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Some characterizations for graphs of a class of knots

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Abstract. This study addresses a research that combines the fields of graph theory and knot theory. Here, the graph parameters of the twist knots, which is an important knot family, are examined. First of all, graphs of twist knots were obtained by coloring method and then the basic graph parameters known for these graphs were examined. Generalizations have been made based on these calculations. In the study process, the methods and techniques in graph theory were utilized to a great extent.

1. Introduction

This study addresses an area of research that combines the fields of graph theory and knot theory. Knot theory has a structural and organic connection with graph theory. It is possible to find common working areas and practices that connect these two disciplines. So that; some concepts and constants defined for graphs can be used for knots (links), and some concepts and constants defined for knots (links) can also be used for graphs. Historically, mathematicians have studied various graph problems, such as classifying which graphs can be embedded in the plane (in other words, drawn so that their edges do not intersect) and finding the minimum number of intersections in a planar drawing for non-planar graphs. This led to the meeting of graphs and regular diagrams of knots in a common working area. In fact, a graph is not much different from a knot (link), not only in terms of its visual aspect however also in that it acts numerous roles in varied research fields. Over the years, graph theory and knot theory have intertwined to produce mutually beneficial results in many situations. The interrelationship of sub-branches of mathematics has generally been productive. Today, important studies are carried out in branches of science such as chemistry and physics, biology, health sciences and some engineering sciences, by using knots (links) and graphs together. As knot theory grows and evolves, its boundaries continue to change [1, 8, 15]. Moreover, today knot theory overlaps with certain areas of mathematical biology and chemistry. In fact, knots find a wide range of use in many different branches of art.

Knot theory is the subfield of algebraic topology that deals primarily with the problem of classifying mathematical knots. Despite the application of powerful algebraic tools such as homological algebra and homotopy groups, the problem of classifying knots is still not fully solved. Accordingly, knot theorists

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have improved their own tools on account of this search. One of the most useful is the concept of the knot invariant, which itself remains the same when the appearance of the knot is changed (without changing it topologically) [14]. There are numerous examples of knot invariants, some borrowed from various subfields of topology. Ideally, any knot invariant should satisfy two properties: It should be able to distinguish between many different knots and be easy to calculate. Knot theory has survived to this day with this strong motivation and has attracted the attention of scientists. In parallel with these developments, knot invariants were defined. The knots were classified with the help of these invariants. As a result, some important knot types and knot families emerged [13].

Although the Scottish physicist Peter Tait was the first to use the idea of a knot graph, the first person who introduced the term knot graph to the literature was Bankwitz. The projection of a knot or link on a 2-dimensional plane partitions the plane into multiple zones. In knot theory studies, it is a often utilized method to partition these zones into two groups: white zones and black zones. Easing the method, Bankwitz presented the concept of knot graph in his work on alternating knots [29]. Later, Aumann included this concept in his study on alternating knots [4]. Yajima and Kinoshita examined the graphical behavior of this concept [29]. These studies continued in the following periods. Murasugi published his work on graph invariants and their applications to knot theory [20]. Additionally, Kurpita and Murasugi published their work on how graphs and techniques in graph theory could be transferred to knot theory [16]. The recent discovery of close connections between a graph and a knot or link in three-dimensional space reveals that many invariants of graphs play very important roles in classical knot theory. A knot is any simple closed curve in the 3-dimensional space $\mathbb{R}^3$ without its points of intersection, and a link is any union of simple closed curves that do not intersect. A graph can be thought of as a shape consisting of points and line segments (topologically it is called a 1-complex) [5]. A graph corresponding to each knot (link) can be assigned, and such a graph is called a knot (link) graph. One convenient representation of knot diagrams was introduced by Peter Tait. Any knot diagram: it defines a plane graph whose vertices are intersection points and whose edges are paths between consecutive intersection points. Exactly one face of this planar graph is unbounded, each of the others is homeomorphic to a 2-dimensional disk. These faces are painted black or white, with the boundless face being black and any two faces sharing a boundary edge having contrasting colors. As a result, a new plane graph whose vertices are white faces and whose edges correspond to intersections can be created in this way. Each edge of this graph can be labeled as a left edge or a right edge, depending on which thread passes over the other when viewing the corresponding transition from one of the edge's endpoints. As usual for left and right edges: it is indicated by labeling the left edges with (+) and the right edges with (-), or by drawing the left edges with solid lines and the right edges with dashed lines. The original knot diagram is the medial graph of this new plane graph, and the type of each crossing is determined by the sign of the corresponding edge. Changing the sign of each edge corresponds to reflecting the knot in a mirror. For any study related to graph theory, planar graphs are considered and the necessary formulas for graph invariants are calculated or proven on them. Since knot theory deals only with planar graphs, this shows that graph-theoretical proofs can be taken instead of some complex algebraic topological proofs in knot theory [20]. Combinatorial knot theory is that it is well known that there is a graph theoretical way of approaching the theory of knots, links and their topology in three-dimensional space. The established formulation of this theory through knot diagrams works with a formal system that is a mixture of diagrams and graphs. This translation of knot diagrams into graph theoretical concepts is important as it provides a direct link between combinatorial knot theory, graph theory, and the foundations of mathematics. Harary and Kauffman showed how to use graph theory to give a simple and effective set-theoretic support to this diagrammatic formal system [10]. In this work, they introduce a planar graph that is naturally related to any knot or link diagram, and show how these graphs can be used to examine the algebraic invariants of knots.

