_\xi Df = [(W_2\gamma) - \frac{1}{4n}(W_2 r)]\xi + 2n\phi W_2 + (\nabla_{W_2}\phi)Df - \phi QW_2. \tag{31}$$

From the commutative formula (Yano [49]), we obtain

$$\mathfrak{I}_V\nabla_{W_2}W_1 - \nabla_{W_2}\mathfrak{I}_V W_1 - \nabla_{[V,W_2]}W_1 = (\mathfrak{I}_V\nabla)(W_2, W_1). \tag{32}$$

Now, we insert $V = \xi$ and $W_1 = Df$ into (32) with the help of $(\mathfrak{I}_\xi\nabla) = 0$ and equation (29) to (31) to acquire

$$\begin{aligned}g((\nabla_{W_2}\phi)W_1, Df) \quad &+ \quad g(\phi QW_2, W_1) + [(\xi\gamma) - \frac{1}{4n}(\xi r)]g(W_1, W_2) \\ &- \quad W_2(\gamma - \frac{1}{4n}r)\eta(W_1) - 2ng(\phi W_2, W_1) = 0.\end{aligned} \tag{33}$$

We utilize formula

$$(\nabla_{W_2}\phi)W_1 + (\nabla_{\phi W_2}\phi)\phi W_1 = 2g(W_2, W_1)\xi - \eta(W_1)(W_2 + \eta(W_2)\xi)$$

by exchanging W_1 and W_2 with ϕW_1 and ϕW_2 into (33) to find

$$\begin{aligned}g(\phi QW_2, W_1) \quad &- \quad 2ng(\phi W_2, W_1) - \xi(f + \gamma - \frac{1}{4n}r)\eta(W_1)\eta(W_2) \\ &+ \quad g(Q\phi W_2, W_1) + 2\xi(f + \gamma - \frac{1}{2n}r)g(W_1, W_2) \\ &- \quad W_2(f + \gamma - \frac{1}{4n}r)\eta(W_1) = 0.\end{aligned} \tag{34}$$

Now, as this expression $f + \gamma - \frac{1}{4n}r$ is constant, so from the equation (34), we get

$$Q\phi W_1 + \phi Q W_1 - 2n\phi W_1 = 0 \tag{35}$$

for all $\chi(M)$.
We take an inner product of (20) with respect to $f + \gamma - \frac{1}{4n}r = constant$ to read

$$g((\nabla_{W_2}Q)W_1 - (\nabla_{W_1}Q)W_2, Df) = 0. \tag{36}$$

Let $\{e_i, \phi e_i, \xi; i = 1, 2, ...n\}$ be an orthonormal ϕ-basis of M such that $Qe_i = \varrho_i e_i$. Making the use of this into (35), we obtain, $Q\phi e_i = (2n - \varrho_i)\phi e_i$, where ϱ is a non-zero function on M. Then the scalar curvature is given by

$$r = g(Q\xi, \xi) + \Sigma_{i=1}^{n}[g(Qe_i, e_i) + g(Q\phi e_i, \phi e_i)] = 2n(2n+1). \tag{37}$$

Now, we interchange W_1 by ξ into identity (36) with the help of (21) to find

$$Q\phi Df - 2n\phi Df = 0. \tag{38}$$

In light of (35) to get

$$\phi Q Df = 2n\phi Df.$$

We utilize the foregoing equation with ϕ by using (8) to get

$$QDf = 2nDf. \tag{39}$$

Now, we take covariant derivative into (38) to yield

$$(\nabla_{W_1}Q)Df - Q^2W_1 + (\gamma - \frac{1}{4n}r + 2n)QW_1 - 2n(\gamma - \frac{1}{4n}r)W_1 = 0. \tag{40}$$

