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Solving boundary value problem and integral equation via bipolar $\mathcal{R}$-metric space

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Abstract. In this paper, we introduce the concept of bipolar $\mathcal{R}$-metric space. We prove some fixed point theorems with the covariant and contravariant mapping. We provide some examples to strength our obtained results. Also, we provide an application to integral equation and boundary value problem.

1. Introduction

In 1906, the concept of metric space was initiated by Frechet [5]. Metric space have been widely generalised through the exclusion or relaxation of certain axioms, the alteration of the metric function or the abstraction of the idea. Now-a-days the fixed point theory has more valuable applications in mathematics [3, 4]. The existence and uniqueness of common coupled fixed point results for the covariant mappings in bipolar metric space was established by Kishore et al. [10]. Mutlu [20] extended certain coupled fixed point theorems, which can be considered as generalization of Banach fixed point theorem to bipolar metric space. The most important finding in fixed point theory, which had an impact on many researchers, was made in 1922 by the Polish mathematician Stefan Banach [1]. In 2016, the notion of bipolar metric space was introduced by Mutlu and Gurdal [19]. Moreover, these authors have investigated some fixed point and coupled fixed point results in this metric space [20]. Researchers have generalized the metric space structure by either weakening the features of the metric or changing the scope and range of the metric since new findings of space and their qualities are always interesting to mathematicians: for example, in 1993,

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b-metric by S. Czerwik [2], in 1994, partial metric by S. G. Matthews [17], in 2014, partial rectangular metric by S. Shukla [25] and many others.

The concept of bipolar metric space is also extended to the settings of other fixed point theorems of metric spaces, such as the Banach contraction principle (see [8, 9, 11–13, 22, 24]). J. Paul [21] has proved common fixed point theorem on bipolar metric space. The study of nonlinear phenomena benefits greatly from the use of fixed point theory. It is an interdisciplinary area of mathematics that has applications in many different areas of mathematics as well as in other disciplines, such as biology, chemistry, physics, engineering, game theory, mathematical economics, initial and boundary value problems in ordinary and partial differential equations, approximation theory, variational inequalities, and others. Many researchers have proved fixed point theorem by using contractive condition in the setting of bipolar metric space (see [6, 10, 14, 16, 18, 23]). A fixed point theorem on $\mathcal{R}$-complete metric space was proved by Javed, Arshad, Baazeem and Nabil [7]. A fixed point theorem for rational contractive mapping on $\mathcal{R}$-metric space was proved by Mani et al. [15]. From this, we extend the work to bipolar $\mathcal{R}$-metric space to prove fixed point theorem.

Throughout this paper, $\mathbb{N}$ stands for set of all natural numbers, $\mathbb{R}$ stands for set of all real numbers, $\mathcal{R}$ stands for binary relation on nonempty set. In Section 2, we present some basic definitions on bipolar metric space and related definitions of our findings. In Section 3, we give our main results on fixed point theorem on bipolar $\mathcal{R}$-metric space. In Section 4, we give an application to integral equation and boundary value problems.

2. Preliminaries

Definition 2.1. ([19]) *Let Θ and Ξ be a two non-empty set and $\aleph\colon \Theta \times \Xi \to [0, +\infty)$ be a mapping satisfying the following properties:*

(a) $\nu = \varpi$ *if* $\aleph(\nu, \varpi) = 0$*;*

(b) $\aleph(\nu, \varpi) = 0$ *if* $\nu = \varpi$*;*

(c) $\aleph(\nu, \varpi) = \aleph(\varpi, \nu)$ *if* $\nu, \varpi \in \Theta \cap \Xi$*;*

(d) $\aleph(\nu, \varpi) \leq \aleph(\nu, \mathfrak{z}) + \aleph(\mathfrak{r}, \mathfrak{z}) + \aleph(\mathfrak{r}, \varpi) \quad \forall \nu, \mathfrak{r} \in \Theta$ *and* $\mathfrak{z}, \varpi \in \Xi$*.*

Then the mapping $\aleph$ is called a bipolar metric on the pair (Θ, Ξ) and the triple $(\Theta, \Xi, \aleph)$ is called a bipolar metric space.

Definition 2.2. ([19]) *The pair of $(\Theta, \aleph)$ be a metric space and $\mathcal{R}$ is a relation on Θ. Then, the triple $(\Theta, \aleph, \mathcal{R})$ is called $\mathcal{R}$-metric space.*

Definition 2.3. ([19]) *Suppose $(\Theta, \Xi, \aleph)$ be a bipolar metric space and $\mathcal{R}$ is a relation on $\Theta \cap \Xi$. Then $(\Theta, \Xi, \aleph, \mathcal{R})$ is called bipolar $\mathcal{R}$-metric space.*

Definition 2.4. ([19]) *Let $(\Theta, \Xi, \aleph, \mathcal{R})$ be a bipolar $\mathcal{R}$-metric space and Θ = set of left point; Ξ = set of right points, $\Theta \cap \Xi$ = set of central point and $\Theta \cap \Xi$ be a non-empty set, then the space is called disjoint.*

Definition 2.5. ([19]) *Let $(\Theta, \Xi, \aleph, \mathcal{R})$ be a bipolar $\mathcal{R}$-metric space. A sequence $\{\nu_\psi\}$ on the set Θ is called left sequence and $\{\varpi_\psi\}$ on Ξ is called right sequence. In a bipolar metric space, a left (or) a right sequence is called simply a sequence.*