Therefore, examining the combinatorial properties of the graph corresponding to a knot, their characterization, and its contribution to knot theory is a problem worth investigating. Throughout this study, the graph parameters of graphs of twist knots, an important knot family, have been examined by making significant use of graphs and techniques in graph theory. Graphs are mathematical tools that are frequently used in mathematics and other branches of science [2, 3].

Twist knots are a family of knots that have been studied in depth in previous years. In 2001, Hoste and Shanahan investigated the topological properties of twist knots [12]. Additionally, various polynomial invariants of twist knots were examined and calculated [9, 21, 27]. Şahin and Şahin characterized the Hosoya, Tutte and Jones polynomials of the graphs of twist knots in their works between 2018-2019 [22–24]. In 2020, A. Şahin presented his work showing the graph coloring numbers of graphs of twist knots for some different types of graph coloring [25]. In 2021, some domination type invariants were characterized for graphs of twist knots [26]. As a result, the properties of the family of twist knots and their corresponding graphs appear as a problem worth examining.

2. Preliminaries

Mathematically, knots are modeled as closed loops, that is, loops with no beginning or end (see Figure 1). Additionally, a knot must not intersect with itself.

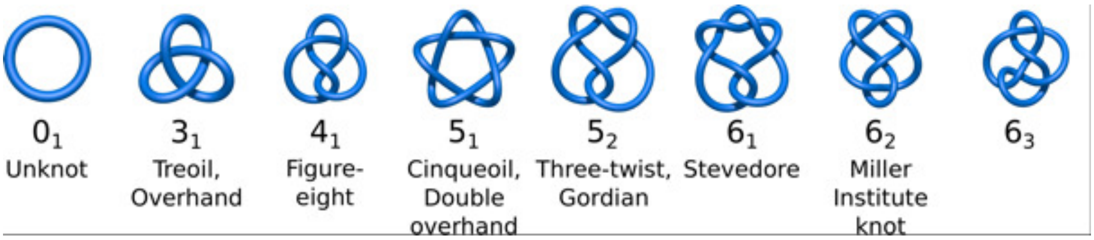

Figure 1: Some examples of mathematical knots [7].

Let $f : X \to Y$ be a function, where X and Y are two topological spaces. If there is a homeomorphism from the topological space X to the subspace $f(X)$ in Y, the space X is said to be embedded in the space Y. A knot K is a continuous embedding of 1-sphere S^1 into 3-sphere S^3 [18]. In other words, a knot is a one-to-one continuous transformation $K : S^1 \to S^3$. Note that $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

A sequential finite combination of knots that do not intersect with each other is called a link. Each knot that makes up the link is called a component of the link (see Figure 2) [13].

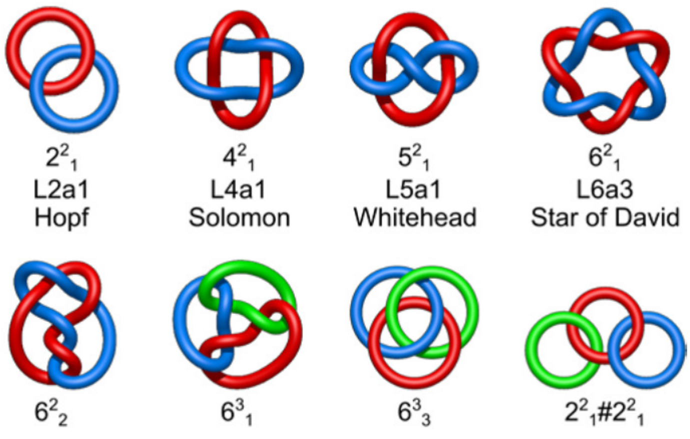

Figure 2: Some examples of links [7].

Unless otherwise stated, it will be assumed that the situations that are valid for knots are also valid for links. In fact, the shapes given as knots so far are regular diagrams in two-dimensional space of the knots embedded in S^3. Almost all of the work on knots is carried out on these regular diagrams, which are considered to be a complete representation of the knot. The narrow region on a regular diagram that shows that the arc of a knot passes from above or below is called the crossing point of the knot (simply crossing). Additionally, knots are named according to their crossing numbers.

