Filomat 39:33 (2025), 11943–11959
https://doi.org/10.2298/FIL2533943B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the existence and multiplicity of solutions for double phase equations with Robin boundary conditions

Omar Benslimane[a,*], Mohamed Badr Benboubker[b]

[a]*Mohammed V University in Rabat, EST of Salé, Rabat, Morocco*
[b]*Sidi Mohamed Ben Abdellah University, EST, Fez, Morocco*

Abstract. This paper studies the following new class of nonlinear double-phase equations with a defined Robin boundary condition (the nonlinear disturbances that drive them to obey a suitable condition at the origin and on the boundary)

$$\begin{cases} -\mathrm{div}\left(\mathcal{A}\left(\nabla u\right)\right) - \mathrm{div}\left(\mathcal{B}\left(\nabla u\right)\right) = \mu f\left(x,u\right) & \text{in } \Omega, \\ \left(\mathcal{A}\left(\nabla u\right) + \mathcal{B}\left(\nabla u\right)\right).\eta + b\left(x\right)|u|^{p(x)-2}u + d\left(x\right)|u|^{q(x)-2}u = g\left(x,u\right) & \text{on } \partial\Omega, \end{cases}$$

where $\mu > 0, \Omega \subset R^N (N \geqslant 2)$ is a bounded open domain with smooth boundary $\partial\Omega$, η is the outer unit normal vector on $\partial\Omega$, $b\,(.)$ and $d\,(.)$ are positive and continuous functions on $\partial\Omega$. At first, via a generalized mountain-pass approach with Critical point theory, we prove that this problem with superlinear nonlinearity has a solution and infinitely many solutions. Furthermore, we rigorously verify the existence of infinitely many solutions by imposing sufficient constraints on the functions f and g. Our results comprehensively expand and generalize specific recent contributions within the existing literature.

1. Introduction

There has been a tremendous increase in the study of mathematical problems with variable exponents in recent years. This increased focus can be attributed to the intricate and captivating mathematical challenges posed by these equations, as well as their versatile application in capturing phenomena that manifest in a variety of fields, including elastic mechanics [10], thermo-rheological fluids [5], electrorheological materials [31], image restoration [12], electrical resistivity, polycrystal plasticity [11], and continuum mechanics [6]. Additionally, nonlinear Robin conditions appear in several physical situations such as climatization [34] or some chemical reactions [13].

This paper seeks to demonstrate the presence and variety of solutions within a particular class of quasi-linear elliptic problems subject to Robin boundary conditions:

$$\begin{cases} -\mathrm{div}\left(\mathcal{A}\left(\nabla u\right)\right) - \mathrm{div}\left(\mathcal{B}\left(\nabla u\right)\right) = \mu f\left(x,u\right) & \text{in } \Omega, \\ \left(\mathcal{A}\left(\nabla u\right) + \mathcal{B}\left(\nabla u\right)\right).\eta + b(x)|u|^{p(x)-2}u + d(x)|u|^{q(x)-2}u = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 35J60; Secondary 35J70.
Keywords. Double phase problem; Variable exponent Sobolev space; $(p(x), q(x))$- Laplacian-like operators; Critical point theory; Multiple solutions Mountain-pass theorem.
Received: 27 August 2025; Revised: 07 October 2025; Accepted: 07 October 2025
Communicated by Maria Alessandra Ragusa
* Corresponding author: Omar Benslimane
Email addresses: omar.benslimane@usmba.ac.ma (Omar Benslimane), simo.ben@hotmail.com (Mohamed Badr Benboubker)
ORCID iDs: https://orcid.org/0000-0002-2730-6259 (Omar Benslimane), https://orcid.org/0000-0002-3176-253X (Mohamed Badr Benboubker)

where $\Omega \subset R^N (N \geqslant 2)$ is a bounded open domain with a smooth boundary $\partial\Omega$. The functions $\mathcal{A}(\nabla u) = a\left(|\nabla u|^{p(x)}\right)|\nabla u|^{p(x)-2}\nabla u$ and $\mathcal{B}(\nabla u) = a\left(|\nabla u|^{q(x)}\right)|\nabla u|^{q(x)-2}\nabla u$ are defined, with $\mu > 0$ and η representing the outer unit normal vector on $\partial\Omega$. The functions $b(\cdot)$ and $d(\cdot)$ are positive and continuous on $\partial\Omega$. Additionally, f and g are Caratheodory functions that satisfy the Growth condition. Moreover, the exponents $p(\cdot)$ and $q(\cdot)$ fulfill specific requirements

$$p(.), q(.) \in C_+\left(\overline{\Omega}\right) = \left\{r(.) \in C\left(\overline{\Omega}\right) \;/\; 1 < r_- < r(.) < r_+ < +\infty \text{ in } \overline{\Omega}\right\} \text{ such that } 1 < q_- \leqslant q_+ < p_- \leqslant p_+ < N, \quad (2)$$

where $q_- := ess \inf\limits_{x\in\overline{\Omega}} q(x)$ and $q_+ := ess \sup\limits_{x\in\overline{\Omega}} q(x)$, idem for p_+ and p_-.

Note that the $(p(.), q(.))-$Laplacian operator in equation (1) is a specific instance of divergence form operators $-\text{div}(\mathcal{A}(\nabla u)) - \text{div}(\mathcal{B}(\nabla u))$, commonly encountered in various nonlinear diffusion problems, especially in the mathe-matical modeling of non-Newtonian fluids. Specifically, when $a(t) = 1 + \dfrac{t}{\sqrt{1+t^2}}$, it gives rise to the generalized Capillary operator, a crucial element in applied fields such as industrial, biomedical, and pharmaceutical applications, as introduced by W. Ni and J. Serrin [27].

Several researchers [2, 3, 7–9, 17, 22, 26, 28] have extensively investigated the existence of solutions for problems such as (1) using the variational technique, and their work has inspired the present paper. The reader can consult [1, 24, 30, 33, 35] and the references for further results.

In [7], the authors employed both variants of the Mountain- Pass theorem to establish the existence and multiplicity of solutions under the Palais-Smale condition despite the absence of the second divergence element in problem (1). While the other authors in [25] proved the existence and multiplicity of solutions to their Dirichlet problem using the Mountain-Pass, Fountain theorems.

In this article, we provide an extension of the result found in [7] for the problem (1), which is denoted by $(p(.), q(.))-$Laplacian-like operators. To begin, we will do an in-depth investigation into the features of the Lagrange functional that are linked to our situation. In the second step of our proof, we will apply both Mountain- Pass theorems to demonstrate the existence of at least a non-trivial solution and the existence of infinitely many pairs solutions to the problem (1).

For the purposes of our investigation, we make the following assumptions:

(H1): $a(.) : \mathbb{R}^+ \to \mathbb{R}$ is a continuous functions satisfy:

$$K_1 \leqslant a(t) \leqslant K_2, \ \forall t \geqslant 0, \text{ where } 0 < K_1 < K_2.$$

(H2): We defined the function $\theta(.) : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ as $\theta(\xi) = A\left(|\xi|^{p(x)}\right)$, where $x \in \Omega$, and $A(.)$ is the antiderivative of $a(.)$, expressed as $A(t) = \int_0^t a(s)\,ds$. This function is strictly convex.

(H3): $b(.)$ and $d(.)$ are positive and continuous functions defined on $\partial\Omega$.

$f(.,.) : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Caratheodory function where $f(x,0) = 0$ and they satisfy the following conditions:

(F1): For all $(x,t) \in \Omega\times\mathbb{R}$, the inequality $|f(x,t)| \leqslant f_1(x)\ |t|^{r(x)-1}$ holds, where $1 \leqslant r_- \leqslant r_+ \leqslant q_-$, f_1 is a measurable, non negative function and $f_1 \in L^{\frac{q(.)}{q(.)-r(.)}}(\Omega)$.

(F2): For all $(x,s) \in \Omega\times\mathbb{R}$, the inequality $|f(x,t)| \geqslant f_2(x)\ |t|^{\alpha(x)-1}$ holds, where $1 \leqslant \alpha_- \leqslant \alpha_+ \leqslant r_-$, $f_2 > 0$ in some non-empty open set $O \subset \Omega$.