As $r = 2n(2n+1)$ is constant, then $divQ = \frac{1}{2} = dr = 0$. Now, we contract the identity (40) to get

$$||Q||^2 = 0.$$

Now, with the help of $r = 2n(2n+1)$, we obtain

$$||Q - \frac{r}{2n+1}I||^2 = 0.$$

i.e., the length of the symmetric tensor $Q - \frac{r}{2n+1}I$ vanishes, we must have $QW_1 = 2nW_1$. Thus M is Einstein with Einstein constant $2n$. Suppose M is complete. Then, from Myers Theorem [31], M is compact. Again, we apply Boyer and Galicki's [6] result *Any compact K-contact Einstein manifold is Sasakian* to conclude that M is Sasakian. Also, we can write the identity (16) as , $\nabla_{W_1}Df = -\zeta W_1$, where $\zeta = 2n + \frac{1}{4n}r - \gamma$, then by Obata's theorem [32], it is isometric to a unit sphere S^{2n+1}. This we can finish the proof. □

Ghosh [25] also acquired some results for contact metric manifold with potential vector field collinear with the Reeb vector field and non-Sasakian (κ, ϖ)-contact manifolds. Motivated by this results, one can ask the following question:
Are there exists a contact metric manifolds, whose metrics are almost Ricci-Bourguignon soliton?
We give the answer of the question in the following way.

Theorem 3.2. *Let $M^{(2n+1)}$ be a complete contact metric manifold where the Reeb vector field ξ is an eigenvector of the Ricci operator at each point of M. If the metric endows an almost Schouten soliton with a non-zero potential vector field is collinear with the Reeb vector field ξ, then the manifold M is compact Einstein Sasakian and the potential vector field is a constant multiple of the Reeb vector field ξ.*

Proof. As by hypothesis, the potential vector field is collinear with the Reeb vector field i.e., $V = \varrho\xi$, where ϱ is a non-zero function on M as V is non-zero. Now, we differentiate it along with the vector field W_1 in light of (5) to get

$$\nabla_{W_1} V = (W_1\varrho)\xi - \varrho(\phi W_1 + \phi h W_1). \tag{41}$$

In view of the identity (1), the preceding equation becomes

$$\begin{aligned} W_2\varrho\eta(W_1) \quad &+ \quad W_1\varrho\eta(W_2) - 2\varrho g(\phi h W_1, W_2) + 2S(W_1, W_2) \\ &= \quad 2(\gamma - \frac{1}{4n}r)g(W_1, W_2). \end{aligned} \tag{42}$$

Now, we plug $W_1 = W_2 = \xi$ into (42) making use of (6) to find

$$\xi\varrho + 2Trl = 2(\gamma - \frac{1}{4n}r). \tag{43}$$

We replace W_2 by ξ into (42) to yield

$$D\varrho + (\xi\varrho)\xi + 2Q\xi - 2(\gamma - \frac{1}{4n}r)\xi = 0. \tag{44}$$

Since, $Q\xi = (Trl)\xi$. So, in light of this fact into (43), the foregoing equation becomes

$$D\varrho = (\xi\varrho)\xi. \tag{45}$$

Now, we differentiate the identity (45) with respect to W_1 by using (5) to acquire

$$\nabla_{W_1} D\varrho = W_1(\xi\varrho)\xi - (\xi\varrho)(\phi W_1 + \phi h W_1). \tag{46}$$

In light of Poincare lemma into (46), we obtain

$$W_1(\xi\varrho)\eta(W_2) - W_2(\xi\varrho)\eta(W_1) + 2(\xi\varrho)d\eta(W_1, W_2) = 0. \tag{47}$$

Now, we choose vector fields W_1 and W_2 are orthogonal to ξ. Also, we know that $d\eta \neq 0$. So, equation (47) yields $\xi\varrho = 0$, which implies $D\varrho = 0$ i. e., ϱ is constant. Now, the identity (42) transforms into

$$2QW_2 + 2\varrho h\phi W_2 - 2(\gamma - \frac{1}{4n}r)W_2 = 0. \tag{48}$$

We contract the identity (48) making use of $Trh\phi = 0$ to find

$$2(2n+1)\gamma = 2[1 + (2n+1)\vartheta]r. \tag{49}$$

We differentiate (48) in view of W_1 to obtain

$$2(\nabla_{W_1} Q)W_2 + 2\varrho(\nabla_{h\phi})W_2 - 2[(W_1\gamma)W_2 - \frac{1}{4n}(W_1 r)W_2] = 0. \tag{50}$$