Let V be a finite, non-empty set of the form $V = \{v_1, v_2, v_3, \ldots, v_n\}$, whose elements are called vertices. The binary structure (V, E) consisting of a finite set of edges in the form $E = \{e = v_i v_j : v_i, v_j \in V\}$ connecting these points or the point itself (which does not give any geometric or positional information but only shows the relationship between the points) is called a graph. A graph is denoted by $G = (V, E)$ or simply G. If there is at least one edge between any vertices v_1 and v_2 of G, it is said that they are adjacent or neighbor. Similarly, edges in any graph that have a common vertex are called neighbor edges. The number of elements of the set V which is the set of vertices in a graph G is called the order of the graph G and is denoted by $|V(G)|$. A sequence with a finite number of elements consisting of neighboring vertices and edges in a graph is called a walk, and the symbol W is used to express the walk, and the length of this walk is determined by the number of edges in the sequence. If each edge and each vertex in a walk occurs once, this walk is called a path and the symbol P is used to denote a path. If a path starts from the same vertex and ends at the same vertex again, it is called a cycle. The symbol C_n is used to denote a cycle with n vertices. Let v be any vertex in a graph G. Accordingly, the degree of v is the number of edges connected to v, and the symbol d_v is used to show the degree of v. In a graph, the vertex with the smallest degree is called the vertex with the minimum degree and the symbol $\delta(G)$ is used to show the degree of this vertex, while the vertex with the largest degree is called the vertex with the maximum degree and the symbol $\Delta(G)$ is used to show it. The degree of the vertex of the graph containing a loop is taken as +2. If $d_v = 0$, v is called isolated vertex, and if $d_v = 1$, v is called pendant (end vertex). If there is always a path between any two arbitrary vertices of a graph, this graph is called a connected graph. Otherwise, this graph is called a disconnected graph.

Let G be a connected and distance-based graph. The distance between any two vertices u and v is the minimum number of paths between vertices u and v, and this distance is denoted by $d(u, v)$. The topological diameter $d(G)$ of a graph G (i.e. longest topological distance in G) is defined as $d(G) = \max_{u,v\in V(G)}\{d(u, v)\}$ [10]. Let $V(G)$ be taken as $S \subseteq V(G)$ and $S \neq \emptyset$, where $V(G)$ is the set of vertices of this graph. If each vertex of the graph G belongs to S or is adjacent to a vertex of S, S is called a dominating set. A graph can have more than one dominating set. The number of elements of the one with the least number of elements among all dominating sets of a graph is called the domination number of this graph, and the domination number of a graph G is denoted by the symbol $\gamma(G)$ [11]. The minimum number of colors required to color the adjacent vertices of the graph G in different colors is called the coloring number and is denoted $\chi(G)$ [5]. Let $V(G)$ be taken as $S \subseteq V(G)$ and $S \neq \emptyset$, where $V(G)$ is the set of vertices of this graph. When the vertices in the set S are taken in pairs, if there is no edge belonging to G between these vertices, S is called an independent cluster. A graph can have multiple independent sets. The number of elements of the one with the highest number of elements among all independent sets of a graph is called the independence number of this graph, and the independence number of a graph G is denoted by the symbol $\beta(G)$ [28]. The set of mutually adjacent vertices of G is called the clique set. The maximum number of elements in the clique sets of a graph G is called the clique number of the graph. This value is denoted by $K(G)$ and is expressed mathematically as follows: $K(G) = max\{\, |V_i| : V_i$ clique set $\}$ [6]. The minimum number of vertices that must be removed from the graph in order to turn the graph G into a graph consisting of unconnected or only isolated vertices is called the vertex connectivity number of the graph and is denoted by $C(G)$. The vertex connectivity number is defined as $C(G) = \min_{S\subseteq V(G)}\{|S| : w(G - S) \geq 2\}$, where $C(G)$ is the number of components of a graph G [28]. Let $V(G)$ be taken as $S \subseteq V(G)$ and $S \neq \emptyset$, where $V(G)$ is the set of vertices of this graph. If at least one vertex of each edge in the graph G is an element of S, S is called a covering set. A graph can have more than one covering set. The number of elements of the one with the least number of elements among all covering sets of a graph is called the cover number of this graph, and the cover number of a G graph is denoted by the symbol $\alpha(G)$ [28]. The division of the vertex set $V(G)$ into maximum clique sets is called the clique cover of the graph G. The minimum number of this fragmentation is called the click cover number and is denoted by (G) [28]. A matching in the graph G is a set of edges where any two have no vertex in common (i.e., they are formed by independent edges). The maximum matching is the matching created with the maximum number of edges possible. The number of edges that exist in the maximum matching of the graph G is called the matching number of G and is denoted by $M(G)$ [6]. Let a graph G be given and the set of vertices of this graph be $V(G)$. The sum $\sum_{u,v\in V(G)} d(u, v)$ is called the Wiener index of the graph G and is denoted by $W(G)$ [19].

A twist knot T_n, where n is a natural number, is defined as a knot that can be drawn in the form given in Figure 3 below. The natural number n is called the tangle number of the twisted knot T_n [17].

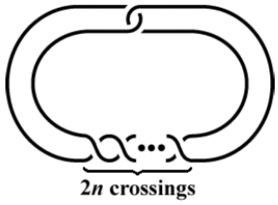

Figure 3: An example of a twist knot.

Figure 4 and Figure 5 show two well-known examples of twist knots [17].

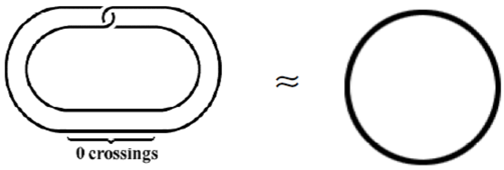

Figure 4: Equivalence of knot T_0 to knot 0_1.