Definition 2.6. ([19]) *Let $(\Theta, \Xi, \aleph, \mathcal{R})$ be a bipolar $\mathcal{R}$-metric space. A sequence $\{\nu_\psi\}$ is called a convergent to a point ν if and only if (ν_ψ) is a left sequence $\lim_{\psi \to +\infty} \aleph(\nu_\psi, \nu) = 0$ and $\nu \in \Xi$ (or) (ν_ψ) is a right sequence, $\lim_{\psi \to +\infty} \aleph(\nu, \nu_\psi) = 0$ and $\nu \in \Theta$.*

Definition 2.7. ([19]) *Let $(\Theta, \Xi, \aleph, \mathcal{R})$ be a bipolar $\mathcal{R}$-metric space. A bisequence (ν_ψ, ϖ_ψ) on $(\Theta, \Xi, \aleph)$ is a sequence on the set $\Theta \times \Xi$, if the sequence (ν_ψ) and (ϖ_ψ) are convergent, then the bisequence (ν_ψ, ϖ_ψ) is called convergent.*

Definition 2.8. ([19]) *Let $(\Theta, \Xi, \aleph, \mathcal{R})$ be a bipolar $\mathcal{R}$-metric space, if bisequence (ν_ψ, ϖ_ψ) is called Cauchy bisequence if* $\lim_{\psi\to+\infty} \aleph(\nu_\psi, \varpi_\psi) = 0$.

Definition 2.9. ([19]) *Let (Θ_1, Ξ_1) and (Θ_2, Ξ_2) be two pair of sets. A map $\Upsilon\colon \Theta_1 \cup \Xi_1 \to \Theta_2 \cup \Xi_2$ is called*

(a) covariant if $\Upsilon(\Theta_1) \subseteq \Theta_2$ and $\Upsilon(\Xi_2) \subseteq \Xi_2$ and it is denoted by $\Upsilon\colon (\Theta_1, \Xi_1) \rightrightarrows (\Theta_2, \Xi_2)$;

(b) contravariant if $\Upsilon(\Theta_1) \subseteq \Theta_2$ and $\Upsilon(\Xi_2) \subseteq \Xi_2$ and it is denoted by $\Upsilon\colon (\Theta_1, \Xi_1) \rightleftarrows (\Theta_2, \Xi_2)$.

Definition 2.10. ([19]) *Let $(\Theta, \Xi, \aleph, \mathcal{R})$ be a bipolar $\mathcal{R}$-metric space is called complete, if every Cauchy bisequence is convergent.*

3. Main Results

Theorem 3.1. *Let $(\Theta, \Xi, \aleph, \mathcal{R})$ be a complete bipolar $\mathcal{R}$-metric space. The mapping $\Upsilon\colon \Theta \cup \Xi \to \Theta \cup \Xi$ be a $\mathcal{R}$-preserving and $\mathcal{R}$-continuous such that*

(i) $\Upsilon(\Theta) \subseteq \Theta$ and $\Upsilon(\Xi) \subseteq \Xi$;

(ii) $\aleph(\Upsilon\nu, \Upsilon\varpi) \le \varrho\aleph(\nu, \varpi)$ for all $\nu \in \Theta, \varpi \in \Xi$ with $(\nu, \varpi) \in \mathcal{R}$ where $\varrho \in (0, 1)$.

Then Υ has a unique fixed point.

Proof. Since $(\Theta, \Xi, \aleph, \mathcal{R})$ is a complete bipolar $\mathcal{R}$-metric space, there exist $\nu_0 \in \Theta$ and $\varpi_0 \in \Xi$ such that $\nu_0\mathcal{R}\nu \quad \forall\nu \in \Theta$ and $\varpi_0\mathcal{R}\varpi \quad \forall\varpi \in \Xi$ and $(\nu_0, \varpi_0) \in \mathcal{R}$.

$$\begin{aligned} \nu_0\mathcal{R}\Upsilon\nu_0 \quad &(\text{or}) \quad \Upsilon\nu_0\mathcal{R}\nu_0 \\ \varpi_0\mathcal{R}\Upsilon\varpi_0 \quad &(\text{or}) \quad \Upsilon\varpi_0\mathcal{R}\varpi_0. \end{aligned}$$

Define sequences $\{\nu_\psi\}$ and $\{\varpi_\psi\}$ by

$$\Upsilon\nu_\psi = \nu_{\psi+1} \quad \text{and} \quad \Upsilon\varpi_\psi = \varpi_{\psi+1}, \quad \forall\psi \in \mathbb{N} \cup \{0\}.$$

Since Υ is $\mathcal{R}$-preserving, hence $\{\nu_\psi\}$ and $\{\varpi_\psi\}$ are $\mathcal{R}$-sequence. That is $\{\nu_\psi, \varpi_\psi\}$ is $\mathcal{R}$-bisequence. Then from condition ((ii)),

$$\aleph(\nu_1, \varpi_1) = \aleph(\Upsilon\nu_0, \Upsilon\varpi_0) \le \varrho\aleph(\nu_0, \varpi_0)$$

proceeding as above,

$$\aleph(\nu_\psi, \varpi_\psi) = \aleph(\Upsilon\nu_{\psi-1}, \Upsilon\varpi_{\psi-1}) \le \varrho^\psi\aleph(\nu_0, \varpi_0)$$

and

$$\aleph(\nu_{\psi+1}, \varpi_\psi) = \aleph(\Upsilon\nu_\psi, \Upsilon\varpi_{\psi-1}) \le \varrho^\psi\aleph(\nu_0, \varpi_0).$$