Again, we contract the identity (50) and making the use of $div(h\phi)W_2 = g(Q\xi, W_2) - 2n\eta(W_2)$ in contact metric manifold to infer

$$(1 + 2\vartheta)(W_2 r) + 2\varrho[Trl - 2n]\eta(W_2) - 2(W_2\gamma) = 0. \tag{51}$$

If we take W_2 is orthogonal to ξ into (51) in view of (49) to find either $2n - 1 = 0$ or $V_2 r = 0$. As $2n - 1 \neq 0$, so we have $V_2 r = 0$.. Now, we interchange W_2 by $\phi^2 W_2$ entails that $Dr = (\xi r)\xi$. Now, we use the previous argument as prior to see that r is constant, which implies from (49) γ is also constant. In light of (43), we get Trl is constant. Now, we get the identity in view of (51) $Trl - 2n = 0$ i. e., $h = 0$. So, M is K-contact and by virtue of (48), we see that M is Einstein. Since M is complete and also Einstein, then in view of the results Sharma [43], we can conclude that M is compact and also applying the result of Boyer and Galicki [6], we finish the proof. □

Now, we have the following remark from the previous theorem and a result of Boyer and Galicki [6],

Remark 3.3. *If a manifold M be a complete K-contact manifold admits an almost Ricci-Bourguignon soliton with non-zero potential vector field V, then the manifold is a compact Sasakian manifold and the soliton is trivial.*

Theorem 3.4. *If a non-Sasakian (κ, ϖ)-contact metric manifold $M^{(2n+1)}$ endows a gradient almost Schouten soliton, then the manifold is flat with dimension three and the soliton vector field is homothetic, and for $n > 1$, M is locally isometric to $E^{n+1} \times S^n(4)$ and the soliton vector field is tangential to the Euclidean factor E^{n+1}.*

Proof. We again apply the expression of Riemannian curvature tensor $R(W_1, W_2)Df = \nabla_{W_1}\nabla_{W_2}Df - \nabla_{W_2}\nabla_{W_1}Df - \nabla_{[W_1,W_2]}Df$ with the help of (16) to deduce

$$R(W_1, W_2)Df = [(W_1\gamma)W_2 - (W_2\gamma)W_1] - [(\nabla_{W_1}Q)W_2 - (\nabla_{W_2}Q)W_1]. \tag{52}$$

We take the covariant derivative of (14) using the identity (52) to find

$$\begin{aligned} R(W_1, W_2)Df &= \{[2(n-1) + \varpi][2(1-\kappa)g(W_2, \phi W_1)\xi + \eta(W_1)[h(\phi W_2 \\ &+ \phi hW_2] - \eta(W_2)[h(\phi W_1 + \phi hW_1] + \varpi\eta(W_1)\phi hW_2 \\ &- \varpi\eta(W_2)\phi hW_1] + [2(1-n) + n(2\kappa + \varpi)][2g(W_2, \phi W_1)\xi \\ &- (\phi W_2 + \phi hW_2)\eta(W_1) + (\phi W_1 + \phi hW_1)\eta(W_2)]\} \\ &+ [(W_1\gamma)W_2 - (W_2\gamma)W_1]. \end{aligned} \tag{53}$$

We take an inner product of the foregoing equation with ξ to yield

$$\begin{aligned} g(R(W_1, W_2)Df, \xi) &= 2(\varpi + 2\kappa - \kappa\varpi + n\varpi)g(W_2, \phi W_1) + [(W_1\gamma)W_2 \\ &- (W_2\gamma)W_1]. \end{aligned} \tag{54}$$

Again, we lay hold of the inner product of (13) with Df to read

$$\begin{aligned} g(R(W_1, W_2)\xi, Df) &= \kappa[\eta(W_2)g(W_1, Df) - \eta(W_1)g(W_2, Df)] \\ &+ \varpi[\eta(W_2)g(hW_1, Df) - \eta(W_1)g(hW_2, Df)]. \end{aligned} \tag{55}$$

