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The Beurling degree of inner matrix functions (II)

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Abstract. In this paper we show that for square-inner matrix functions, quasi-equivalence preserves the Beurling degree by using the Moore–Nordgren theorem.

1. Introduction

Let D and E be separable complex Hilbert spaces. Write $\mathcal{B}(D, E)$ for the set of all bounded linear operators from D into E and abbreviate $\mathcal{B}(E, E)$ to $\mathcal{B}(E)$. For an operator $T \in \mathcal{B}(E)$, an *orbit* of $x \in E$ under T is defined by

$$\mathcal{O}_x(T) := \left\{T^n x : n \geq 0\right\} = \{x, Tx, T^2x, \dots\}.$$

If $\bigvee \mathcal{O}_x(T) = E$, then x is a cyclic vector for T. For example, if $x = 1 \in H^2$ then $\bigvee \mathcal{O}_1(S) = H^2$. The *spectral multiplicity*, denoted by μ_T, of an operator $T \in \mathcal{B}(E)$ is defined by

$$\mu_T := \inf\left\{\text{card}\, F : \bigvee_{x \in F} \mathcal{O}_x(T) = E,\ F \subseteq E\right\},$$

For a Banach space $\mathcal{X}$, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and let $L^\infty_{\mathcal{X}} \equiv L^\infty_{\mathcal{X}}(\mathbb{T})$ be the set of $\mathcal{X}$-valued essentially bounded measurable functions on $\mathbb{T}$. We also let $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy space and $H^\infty_{\mathcal{X}} \equiv H^\infty_{\mathcal{X}}(\mathbb{T}) = L^\infty_{\mathcal{X}} \cap H^2_{\mathcal{X}}$. We observe that $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$.

Let $M_{N\times r}$ denote the set of all $N \times r$ complex matrices and write $M_N := M_{N\times N}$. Write I_N for the $N \times N$ identity matrix. A matrix-valued function $\Delta \in H^\infty_{M_{N\times r}}$ is called an *inner* matrix function if $\Delta(z)$ is an isometry as an operator from $\mathbb{C}^r$ into $\mathbb{C}^N$ for almost all $z \in \mathbb{T}$, i.e., $\Delta^*\Delta = I_r$ a.e. on $\mathbb{T}$. Thus, for $\Delta \in H^\infty_{M_{N\times r}}$ to be inner,

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$N \geq r$. To avoid triviality, we assume that inner matrix function is not unitary matrix. For an inner matrix function Δ in $H^\infty_{M_{N\times r}}$, let

$$\mathcal{H}(\Delta) := H^2_{\mathbb{C}^N} \ominus \Delta H^2_{\mathbb{C}^r}.$$

The space $\mathcal{H}(\Delta)$ is often called a model space or a de Branges–Rovnyak space [2], [5], [7]. Let $S_{\mathbb{C}^N}$ be the shift operator on $H^2_{\mathbb{C}^N}$, i.e.,

$$(S_{\mathbb{C}^N} f)(z) = zf(z) \quad \text{for each } f \in H^2_{\mathbb{C}^N}.$$

By the Beurling–Lax Theorem (cf. [5]), every invariant subspace of $S^*_{\mathbb{C}^N}$ is of the form $\mathcal{H}(\Delta)$ for some inner function $\Delta \in H^\infty_{M_{N\times r}}$. Thus the restriction $S^*_{\mathbb{C}^N}|_{\mathcal{H}(\Delta)}$ of $S^*_{\mathbb{C}^N}$ to its invariant subspace $\mathcal{H}(\Delta)$ is in $\mathcal{B}(\mathcal{H}(\Delta))$. Often, $S^*_{\mathbb{C}^N}|_{\mathcal{H}(\Delta)}$ is called the truncated backward shift operator.