$g(.,.) : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ is a Caratheodory function, with $g(x,0) = 0$. The following conditions are satisfied:

(G1): For all $(x,t) \in \partial\Omega\times\mathbb{R}$, the inequality $|g(x,t)| \leqslant g_1(x)\ |t|^{s(x)-1}$ holds, where $1 \leqslant p_+ \leqslant s_- \leqslant s_+$, and $s(.) \leqslant p^\partial(.)$ on $\overline{\Omega}$. Here g_1 is a measurable, non negative function, and there exists a positive constant C_g such that $0 \leqslant g_1(.) \leqslant C_g$.

(G2): For all $(x,t) \in \partial\Omega \times \mathbb{R}$, the limit $\lim_{t\to 0} \frac{g(x,t)\,t}{|t|^{p_+-1}} = 0$ holds.

(G3): There exists $\lambda > p_+ \frac{K_2}{K_1}$ such that $\lambda G(x,t) \leqslant g(x,t)\,t$, for all $(x,t) \in \partial\Omega\times\mathbb{R}$, where $G(x,u) = \int_0^u g(x,s)\,ds$, ($K_1, K_2$ are mentioned in (H1)).

The present paper is structured in the following manner: In Section 2, we present an overview of the fundamental features of the function spaces $L^{p(.)}(\Omega)$ and $W^{1,p(.)}(\Omega)$, as well as the Nemytsky operator. Additionally, we provide some assertions and lemmas that are relevant and useful in this context. Section 3 establishes the existence and multiplicity of weak solutions to the problem (1).

2. Preliminaries

This section briefly overviews recent work in the Generalized Lebesgue space $L^{p(.)}(\Omega)$ and Generalized Sobolev space $W^{1,p(.)}(\Omega)$. See Refs. [14, 16, 19, 21, 37] for further reading.

Let $p(.) \in C_+(\Omega)$ the variable exponent appearing in (1) , we define the Generalized Lebesgue space $L^{p(.)}(\Omega)$ by:

$$L^{p(.)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ mesurable } / \ \rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx < +\infty \right\}.$$

where $\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx$ is the convex and continuous modular of u in $L^{p(.)}(\Omega)$. Moreover, as $p(.) \in C_+(\Omega)$, then $\rho_{p(.)}(.)$ is weakly lower semi-continuous.

Since $p(.) \in C_+(\Omega)$ then $L^{p(.)}(\Omega)$ is a uniformly convex, separable and Banach space, equipped by the following Luxemburg norm

$$\|u\|_{p(.)} = \inf\left\{ \lambda > 0 \ / \ \rho_{p(.)}\left(\frac{u}{\lambda}\right) \leqslant 1 \right\}.$$

Remark 2.1. *$L^{p(.)}(\Omega)$ is a reflexive space, and its dual space is isomorphic to $L^{p'(.)}(\Omega)$ where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$ with $p'(.)$ is the conjugate exponent of $p(.)$.*

Definition 2.2. *For all $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$, we define the Hölder inequality by*

$$\left| \int_\Omega uv\,dx \right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \|u\|_{p(.)} \|v\|_{p'(.)}.$$

The definition of the Generalized Sobolev space is as follows:

$$W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) \ / \ |\nabla u| \in L^{p(.)}(\Omega) \right\}.$$

Moreover, $W^{1,p(.)}(\Omega)$ is a reflexive space, and its dual space is isomorphic to $W^{-1,p'(.)}(\Omega)$.
Since $p(.) \in C_+(\Omega)$ then $W^{1,p(.)}(\Omega)$ is a uniformly convex, separable and Banach space, which is endowed with the following norm:

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

In what follows, we introduce the following notation:

$$\rho_{1,p(.)}(u) = \int_\Omega |u|^{p(x)}\,dx + \int_\Omega |\nabla u|^{p(x)}\,dx$$

for $u \in M(\Omega)$ a measurable function.

Proposition 2.3. *[36]. If $u \in W^{1,p(.)}(\Omega)$, then the following properties hold true:*

(i) $\|u\|_{1,p(.)} > 1 \Rightarrow \|u\|_{1,p(.)}^{p_-} \leqslant \rho_{1,p(.)}(u) \leqslant \|u\|_{1,p(.)}^{p_+}$,

(ii) $\|u\|_{1,p(.)} < 1 \Rightarrow \|u\|_{1,p(.)}^{p_+} \leqslant \rho_{1,p(.)}(u) \leqslant \|u\|_{1,p(.)}^{p_-}$,

(iii) $\|u\|_{1,p(.)} < 1$ *(respectively* $= 1; > 1$*)* $\Leftrightarrow \rho_{1,p(.)}(u) < 1$ *(respectively* $= 1; > 1$*).*

We proceed to define two further norms in space $W^{1,p(.)}(\Omega)$:

Proposition 2.4. *(See theorem 2.1 in [15]). For any $u \in W^{1,p(.)}(\Omega)$,*
if $b(.) \in L^\infty(\partial\Omega)$ with $b_- > 0$ then

$$\|u\|_* = inf\left\{\lambda > 0/ \int_\Omega \left|\frac{\nabla u}{\lambda}\right|^{p(x)} dx + \int_{\partial\Omega} b(x)\left|\frac{u}{\lambda}\right|^{p(x)} d\sigma \leqslant 1\right\}$$

is a norm in $W^{1,p(.)}(\Omega)$ that is equivalent to the standard norm $\|u\|_{1,p(.)}$. Moreover, the modular

$$\rho^*_{1,p(.)}(u) = \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} b(x)|u|^{p(x)} d\sigma$$

verifies the same properties mentioned in the proposition 2.3 (See proposition 2.4 in [15]).

Next, we will present some results of continuous and compact embedding:

Proposition 2.5. *(See theorem 2.2 in [21]). Let Ω be a bounded domain of $\mathbb{R}^\mathbb{N}$ and $p(.), q(.) \in C_+\left(\overline{\Omega}\right)$.*
If $p(.) < N$ and $q(.) < p^*(.)$ in $\overline{\Omega}$ with

$$p^*(x) := \begin{cases} \dfrac{Np(x)}{N - p(x)}, & if \quad p(x) < N \\ \infty, \quad if \quad p(x) \geqslant N, \end{cases}$$

then, there is a continuous and compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega)$.

Proposition 2.6. *(See [19, corollary 2.2]). Let Ω be an open bounded domain with Lipschitz boundary and $p(.) \in C(\bar{\Omega})$ where $1 < p_- \leqslant p_+ < N$.*
If $q(.) \in C(\partial\Omega)$ satisfies the condition $1 \leqslant q(x) < p^\partial(x)$, for all $x \in \partial\Omega$, with

$$p^\partial(x) := (p(x))^\partial := \begin{cases} \dfrac{(N-1)p(x)}{N - p(x)}, \ if \ p(x) < N \\ \infty, \ if \ p(x) \geqslant N. \end{cases}$$

Then, there is a continuous and compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{q(.)}(\partial\Omega)$. In particular, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p(.)}(\partial\Omega)$.

The data $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ appeared in our problem is a Caratheodory function (i.e. $x \longmapsto f(x,s)$ is measurable for $s \in \mathbb{R}$ and $s \longmapsto f(x,s)$ is continuous for a.e $x \in \Omega$). For this, we introduced an operator associated with this function N_f, namely the Nemytsky operator: $N_f : M(\Omega) \longrightarrow M(\Omega)$ defined by $\left(N_f(u)\right)(x) = f(x, u(x)), \ \forall x \in \Omega$.

Proposition 2.7. *(See [20, theorem 2.1]). If $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function and satisfies*

$$|f(x,s)| \leqslant a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}}, \ \forall x \in \Omega, \ \forall s \in \mathbb{R},$$

where $p_1(.), p_2(.) \in C_+\left(\overline{\Omega}\right)$, $a(.) \in L^{p_2(.)}(\Omega, \mathbb{R}^+)$ and $b \geqslant 0$ is constant. Then the Nemytsky operator from $L^{p_1(.)}(\Omega)$ to $L^{p_2(.)}(\Omega)$ defined by $\left(N_f(u)\right)(x) = f(x, u(x)), \ \forall x \in \Omega$ is a continuous and bounded operator.