Now, we add previous two identities to obtain

$$\begin{aligned} \varpi[\varpi(W_2)g(hW_1, Df) &- \eta(W_1)g(hW_2, Df)] + \kappa[\eta(W_2)g(W_1, Df) \\ &- \eta(W_1)g(W_2, Df)] + 2(\varpi + 2\kappa - \kappa\varpi \\ &+ n\varpi)g(W_2, \phi W_1) + [(V_1\gamma)\eta(W_2) \\ &- (V_2\gamma)\eta(W_1)] = 0. \end{aligned} \tag{56}$$

We replace W_1 and W_2 by ϕW_1 and ϕW_2 into (56) and in light of $R(\phi W_1, \phi W_2)\xi = 0$ to yield

$$\kappa = \frac{\varpi(1+n)}{\varpi - 2}. \tag{57}$$

Now, we substitute $W_2 = \xi$ into (56) to obtain

$$(\kappa + \varpi h)Df + 2(D\gamma) - [\kappa(\xi f) + 2(\xi\gamma)]\xi = 0. \tag{58}$$

We replace W_1 by Df into (14) to invoke

$$QDf = -4n(D\gamma). \tag{59}$$

Making use of (59), identity (58) becomes

$$2n(\kappa + \varpi h)Df - QDf - 2n[\kappa(\xi f) + 2(\xi\gamma)]\xi = 0. \tag{60}$$

We take an inner product with ξ, the preceding equation becomes

$$\kappa(\xi f) + 2(\xi\gamma) = 0. \tag{61}$$

Then the identity (60) becomes

$$2n(\kappa + \varpi h)Df - QDf = 0. \tag{62}$$

We differentiate (62) with respect to ξ to yield

$$(2n\varpi^2 - \varpi[2(n-1) + \varpi])\phi hDf - 2n\varpi h(\gamma - \frac{1}{4n}r - 2n\kappa)\xi = 0. \quad (63)$$

Now, we take an inner product the identity (63) with ξ to achieve

$$\varpi h(\gamma - \frac{1}{4n}r - 2n\kappa) = 0. \quad (64)$$

Then (63) becomes

$$2n\varpi^2 - \varpi[2(n-1) + \varpi])\phi hDf = 0. \quad (65)$$

Now, operating h into (65) in view of (12), we get

$$(\kappa - 1)\varpi[2(n-1) + \varpi - 2n\varpi]\phi Df = 0. \quad (66)$$

Thus we have two case in the following.

Case 1. If ϖ is zero, then the identity (57) provides $\kappa = 0$. So, $R(W_1, W_2)\xi = 0$. Therefore, proved that a $(2n+1)$-dimensional contact metric manifold M^{2n+1} is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat if $n = 1$.

Case 2. Let $\phi Df = 0$. Then, we operate h on it to find $Df = (\xi f)\xi$. We differentiate covariantly the preceding equation along with the vector field W_1 in light of (5) to yield

$$\nabla_{W_1} Df = W_1(\xi f)\xi - (\xi f)(\phi W_1 + \phi h W_1). \quad (67)$$

Now, we apply Poincare lemma into (67) to acquire

$$W_1(\xi f)\eta(W_2) - W_2(\xi f)\eta(W_1) + (\xi f)d\eta(W_1, W_2) = 0. \quad (68)$$

If we take W_1 and W_2 are orthogonal to ξ, then $\xi f = 0$ as $d\eta \neq 0$. So, $Df = 0$, which implies f is constant. Now, (16) yields

$$QW_2 = (\gamma - \frac{1}{4n}r)W_2, \quad (69)$$

i. e., the manifold is Einstein. We trace the previous equation to get

$$r = (2n+1)(\gamma - \frac{1}{4n}r). \quad (70)$$

Now, using the Theorem 4.1 of Ghosh [25], we can find the scalar curvature $r = 2n\kappa(2n+1)$ and for $n = 1$, M is locally flat and making use of $\varpi = -2(n-1)$(as M is non-Sasakian) into (57) to find $\kappa = n - \frac{1}{n} > 1$, which gives contradiction for $n > 1$. Since the manifold is flat in dimension three with γ is constant, in view of (16), we can say that the vector field is homothetic.