For $\psi < \tau, \ \forall\psi, \tau \in \mathbb{N}$,

$$\begin{aligned} \aleph(\nu_{\psi+\tau}, \varpi_\psi) &\le [\aleph(\nu_{\psi+\tau}, \varpi_{\psi+1}) + \aleph(\nu_\psi, \varpi_{\psi+1}) + \aleph(\nu_\psi, \varpi_\psi)] \\ &\le \aleph(\nu_{\psi+\tau}, \varpi_{\psi+1}) + 2\varrho^\psi\aleph(\nu_0, \varpi_0) \\ &\le \aleph(\nu_{\psi+\tau}, \varpi_{\psi+2}) + \aleph(\nu_{\psi+1}, \varpi_{\psi+2}) + \aleph(\nu_{\psi+1}, \varpi_{\psi+3}) + 2\varrho^\psi\aleph(\nu_0, \varpi_0) \\ &\le \aleph(\nu_{\psi+\tau}, \varpi_{\psi+3}) + 2^2\varrho^\psi\aleph(\nu_0, \varpi_0) \\ &\ \ \vdots \\ &\le \aleph(\nu_{\psi+\tau}, \varpi_{\psi+\omega}) + 2^{\tau+1}\varrho^\psi\aleph(\nu_0, \varpi_0) \\ &\le 2\varrho^\psi \sum_{\pi=0}^{\infty} 2^\pi\aleph(\nu_0, \varpi_0) < \mathcal{K}_\psi. \end{aligned}$$

Similarly, we get

$$\aleph(\nu_\psi, \varpi_{\psi+\tau}) < \mathcal{K}_\psi.$$

Then

$$\aleph(\nu_\psi, \varpi_\tau) \leq \aleph(\nu_\psi, \varpi_{\psi 0}) + \aleph(\nu_{\psi 0}, \varpi_{\psi 0}) + \aleph(\nu_{\psi 0}, \varpi_\tau) \leq 3\mathcal{K}_{\psi 0} < \epsilon,$$

where $\epsilon > 0$. Here $\{\nu_\psi, \varpi_\psi\}$ converges to zero. Hence $\{\nu_\psi, \varpi_\psi\}$ is a Cauchy $\mathcal{R}$-bisequence. Since $(\Theta, \Xi, \aleph, \mathcal{R})$ is a bipolar $\mathcal{R}$-metric space, we get $\{\nu_\psi, \varpi_\psi\}$ is a convergent Cauchy $\mathcal{R}$-bisequence. So the both sequences $\{\nu_\psi\}$ and $\{\varpi_\psi\}$ have a unique limit.

From (4.1), we have

$$\aleph(\Upsilon\nu, \nu) \leq \aleph(\Upsilon\nu, \Upsilon\varpi_\psi) + \aleph(\Upsilon\nu_\psi, \Upsilon\varpi_\psi) + \aleph(\Upsilon\nu_\psi, \nu).$$

Taking $\psi \to \infty$ in above equation,

$$\aleph(\Upsilon\nu, \nu) \to 0.$$

So $\Upsilon\nu = \nu$. Hence ν is a fixed point of Υ.

Now we prove the uniqueness. Let us choose another fixed point $\omega \in \Theta \cap \Xi$ of Υ. Since the both sequences $\{\nu_\psi\}$ and $\{\varpi_\psi\}$ have the same limit.

$$\nu_0\mathcal{R}\nu, \quad \nu_0\mathcal{R}\omega \quad \text{and} \quad \varpi_0\mathcal{R}\nu, \quad \varpi_0\mathcal{R}\omega.$$

Since Υ is $\mathcal{R}$-preserving, we get

$$\Upsilon\nu_0\mathcal{R}\Upsilon\nu, \quad \Upsilon\nu_0\mathcal{R}\Upsilon\omega \quad \text{and} \quad \Upsilon\varpi_0\mathcal{R}\Upsilon\nu, \quad \Upsilon\varpi_0\mathcal{R}\Upsilon\omega.$$

By simple induction,

$$\Upsilon\nu_\psi\mathcal{R}\Upsilon\nu, \quad \Upsilon\nu_\psi\mathcal{R}\Upsilon\omega \quad \text{and} \quad \Upsilon\varpi_\psi\mathcal{R}\Upsilon\nu, \quad \Upsilon\varpi_\psi\mathcal{R}\Upsilon\omega.$$

From condition ((ii)), we have

$$\aleph(\Upsilon\nu_\psi, \Upsilon\nu) \leq \varrho\aleph(\nu_\psi, \nu) \quad \text{and} \quad \aleph(\Upsilon\nu_\psi, \Upsilon\omega) \leq \varrho\aleph(\nu_\psi, \omega)$$

similarly,

$$\aleph(\Upsilon\varpi_\psi, \Upsilon\nu) \leq \varrho\aleph(\varpi_\psi, \nu) \quad \text{and} \quad \aleph(\Upsilon\varpi_\psi, \Upsilon\omega) \leq \varrho\aleph(\varpi_\psi, \omega).$$

Then,

$$\begin{aligned}\aleph(\nu, \omega) &= \aleph(\Upsilon\nu, \Upsilon\omega) \\ &\leq \varrho[\aleph(\Upsilon\nu, \Upsilon\nu_\psi) + \aleph(\Upsilon\nu_\psi, \Upsilon\mathfrak{z}_\psi) + \aleph(\Upsilon\mathfrak{z}_\psi, \Upsilon\omega)] \\ &\leq \varrho[\aleph(\nu, \nu_\psi) + \aleph(\nu_\psi, \mathfrak{z}_\psi) + \aleph(\mathfrak{z}_\psi, \omega)].\end{aligned}$$