3. Existence result

In this section, we will prove the existence results of weak solutions problem (1). From the divergence terms of this problem and the condition (2), it is easy to notice that the solutions space is $E = W^{1,p(.)}(\Omega) \cap W^{1,q(.)}(\Omega)$ and its norm is: $\|.\| = \|.\|_{1,p(.)}$.
We give the definition of weak solution of this problem :

Definition 3.1. *We say that $u \in E$ is a weak solution of problem (1), if for all $v \in E$ such that:*

$$\int_\Omega \mathcal{A}(\nabla u)\nabla v dx + \int_\Omega \mathcal{B}(\nabla u)\nabla v dx + \int_{\partial\Omega} b(x)|u|^{p(x)-2}uv d\sigma + \int_{\partial\Omega} d(x)|u|^{q(x)-2}uv d\sigma \\ = \mu \int_\Omega f(x,u)v dx + \int_{\partial\Omega} g(x,u)v d\sigma. \tag{3}$$

The functional $J_\mu : E \longrightarrow \mathbb{R}$ associated with problem (1) is introduced as follows :

$$J_\mu(u) = \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})dx + \int_\Omega \frac{1}{q(x)} A(|\nabla u|^{q(x)})dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x)|u|^{p(x)} d\sigma + \int_{\partial\Omega} \frac{1}{p(x)} d(x)|u|^{q(x)} d\sigma \\ - \mu \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma. \tag{4}$$

Where $F(x,u) = \int_0^u f(x,s)ds,\ \forall x \in \Omega$, $G(x,u) = \int_0^u g(x,s)ds,\ \forall x \in \partial\Omega$, σ is the measure on the boundary $\partial\Omega$.

3.1. Existence of non-trivial weak solutions

Theorem 3.2. *Assume that (H1)–(H3),(F1), and (G1)–(G3) hold. Then, there exists a least one non-trivial weak solution of the problem (1).*

We apply the mountain pass approach to the functional J_μ under the Palais-Smale condition (i.e $(P-S)$-condition). Here, we recall the definition of $(P-S)$-condition:

Definition 3.3. *Let E be a real Banach space, E' is its topological dual and let be a functional $J \in C^1(E,\mathbb{R})$. We say that J satisfies a Palais-Smale condition, if for every sequence $(u_n)_n$ in E such that :*

$$(J(u_n))_n \text{ is bouned and } \lim_{n\to+\infty} \|J'(u_n)\|_{E'} = 0, \tag{5}$$

has a convergent sub-sequence.
Moreover, if instead of $(J(u_n))_n$ we have $\lim_{n\to+\infty} J(u_n) = c$, We say that J satisfies Palais-Smale condition for level c (i.e. local condition), noted $(P-S)_c$.

We will consider the following version of the Mountain Pass Theorem :.

Theorem 3.4. *(See [29, Theorem 3.2]). Consider a real Banach space E and a functional $J \in C^1(E,\mathbb{R})$ that satisfies a Palais-Smale condition. Assume $J(0) = 0$, and the following conditions are met:*
- *(P1): $\exists R > 0,\ \exists \rho > 0$ such that $J(u) \geqslant \rho,\ for\ \|u\| = R$,*
- *(P2): $\exists e \in E$ such that $J(e) \leqslant 0$ for $\|u\| > R$. Then,*

J has a critical value c with $c \geqslant \rho$, such that $c = \inf_{\gamma\in\Gamma} \max_{v\in\gamma([0,1])} J(v)$, where $\Gamma = \{\gamma \in C([0,1];E)/\gamma(0) = 0,\ \gamma(1) = e\}$.

The Theorem 3.2 can be proven by establishing the following three propositions:

Proposition 3.5. *The functional J_μ is the class $C^1(E,\mathbb{R})$ and $J_\mu(0) = 0$.*

Proof. for $u \in E$, we put

$$L_1(u) = \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x)|u|^{p(x)}d\sigma,$$

$$L_2(u) = \int_\Omega \frac{1}{q(x)} A(|\nabla u|^{q(x)})dx + \int_{\partial\Omega} \frac{1}{p(x)} d(x)|u|^{q(x)}d\sigma,$$

$$\Phi(u) = \int_\Omega F(x,u)dx,$$

and

$$\Psi(u) = \int_{\partial\Omega} G(x,u)d\sigma,$$

we have

$$J_\mu = L_1 + L_2 - \mu\Phi - \Psi.$$

The proof of this proposition is organized as follows:

• **Step 1 : the functional J_μ is well defined on E and $J_\mu(0) = 0$:**

According to (H1) and (H3), we conclude that

$$L_1(u) = \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x)|u|^{p(x)}d\sigma \leqslant \int_\Omega \frac{1}{p(x)} \left[\int_0^{|\nabla u|^{p(x)}} a(t)dt\right] dx + \int_{\partial\Omega} \frac{1}{p(x)} \|b\|_\infty |u|^{p(x)}d\sigma.$$

Then

$$L_1(u) \leqslant \frac{K_2}{p_-} \int_\Omega |\nabla u|^{p(x)}dx + \frac{\|b\|_\infty}{p_-} \int_{\partial\Omega} |u|^{p(x)}d\sigma \leqslant K_3 \rho_{1,p(.)}(u), \ where \ K_3 = \frac{1}{p_-}\max(K_2, \|b\|_\infty) > 0.$$

Since $u \in E$, we conclude $L_1(u) < +\infty$, thus L_1 is a functional well defined.
In the same way, we can prove L_2 is also a functional well defined.
From (F1), we have

$$\Phi(u) = \int_\Omega F(x,u)dx = \int_\Omega \left[\int_0^u f(x,s)ds\right] dx \leqslant \int_\Omega \frac{1}{r(x)} |f_1(x)||u|^{r(x)}dx,$$

since $u \in E \hookrightarrow L^{q(.)}(\Omega)$ and $f_1 \in L^{\frac{q(.)}{q(.)-r(.)}}(\Omega)$, from the Hölder inequality, that implies

$$\Phi(u) \leqslant \frac{1}{r_-} \|f_1\|_{\frac{q(.)}{q(.)-r(.)}} \||u|^{r(.)}\|_{\frac{q(.)}{r(.)}} < +\infty.$$

Then Φ is a functional well defined.
From (G1), we have

$$\Psi(u) = \int_{\partial\Omega} G(x,u)d\sigma = \int_{\partial\Omega} \left[\int_0^u g(x,s)ds\right] d\sigma \leqslant \int_{\partial\Omega} \frac{1}{s(x)} |g_1(x)||u|^{s(x)}d\sigma \leqslant \frac{C_g}{s_-} \int_{\partial\Omega} |u|^{s(x)}d\sigma \leqslant \frac{C_g}{s_-} \int_{\partial\Omega} |u|^{p^\partial(x)}d\sigma,$$

then,

$$\Psi(u) \leqslant \frac{C_g}{s_-} \rho_{p^\partial(.),\partial\Omega}(u).$$

From proposition 2.3 in [21] and the proposition 2.6, we have that

$$\Psi(u) \leqslant c\|u\| < +\infty,$$

where $c > 0$. Hence, Ψ is a functional well defined.
From the hypothesis (H2) and $f(x, 0) = g(x, 0) = 0$, it's easy to verify that $J_\mu(0) = 0$.
In summary, the functional J_μ is well defined and $J_\mu(0) = 0$.