Case 3. If $2(n-1) + \varpi - 2n\varpi = 0$, then $\varpi = \frac{2(n-1)}{2n-1}$. Now, we insert this expression into (57) to yield $\kappa = \frac{1}{n} - n$. For $n = 1$ it implies $\varpi = \kappa = 0$ and hence the manifold becomes flat. In light of (14) into (62), we get

$$-2(n-1) + n(2\kappa + \varpi)](Df - (\xi f)\xi) + [2n\varpi - 2(n-1) - \varpi]hDf = 0. \quad (71)$$

Now, we put $\varpi = \frac{2(n-1)}{2n-1}$ and $\kappa = \frac{1}{n} - n$ into (71) to obtain $Df = (\xi f)\xi$. Now, for $n > 1$ we use the condition of **Case 2**. As $QW_1 = 2nkW_1$, setting covariant derivative entails that $\nabla Q = 0$ and the identity (52) provides

$$R(W_1, W_2)Df = 2[(W_1\gamma)W_2 - (W_2\gamma)W_1]. \quad (72)$$

As $R(W_1, W_2)\xi = 0$ and we take an inner product of the identity (72) with ξ by interchanging W_2 by ξ entails that $V_1\gamma = (\xi\gamma)\eta(W_1)$. Now, we proceed as in the **Case 2**, it is easy to say that γ is constant and in higher dimension from(72) implies $R(W_1, W_2)Df = 0$ i.e., Df is tangent to the flat factor E^{n+1}. This completes the proof. □

We get very familiar result from [3] that a compact almost Ricci soliton with constant scalar curvature is isometric to a sphere. From [33], we can say that in dimension greater that three, a contact metric manifold of constant curvature is a Sasakian manifold of constant curvature. Also, the scalar curvature is constant for a (κ, ϖ)-contact manifold. So, we have the following:

Remark 3.5. *If a compact (κ, ϖ)-contact manifold endows an almost Schouten soliton, then it is either flat or Sasakian of dimension 3 and it is isometric to a unit sphere S^{2n+1} for dimension greater than 3.*

Conflict of interest

The author states that there is no conflict of interest.

Data Availability

No data is required in this manuscript.

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References

[1] W. M. Boothby and H. C. Wang: *On Contact Manifolds*, Annals of Mathematics, Second Series, Vol. **68**, No. 3 (Nov., 1958), pp. 721-734 (14 pages).
[2] V. Borges.: *On complete gradient Schouten solitons*, Nonlinear Anal. (2022). https://doi.org/10.2016/j.na.2022.112883
[3] A. Barros, R. Batista and E. Ribeiro Jr.: *Compact almost Ricci solitons with constant scalar curvature are gradient*, Monatsh. Math. **174**(1), 29–39 (2014).
[4] D. E. Blair.: *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhauser, Boston (2002).
[5] D. E. Blair, T. Koufogiorgos and J. Papantoniou.: *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. **91**(1-3), 189-214 (1995).
[6] C. P. Boyer and K. Galicki.: *Einstein manifolds and contact geometry*, Proc. Amer. Math. Soc. **129**, 2419–2430 (2001). 1979), Lecture Notes in Math., Springer, Berlin-New York(1981), Vol. **838**, pp. 42–63.
[7] J. T. Cho and R. Sharma.: *Contact geometry and Ricci solitons*, Int. J. Geom. Methods Math. Phy. **7**(6), 951–960 (2010).
[8] G. Catino, L. Mazzieri.: *Gradient Einstein solitons*, Nonlinear Anal. **132**, 66–94. (2016) J. Math. **287** (2017), 337–370.
[9] S. K. Chaubey and Y. J. Suh: *Ricci-Bourguignon solitons and Fisher-Marsden conjecture on generalized Sasakian-space-forms with β-Kenmotsu structure*, to appear in J. of Korean Math. Soc., (2023).
[10] S. Dey and S. Roy.: *$*$-η-Ricci Soliton within the framework of Sasakian manifold*, Journal of Dynamical Systems & Geometric Theories, (2020), Vol-**18**(2), pp-163-181.
[11] S. Dey, S. Sarkar and A. Bhattacharyya.: *$*$-η Ricci soliton and contact geometry*, Ricerche di Matematica, **73**, 1203–1221, (2024)
[12] S. Dey and S. Uddin.: *Conformal η-Ricci almost solitons on Kenmotsu manifolds*, International Journal of Geometric Methods in Modern Physics, Vol. **19**, No. 08, 2250121(2022),, https://doi.org/10.1142/S0219887822501213 (2022).
[13] S. Dey:*Conformal Ricci soliton and almost conformal Ricci soliton in paracontact geometry*, International Journal of Geometric Methods in Modern Physics, **20**(03), 2350041, 2023.
[14] S. Dey: *Certain results of κ-almost gradient Ricci-Bourguignon soliton on pseudo-Riemannian manifolds*, Journal of Geometry and Physics, **184**(2022), 104725.
[15] S. Dey: *Certain results on gradient almost η-Ricci-Bourguignon soliton*, Quaestiones Mathematicae, **47(2)** (2024), 263–284.
[16] S .Dey and Y. J. Suh: *On K-contact metric manifolds satisfying an almost gradient Ricci-Bourguignon soliton*, Reviews in Mathematical Physics, **37(02)**, 2450032 (2025).
[17] S .Dey and A. Ali: *Certain paracontact metrics satisfying gradient ρ-Ricci-Bourguignon almost solitons*, International Journal of Geometric Methods in Modern Physics, **22(03**, 2450288 (2025).
[18] S. Dey, F. Alghamdi and S. Uddin: *Characterization of almost contact metrics admitting certain types of solitons* to appear in International Journal of Geometric Methods in Modern Physics (Online Ready), https://doi.org/10.1142/S0219887825501695 (2025).
[19] S .Dey and Y. J. Suh: *Geometry of almost contact metrics as an almost $*$-η-Ricci-Bourguignon solitons*, Reviews in Mathematical Physics, **35**(7) (2023), 2350012.
[20] S. Dey and S. Roy.:*Characterization of general relativistic spacetime equipped with η-Ricci-Bourguignon soliton*, Journal of Geometry and Physics, **178**(2022), 104578, https://doi.org/10.1016/j.geomphys.2022.104578.
[21] K. L. Duggal.: *Almost Ricci solitons and physical applications*, International Electronic Journal of Geometry, **10**(2), 2017, 1-10.