Taking $\psi \to \infty$, we get

$$\aleph(\nu, \omega) \to 0.$$

Thus $\nu = \omega$. Hence Υ has a unique fixed point. □

Theorem 3.2. *Let $(\Theta, \Xi, \aleph, \mathcal{R})$ be a complete bipolar $\mathcal{R}$-metric space and the contravariant contraction $\Upsilon\colon (\Theta, \Xi, \aleph, \mathcal{R}) \rightleftarrows (\Theta, \Xi, \aleph, \mathcal{R})$. The mapping $\Upsilon\colon \Theta \cup \Xi \to \Theta \cup \Xi$ has a unique fixed point.*

Proof. Since Υ is a contravariant contraction, there exist a $\wp \in (0, 1)$ such that

$$\aleph(\Upsilon\varpi, \Upsilon\nu) \leq \wp \cdot \aleph(\nu, \varpi) \quad \forall(\nu, \varpi) \in \Theta \times \Xi.$$

Let $\nu_0 \in \Theta$, define $\Upsilon\nu_\psi = \varpi_\psi$ and $\Upsilon\varpi_\psi = \nu_{\psi+1}$ for all $\psi \in \mathbb{N}$. Then (ν_ψ, ϖ_ψ) is a bisequence on $(\Theta, \Xi, \aleph, \mathcal{R})$.

$$\begin{aligned}
\aleph(\nu_\psi, \varpi_\psi) &= \aleph(\Upsilon\varpi_{\psi-1}, \Upsilon\nu_\psi) \\
&\leq \wp \cdot \aleph(\nu_\psi, \varpi_\psi) \\
&= \aleph(\Upsilon\varpi_\psi, \Upsilon\nu_{\psi+1}) \\
&\leq \wp \cdot \wp\aleph(\nu_{\psi+1}, \varpi_{\psi+1}) \\
&\vdots \\
&\leq \wp^\psi \cdot \aleph(\nu_0, \varpi_0),
\end{aligned}$$

and

$$\begin{aligned}
\aleph(\nu_{\psi+1}, \varpi_\psi) &= \aleph(\Upsilon\varpi_\psi, \Upsilon\nu_\psi) \\
&\leq \wp \cdot \aleph(\nu_{\psi+1}, \varpi_\psi) \\
&= \aleph(\Upsilon\varpi_{\psi+1}, \Upsilon\nu_{\psi+2}) \\
&\leq \wp^2\aleph(\nu_{\psi+2}, \varpi_{\psi+2}) \\
&\vdots \\
&\leq \wp^{\psi+1} \cdot \aleph(\nu_0, \varpi_0).
\end{aligned}$$

For $\psi < \tau,\ \psi, \tau \in \mathbb{N}$, we have

$$\begin{aligned}
\aleph(\nu_{\psi+\tau}, \varpi_\psi) &\leq \aleph(\nu_{\psi+\tau}, \varpi_{\psi+1}) + \aleph(\nu_\psi, \varpi_{\psi+1}) + \aleph(\nu_\psi, \varpi_\psi) \\
&\leq \aleph(\nu_{\psi+\tau}, \varpi_{\psi+1}) + (\wp^\psi + \wp^{\psi+1})\aleph(\nu_0, \varpi_0) \\
&\leq \aleph(\nu_{\psi+\tau}, \varpi_{\psi+2}) + \aleph(\nu_{\psi+1}, \varpi_{\psi+2}) + \aleph(\nu_{\psi+1}, \varpi_{\psi+3}) + (\wp^\psi + \wp^{\psi+1})\aleph(\nu_0, \varpi_0) \\
&\leq \aleph(\nu_{\psi+\tau}, \varpi_{\psi+3}) + (\wp^\psi + \wp^{\psi+1} + \wp^{\psi+2} + \wp^{\psi+3})\aleph(\nu_0, \varpi_0) \\
&\vdots \\
&\leq (\wp^\psi + \cdots + \wp^{\psi+\tau})\aleph(\nu_0, \varpi_0) \\
&\leq \wp^\psi \sum_{\pi=0}^{\infty} \wp^\pi \aleph(\nu_0, \varpi_0) \\
&< \mathcal{K}_\psi.
\end{aligned}$$

Similarly, $\aleph(\nu_\psi, \varpi_{\psi+\tau}) < \mathcal{K}_\psi$. Then

$$\aleph(\nu_\psi, \varpi_\tau) \leq \aleph(\nu_\psi, \varpi_{\psi0}) + \aleph(\nu_{\psi0}, \varpi_{\psi0}) + \aleph(\nu_{\psi0}, \varpi_\tau) \leq 3\mathcal{K}_{\psi0} < \epsilon,$$

where $\epsilon > 0$. Here $\{\nu_\psi, \varpi_\psi\}$ converges to zero. Hence $\{\nu_\psi, \varpi_\psi\}$ is a Cauchy $\mathcal{R}$-bisequence. Since $(\Theta, \Xi, \aleph, \mathcal{R})$ is a bipolar $\mathcal{R}$-metric space, we get $\{\nu_\psi, \varpi_\psi\}$ is a convergent Cauchy $\mathcal{R}$-bisequence. So the both sequences $\{\nu_\psi\}$ and $\{\varpi_\psi\}$ have a unique limit.

From (3), we have

$$\aleph(\nu, \Upsilon\nu) \leq \aleph(\nu, \Upsilon\varpi_\psi) + \aleph(\Upsilon\nu_\psi, \Upsilon\varpi_\psi) + \aleph(\Upsilon\nu_\psi, \Upsilon\nu).$$

Taking $\psi \to \infty$ in above equation, we find $\aleph(\nu, \Upsilon\nu) \to 0$. So that $\Upsilon\nu = \nu$. Hence ν is a fixed point of Υ.