For $\Phi \in L^2_{M_{N\times r}}$, the Hankel operator $H_\Phi : H^2_{\mathbb{C}^r} \to H^2_{\mathbb{C}^N}$ is a densely defined operator defined by

$$H_\Phi p := JP_-(\Phi p) \quad (p \in \mathcal{P}_{\mathbb{C}^r}),$$

where $\mathcal{P}_{\mathbb{C}^r}$ is the set of $\mathbb{C}^r$-valued polynomials, P_- is the orthogonal projection from $L^2_{\mathbb{C}^N}$ onto $L^2_{\mathbb{C}^N} \ominus H^2_{\mathbb{C}^N}$, and J denotes the unitary operator from $L^2_{\mathbb{C}^N}$ to $L^2_{\mathbb{C}^N}$ given by $(Jg)(z) := \overline{z}g(\overline{z})$ for $g \in L^2_{\mathbb{C}^N}$. For a function $\Phi \in L^2_{M_{N\times r}}$, write

$$\breve{\Phi}(z) := \Phi(\overline{z}) \quad \text{and} \quad \widetilde{\Phi} := \Phi(\overline{z})^*.$$

It is known that $\ker H^*_{\breve{\Phi}}$ is invariant for $S_{\mathbb{C}^N}$. Thus, by the Beurling–Lax Theorem, $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$ for some inner function $\Delta \in H^\infty_{M_{N\times r}}$.

Definition 1.1. *[2] Let Δ be an inner matrix function in $H^\infty_{M_{N\times r}}$. Then the Beurling degree of Δ, denoted by $\deg_B(\Delta)$, is defined by*

$$\deg_B(\Delta) := \min\Big\{m : \ \ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r} \textit{ for some } \Phi \in L^2_{M_{N\times m}}\Big\}.$$

If $\Phi \in L^\infty_{M_{N\times m}}$, then we can easily check that $H^*_{\breve{\Phi}} = H_{\Phi^*}$. Note that $\deg_B(\Delta) \leq r+1$ (cf. [2, Corollary 4.2]). Also, $\deg_B(\Delta) \leq N$, because if $N = r$, then $\ker H_{\Delta^*} = \Delta H^2_{\mathbb{C}^N}$. For a subset F of $H^2_{\mathbb{C}^N}$, let E^*_F denote the smallest $S^*_{\mathbb{C}^N}$-invariant subspace containing F, i.e.,

$$E^*_F = \bigvee \Big\{ S^{*n}_{\mathbb{C}^N} F : \ n \geq 0 \Big\}.$$

Then by the Beurling–Lax Theorem, $E^*_F = \mathcal{H}(\Delta)$ for an inner function Δ with values in $M_{N\times r}$. Now, given a backward shift-invariant subspace $\mathcal{H}(\Delta)$, we may ask:

Question 1.2. *What is the smallest number of vectors in F satisfying $\mathcal{H}(\Delta) = E^*_F$?*

We here observe that Question 1.2 is identical to the problem of finding the spectral multiplicity of the truncated backward shift operator $S^*_{\mathbb{C}^N}|_{\mathcal{H}(\Delta)}$.

If $\Phi \in H^2_{M_{N\times r}}$ and $\{e_k : k = 1, 2, \cdots, r\}$ is an orthonormal basis for $\mathbb{C}^r$, write

$$\phi_k := \Phi e_k \in H^2_{\mathbb{C}^N}.$$

We then define

$$\{\Phi\} := \{\phi_1, \ldots, \phi_r\} \subseteq H^2_{\mathbb{C}^N}.$$

Hence, $\{\Phi\}$ may be regarded as the set of "column" vectors ϕ_k (in $H^2_{\mathbb{C}^N}$) of Φ. It was known ([2, Lemma 2.9]) that if $\Phi \in M_{N\times r}$, then

$$E^*_{\{\Phi\}} = \text{cl ran}\, H_{\overline{z}\breve{\Phi}}. \tag{1}$$

Also, an answer to Question 1.2 was given in [2, Theorem 4.6]. Indeed, it was shown that there is a connection between the Beurling degree and the spectral multiplicity: more concretely, given an inner function Δ with values in $M_{N\times r}$, if $T := S^*_{\mathbb{C}^N}|_{\mathcal{H}(\Delta)}$, then

$$\mu_T = \deg_B(\Delta). \tag{2}$$

For an operator $A_k \in \mathcal{B}(E_k)$ (E_k is a Hilbert space for each $k = 1, 2$), we say that A_1 is a *deformation* of A_2, and write $A_1 \prec A_2$, if there exists an operator $T \in \mathcal{B}(E_1, E_2)$ such that T is injective and has dense range, and $TA_1 = A_2T$. If $A_1 \prec A_2$ and $A_2 \prec A_1$, we say that A_1 and A_2 are *quasi-similar*.