• **Step 2 : the functional J_μ is Gateaux derivative:**

Let $u, v \in E$ and $h > 0$ small enough, we have

$$L_1(u + hv) - L_1(u) = \int_\Omega \frac{1}{p(x)} \left[A(|\nabla u + h\nabla v|^{p(x)}) - A(|\nabla u|^{p(x)})\right] dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x) \left[|u + hv|^{p(x)} - |u|^{p(x)}\right] d\sigma,$$

from (H2), that implies

$$L_1(u + hv) - L_1(u) \leqslant \int_\Omega \frac{1}{p(x)} \left[\int_{|\nabla u|^{p(x)}}^{|\nabla u + h\nabla v|^{p(x)}} a(t)dt\right] dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x) \left[|u + hv|^{p(x)} - |u|^{p(x)}\right] d\sigma,$$

we apply The Mean Value Theorem on the function $a(.)$, then there exists a positive constant $\theta_0 = \theta_0(\nabla u, \nabla u + h\nabla v)$ in $]0, 1[$ such that

$$\begin{aligned} L_1(u + hv) - L_1(u) \leqslant &\int_\Omega \frac{1}{p(x)} \left[|\nabla u + h\nabla v|^{p(x)} - |\nabla u|^{p(x)}\right] a\left(|\nabla u|^{p(x)} + \theta_0(|\nabla u + h\nabla v|^{p(x)} - |\nabla u|^{p(x)})\right) dx \\ &+ \int_{\partial\Omega} \frac{1}{p(x)} b(x) \left[|u + hv|^{p(x)} - |u|^{p(x)}\right] d\sigma. \end{aligned}$$

We apply once again, The Mean Value Theorem on the function $\phi(t) = |t|^{p(x)}$ $\forall t \in \mathbb{R}^{\mathbb{N}}$, then there exist two positives constants $\theta_1 = \theta_1(\nabla u, \nabla u + h\nabla v), \theta_2 = \theta_2(u, u + hv)$ *in* $]0, 1[$ such that

$$\begin{aligned} L_1(u + hv) - L_1(u) \leqslant &\int_\Omega h|\nabla u + h\theta_1\nabla v|^{p(x)-2}(\nabla u + h\theta_1\nabla v)a(|\nabla u|^{p(x)} + \theta_0\Big(|\nabla u + h\nabla v|^{p(x)} - |\nabla u|^{p(x)}\Big)\nabla v dx \\ &+ \int_{\partial\Omega} hb(x)|u + h\theta_2 v|^{p(x)-2}(u + h\theta_2 v)v d\sigma \end{aligned}.$$

From Lebesgue's dominated convergence theorem, we obtain

$$\lim_{h\to 0} \frac{L_1(u + hv) - L_1(u)}{h} = \int_\Omega a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\nabla v dx = \int_\Omega \mathcal{A}(\nabla u)\nabla v dx + \int_{\partial\Omega} b(x)|u|^{p(x)-2}uv d\sigma = l_u(v),$$

by Hölder's inequality, it's easy to verify that l_u is a continuous linear functional on E with respect to v, thus the functional L_1 is Gateaux derivative where its Gateaux derivative L_1' defined from E to its dual E' by

$$< L_1'(u), v >= \int_\Omega \mathcal{A}(\nabla u)\nabla v dx + \int_{\partial\Omega} b(x)|u|^{p(x)-2}uv d\sigma \quad \forall v \in E.$$

with $< ., . >$ is the duality bracket between E and its dual E'.
In the same way, we can prove the functional L_2 is also Gateaux derivative where its Gateaux derivative L_2' defined from E to its dual E' by:

$$< L_2'(u), v >= \int_\Omega \mathcal{B}(\nabla u)\nabla v dx + \int_{\partial\Omega} d(x)|u|^{q(x)-2}uv d\sigma \quad \forall v \in E.$$

On the other hand, by applying the Mean Value Theorem on the function F, then there exists a positive constant $\theta_3 = \theta_1(\nabla u, \nabla u + h\nabla v)$ *in* $]0, 1[$ such that

$$\Phi(u + hv) - \Phi(u + hv) = \int_\Omega \Big(F(x, u + hv) - F(x, u)\Big)dx = \int_\Omega hvf(x, u + h\theta_3 v)dx,$$

by using Lebesgue's dominated convergence theorem, we obtain

$$\lim_{h\to 0}\frac{\Phi(u+hv)-\Phi(u+hv)}{h}=\int_\Omega f(x,u)v dx.$$

By (F1) and the Hölder inequality, it's easy to verify that $v\longrightarrow\int_\Omega f(x,u)vdx$ is continuous linear functional with respect to v, thus the functional Φ is Gateaux derivative and its the functional Gateaux derivative Φ' defined from E to its dual E' by:

$$<\Phi'(u),v>=\int_\Omega f(x,u)vdx,\ \forall v\in E.$$

In the same way, we can prove the functional Ψ is also Gateaux derivative where its the functional Gateaux derivative Ψ' defined from E to its dual E' by:

$$<\Psi'(u),v>=\int_{\partial\Omega} g(x,u)vd\sigma,\ \forall v\in E.$$

Since a functional $L_1,\ L_2,\ \Phi$ *and* Ψ are Gateaux derivatives, hence $J_\mu=L_1+L_2-\mu\Phi-\Psi$ is also Gateaux derivative and its the functional Gateaux derivative J'_μ defined from E to its dual E' by:

$$<J'_\mu(u),v>=<L'_1(u),v>+<L'_2(u),v>-\mu<\Phi'(u),v>-<\Psi'(u),v>,\ \forall v\in E.$$

• **Step 3 : The Gateaux derivative J'_μ is continuous:**

Let $v\in E$ and $(u_n)_n\in E$ such that $(u_n)_n\longrightarrow u$ strongly in E,
Firstly, we have

$$\left|\left\langle L'_1(u_n)-L'_1(u),v\right\rangle\right|\leqslant\int_\Omega\left|\mathcal{A}(\nabla u_n)-\mathcal{A}(\nabla u)\right||\nabla v|dx+\int_{\partial\Omega}|b(x)|\left||u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\right||v|d\sigma.$$

From (H1) and (H3), we conclude that

$$\left|\left\langle L'_1(u_n)-L'_1(u),v\right\rangle\right|\leqslant\int_\Omega\left|K_2|\nabla u_n|^{p(x)-2}\nabla u_n-K_1|\nabla u|^{p(x)-2}\nabla u\right||\nabla v|dx+\|b\|_\infty\int_{\partial\Omega}\left||u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\right||v|d\sigma,$$

from the Hölder inequality, we obtain

$$\left|\left\langle L'_1(u_n)-L'_1(u),v\right\rangle\right|\leqslant C\Big[\|\,|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\,\|_{p'(.)}\|\nabla v\|_{p(.)}+\||u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\,\|_{p'(.)}\|v\|_{p(.)}\Big],$$

where $C>0$. Then

$$\sup_{\|v\|\leqslant 1}\left|\left\langle L'_1(u_n)-L'_1(u),v\right\rangle\right|\leqslant C\left[\|\,|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\,\|_{p'(.)}+\||u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u\,\|_{p'(.)}\right],$$

Since $(u_n)_n\longrightarrow u$ strongly in E, therefore, $|\nabla u_n|^{p(x)-2}\nabla u_n\longrightarrow|\nabla u|^{p(x)-2}\nabla u$ and $|u_n|^{p(x)-2}u_n\longrightarrow|u|^{p(x)-2}u$ strongly in $L^{p'(.)}(\Omega)$, thus

$$\lim_{n\to+\infty}\|L'_1(u_n)-L'_1(u)\|_{E'}=\lim_{n\to+\infty}\sup_{\|v\|\leqslant 1}\left|\left\langle L'_1(u_n)-L'_1(u),v\right\rangle\right|=0,$$

so the functional L'_1 is continuous.
In the same way and using (2), we can prove the functional L'_2 is also continuous.
Secondly, from the hypothesis (F1), we have

$$\left|\left\langle\Phi'(u_n)-\Phi'(u),v\right\rangle\right|\leqslant\int_\Omega\left|f(x,u_n)-f(x,u)\right||v|dx\leqslant\int_\Omega|f_1(x)|\left||u_n|^{r(x)-1}-|u|^{r(x)-1}\right||v|dx,$$

since $f_1 \in L^{\frac{q(.)}{q(.)-r(.)}}(\Omega)$, $u, v, u_n \in E \hookrightarrow W^{1,q(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$, $|u_n|^{r(x)-1}, |u|^{r(x)-1} \in L^{\frac{q(.)}{r(.)-1}}(\Omega)$ and $\frac{q(.)-r(.)}{q(.)} + \frac{1}{q(.)} + \frac{r(.)-1}{q(.)} = 1$, from generalized Hölder's inequality (see [14]), we obtain

$$\left|\left\langle \Phi'(u_n) - \Phi'(u), v\right\rangle\right| \leqslant \|f_1\|_{\frac{q(.)}{q(.)-r(.)}} \||u_n|^{r(x)-1} - |u|^{r(x)-1}\|_{\frac{q(.)}{r(.)-1}} \|v\|_{q(.)} \leqslant c_1\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} \||u_n|^{r(x)-1} - |u|^{r(x)-1}\|_{\frac{q(.)}{r(.)-1}} \|v\|,$$

where $c > 0$, then

$$\|\Phi'(u_n) - \Phi'(u)\|_{E'} = \sup_{\|v\|\leqslant 1} \left|\left\langle \Phi'(u_n) - \Phi'(u), v\right\rangle\right| \leqslant c_1\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} \||u_n|^{r(x)-1} - |u|^{r(x)-1}\|_{\frac{q(.)}{r(.)-1}}.$$