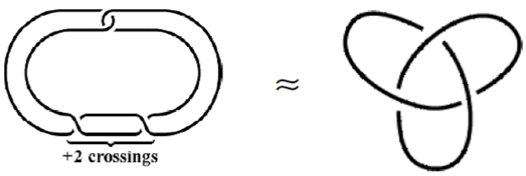

Figure 5: Equivalence of knot T_2 to knot 3_1.

In the Figures 6 – 10 below, the steps to obtain the graphs of some twist knots by coloring method are given respectively. For more detailed information, see [25].

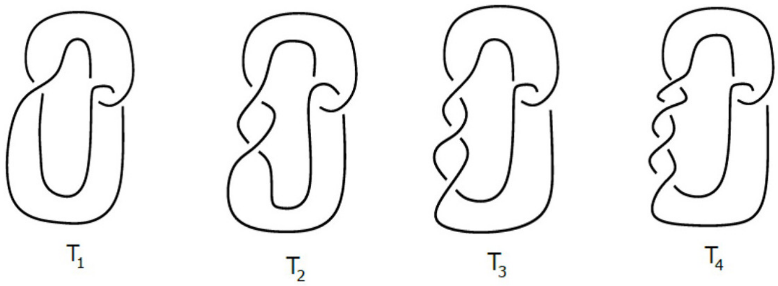

Figure 6: Step 1.

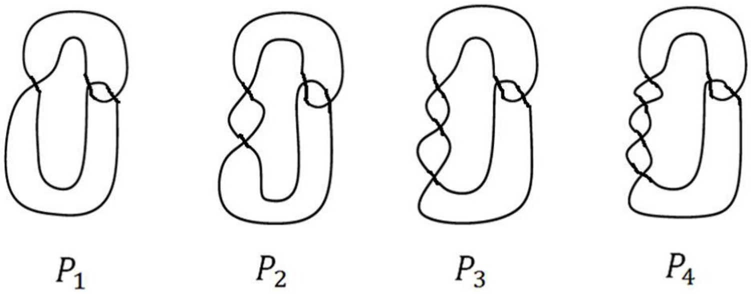

Figure 7: Step 2.

3. Main Results

Theorem 3.1. *For natural numbers $n = 1, 2, 3, 4, \dots$, the diameter of graphs of twist knots is defined as:*

$$d(G_{2n-1}) = d(G_{2n}) = n.$$

Proof. In each twist knot graph, there are two parallel paths between the first two vertices and one path between the remaining vertices, and they have one more edge than the vertex. Let the vertices of the graphs of twist knots be named with sequentially numbered vertices in a clockwise direction, starting from the top vertex.

It is obvious that for $n = 1, 2$, the diameters of graphs G_1 and G_2 will be 1. For $n \geq 3$, the proof will be completed in two steps. Firstly, let n be odd. In this case, when we move clockwise over the first vertex of the graph G_n the vertex with the maximum distance from v_1 becomes the vertex with numbered $\frac{n+3}{2}$. For $n = 3, 5, 7, 9, \dots$ the difference of the numbers of these vertices, $\frac{n+3}{2} - 1$, gives the diameter of the graph G_n.

Now let's show the case where n is even. In this case, when we move clockwise over the first vertex of the graph G_n the vertex with the maximum distance from v_1 becomes the vertex with numbered $\frac{n+2}{2}$. For $n = 2, 4, 6, 8, \dots$ the difference of the numbers of these vertices, $\frac{n+2}{2} - 1$, gives the diameter of the graph G_n. Thus, the proof is completed. □

Theorem 3.2. *For $n = 3k$ and $k = 1, 2, 3, 4, \dots$, the domination number of graphs of twist knots is defined as:*

$$\gamma(G_n) = \gamma(G_{n+1}) = \gamma(G_{n+2}) = k + 1,$$

where $\gamma(G_1) = \gamma(G_2) = 1$.

Proof. Let's recall the definition of domination. Suppose that G be a connected graph. Let $V(G)$ be taken as $S \subseteq V(G)$ and $S \neq \emptyset$, where $V(G)$ is the set of vertices of this graph. If each vertex of the graph G belongs to

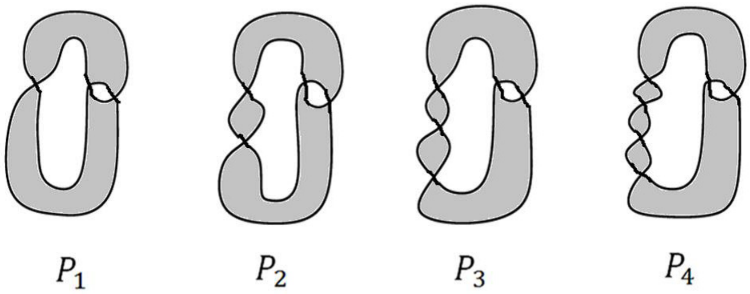

Figure 8: Step 3.