For an inner matrix function $\Delta \in H^\infty_{M_{N\times r}}$, write

$$\mathbf{S}_\Delta := P_{\mathcal{H}(\Delta)} S_{\mathbb{C}^N}|_{\mathcal{H}(\Delta)}.$$

We recall that the Model Theorem (cf. [5]) states that if $T \in \mathcal{B}(E)$ is a contraction (i.e., $\|T\| \le 1$) satisfying $\lim_{n\to\infty} T^{*n}x = 0$ for each $x \in E$, then T is unitarily equivalent to $\mathbf{S}_\Delta$ for some inner function Δ. In this case, $\mathbf{S}_\Delta$ is called the *model operator* of T and Δ is called the *characteristic function* of T. Also, for the functions Θ_1 and Θ_2 in $H^\infty_{M_N}$ are called *quasi-equivalent* if there exist functions X and Y in $H^\infty_{M_N}$ such that $X\Theta_1 = \Theta_2 Y$ and such that the inner parts $(\det X)^i$ and $(\det Y)^i$ of the corresponding determinants are coprime to $(\det \Theta_k)^i$, $k = 1, 2$.

In [3, Lemma 2.5], the authors have shown that if Δ_1 and Δ_2 are quasi-equivalent then $\deg_B(\Delta_1)$ and $\deg_B(\Delta_2)$ coincide. In this paper we prove the same result, without using the equality (2), by a direct use of the Moore–Nordgren theorem.

2. The main result

We begin with the Moore–Nordgren theorem. The following lemma is the crucial point of the Moore–Nordgren theory.

Lemma 2.1. (**Moore–Nordgren Theorem**) *[4], [5], [6] Let Δ_k ($k = 1, 2$) be an inner matrix function in $H^\infty_{M_N}$. If Δ_1 and Δ_2 are quasi-equivalent then $\mathbf{S}_{\Delta_1}$ and $\mathbf{S}_{\Delta_2}$ are quasi-similar.*

Lemma 2.2. *[2] Let Δ be an inner matrix function in $H^\infty_{M_N}$. Then, for $f \in H^2_{\mathbb{C}^N}$,*

$$f \in \mathcal{H}(\Delta) \iff \Delta^* f \in L^2_{\mathbb{C}^N} \ominus H^2_{\mathbb{C}^N}.$$

We are ready for proving:

Theorem 2.3. *Let Δ_i ($i = 1, 2$) be an inner matrix function in $H^\infty_{M_N}$. If $\mathbf{S}_{\widetilde{\Delta}_1} \prec \mathbf{S}_{\widetilde{\Delta}_2}$, then*

$$deg_B(\Delta_2) \le deg_B(\Delta_1).$$

Proof. Suppose $\mathbf{S}_{\widetilde{\Delta}_1}$ is a deformation of $\mathbf{S}_{\widetilde{\Delta}_2}$. Then there exists an operator $T : \mathcal{H}(\widetilde{\Delta}_1) \to \mathcal{H}(\widetilde{\Delta}_2)$ such that T is injective and has dense range, and

$$T\mathbf{S}_{\widetilde{\Delta}_1} = \mathbf{S}_{\widetilde{\Delta}_2} T. \tag{3}$$

For each $k = 1, 2$, let

$$V_k f := \bar{z}\widetilde{\Delta}_k \breve{f} \quad \text{for } f \in \mathcal{H}(\Delta_k).$$