Since $(u_n)_n \longrightarrow u$ strongly in E, then $|u_n|^{r(x)-1} \longrightarrow |u|^{r(x)-1}$ strongly in $L^{\frac{q(.)}{r(.)-1}}(\Omega)$, thus $\lim_{n\to+\infty} \|\Phi'(u_n)-\Phi'(u)\|_{E'} = 0$, so the functional Φ' is continuous.
Thirdly, from (G1), we have :

$$\left|\left\langle \Psi'(u_n) - \Psi'(u), v\right\rangle\right| \leqslant \int_{\partial\Omega} \left|g(x, u_n) - g(x, u)\right| |v| d\sigma \leqslant \int_{\partial\Omega} |g_1(x)|\left||u_n|^{s(x)-1} - |u|^{s(x)-1}\right| |v| d\sigma,$$

then

$$\left|\left\langle \Psi'(u_n) - \Psi'(u), v\right\rangle\right| \leqslant C_g \int_{\partial\Omega} \left||u_n|^{s(x)-1} - |u|^{s(x)-1}\right| |v| d\sigma \leqslant C_g \int_{\partial\Omega} \left||u_n|^{p^\partial(x)-1} - |u|^{p^\partial(x)-1}\right| |v| d\sigma.$$

Since $u, v, u_n \in E \hookrightarrow W^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p^\partial(.)}(\partial\Omega)$ compactly (see proposition 2.6), $|u_n|^{p^\partial(x)-1}, |u|^{p^\partial(x)-1} \in L^{\frac{p^\partial(.)}{p^\partial(.)-1}}(\partial\Omega)$, from Hölder's inequality, we obtain

$$\left|\left\langle \Psi'(u_n) - \Psi'(u), v\right\rangle\right| \leqslant C_g \||u_n|^{p^\partial(x)-1} - |u|^{p^\partial(x)-1}\|_{\frac{p^\partial(.)}{p^\partial(.)-1}} \|v\|_{p^\partial(.)} \leqslant c_2 C_g \||u_n|^{p^\partial(x)-1} - |u|^{p^\partial(x)-1}\|_{\frac{p^\partial(.)}{p^\partial(.)-1}} \|v\|,$$

then

$$\|\Psi'(u_n) - \Psi'(u)\|_{E'} = \sup_{\|v\|\leqslant 1} \left|\left\langle \Psi'(u_n) - \Psi'(u), v\right\rangle\right| \leqslant c_2 C_g \||u_n|^{p^\partial(x)-1} - |u|^{p^\partial(x)-1}\|_{\frac{p^\partial(.)}{p^\partial(.)-1}},$$

Since $(u_n)_n \longrightarrow u$ strongly in E and $u, u_n \in E \hookrightarrow W^{1,p(.)}(\Omega) \hookrightarrow L^{p^\partial(.)}(\partial\Omega)$, then $|u_n|^{p^\partial(x)-1} \longrightarrow |u|^{p^\partial(x)-1}$ strongly in $L^{\frac{p(.)}{p(.)-r(.)}}(\partial\Omega)$, thus $\lim_{n\to+\infty} \|\Psi'(u_n) - \Psi'(u)\|_{E'} = 0$, so the functional Φ' is continuous.
In summary, we conclude that the functional J'_μ is continuous.
Since the functional J_μ is Gateaux derivative and its Gateaux derivative J'_μ is continuous, hence J_μ is of class $C^1(E, \mathbb{R})$. □

Proposition 3.6. *We assume that the hypothesis (H1)–(H3), (F1), (G1), and (G3) hold. Then the functional J_μ satisfies the Palais-Smale condition on E.*

Proof. Let $(u_n)_n$ a sequence in E. We assume that J_μ is bounded (i.e. $\exists M > 0 : |J_\mu(u_n)| \leqslant M, \forall (u_n)_n \in E$) and $\lim_{n\to+\infty} \|J'_\mu(u_n)\|_{E'} = 0$,
• **Firstly, we will prove that $(u_n)_n$ is bounded:**
By the contrary, we assume that $(u_n)_n$ is unbounded, we take $\|u_n\| > 1$ for any integer n. We have

$$M \geqslant J_\mu(u_n) = \int_\Omega \frac{1}{p(x)} A(|\nabla u_n|^{p(x)}) dx + \int_\Omega \frac{1}{q(x)} A(|\nabla u_n|^{q(x)}) dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x)|u_n|^{p(x)} d\sigma + \int_{\partial\Omega} \frac{1}{q(x)} d(x)|u_n|^{q(x)} d\sigma$$
$$- \mu \int_\Omega F(x, u_n) dx - \int_{\partial\Omega} G(x, u_n) d\sigma,$$

from (H1), (H2) and (F1), we obtain

$$M \geqslant \frac{K_1}{p_+} \int_\Omega |\nabla u_n|^{p(x)} dx + \frac{K_1}{q_+} \int_\Omega |\nabla u_n|^{q(x)} dx + \frac{1}{p_+} \int_{\partial\Omega} b(x)|u_n|^{p(x)} d\sigma + \frac{1}{q_+} \int_{\partial\Omega} d(x)|u_n|^{q(x)} d\sigma$$
$$- \frac{\mu}{r_-} \int_\Omega f_1(x)|u_n|^{r(x)} dx - \int_{\partial\Omega} G(x, u_n) d\sigma,$$

from Hölder's inequality and (F1), we conclude that $\int_\Omega f_1(x)|u_n|^{r(x)}dx \leqslant \|f_1\|_{\frac{q(.)}{q(.)-r(.)}} \||u|^{r(.)}\|_{\frac{q(.)}{r(.)}}$.
As $L^{q(.)}(\Omega) \hookrightarrow L^{\frac{q(.)}{r(.)}}(\Omega)$, and $E \hookrightarrow L^{q(.)}(\Omega)$, then, there exist positive constants $c_1,\ c_2$, such that

$$\int_\Omega f_1(x)|u_n|^{r(x)}dx \leqslant c_1\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} \||u_n|^{r(.)}\|_{q(.)} \leqslant c_2\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} \||u_n|^{r(.)}\|,$$

therefore

$$\int_\Omega f_1(x)|u_n|^{r(x)}dx \leqslant c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} \|u_n\|^{r_+}, \tag{6}$$

from (6), the last inequality of M implies

$$\begin{aligned} M \geqslant\ & \frac{K_1}{p_+}\int_\Omega |\nabla u_n|^{p(x)}dx + \frac{K_1}{q_+}\int_\Omega |\nabla u_n|^{q(x)}dx + \frac{1}{p_+}\int_{\partial\Omega} b(x)|u_n|^{p(x)}d\sigma + \frac{1}{q_+}\int_{\partial\Omega} d(x)|u_n|^{q(x)}d\sigma \\ & - \frac{\mu}{r_-}c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}\|u_n\|^{r_+} - \int_{\partial\Omega} G(x,u_n)d\sigma. \end{aligned} \tag{7}$$