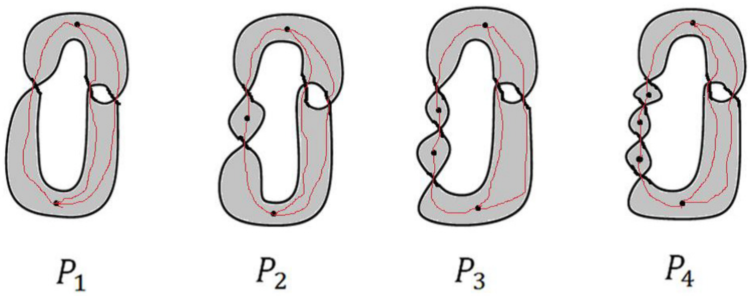

Figure 9: Step 4.

S or is adjacent to a vertex of S, S is called a dominating set. A graph can have more than one dominating set. The number of elements of the one with the least number of elements among all dominating sets of a graph is called the domination number of this graph. So accordingly, all remaining vertices are dominated by selecting a single vertex in the graphs G_1 and G_2. Thus, the domination number for graphs G_1 and G_2 is 1. In other words, $\gamma(G_1) = \gamma(G_2) = 1$.

Now, let us show the proof of the domination numbers of the graphs formed for $n \geq 3$. In graphs of twist knots, all vertices' degrees except the first two are two. In order to dominate these graphs with a minimum number of vertices, the first vertex, that is v_1, can be taken as an element of the dominant set of all graphs. This first vertex dominates the second and last vertex of the graph. For all subsequent vertices, one vertex will dominate at most two vertices. Therefore, among these remaining vertices, groups with a maximum number of three elements will be formed. Unlike other groups, 1 or 2 vertices may remain in the last group. In order for these remaining vertices to be dominated only one single vertex must be included in the dominating set along with a vertex from each group of three. A different vertex will be included in the minimum numbered dominating set of each graph indexed $n = 3k$ for $k \in \mathbb{N}^+$. That is, for $k \in \mathbb{N}^+$, the cardinality of the minimum numbered dominating sets of graphs with indexes $n = 3k$, $n = 3k+1$, $n = 3k+2$ will be equal. As a result, the domination numbers of graphs indexed in this way will be equal. Thus, the proof is completed. □

Theorem 3.3. ([25]) The chromatic number of graphs of twist knots is defined as:

$$\chi(G_n) = \begin{cases} 2, & \text{if } n \text{ is odd} \\ 3, & \text{if } n \text{ is even} \end{cases}.$$

Corollary 3.4. ([25]) Since a graph G is called bipartite if $\chi(G)$ is two, at the status where n is odd, the graphs G_n of twist knots are bipartite graphs.

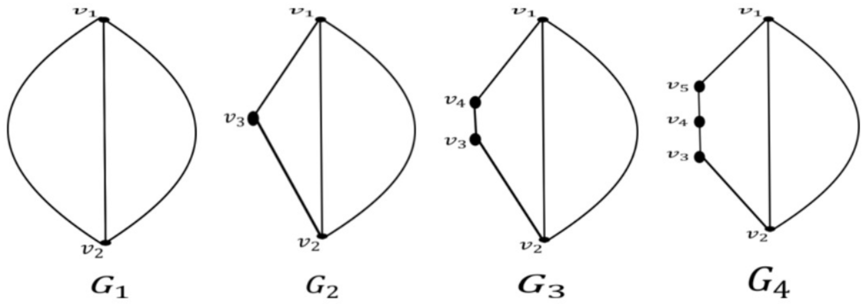

Figure 10: Step 5.

Theorem 3.5. *The independence number of graphs of twist knots is defined as:*

$$\beta(G_n) = \beta(G_{n+1}) = k,$$

such that for $k \in \mathbb{N}^+$, $n = 2k - 1$.

Proof. Let G be a graph and S be any subset of the set of vertices of this graph. If the vertices selected from the set S are consecutive vertices, these vertices form an edge in the G graph. For this reason, if the vertex to be selected from the S set is grouped two consecutive vertices at a time, there will be no edge relationship between these vertices. For $k \in \mathbb{N}^+$, the maximum independent set, with maximum elements, of the graph G_n will be $\{v_1, v_3, \ldots, v_{2k-1}\}$, which will be a set with k elements.

For $k \in \mathbb{N}^+$ and $n = 2k - 1$, in the independent set, with maximum elements, of the graph G_{n+1} the vertex v_{n+2} is not included as an element since there will be an edge between the vertex v_1 and the vertex v_{n+2} also, since the vertex v_{n+1} is double indexed it will not be an element of the independent set. The independent set, with maximum elements, of the graph G_n will be $\{v_1, v_3, \ldots, v_{2k-1}\}$, which will be a set with k elements. Thus, the independence numbers of the graphs will be $\beta(G_n) = \beta(G_{n+1}) = k$, where for $k \in \mathbb{N}^+$, $n = 2k - 1$. So, the proof is completed. □

Theorem 3.6. *The clique number of graphs of twist knots is defined as:*

$$K(G_n) = \begin{cases} 3, & \text{for } n = 2 \\ 2, & \text{for other cases} \end{cases}.$$