In the next, we prove the uniqueness. Let us choose another fixed point $\omega \in \Theta \cap \Xi$ of Υ. Since the both sequences $\{\nu_\psi\}$ and $\{\varpi_\psi\}$ have the same limit.

$$\nu_0\mathcal{R}\nu, \quad \nu_0\mathcal{R}\omega \quad \text{and} \quad \varpi_0\mathcal{R}\nu, \quad \varpi_0\mathcal{R}\omega.$$

Since Υ is $\mathcal{R}$-preserving, we get

$$\Upsilon\nu_0\mathcal{R}\Upsilon\nu, \quad \Upsilon\nu_0\mathcal{R}\Upsilon\omega \quad \text{and} \quad \Upsilon\varpi_0\mathcal{R}\Upsilon\nu, \quad \Upsilon\varpi_0\mathcal{R}\Upsilon\omega.$$

By simple induction,

$$\Upsilon\nu_\psi\mathcal{R}\Upsilon\nu, \quad \Upsilon\nu_\psi\mathcal{R}\Upsilon\omega \quad \text{and} \quad \Upsilon\varpi_\psi\mathcal{R}\Upsilon\nu, \quad \Upsilon\varpi_\psi\mathcal{R}\Upsilon\omega.$$

Then,

$$\begin{aligned}\aleph(\nu,\omega) &= \aleph(\Upsilon\omega,\Upsilon\nu)\\ &\leq \wp[\aleph(\omega,\nu_\psi) + \aleph(\nu_\psi,\mathfrak{z}_\psi) + \aleph(\mathfrak{z}_\psi,\nu)].\end{aligned}$$

Taking $\psi \to \infty$, we get $\aleph(\nu,\omega) \to 0$, i.e., $\nu = \omega$. Therefore Υ has a unique fixed point. □

Theorem 3.3. *Let $(\Theta,\Xi,\aleph,\mathcal{R})$ be a complete bipolar $\mathcal{R}$-metric space and the contravariant contractions $\Upsilon\colon(\Theta,\Xi,\aleph,\mathcal{R}) \rightleftarrows (\Theta,\Xi,\aleph,\mathcal{R})$ and let $\ell \in (0,\frac{1}{2})$ such that*

$$\aleph(\Upsilon\nu,\Upsilon\varpi) \leq \ell[\aleph(\nu,\Upsilon\nu) + \aleph(\varpi,\Upsilon\varpi)], \ \forall(\nu,\varpi) \in \Theta\times\Xi.$$

Then the mapping $\Upsilon\colon\Theta\cup\Xi \to \Theta\cup\Xi$ has a unique fixed point.

Proof. Let $\nu_0 \in \Theta, \varpi_0 \in \Xi$, define $\Upsilon\nu_\psi = \varpi_\psi$ and $\Upsilon\varpi_\psi = \nu_{\psi+1}$.

$$\begin{aligned}\aleph(\nu_\psi,\varpi_\psi) &= \aleph(\Upsilon\varpi_{\psi-1},\Upsilon\nu_\psi)\\ &\leq \ell[\aleph(\nu_\psi,\Upsilon\nu_\psi) + \aleph(\varpi_{\psi-1},\Upsilon\varpi_{\psi-1})]\\ &\leq \ell[\aleph(\nu_\psi,\varpi_\psi) + \aleph(\varpi_{\psi-1},\nu_\psi)]\\ \aleph(\nu_\psi,\varpi_\psi) - \ell\aleph(\nu_\psi,\varpi_\psi) &\leq \ell\aleph(\varpi_{\psi-1},\nu_\psi)\\ (1-\ell)\aleph(\nu_\psi,\varpi_\psi) &= \ell\aleph(\nu_\psi,\varpi_{\psi-1})\\ \aleph(\nu_\psi,\varpi_\psi) &\leq \frac{\ell}{1-\ell}\aleph(\nu_\psi,\varpi_{\psi-1})\\ \aleph(\nu_\psi,\varpi_\psi) &\leq \ell^\psi\aleph(\nu_\psi,\varpi_{\psi-1}),\end{aligned}$$

where $\ell^\psi = \frac{\ell}{1-\ell}$.

$$\begin{aligned}\aleph(\nu_{\psi+1},\varpi_\psi) &= \aleph(\Upsilon\varpi_\psi,\Upsilon\nu_\psi)\\ &\leq \ell[\aleph(\nu_{\psi+1},\Upsilon\nu_\psi) + \aleph(\varpi_{\psi-1},\Upsilon\varpi_\psi)]\\ \aleph(\nu_{\psi+1},\varpi_\psi) - \ell\aleph(\nu_{\psi+1},\varpi_\psi) &\leq \ell\aleph(\varpi_{\psi-1},\nu_{\psi+1})\\ (1-\ell)\aleph(\nu_{\psi+1},\varpi_\psi) &= \ell\aleph(\nu_{\psi+1},\varpi_{\psi-1})\\ \aleph(\nu_{\psi+1},\varpi_\psi) &\leq \frac{\ell}{1-\ell}\aleph(\nu_{\psi+1},\varpi_{\psi-1}).\end{aligned}$$

Now, we have

$$\begin{aligned}\aleph(\nu_\psi,\varpi_\psi) &\leq \ell^\psi\aleph(\nu_0,\varpi_0)\\ \aleph(\nu_{\psi+1},\varpi_\psi) &\leq \ell^{\psi+1}\aleph(\nu_0,\varpi_0)\end{aligned}$$