On another hand, we have

$$\begin{aligned} \left\langle J'_\mu(u_n), u_n\right\rangle =\ & \int_\Omega a(|\nabla u_n|^{p(x)})|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} b(x)|u_n|^{p(x)}d\sigma + \int_\Omega a(|\nabla u_n|^{q(x)})|\nabla u_n|^{q(x)}dx + \int_{\partial\Omega} d(x)|u_n|^{q(x)}d\sigma \\ & - \mu\int_\Omega f(x,u_n)u_n dx - \int_{\partial\Omega} g(x,u_n)u_n d\sigma, \end{aligned}$$

by (H1),(F1), (G3), we have

$$\begin{aligned} \left\langle J'_\mu(u_n), u_n\right\rangle \leqslant\ & K_2\int_\Omega |\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} b(x)|u_n|^{p(x)}d\sigma + K_2\int_\Omega |\nabla u_n|^{q(x)}dx + \int_{\partial\Omega} d(x)|u_n|^{q(x)}d\sigma \\ & + \mu\int_\Omega f_1(x)|u_n|^{r(x)}dx - \lambda\int_{\partial\Omega} G(x,u_n)d\sigma, \end{aligned}$$

from (6), we deduce that

$$\begin{aligned} -\left\langle J'_\mu(u_n), u_n\right\rangle \geqslant\ & -K_2\int_\Omega |\nabla u_n|^{p(x)}dx - \int_{\partial\Omega} b(x)|u_n|^{p(x)}d\sigma - K_2\int_\Omega |\nabla u_n|^{q(x)}dx - \int_{\partial\Omega} d(x)|u_n|^{q(x)}d\sigma \\ & - \mu c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}\|u_n\|^{r_+} + \lambda\int_{\partial\Omega} G(x,u_n)d\sigma, \end{aligned} \tag{8}$$

from (7) and (8), we conclude that

$$\begin{aligned} M - \frac{1}{\lambda}\left\langle J'_\mu(u_n), u_n\right\rangle \geqslant\ & \left(\frac{K_1}{p_+} - \frac{K_2}{\lambda}\right)\int_\Omega |\nabla u_n|^{p(x)}dx + \left(\frac{K_1}{q_+} - \frac{K_2}{\lambda}\right)\int_\Omega |\nabla u_n|^{q(x)}dx + \left(\frac{1}{p_+} - \frac{1}{\lambda}\right)\int_{\partial\Omega} b(x)|u_n|^{p(x)}d\sigma \\ & + (\frac{1}{q_+} - \frac{1}{\lambda})\int_{\partial\Omega} d(x)|u_n|^{q(x)}d\sigma - \mu c_3(\frac{1}{r_-} + \frac{1}{\lambda})\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}\|u_n\|^{r_+}. \end{aligned}$$

In (G3), we have $\lambda > p_+\dfrac{K_2}{K_1}$,we conclude

$$\begin{aligned} M - \frac{1}{\lambda}\left\langle J'_\mu(u_n), u_n\right\rangle \geqslant\ & \min\left(\frac{K_1}{p_+} - \frac{K_2}{\lambda}, \frac{1}{p_+} - \frac{1}{\lambda}\right)\left[\int_\Omega |\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} b(x)|u_n|^{q(x)}d\sigma\right] \\ & + \min\left(\frac{K_1}{q_+} - \frac{K_2}{\lambda}, \frac{1}{q_+} - \frac{1}{\lambda}\right)\left[\int_\Omega |\nabla u_n|^{q(x)}dx + \int_{\partial\Omega} d(x)|u_n|^{q(x)}d\sigma\right]. \\ & - \mu c_3(\frac{1}{r_-} + \frac{1}{\lambda})\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}\|u_n\|^{r_+}. \end{aligned}$$

We put the positive constants
$c_4 = \mu c_3(\frac{1}{r_-} + \frac{1}{\lambda})\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}$, $c_5 = \min\left(\frac{K_1}{p_+} - \frac{K_2}{\lambda}, \frac{1}{p_+} - \frac{1}{\lambda}\right)$ and $c_6 = \min\left(\frac{K_1}{q_+} - \frac{K_2}{\lambda}, \frac{1}{q_+} - \frac{1}{\lambda}\right)$.
From proposition 2.4, we conclude that

$$M \geqslant \frac{1}{\lambda}\left\langle J'_\mu(u_n), u_n\right\rangle + c_5\rho^*_{1,p(.)}(u_n) + c_6\rho^*_{1,q(.)}(u_n) - c_4\|u_n\|^{r_+},$$

since J_μ is of class $C^1(E,\mathbb{R})$ and $\rho^*_{1,q(.)}(u_n) \geqslant 0$, that implies

$$M \geqslant -\frac{1}{\lambda}\|J'_\mu(u_n)\|_{E'}\ \|u_n\| + c_5\rho^*_{1,p(.)}(u_n) - c_4\|u_n\|^{r_+}.$$

From proposition 2.4, we conclude that

$$M \geqslant -\frac{1}{\lambda}\|J'_\mu(u_n)\|_{E'}\ \|u_n\| + c_5\|u_n\|^{p_-} - c_4\|u_n\|^{r_+},$$

then

$$M \geqslant \|u_n\|\left[-\frac{1}{\lambda}\|J'_\mu(u_n)\| + \|u_n\|^{r_+-1}\left(c_5\|u_n\|^{p_--r_+} - c_4\right)\right].$$

Letting $n \longrightarrow +\infty$, we obtain $M \geqslant +\infty$, this is a contradiction. So $(u_n)_n$ is bounded in E.
• **Secondly, we prove that any** $(P-S)-$ **sequence has a convergent sub-sequence:**
To demonstrate this, we need the following lemma :

Lemma 3.7. *(see [25], [23]) The functional* $L'_1 + L'_2 : E \longrightarrow E'$ *is of type* (S_+). *(i.e. if* $(u_n)_n \rightharpoonup u$ *weakly in E and* $\limsup_{n\to+\infty}\left\langle (L'_1+L'_2)(u_n), u_n - u\right\rangle$, *then* $(u_n)_n \longrightarrow u$ *strongly in E).*

Given that E is a Banach and reflexive space, there exists $u \in E$ such that, considering a subsequence denoted as $(u_n)_n$, we have $(u_n)_n \rightharpoonup u$ weakly in E, then by the compact embedding $E \hookrightarrow\hookrightarrow L^{p(.)}(\Omega)$, we obtain

$$(u_n)_n \longrightarrow u \text{ a.e. in } \Omega \text{ and } (u_n)_n \longrightarrow u \text{ in } L^{p(.)}(\Omega). \tag{9}$$

On other hand, we have,

$$\left\langle L'_1(u_n) + L'_2(u_n), u_n - u\right\rangle = \int_\Omega \mathcal{A}(\nabla u_n)(\nabla u_n - \nabla u)dx + \int_{\partial\Omega} b(x)|u_n|^{p(x)-2}u_n(u_n-u)d\sigma + \int_\Omega \mathcal{B}(\nabla u_n - \nabla u)dx + \int_{\partial\Omega} d(x)|u_n|^{q(x)-2}u_n(u_n-u)d\sigma$$

and

$$< J'_\mu(u), v > = < L'_1(u), v > + < L'_2(u), v > -\mu < \Phi'(u), v > - < \Psi'(u), v >,\ \forall v \in E,$$

then

$$\limsup_{n\to+\infty}\left\langle L'_1(u_n) + L'_2(u_n), u_n - u\right\rangle = \limsup_{n\to+\infty}\left\langle J'_\mu(u_n), u_n - u\right\rangle + \mu\limsup_{n\to+\infty}\int_\Omega f(x,u_n)(u_n-u)dx + \limsup_{n\to+\infty}\int_{\partial\Omega} g(x,u_n)(u_n-u)d\sigma,$$

Since $\lim_{n\to+\infty}\|J'_\mu(u_n)\|_{E'} = 0$ and from (F1) and (G1), we conclude that

$$\limsup_{n\to+\infty}\left\langle L'_1(u_n)+L'_2(u_n), u_n-u\right\rangle \leqslant \mu\limsup_{n\to+\infty}\int_\Omega f_1(x)|u_n|^{r(x)-1}(u_n-u)dx + \limsup_{n\to+\infty}\int_{\partial\Omega} g_1(x)|u_n|^{s(x)-1}(u_n-u)d\sigma, \tag{10}$$

we have

$$\int_{\Omega} f_1(x)|u_n|^{r(x)-1}(u_n - u)dx \leqslant \int_{\Omega} |f_1(x)|\,|u_n|^{r(x)-1}|u_n - u|dx,$$

since $f_1 \in L^{\frac{q(.)}{q(.)-r(.)}}(\Omega)$, $u_n, u \in E \subset L^{q(.)}(\Omega)$, $|u_n|^{r(x)-1} \in L^{\frac{q(.)}{r(.)-1}}(\Omega)$ $\frac{q(.)-r(.)}{q(.)} + \frac{1}{q(.)} + \frac{r(.)-1}{q(.)} = 1$, from generalized Hölder's inequality (see [14]), we obtain

$$\int_{\Omega} f_1(x)|u_n|^{r(x)-1}(u_n - u)dx \leqslant c_8\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} \max(\|u_n\|^{r_- -1}_{\frac{q(.)}{r(.)-1}}, \|u_n\|^{r_+ -1}_{\frac{q(.)}{r(.)-1}})\|u_n - u\|_{q(.)},$$

where c_8 is a positive constant.
From (9), we conclude that

$$\limsup_{n\to+\infty} \int_{\Omega} f_1(x)|u_n|^{r(x)-1}(u_n - u)dx = 0.$$

Similarly, we can prove that

$$\limsup_{n\to+\infty} \int_{\partial\Omega} g_1(x)|u_n|^{s(x)-1}(u_n - u)d\sigma = 0.$$