If $f \in \mathcal{H}(\Delta_k)$, then $\widetilde{\Delta}^*_k V_k f = \bar{z}\breve{f} \in L^2_{\mathbb{C}^N} \ominus H^2_{\mathbb{C}^N}$. Thus, by Lemma 2.2, V_k is an isometry from $\mathcal{H}(\Delta_k)$ onto $\mathcal{H}(\widetilde{\Delta}_k)$. For $g \in \mathcal{H}(\widetilde{\Delta}_k)$, let $f := \bar{z}\Delta_k \breve{g}$. Then, by Lemma 2.2, $f \in \mathcal{H}(\Delta_k)$ and

$$V_k f = \bar{z}\widetilde{\Delta}_k z \breve{\Delta}_k g = g,$$

which implies that V_k is a surjection. Thus V_k is a unitary operator from $\mathcal{H}(\Delta_k)$ onto $\mathcal{H}(\widetilde{\Delta}_k)$. We claim that

$$V_2^{-1}TV_1S^*_{\mathbb{C}^N}|_{\mathcal{H}(\Delta_1)} = S^*_{\mathbb{C}^N}|_{\mathcal{H}(\Delta_2)}V_2^{-1}TV_1. \tag{4}$$

If $f \in \mathcal{H}(\Delta_k)$, then

$$\mathbf{S}_{\widetilde{\Delta}_k}V_kf = P_{\mathcal{H}(\widetilde{\Delta}_k)}(\widetilde{\Delta}_k\breve{f}) = (I_n - \widetilde{\Delta}_kP_+\breve{\Delta}_k)(\widetilde{\Delta}_k\breve{f}) = \widetilde{\Delta}_k(\breve{f} - \widehat{f}(0)),$$

where P_+ is the orthogonal projection from $L^2_{\mathbb{C}^N}$ onto $H^2_{\mathbb{C}^N}$ and $\widehat{f}(0)$ denotes the 0-th Fourier coefficient of f. It thus follows that$\mathbf{S}_{\widetilde{\Delta}_k} = V_kS^*_{\mathbb{C}^N}|_{\mathcal{H}(\Delta_k)}V_k^{-1}$ for $k = 1, 2$. Thus, by (3), we have that

$$TV_1S^*_{\mathbb{C}^N}|_{\mathcal{H}(\Delta_1)}V_1^{-1} = V_2S^*_{\mathbb{C}^N}|_{\mathcal{H}(\Delta_2)}V_2^{-1}T,$$

which proves (4). We now write $C := V_2^{-1}TV_1$. It is clear that $C \in \mathcal{B}(\mathcal{H}(\Delta_1), \mathcal{H}(\Delta_2))$ is injective and has dense range. Also, it follows from (4) that

$$C\Big(S^*_{\mathbb{C}^N}|_{\mathcal{H}(\Delta_1)}\Big)^n = \Big(S^*_{\mathbb{C}^N}|_{\mathcal{H}(\Delta_2)}\Big)^nC \quad \text{for all } n = 0, 1, 2, \ldots. \tag{5}$$

Let $r := \deg_B(\Delta_1)$. Then there exists a function $\Phi \in L^2_{M_{N\times r}}$ such that $\ker H^*_{\breve{\Phi}} = \Delta_1H^2_{\mathbb{C}^N}$. Then we may assume that $\widehat{\Phi}(n) = 0$ for all $n \le 0$. Thus if we put $F := \overline{z}\Phi \in H^2_{M_{N\times r}}$, then it follows from (1) that

$$E^*_{\{F\}} = \text{cl ran}\, H_{\overline{z}\breve{F}} = \text{cl ran}\, H_{\breve{\Phi}} = \Big(\ker H^*_{\breve{\Phi}}\Big)^\perp = \mathcal{H}(\Delta_1). \tag{6}$$