Proof. Let G be a graph and S be any subset of the set of vertices of this graph. In graphs of twist knots, all vertices' degrees except the first two are two. The sets consisting of only two consecutive vertices selected from the set S will be the set of mutually adjacent vertices of the graph G. Therefore, the maximum number of elements in the clique sets of the graph G_n will be 2, where $n \in \mathbb{N}^+$. Specifically, the maximum number of elements in the clique sets of graph G_2 is 3, that is, it is $\{v_1, v_2, v_3\}$. Thus the proof is completed. □

Theorem 3.7. *The vertex connectivity number of graphs of twist knots is defined as:*

$$K(G_n) = \begin{cases} 1, & \text{for } n = 1 \\ 2, & \text{for other cases} \end{cases}.$$

Proof. Let G be a graph and S be any subset of the set of vertices of this graph. In graphs of twist knots, all vertices' degrees except the first two are two. When one vertex with three-degree and one vertex with two-degree which is successors of this vertex are removed from the set S, the graph G can be turned into a graph consisting of unconnected or isolated vertices by removing the minimum number of vertices. Therefore, for $n = 2, 3, 4, \ldots$ the vertex connectivity number of graph G_n will be 2. Specifically, for $n = 1$, the vertex connectivity number of graph G_1 is 1. Thus, the proof is completed. □

Theorem 3.8. *The independence number of graphs of twist knots is defined as:*

$$\alpha\left(G_n\right)=\alpha\left(G_{n+1}\right)=\frac{n+2}{2}, n=2,4,6,8,\ldots$$

where $\alpha\left(G_1\right)=1$.

Proof. Since the covering set $\{v_1\}$ for the graph G_1 contains at least one endpoint of the edge between the vertices v_1 and v_2, the number of elements of the covering set with the least elements of the graph G_1 is 1. Thus, the cover number of graph G_1 is 1.

Since the vertex v_1 in graph G_n will have at least one endpoint of the edge between the last vertex and the second vertex, the vertex v_1 can be included in the covering set of graph G_n. The vertex v_3, which is two successors of the vertex v_1, will also be an element of the cover set since it will represent at least one end point of two edges. This similar situation will be valid for two consecutive vertices after v_3 and for all other vertices that satisfy the same condition. Therefore for $n=2,4,6,8,\ldots$, the covering sets of the graphs G_n with indexed even and G_{n+1} with indexed odd will be a set with indexed odd such that it is $\{v_1,v_3,v_5,\ldots,v_{2n-1}\}$, and consisting of $\frac{n+2}{2}$ vertices. So, the cover number of the graphs of twist knots will be $\alpha\left(G_n\right)=\alpha\left(G_{n+1}\right)=\frac{n+2}{2}$, $n=2,4,6,8,\ldots$. □

Theorem 3.9. *The number of clique cover number of graphs of twist knots are defined as:*

$$\theta\left(G_{2n-1}\right)=\theta\left(G_{2n}\right)=n,$$

where $n\in\mathbb{N}^+$.

Proof. When $n=1,2$ a single clique set will be formed for graphs G_1 and G_2. Therefore, $\theta\left(G_1\right)=\theta\left(G_2\right)=1$.

For $n\geq 3$, in the graphs G_n , there is a path among all vertices except the first two vertices, only between two consecutive vertices. Therefore, the fragmentation of the vertex set of the graph G_n into maximum clique sets is $\{v_1,v_{n+1}\},\{v_2,v_n\}$, $\{v_3,v_{n-1}\},\ldots$ such that if n is odd its number is $\frac{n+1}{2}$ pieces if n is even its number is $\frac{n}{2}$ pieces. This shows that for $n\geq 3$, $\theta\left(G_{2n-1}\right)=\theta\left(G_{2n}\right)=n$. □

Theorem 3.10. *The number of matching number of graphs of twist knots are defined as:*

$$M\left(G_{2n-1}\right)=M\left(G_{2n}\right)=n,$$

where $n\in\mathbb{N}^+$.

Proof. It is known that in the graphs of twist knots, there are $n+2$ edges for $n\in\mathbb{N}^+$ and two of these edges are parallel between the first two vertices. Accordingly, in the matchings of these graphs (except for the graph G_1), the one with the maximum number of elements can be created by ignoring the edges between v_1 and v_2.

The matchings, which have maximum elements, of G_n graphs for $n\geq 2$ are obtained by edge sets such that if n is odd it is $\{v_1v_{n+1}\ ,v_2v_3\ ,\ v_4v_5\ ,\ v_6v_7\ ,\ \ldots,\ v_{n-1}v_n\}$ with $\frac{n+1}{2}$ elements and if n is even it is $\{v_1v_{n+1}\ ,v_2v_3\ ,\ v_4v_5\ ,\ v_6v_7\ ,\ \ldots,\ v_{n-2}v_{n-1}\}$ with $\frac{n}{2}$ elements. Thus, this shows that for $n\in\mathbb{N}^+$, $M\left(G_{2n-1}\right)=M\left(G_{2n}\right)=n$. □