For $\psi < \tau,\ \psi, \tau \in \mathbb{N}$, we get

$$\begin{aligned}
\aleph(\nu_{\psi+\tau}, \varpi_{\psi}) &\leq \aleph(\nu_{\psi+\tau}, \varpi_{\psi+1}) + \aleph(\nu_{\psi}, \varpi_{\psi+1}) + \aleph(\nu_{\psi}, \varpi_{\psi}) \\
&\leq \aleph(\nu_{\psi+\tau}, \varpi_{\psi+1}) + (\ell^{\psi} + \ell^{\psi+1})\aleph(\nu_0, \varpi_0) \\
&\leq \aleph(\nu_{\psi+\tau}, \varpi_{\psi+2}) + \aleph(\nu_{\psi+1}, \varpi_{\psi+2}) + \aleph(\nu_{\psi+1}, \varpi_{\psi+3}) + (\ell^{\psi} + \ell^{\psi+1})\aleph(\nu_0, \varpi_0) \\
&\leq \aleph(\nu_{\psi+\tau}, \varpi_{\psi+3}) + (\ell^{\psi} + \ell^{\psi+1} + \ell^{\psi+2} + \ell^{\psi+3})\aleph(\nu_0, \varpi_0) \\
&\ \ \vdots \\
&\leq (\ell^{\psi} + \cdots + \ell^{\psi+\tau})\aleph(\nu_0, \varpi_0) \\
&\leq \ell^{\psi} \sum_{\pi=0}^{\infty} \ell^{\pi} \aleph(\nu_0, \varpi_0) \\
&< \mathcal{K}_{\psi}.
\end{aligned}$$

Similarly, $\aleph(\nu_{\psi}, \varpi_{\psi+\tau}) < \mathcal{K}_{\psi}$. Then,

$$\aleph(\nu_{\psi}, \varpi_{\tau}) \leq \aleph(\nu_{\psi}, \varpi_{\psi 0}) + \aleph(\nu_{\psi 0}, \varpi_{\psi 0}) + \aleph(\nu_{\psi 0}, \varpi_{\tau}) \leq 3\mathcal{K}_{\psi 0} < \epsilon,$$

where $\epsilon > 0$. Here $\{\nu_{\psi}, \varpi_{\psi}\}$ converges to zero. Hence $\{\nu_{\psi}, \varpi_{\psi}\}$ is a Cauchy $\mathcal{R}$-bisequence. Since $(\Theta, \Xi, \aleph, \mathcal{R})$ is a bipolar $\mathcal{R}$-metric space, we obtain $\{\nu_{\psi}, \varpi_{\psi}\}$ is a convergent Cauchy $\mathcal{R}$-bisequence. So the sequences $\{\nu_{\psi}\}$ and $\{\varpi_{\psi}\}$ have a same limit. From (3),

$$\aleph(\Upsilon\nu, \nu) \leq \aleph(\Upsilon\nu, \Upsilon\varpi_{\psi}) + \aleph(\Upsilon\nu_{\psi}, \Upsilon\varpi_{\psi}) + \aleph(\Upsilon\nu_{\psi}, \nu).$$

Taking $\psi \to \infty$ in above equation, we find $\aleph(\Upsilon\nu, \nu) \to 0$. So $\Upsilon\nu = \nu$. Thus ν is a fixed point of Υ.

To prove uniqueness, let us choose another fixed point $\omega \in \Theta \cap \Xi$ of Υ. So

$$\Upsilon\nu_{\psi}\mathcal{R}\Upsilon\nu, \quad \Upsilon\nu_{\psi}\mathcal{R}\Upsilon\omega \quad \text{and} \quad \Upsilon\varpi_{\psi}\mathcal{R}\Upsilon\nu, \quad \Upsilon\varpi_{\psi}\mathcal{R}\Upsilon\omega.$$

Then,

$$\begin{aligned}
\aleph(\nu, \omega) &= \aleph(\Upsilon\nu, \Upsilon\omega) \\
&\leq \ell[\aleph(\nu, \Upsilon\nu) + \aleph(\omega, \Upsilon\omega)].
\end{aligned}$$

such that $\aleph(\nu, \omega) \to 0$, i.e., $\nu = \omega$. Hence Υ has a unique fixed point. □

Example 3.4. *Let* $\Theta = \{\mathcal{U}_{\psi}(\mathbb{R})\colon \mathcal{U}_{\psi}(\mathbb{R})$ *be an upper triangular matrices over* $\mathbb{R}\}$, $\Xi = \{\mathcal{V}_{\psi}(\mathbb{R})\colon \mathcal{V}_{\psi}(\mathbb{R})$ *be a lower triangular matrices over* $\mathbb{R}\}$ *and the map* $\aleph\colon \Theta \times \Xi \to [0, +\infty)$ *defined by*

$$\aleph(\mathcal{M}, \mathcal{N}) = \sum_{i,j=1}^{\psi} |\nu_{ij} - \varpi_{ij}|,$$

$\forall \mathcal{M} = (\nu_{ij})_{\psi\times\psi} \in \Theta$ *and* $\mathcal{N} = (\varpi_{ij})_{\psi\times\psi} \in \Xi$ *and* $(\nu, \varpi) \in \mathbb{R}$ *iff* $\nu, \varpi \geq 0$. *Then* $(\Theta, \Xi, \aleph, \mathcal{R})$ *is a complete bipolar* $\mathcal{R}$*-metric space.*

$$\Upsilon(\mathcal{M}) = \left(\frac{\nu_{ij}}{4}\right)_{\psi\times\psi}$$

$\forall \mathcal{M} = (\nu_{ij})_{\psi\times\psi} \in \mathcal{U}_{\psi}(\mathbb{R}) \cup \mathcal{V}_{\psi}(\mathbb{R})$. *Now,*

$$\begin{aligned}
\aleph(\Upsilon(\mathcal{M}), \Upsilon(\mathcal{N})) &= \frac{1}{16} \sum_{i,j=1}^{\psi} |\nu_{ij} - \varpi_{ij}| \\
&\leq \frac{1}{2} \sum_{i,j=1}^{\psi} |\nu_{ij} - \varpi_{ij}| \\
&= \wp\aleph(\mathcal{M}, \mathcal{N}).
\end{aligned}$$