Let $\{e_k : k = 1, 2, \cdots, r\}$ be the canonical basis for $\mathbb{C}^r$, and let

$$\Psi := [CFe_1, CFe_2, \cdots, CFe_r] \in L^2_{M_{N\times r}}.$$

Note that $Fe_j \in \mathcal{H}(\Delta_1)$ and $CFe_j \in \mathcal{H}(\Delta_2)$ for all $j = 1, 2, \cdots, r$. Thus it follows from (1), (5) and (6) that

$$\begin{aligned}\text{cl ran}\, H_{\overline{z}\breve{\Psi}} &= \bigvee\Big\{S^{*n}_{\mathbb{C}^N}CFe_j : n \ge 0,\ j = 1, 2, \cdots, r\Big\}\\ &= \bigvee\Big\{CS^{*n}_{\mathbb{C}^N}Fe_j : n \ge 0,\ j = 1, 2, \cdots, r\Big\}\\ &= \text{cl}\, C\mathcal{H}(\Delta_1)\\ &= \mathcal{H}(\Delta_2).\end{aligned}$$

Thus $\ker H^*_{\overline{z}\breve{\Psi}} = \Big(\text{cl ran}\, H_{\overline{z}\breve{\Psi}}\Big)^\perp = \Delta_2H^2_{\mathbb{C}^N}$, which implies that $\deg_B(\Delta_2) \le \deg_B(\Delta_1)$. This completes the proof. □

However, the converse of Theorem 2.3 is not true in general, as we see in the following example.

Example 2.4. *Let*

$$\Delta_k = \begin{bmatrix} z^k & 0 \\ 0 & z^k \end{bmatrix} \quad (k = 1, 2).$$

Then $\deg_B(\Delta_k) \le 2$. *Suppose that* $\deg_B(\Delta_1) = 1$, *then there exists* $\Phi \in L^2_{M_{2\times 1}}$ *such that* $\ker H^*_{\breve{\Phi}} = \Delta_1H^2_{\mathbb{C}^2}$. *Then, by [2, Theorem 3.1],* Φ *can be written as*

$$\Phi = \Delta_1A^* + B,$$

where $A \in H^2_{M_{1\times 2}}$ *is such that* Δ_1 *and* A *are right coprime. Observe that*

$$\widetilde{\Delta}_1H^2_{\mathbb{C}^2} \bigvee \widetilde{A}H^2_{\mathbb{C}^1} \ne H^2_{\mathbb{C}^2}.$$

Thus Δ_1 *and* A *are not right coprime, which is a contradiction. Hence* $deg_B(\Delta_1) = 2$. *Similarly, we can show that* $deg_B(\Delta_2) = 2$. *Note that*

$$\mathbf{S}_{\Delta_1} = P_{\mathcal{H}(\Delta_1)}S_{\mathbb{C}^2}|_{\mathcal{H}(\Delta_1)} = 0 \quad \textit{and} \quad (\mathbf{S}_{\Delta_2}f)(z) = \widehat{f}(0)z \ne 0.$$

Thus $\mathbf{S}_{\Delta_1}$ *is not a deformation of* $\mathbf{S}_{\Delta_2}$.

Corollary 2.5. *Let Δ_i ($i = 1, 2$) be an inner matrix function in $H^\infty_{M_N}$. If $\mathbf{S}_{\widetilde{\Delta}_1}$ and $\mathbf{S}_{\widetilde{\Delta}_2}$ are quasi-similar then*

$$deg_B(\Delta_1) = deg_B(\Delta_2).$$

Example 2.6. *Let*

$$\Delta_1 := \begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Delta_2 := \frac{1}{\sqrt{2}} \begin{bmatrix} z^2 & -z \\ z & 1 \end{bmatrix}.$$

Put $\Phi := \begin{bmatrix} z^2 \\ 1 \end{bmatrix} \in H^\infty_{M_{1\times 2}}$. *Observe that*

$$\begin{aligned} \begin{bmatrix} f \\ g \end{bmatrix} \in \ker H^*_{\breve{\Phi}} &\Longrightarrow \overline{z}^2 f + g \in H^2 \\ &\Longrightarrow f \in z^2 H^2. \end{aligned}$$