Theorem 3.11. *For* $n\in\mathbb{N}^+$, *the Wiener index of graphs of twist knots is defined as:*

$$W(G_n)=\begin{cases} n^3, & \text{if } n \text{ odd} \\ \frac{n(n+1)(2n+1)}{2}, & \text{if } n \text{ even} \end{cases}.$$

Proof. In [22], for $n=1,2,3,4,\ldots$, the Hosoya polynomials of graphs of twist knots are generalized as: for graphs with double indexed it is

$$H\left(G_{2n},y\right)=(2n+1)\,y^0+(2n+1)\,y^1+(2n+1)\,y^2+\cdots+(2n+1)\,y^{n-1}+(2n+1)\,y^n$$

and for graphs with odd indexed it is

$$H\left(G_{2n-1},y\right)=2ny^{0}+2ny^{1}+2ny^{2}+\cdots+2ny^{n-1}+ny^{n}.$$

It is known that if the first derivative of the Hosoya polynomial of any graph is taken with respect to the independent variable and 1 is written instead of the independent variable in the derivative polynomial, the Wiener index value of this graph will be obtained. Accordingly, in order to complete the proof, it is sufficient to apply this differentiation process in the above generalizations.

If n is even, it is

$$H\left(G_{2n},y\right)=(2n+1)\,y^{0}+(2n+1)\,y^{1}+(2n+1)\,y^{2}+\cdots+(2n+1)\,y^{n-1}+(2n+1)\,y^{n}$$

$$\left[H\left(G_{2n},y\right)\right]'=0+(2n+1)+2\,(2n+1)\,y^{1}+\cdots+(n-1)\,(2n+1)\,y^{n-2}+n\,(2n+1)\,y^{n-1}$$

$$\left[H\left(G_{2n},1\right)\right]'=(2n+1)+2\,(2n+1)\,1+\cdots+(n-1)\,(2n+1)\,1^{n-2}+n\,(2n+1)\,1^{n-1}$$

$$\left[H\left(G_{2n},1\right)\right]'=(2n+1)\,(1+2+3+\cdots+n)$$

$$W\left(G_{n}\right)=\frac{n(n+1)\,(2n+1)}{2}.$$

If n is odd, it is

$$H\left(G_{2n-1},y\right)=2ny^{0}+2ny^{1}+2ny^{2}+\cdots+2ny^{n-1}+ny^{n}$$

$$\left[H\left(G_{2n-1},y\right)\right]'=0+2n+4ny+\cdots+2n(n-1)y^{n-2}+n^{2}y^{n-1}$$

$$\left[H\left(G_{2n-1},1\right)\right]'=2n+4n1+\cdots+2n(n-1)1^{n-2}+n^{2}1^{n-1}$$

$$\left[H\left(G_{2n-1},1\right)\right]'=2n\,(1+2+3+\cdots+n-1)+n^{2}$$

$$W\left(G_{n}\right)=2n\frac{(n-1)n}{2}+n^{2}$$

$$W\left(G_{n}\right)=n^{3}-n^{2}+n^{2}$$

$$W\left(G_{n}\right)=n^{3}.$$

□

4. Applications

For the applications we will present here, we can consider the planar graphs G_7 and G_8 which correspond to the twist knots T_7 and T_8 (see Figure 11).

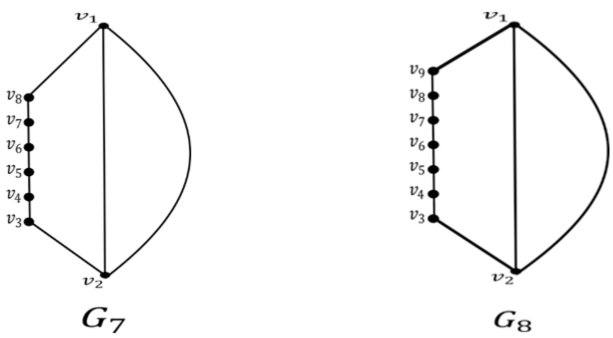

Figure 11: The planar graphs G_7 and G_8

The diameter of the graph G_7 is found as follows. First, the distances between the vertices of the graph are determined. These distances are as follows:
$d(v_1, v_2) = 1$, $d(v_1, v_3) = 2$, $d(v_1, v_4) = 3$, $d(v_1, v_5) = 4$, $d(v_1, v_6) = 3$, $d(v_1, v_7) = 2$, $d(v_1, v_8) = 1$;
$d(v_2, v_3) = 1$, $d(v_2, v_4) = 2$, $d(v_2, v_5) = 3$, $d(v_2, v_6) = 4$, $d(v_2, v_7) = 3$, $d(v_2, v_8) = 2$;
$d(v_3, v_4) = 1$, $d(v_3, v_5) = 2$, $d(v_3, v_6) = 3$, $d(v_3, v_7) = 4$, $d(v_3, v_8) = 3$;
$d(v_4, v_5) = 1$, $d(v_4, v_6) = 2$, $d(v_4, v_7) = 3$, $d(v_4, v_8) = 4$;
$d(v_5, v_6) = 1$, $d(v_5, v_7) = 2$, $d(v_5, v_8) = 3$;
$d(v_6, v_7) = 1$, $d(v_6, v_8) = 2$;
$d(v_7, v_8) = 1$.