$\forall \mathcal{M} = (\nu_{ij})_{\psi\times\psi} \in \Theta$ and $\mathcal{N} = (\varpi_{ij})_{\psi\times\psi} \in \Xi$. All the conditions of Theorem 3.2 are satisfied with $\wp = \frac{1}{2}$. Hence, Υ has a unique fixed point $(0_{\psi\times\psi}, 0_{\psi\times\psi}) \in \mathcal{U}_\psi(\mathbb{R}) \cup \mathcal{V}_\psi(\mathbb{R})$, where $0_{\psi\times\psi}$ is the null matrix.

Example 3.5. *Let $\Theta = \{0,1,2,3\}$ and $\Xi = \{\frac{1}{6}, \frac{1}{2}, 1, 4, 5, 6\}$ be equipped with*

$$\aleph(\nu, \varpi) = |\nu - \varpi|$$

for all $\nu \in \Theta, \varpi \in \Xi$ and $(\nu, \varpi) \in \mathcal{R}$ if and only if $\nu, \varpi \geq 0$. Then $(\Theta, \Xi, \aleph, \mathcal{R})$ is a complete bipolar $\mathcal{R}$-metric space. Define $\Upsilon\colon \Theta \cup \Xi \rightleftarrows \Theta \cup \Xi$ by

$$\Upsilon(\nu) = \begin{cases} \frac{1}{6}, & \text{if} \quad \nu \in \{1,3\} \\ 0, & \text{if} \quad \nu \in \{0, \frac{1}{6}, \frac{1}{2}, 1, 4\} \end{cases}$$

for all $\nu \in \Theta \cup \Xi$. Let $\nu \in \Theta, \varpi \in \Xi$. Then, we can easily get

$$\aleph(\Upsilon\nu, \Upsilon\varpi) \leq \frac{1}{2}\Big[\aleph(\nu, \Upsilon\nu) + \aleph(\varpi, \Upsilon\varpi)\Big].$$

All the conditions of Theorem 3.3 are satisfied with $\ell = \frac{1}{2}$. Thus Υ has a unique fixed point as $\nu = 0$.

4. Application

To solve an integral equation analytically, we use the obtained results from Theorem 3.1.

Theorem 4.1. *Suppose we have the integral equation as follows:*

$$\nu(\mho) = \mathfrak{h}(\mho) + \int_{\Theta\cup\Xi} \mathcal{G}(\mho, \nabla, \nu(\nabla))d\nabla, \quad \nu \in \Theta \cup \Xi.$$

where $\Theta \cup \Xi$ is a Lebesgue measurable set. Let us assume that

(i) $\mathcal{G}\colon (\Theta^2 \cup \Xi^2) \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $\mathfrak{b} \in \mathcal{L}^\infty(\Theta) \cup \mathcal{L}^\infty(\Xi)$,

(ii) there is a continuous function $\theta\colon \Theta^2 \cup \Xi^2 \to \mathbb{R}^+$ and $\varrho \in (0,1)$ such that

$$|\mathcal{G}(\mho, \nabla, \nu(\nabla)) - \mathcal{G}(\mho, \nabla, \varpi(\nabla))| \leq \varrho\theta(\mho, \nabla)(|\nu(\nabla) - \varpi(\nabla)|)$$

for all $\nu, \varpi \in \Theta^2 \cup \Xi^2$,

(iii) $\left|\int_{\Theta\cup\Xi} \theta(\mho, \nabla)d\nabla\right| \leq 1$, *i.e.*, $\sup_{\mho\in\Theta\cup\Xi} \int_{\Theta\cup\Xi} \theta(\mho, \nabla)d\nabla \leq 1$.

Then the integral equation has a unique solution in $\mathcal{L}^\infty(\Theta) \cup \mathcal{L}^\infty(\Xi)$.

Proof. Let $\Theta = \mathcal{L}^\infty(\Theta)$ and $\Xi = \mathcal{L}^\infty(\Xi)$ be two normed linear spaces, where Θ, Ξ are Lebesgue measurable sets and $\tau(\Theta \cup \Xi) < \infty$. Consider $\aleph\colon \Theta \times \Xi \to \mathbb{R}^+$ to be defined by $\aleph(\nu, \varpi) = |\nu - \varpi|$ and $(\nu, \varpi) \geq 0$ iff $(\nu, \varpi) \in \mathcal{R}$. Then $(\Theta, \Xi, \aleph, \mathcal{R})$ is a complete bipolar $\mathcal{R}$-metric space. Define the covariant mapping $\Upsilon\colon \mathcal{L}^\infty(\Theta) \cup \mathcal{L}^\infty(\Xi) \to \mathcal{L}^\infty(\Theta) \cup \mathcal{L}^\infty(\Xi)$ by

$$\Upsilon(\nu(\mho)) = \mathfrak{h}(\mho) + \int_{\Theta\cup\Xi} \mathcal{G}(\mho, \nabla, \nu(\nabla))d\nabla, \quad \nu \in \Theta \cup \Xi.$$