From (10), we conclude that $\limsup_{n\to+\infty} \left\langle L_1'(u_n) + L_2'(u_n), u_n - u\right\rangle = 0$, from the lemma 3.7, we conclude that : $(u_n)_n \longrightarrow u$ strongly in E. □

Proposition 3.8. *The functional J_μ satisfies the geometrical conditions (P1) and (P2) of Moutain Pass.*

Proof. • For $u \in E$ with $\|u\| \leqslant 1$,We have

$$J_\mu(u) = \int_{\Omega} \frac{1}{p(x)} A(|\nabla u|^{p(x)})dx + \int_{\Omega} \frac{1}{q(x)} A(|\nabla u|^{q(x)})dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x)|u|^{p(x)}d\sigma + \int_{\partial\Omega} \frac{1}{q(x)} d(x)|u|^{q(x)}d\sigma$$
$$- \mu \int_{\Omega} F(x, u)dx - \int_{\partial\Omega} G(x, u)d\sigma,$$

from (H1), (H2), (F1), (G3) and (6) we obtain

$$J_\mu(u) \geqslant \frac{K_1}{p_+}\int_{\Omega} |\nabla u|^{p(x)}dx + \frac{K_1}{p_+}\int_{\Omega} |\nabla u|^{p(x)}dx + \frac{1}{p_+}\int_{\partial\Omega} b(x)|u|^{p(x)}d\sigma + \frac{1}{p_+}\int_{\partial\Omega} d(x)|u|^{p(x)}d\sigma$$
$$- \frac{\mu}{r_-} c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} |||u|||^{r_+} - \frac{1}{\lambda}\int_{\partial\Omega} g(x, u)ud\sigma,$$
$$\geqslant 2\frac{K_1}{p_+}\int_{\Omega} |\nabla u|^{p(x)}dx + \frac{1}{p_+}\int_{\partial\Omega} (b(x) + d(x))|u|^{p(x)}d\sigma - \frac{\mu}{r_-} c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} |||u|||^{r_+} - \frac{1}{\lambda}\int_{\partial\Omega} g(x, u)ud\sigma,$$

from (H3) and proposition 2.4, we conclude

$$J_\mu(u) \geqslant \frac{2}{p_+}\min(K_1, 1)\rho^*_{1,p(.)}(u) - \frac{\mu}{r_-} c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} \|u\|^{r_+} - \frac{1}{\lambda}\int_{\partial\Omega} g(x, u)ud\sigma,$$

From (G1), (G2) and (G3), we apply the limit definition, we have: for all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that, for all $|u| \leqslant C_\epsilon$:

$$\frac{1}{\lambda} g(x, u)u \leqslant \frac{\epsilon}{\lambda}|u|^{p_+} + C_\epsilon|u|^{s(x)}. \tag{11}$$

We deduce that

$$J_\mu(u) \geqslant \frac{2}{p_+}\min(K_1, 1)\rho^*_{1,p(.)}(u) - \frac{\mu}{r_-} c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} \|u\|^{r_+} - \frac{\epsilon}{\lambda}\int_{\partial\Omega} |u|^{p_+}d\sigma - C_\epsilon \int_{\partial\Omega} |u|^{s(x)}d\sigma.$$

Since $\partial\Omega \in \Omega$ and from proposition 2.4, we have

$$J_\mu(u) \geqslant \frac{2}{p_+}\min(K_1,1)\|u\|^{p_+} - \frac{\mu}{r_-}c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}\|u\|^{r_+} - \frac{\epsilon}{\lambda}\|u\|^{p_+}_{p_+} - C_\epsilon\|u\|^{s_-}_{s(.)},$$

since the embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{s(.)}(\Omega)$ of constant $c_{10} > 0$, the embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{p_+}(\Omega)$ of constant $c_9 > 0$, the above inequality implies that

$$J_\mu(u) \geqslant \left(\frac{2}{p_+}\min(K_1,1) - \frac{\epsilon}{\lambda}c_9\right)\|u\|^{p_+} - \frac{\mu}{r_-}c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}\|u\|^{r_+} - C_\epsilon c_{10}\|u\|^{s_-}, \tag{12}$$

since $s_- > r_+$, that yields

$$J_\mu(u) \geqslant \left(\frac{2}{p_+}\min(K_1,1) - \frac{\epsilon}{\lambda}c_9\right)\|u\|^{p_+} - \left(\frac{\mu}{r_-}c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} + C_\epsilon c_{10}\right)\|u\|^{r_+}.$$

We put $c_{11} = \frac{2}{p_+}\min(K_1,1) - c_9\frac{\epsilon}{\lambda}$ and $c_{12} = \mu\frac{c_3}{r_-}\|f_1\|_{\frac{q(.)}{q(.)-r(.)}} + C_\epsilon c_{10}$ then, for $\|u\| = R$, we have

$$J_\mu(u) \geqslant R^{r_+}\phi(R) = R^{r_+}\left(c_{11}R^{p_+-r_+} - c_{12}\right) \textit{ with } 0 < R \leqslant 1 \tag{13}$$

for ϵ small enough that we want, there exist $R > R_m = \left(\frac{c_{12}}{c_{11}}\right)^{\frac{1}{p_+-r_+}}$ and $\rho = R^{r_+}\phi(R) > 0$, such that $J_\mu(u) \geqslant \rho$ for $\|u\| = R$, so, the proof of condition (P1) of Mountain-Pass theorem is complete.

• for $u \in E$ with $\|u\| > 1$, and $t > 1$, from (H1), (H2), (2), and (6), we deduce that

$$\begin{aligned} J_\mu(tu) \leqslant |t|^{p_+}&\left[\frac{K_2}{p_-}\int_\Omega|\nabla u|^{p(x)}dx + \frac{K_2}{q_-}\int_\Omega|\nabla u|^{q(x)}dx + \frac{1}{p_-}\int_{\partial\Omega}b(x)|u|^{p(x)}d\sigma + \frac{1}{q_-}\int_{\partial\Omega}d(x)|u|^{q(x)}d\sigma\right], \\ &+ \frac{\mu|t|^{r_+}}{r_-}c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}\|u\|^{r_+} - \int_{\partial\Omega}G(x,tu)d\sigma \end{aligned}$$

then,

$$J_\mu(tu) \leqslant |t|^{p_+}\left[2\frac{K_2}{q_-}\int_\Omega|\nabla u|^{p(x)}dx + \frac{1}{q_-}\int_{\partial\Omega}(b(x)+d(x))|u|^{p(x)}d\sigma\right] + \frac{\mu|t|^{r_+}}{r_-}c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}\|u\|^{r_+} - \int_{\partial\Omega}G(x,tu)d\sigma,$$

from proposition 2.4 and $r_+ \leqslant p_+$, we obtain

$$J_\mu(tu) \leqslant \left[\max\left(2\frac{K_2}{q_-},\frac{1}{q_-}\right)\rho^*_{1,p(.)}(u) + \frac{\mu}{r_-}c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}\|u\|^{r_+}\right]|t|^{p_+} - \int_{\partial\Omega}G(x,tu)d\sigma.$$

We put $l(y) = y^{-\lambda}G(x,ty) - G(x,t), \forall y \geqslant 1$, from (G3), we have $l'(y) = y^{-\lambda-1}[-\lambda G(x,ty) + tyg(x,ty)] \geqslant 0$, then $l(y) \geqslant l(1)$, so $G(x,ty) \geqslant y^\lambda G(x,t)$, $\forall(x,t) \in \partial\Omega\times\mathbb{R}$.
From (11), we conclude that

$$J_\mu(tu) \leqslant c_{13}|t|^{p_+} - |t|^\lambda\int_{\partial\Omega}G(x,u)d\sigma \leqslant c_{13}|t|^{p_+} - |t|^\lambda c_0\int_{\partial\Omega}\left[\frac{\epsilon}{\lambda}|u|^{p_+} + C_\epsilon|u|^{s(x)}\right]d\sigma \leqslant c_{13}|t|^{p_+} - c_{14}|t|^\lambda, \tag{14}$$

where $c_{13} = \left[\max\left(2\frac{K_2}{q_-},\frac{1}{q_-}\right)\rho^*_{1,p(.)}(u) + \frac{\mu}{r_-}c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}\|u\|^{r_+}\right]$, $c_{14} = c_0\int_{\partial\Omega}\left[\frac{\epsilon}{\lambda}|u|^{p_+} + C_\epsilon|u|^{s(x)}\right]d\sigma$, then

$$J_\mu(tu) \leqslant |t|^{p_+}[c_{13} - c_{14}|t|^{\lambda-p_+}],$$

We pass to limit, we obtain

$$\lim_{t\to+\infty}J_\mu(tu) = -\infty,$$

then $J_\mu(tu) < 0$, thus the proof of condition (P2) of Mountain- Pass theorem is complete. The proof of proposition 3.8 is complete. □

According to propositions 3.5, 3.6, 3.8, we may apply theorem 3.4 to conclude that the problem (1) has at least one nontrivial weak solution.