[22] D. Ganguly, S. Dey, A. Ali and A. Bhattacharyya.: *Conformal Ricci soliton and Quasi-Yamabe soliton on generalized Sasakian space form*, Journal of Geometry and Physics, Vol. **169** (2021) 104339, https://doi.org/10.1016/j.geomphys.2021.104339.

[23] D. Ganguly, S. Dey and A. Bhattacharyya.: *On trans-Sasakian 3-manifolds as η-Einstein solitons*, Carpathian Mathematical Publications, **13(2)** (2021), pp:460-474, https://doi.org/10.15330/cmp.13.2.460-474.

[24] A. Ghosh.: *Kenmotsu 3-metric as a Ricci soliton*, Chaos Solitons Fractals, **44**, 647-650 (2011).

[25] A. Ghosh.: *Certain contact metrics as Ricci almost solitons*, Results Math. **65**, 81–94 (2014)

[26] A. Ghosh.: *Ricci solitons and contact metric manifolds*, Glasgow Math. J. (2012) (published on line).

[27] A. Ghosh, R. Sharma and J. T. Cho.: *Contact metric manifolds with η-parallel torsion tensor*, Ann. Glob Anal. Geom. **34**, 287–299 (2008).
1055.

[28] Y. L. Li, D. Ganguly, S. Dey and A. Bhattacharyya.: *Conformal η-Ricci solitons within the framework of indefinite Kenmotsu manifolds*, AIMS Math. 2022, **7(4)**, 5408–5430.

[29] Y. L. Li, S. Dey, S. Pahan and A. Ali: *Geometry of conformal η-Ricci solitons and conformal η-Ricci almost solitons on paracontact geometry*, Open Mathematics 2022; **20(1)**: 574–589.

[30] Y. L. Li, S. Mondal, S. Dey, A. Bhattacharyya and A. Ali.: *A Study of Conformal η-Einstein Solitons on Trans-Sasakian 3-Manifold*, Journal of Nonlinear Mathematical Physics, https://doi.org/10.1007/s44198-022-00088-z (2022).