Then,

$$\begin{aligned}
\aleph(\Upsilon\nu(\mho),\Upsilon\varpi(\mho)) &= |\Upsilon\nu(\mho)-\Upsilon\varpi(\mho)| \\
&= \left|\mathfrak{h}(\mho)+\int_{\Theta\cup\Xi}\mathcal{G}(\mho,\nabla,\nu(\nabla))d\nabla-\mathfrak{h}(\mho)-\int_{\Theta\cup\Xi}\mathcal{G}(\mho,\nabla,\varpi(\nabla))d\nabla\right| \\
&\le \int_{\Theta\cup\Xi}|\mathcal{G}(\mho,\nabla,\nu(\nabla))-\mathcal{G}(\mho,\nabla,\varpi(\nabla))|d\nabla \\
&\le \int_{\Theta\cup\Xi}\varrho\theta(\mho,\nabla)(|\nu(\nabla)-\varpi(\nabla)|)d\nabla \\
&\le \varrho|\nu(\nabla)-\varpi(\nabla)| \\
&= \varrho\aleph(\nu,\varpi)
\end{aligned}$$

All the conditions of Theorem 3.1 are satisfied. Thus Υ has a unique solution to integral equation. □

In the next subsection of application, we deal with the boundary value problem of our obtained results as follows:

4.1. Application to boundary value problem

Now, from our obtained results, we can apply the application that is more specific and easy to find the existence on unique solution on boundary value problem:

$$\frac{d^2\nu}{d\mho^2}+\mathfrak{g}(\mho,\nu(\mho))=0,\quad \mho\in[0,1],\quad \nu(0)=\nu(1)=0, \tag{4.1}$$

where $\mathfrak{g}\colon[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.
From the boundary value problem is equivalent to the integral equation:

$$\nu(\mho)=\int_0^1\mathcal{G}(\mho,\nabla)\mathfrak{g}(\nabla,\nu(\nabla))d\nabla \tag{4.2}$$

where the Green's function $\mathcal{G}(\mho,\nabla)$ is defined by

$$\mathcal{G}(\mho,\nabla)=\begin{cases}\mho(1-\nabla), & \text{if } \ 0\le\mho\le\nabla\le1\\ \nabla(1-\mho), & \text{if } \ 0\le\nabla\le\mho\le1.\end{cases}$$

From the properties of the Green's function $\mathcal{G}$ is defined by:

(a) $\mathcal{G}(\mho,\nabla)\ge0$ for all $\mho,\nabla\in[0,1]$:

(b) $\sup_{0\le\mho\le1}\int_0^1\mathcal{G}(\mho,\nabla)d\nabla\le1$.

Theorem 4.2. *If the function* $\mathfrak{g}$ *is such that*

$$|\mathfrak{g}(\nabla,\mathfrak{a})-\mathfrak{g}(\nabla,\mathfrak{b})|\le\varrho|\mathfrak{a}-\mathfrak{b}|$$

for all $\nabla\in[0,1]$ *and for all* $\mathfrak{a},\mathfrak{b}\in\mathbb{R}$ *and* $\varrho\in(0,1)$, *then the boundary value problem* (4.1) *has a unique solution.*

Proof. Let $\Theta=(C[0,1],(-\infty,0])$ and $\Xi=(C[0,1],[0,\infty))$ be two sets of all continuous functions on $[0,1]$. For $\Theta=(C[0,1],(-\infty,0])$ lies in $(-\infty,0]$ and for $\Xi=(C[0,1],[0,\infty))$ lies in $[0,\infty)$, consider $\aleph\colon\Theta\times\Xi\to\mathbb{R}^+$ defined by

$$\aleph(\nu,\varpi)=\sup_{\nabla\in[0,1]}|\nu(\nabla)-\varpi(\nabla)|.$$

Define $(\nu,\varpi) \geq 0$ if and only if $(\nu,\varpi) \in \mathcal{R}$. Now we consider a mapping Υ defined on $\mathcal{S}$ by

$$\Upsilon\nu(\mho) = \int_0^1 \mathcal{G}(\mho,\nabla)\mathfrak{g}(\nabla,\nu(\nabla))d\nabla, \quad \text{for all} \quad \mho \in [0,1].$$

Then, $\nu,\varpi \in \mathcal{S}$ and for all $\mho \in [0,1]$, we have

$$\begin{aligned}
|\Upsilon\nu(\mho) - \Upsilon\varpi(\mho)| &= \left| \int_0^1 \mathcal{G}(\mho,\nabla)\mathfrak{g}(\nabla,\nu(\nabla))d\nabla - \int_0^1 \mathcal{G}(\mho,\nabla)\mathfrak{g}(\nabla,\varpi(\nabla))d\nabla \right| \\
&\leq \int_0^1 \mathcal{G}(\mho,\nabla)|\mathfrak{g}(\nabla,\nu(\nabla)) - \mathfrak{g}(\nabla,\varpi(\nabla))|d\nabla \\
&\leq \varrho \int_0^1 \mathcal{G}(\mho,\nabla)|\nu(\nabla) - \varpi(\nabla)|d\nabla \\
&\leq \varrho\aleph(\nu,\varpi) \int_0^1 \mathcal{G}(\mho,\nabla)d\nabla \\
&\leq \varrho\aleph(\nu,\varpi).
\end{aligned}$$

Thus,

$$\aleph(\Upsilon\nu,\Upsilon\varpi) \leq \varrho\aleph(\nu,\varpi).$$

Thus, all conditions in Theorem 3.1 are satisfied and Υ has a unique fixed point and unique solution to the boundary value problem. □

5. Conclusion

In this article, we introduce the notion of bipolar $\mathcal{R}$-metric space and proved fixed point theorems. Moreover, we provide some suitable examples of our main results. Based on our results, we apply to find analytical solution to integral equation and boundary value problems.

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