Thus we see that $\ker H^*_{\breve{\Phi}} = \Delta_1 H^2_{\mathbb{C}^2}$, *and hence* $deg_B(\Delta_1) = 1$. *Put* $e_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 := \begin{bmatrix} z \\ 0 \end{bmatrix}$, *and* $e_3 := \frac{1}{\sqrt{2}} \begin{bmatrix} z \\ 1 \end{bmatrix}$. *Then* $\mathcal{H}(\Delta_1) = \mathrm{span}\{e_1, e_2\}$ *and* $\mathcal{H}(\Delta_2) = \mathrm{span}\{e_1, e_3\}$. *Note that* $S_{\Delta_1} : e_1 \mapsto e_2$, $e_2 \mapsto 0$ *and*

$$S_{\Delta_2} e_1 = P_{\mathcal{H}(\Delta_2)} z e_1 = \langle z e_1, e_1 \rangle e_1 + \langle z e_1, e_3 \rangle e_3 = \frac{1}{\sqrt{2}} e_3.$$

Similarly, we have that $S_{\Delta_2} e_3 = 0$. *Define* $T : \mathcal{H}(\Delta_1) \to \mathcal{H}(\Delta_2)$ *linear by*

$$T : e_1 \mapsto e_1, \quad e_2 \mapsto \tfrac{1}{\sqrt{2}} e_3.$$

Then T is invertible and $T S_{\Delta_1} = S_{\Delta_2} T$. *Thus, by Corollary 2.5,* $deg_B(\Delta_2) = deg_B(\Delta_1) = 1$.

The following lemma is elementary:

Lemma 2.7. *Let $\Delta_k \in H^\infty_{M_N}$ ($k = 1, 2$). If Δ_1 and Δ_2 are quasi-equivalent then $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$ are quasi-equivalent.*

Proof. Suppose that Δ_1 and Δ_2 are quasi-equivalent. Then there exist functions $X, Y \in H^\infty_{M_N}$ such that $X\Delta_1 = \Delta_2 Y$ and such that the inner parts $(\det X)^i$ and $(\det Y)^i$ of the corresponding determinants are coprime to $(\det \Delta_k)^i$ ($k = 1, 2$). Observe that

$$(\det \widetilde{A})^i = (\widetilde{\det A})^i = [\widetilde{(\det A)^i}] \quad \text{for all } A \in H^\infty_{M_N}.$$

Thus $(\det \widetilde{X})^i$ and $(\det \widetilde{Y})^i$ are coprime to $(\det \widetilde{\Delta}_k)^i$ ($k = 1, 2$), and hence $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$ are quasi-equivalent. □

Corollary 2.8. *Let Δ_i ($i = 1, 2$) be an inner matrix function in $H^\infty_{M_N}$. If Δ_1 and Δ_2 are quasi-equivalent then*

$$deg_B(\Delta_1) = deg_B(\Delta_2).$$

Proof. It follows from Lemma 2.1, Corollary 2.5 and Lemma 2.7. □

References

[1] R. E. Curto, I. S. Hwang, and W. Y. Lee, *Matrix functions of bounded type: An interplay between function theory and operator theory*, Mem. Amer. Math. Soc. **260** (2019), no. 1253, v+100 pp.

[2] R. E. Curto, I. S. Hwang, and W. Y. Lee, *The Beurling–Lax–Halmos theorem for infinite multiplicity*, J. Funct. Anal. **280** (2021), no. 6, Art. ID 108884, 101 pp.

[3] I. S. Hwang, W. Y. Lee and J. Park, *The Beurling degree of inner matrix functions*, Linear Multilinear Algebra **77**(3) (2023), 472–479.

[4] B. Moore III and E. A. Nordgren, *On quasi-equivalence and quasi-similarity*, Acta. Sci. Math. (Szeged) **34** (1973), 311–316.

[5] N. K. Nikol'skiĭ, *Treatise on the Shift Operator*, Springer-Verlag, Berlin, 1986.

[6] E. A. Nordgren, *On quasi-equivalence of matrices over H^∞*, Acta. Sci. Math. (Szeged) **34** (1973), 301–310.

[7] B. Sz.-Nagy, C. Foiaş, H. Bercovici, and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, 2nd ed., Springer, New York, 2010.