[31] S. B. Myers.: *Connections between differential geometry and topology*, Duke Math. J. **1**, 376–391 (1935).

[32] M. Obata.: *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14**, 333–340 (1962)

[33] Z. Olszak.: *On contact metric manifolds*, Tohoku Math. J. **34**, 247–253 (1979).

[34] B. J. Papantoniou.: *Contact Riemannian manifolds satisfying $R(\xi, W_1) \cdot R = 0$ and $\xi \in (\kappa, \mu)$-nullity distribution*, Yokohama Math. J. **40**(2), 149-161 (1993).

[35] D.S. Patra.: *K-contact metrics as Ricci almost solitons*, Beitr. Algebra Geom. (2020). https://doi.org/10.1007/s13366-020-00539-y

[36] G. Perelman.: *The entropy formula for the Ricci flow and its geometric applications*, Preprint, http://arXiv. orgabsmath.DG/02111159

[37] R. Pina and I. Menezes.: *Rigidity results on gradient Schouten solitons*, arxiv:2010.06729V1 [math.DG] (2020).

[38] S. Roy, S. Dey, A. Bhattacharyya and S. K. Hui.: *$*$-Conformal η-Ricci Soliton on Sasakian manifold*, Asian-European Journal of Mathematics, Vol. **15**, No. 2(2022) 2250035, https://doi.org/10.1142/S1793557122500358.

[39] S. Roy, S. Dey and A. Bhattacharyya.: *Yamabe Solitons on $(LCS)_n$-manifolds*, Journal of Dynamical Systems & Geometric Theories, (2020), Vol-**18**(2), pp-261-279.

[40] S. Roy, S. Dey and A. Bhattacharyya.: *A Kenmotsu metric as a conformal η-Einstein soliton*, Carpathian Mathematical Publications, **13**(1) (2021), pp: 110-118, https://doi.org/10.15330/cmp.13.1.110-118.

[41] S. Sarkar, S. Dey and X. Chen.: *Certain results of conformal and $*$-conformal Ricci soliton on para-cosymplectic and para-Kenmotsu manifolds*, Filomat **35:15** (2021), 5001–5015, https://doi.org/10.2298/FIL2115001S.

[42] S. Sarkar, S. Dey, Ali H. Alkhaldi and A. Bhattacharyya: *Geometry of para-Sasakian metric as an almost conformal η-Ricci soliton*, Journal of Geometry and Physics, **181 (2022)**, 104651, https://doi.org/10.1016/j.geomphys.2022.104651.

[43] R. Sharma.: *Certain results on K-contact and (κ, μ)-contact manifolds*, J. Geom., **89**, 2008, 138-147.

[44] Y. J. Suh: *Ricci-Bourguignon solitons on real hypersurfaces in the complex hyperbolic quadric*, Rev. real/Acad. Cienc. Exactas. Fis. Nat. Ser. A-Mat., Vol. **116** (2022), 110 (23 pages).

[45] S. Tanno.: *Ricci curvature of contact Riemannian manifolds*, Tohoku. Math. J. **40**, 441-448 (1988).

[46] V. Venkatesha and H. A. Kumara.: *Ricci solitons and geometrical structure in a perfect fluid spacetime with torse-forming vector field*, Afrika Mathematika **30**, 725–736(2019).

[47] V. Venkatesha, and H. A. Kumara.: *Gradient ρ-Einstein soliton on almost Kenmotsu manifolds*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **65** (2019), 375–388.

[48] E. Woolgar.: *Some applications of Ricci flow in physics*, Canadian Journal of Physics, **86**(4), 2008, 645-651.

[49] K. Yano.: *Integral Formulas in Riemannian Geometry*, Marcel Dekker, New York (1970).

[50] H. İ. Yoldaş: *On Kenmotsu manifolds admitting η-Ricci-Yamabe solitons*, Int. J. geom. Methods Mod. Phys. **18**(12) (2021), 2150189.

[51] H. İ Yoldaş, Ş. E. Meriç and E. Yaşar: *Some Special vector fields on a cosymplectic manifold admitting a Ricci soliton*, Miskolc Mathematical Notes, Vol. 22 (2021), No. 2, pp. 1039–1050.