Since the maximum value between these distances is 4 units, the diameter of the graph G_7 is found as $d(G_7) = 4$.

The domination numbers of the graphs G_7 and G_8 are found as follows. For these calculations, first, the dominating sets with the fewest elements of these graphs are determined. For the graph G_7, the dominating set with the fewest elements is $D = \{v_1, v_4, v_6\}$, and $\gamma(G_7) = 3$. For the graph G_8, the dominating set with the fewest elements is $D = \{v_1, v_4, v_7\}$, and $\gamma(G_8) = 3$.

The chromatic number for the graphs G_7 and G_8 is found as follows. For these calculations, a suitable coloring set containing the minimum number of colors must be assigned to the vertices of these graphs. A suitable coloring for the graph G_7 is $\{v_1 = blue, v_2 = red, v_3 = blue, v_4 = red, v_5 = blue, v_6 = red, v_7 = blue, v_8 = red\}$, and the chromatic number is found as $\chi(G_7) = 2$. A suitable coloring for the graph G_8 is $\{v_1 = blue, v_2 = red, v_3 = blue, v_4 = red, v_5 = blue, v_6 = yellow, v_7 = red, v_8 = blue, v_9 = red\}$, and the chromatic number is found as $\chi(G_8) = 3$.

The independence number of the graphs G_7 and G_8 is found as follows. For these calculations, first, the independent sets with the largest number of elements in these graphs are determined. For the graph G_7, the independent set with the largest number of elements is $V_1 = \{v_1, v_3, v_5, v_7\}$, and $\beta(G_7) = 4$. For the graph G_8, the independent set with the largest number of elements is $V_1 = \{v_1, v_3, v_5, v_7\}$, and $\beta(G_8) = 4$.

The clique number of the graphs G_7 and G_8 is found as follows. For the graph G_7, $V_1 = \{v_1, v_2\}$, $V_2 = \{v_2, v_3\}$ are clique sets with the largest number of elements, and $K(G_7) = 2$. For the graph G_8, $V_1 = \{v_1, v_2\}$, $V_2 = \{v_2, v_3\}$ are clique sets with the largest number of elements, and $K(G_8) = 2$.

The vertex connectivity number of the graphs G_7 and G_8 is found as follows. The set with the least number of vertices that must be removed to make the graph G_7 disconnected is the set $V_1 = \{v_1, v_3\}$, and $C(G_7) = 2$. The set with the least number of vertices that must be removed to make the graph G_8 disconnected is the set $V_1 = \{v_1, v_3\}$, and $C(G_8) = 2$.

The covering number of the graphs G_7 and G_8 is found as follows. For the graph G_7, the covering set with the smallest number of elements is $V_1 = \{v_1, v_3, v_5, v_7\}$, and $\alpha(G_7) = 4$. For the graph G_8, the covering set with the smallest number of elements is $V_1 = \{v_1, v_3, v_5, v_7, v_9\}$, and $\alpha(G_8) = 5$.

The clique covering numbers for the graphs G_7 and G_8 are found as follows. For the graph G_7, the clique covering sets are $V_1 = \{v_1, v_8\}$, $V_2 = \{v_2, v_3\}$, $V_3 = \{v_4, v_5\}$, $V_4 = \{v_6, v_7\}$, and $\theta(G_7) = 4$. For the graph G_8 , the clique covering sets are $V_1 = \{v_1, v_8\}$, $V_2 = \{v_2, v_3\}$, $V_3 = \{v_4, v_5\}$, $V_4 = \{v_6, v_7\}$, and $\theta(G_8) = 4$.

The matching number of the graphs G_7 and G_8 is found as follows. For the graph G_7, the set of edges with the maximum number of elements is $V_1 = \{v_1v_8, v_2v_3, v_4v_5, v_6v_7\}$, and the matching number is found as $M(G_7) = 4$. For the graph G_8, the set of edges with the maximum number of elements is $V_1 = \{v_1v_9, v_3v_4, v_5v_6, v_7v_8\}$, and the matching number is found as $M(G_8) = 4$.

The Wiener indices of the graphs G_7 and G_8 are found as follows. The Wiener index for the graph G_7 is found as $W(G_7) = 7^3 = 343$. The Wiener index for the graph G_8 is found as $W(G_8) = \frac{8.(8+1).(2.8+1)}{2} = 4.9.17 = 612$.

5. Conclusion

In this study, well-known basic graph parameters of graphs of twist knots are calculated. Diameter, domination number, chromatic number, independence number, clique number, vertex connectivity number, cover number, clique cover number, matching number and Wiener index values of these knot graphs were calculated. The graphs of twist knots were characterized according to these considered parameters. It is envisaged that this study may pave the way for similar studies on graphs of different knot types. In addition, problems such as algebraic properties of neighborhood and coincidence matrices the graphs of twist knots, parameter values of graph multiplications of the graphs of twist knots can be suggested as study subjects for researchers interested in this field in future.

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