3.2. Existence of infinity weak solutions

Theorem 3.9. *Assume that (H1)–(H3),(F1),(F2), and (G1)–(G3) hold and*
• *(H4): there exists a nonempty open set $V \in \partial\Omega$ with $G(x,t) > 0$ for all $(x,t) \in V \times \mathbb{R}^+$,*
• *(H5): f and g are odds (i.e. $f(x,-t) = -f(x,t)$, $\forall(x,t) \in \Omega \times \mathbb{R}$ and $g(x,-t) = -g(x,t)$, $\forall(x,t) \in \partial\Omega \times \mathbb{R}$),*
Then, the problem (1) has infinitely many of solutions for all $\mu > 0$.

In establishing Theorem 3.9, we will utilize the following form of the symmetric mountain pass theorem.

Theorem 3.10. *(see [32, Theorem 2.1]) Let E is a real Banach space and $E = Y \oplus Z$, where Y is finite dimensional. Suppose $J \in C^1(E,\mathbb{R})$ is an even functional (i.e.$J(-u) = J(u)$, $\forall u \in E$) satisfying (PS)-condition, $J(0) = 0$ and that the following conditions hold:*
• *(S1): there exists $R > 0$ such that $J_{|\partial B_R \cap Z} \geqslant 0$,*
• *(S2): there exists a finite dimensional subspace $W \subset E$ with $\dim Y < \dim W < \infty$ and there exists $M > 0$ such that $\max_{u\in W} J(u) < M$,*
• *(S3): considering $M > 0$ given by the (S2), J satisfies the condition $(PS)_c$ for $0 \leqslant c \leqslant M$,*
Then J possesses at least $(\dim W - \dim Y)$- pairs of nontrivial critical points.

Remark 3.11. *(see[18]),*
Since E is a reflexive and separable Banach space, then there are $(e_j)_{j\in\mathbb{N}^*} \subseteq E$ and $(e_j^*)_{j\in\mathbb{N}^*} \subseteq E'$ such that
$E = \overline{span\{e_j/\ j \in \mathbb{N}^*\}}$ and $E' = \overline{span\{e_j^*/\ j \in \mathbb{N}^*\}}^{\omega^*}$ and $e_i^*(e_j) = \delta_{ij}$, for $i, j \in \mathbb{N}^*$.
For $k \in \mathbb{N}^*$, we denote $X_k = span\{e_k\}$, $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$.

For to prove the Theorem 3.9, it suffices to prove the following two propositions and by symmetric mountain pass theorem 3.10 we obtain the conclusion of Theorem 3.9.

Proposition 3.12. *Assume that (H1), (H2), (H3) (F1), (G1), (G2) and (G3) hold. Then, there exists $\tilde{\mu} > 0$, $k \in \mathbb{N}$, and $R, \theta > 0$ such that $J_{\mu_{|\partial B_R \cap Z_k}} \geqslant \theta$ for all $0 < \mu < \tilde{\mu}$.*

Proof. We put $R = \|u\|$, with $0 < R \leqslant 1$, from (12), we have

$$J_\mu(u) \geqslant c_{11}\|u\|^{p_+} - c_{15}\mu\|u\|^{r_+} - C_\epsilon c_{10}\|u\|^{s_-} = R^{p_+}\left(c_{11} - c_{15}\mu R^{r_+-p_+}\right) - C_\epsilon c_{10} R^{s_-},$$

where $c_{11} = \dfrac{2}{p_+}\min(K_1, 1) - c_9\frac{\epsilon}{\lambda}$ and $c_{15} = \dfrac{c_3}{r_-}\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}$, therefore, for ϵ small enough that we want, we obtain

$$J_\mu(u) > 0 \ \ if \ \ 0 < \mu < \frac{c_{11} r_-}{c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}} R^{p_+-r_+}.$$

Thus, there exist $\theta > 0$ and $\tilde{\mu} = \dfrac{c_{11} r_-}{c_3\|f_1\|_{\frac{q(.)}{q(.)-r(.)}}} R^{p_+-r_+} > 0$ such that $J_\mu(u) \geqslant \theta$, for all $0 < \mu < \tilde{\mu}$,
which complete the proof of proposition 3.12. □

Proposition 3.13. *Under the assumptions of (H1)-(H3), (F2), (G1)–(G3), and (H4), for any $m \in \mathbb{N}$, there exists a subspace W in E and a constant $M_m > 0$ independent of μ. This subspace satisfies $\dim W = m$ and $\max_{u\in W} J_\mu(u) < M_m$.*

Proof. Define O and V as in (F2) and (H4), respectively. The space W can be constructed using the same method as in Lemma 4.3 in [32]. So, we consider $v_1, ..., v_m$ such that $v_i \in \mathcal{D}(\Omega)$, $supp\ v_i \cap supp\ v_j = \emptyset$,

$supp\ v_i \cap O \neq \emptyset$ and $supp\ v_i \cap V \neq \emptyset$, for all $i, j \in \{1, ..., m\}$ and $i \neq j$.
We have

$$J_\mu(u) = \int_\Omega \frac{1}{p(x)} A(|\nabla u|^{p(x)})dx + \int_\Omega \frac{1}{q(x)} A(|\nabla u|^{q(x)})dx + \int_{\partial\Omega} \frac{1}{p(x)} b(x)|u|^{p(x)}d\sigma + \int_{\partial\Omega} \frac{1}{p(x)} d(x)|u|^{q(x)}d\sigma - \mu \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma,$$

by (H1), (H2) and (2), we deduce that

$$J_\mu(u) \leqslant \frac{K_2}{p_-}\int_\Omega |\nabla u_n|^{p(x)}dx + \frac{K_2}{q_-}\int_\Omega |\nabla u_n|^{q(x)}dx + \frac{1}{p_-}\int_{\partial\Omega} b(x)|u_n|^{p(x)}d\sigma + \frac{1}{q_-}\int_{\partial\Omega} d(x)|u_n|^{q(x)}d\sigma - \mu\int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma,$$

by proposition 2.4, we obtain

$$J_\mu(u) \leqslant \frac{1}{q_-}\max(K_2, 1)\left(\rho^*_{1,p(.)}(u) + \rho^*_{1,q(.)}(u)\right) - \mu\int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma,$$

since $E \hookrightarrow W^{1,q(.)}(\Omega)$, then

$$J_\mu(u) \leqslant c_{16}\frac{1}{q_-}\max(K_2, 1)\max(\|u\|^{p_-}, \|u\|^{p_+}) - \mu\int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma,$$

where $c_{16} > 0$. For $u \in W$, since $supp\ u \cap O \neq \emptyset$, we obtain

$$J_\mu(u) \leqslant c_{16}\frac{1}{q_-}\max(K_2, 1)\max(\|u\|^{p_-}, \|u\|^{p_+}) - \int_{\partial\Omega} G(x,u)d\sigma = \tilde{J}(u), \tag{15}$$

then

$$\max_{u\in W\setminus\{0\}} J_\mu(u) \leqslant \max_{u\in W\setminus\{0\}} \tilde{J}(u) = \max_{u\in\partial B_1(0)\cap W\setminus\{0\}} \tilde{J}(u),$$

for $t > 0$ large enough, $u \in \partial B_1(0) \cap W \setminus \{0\}$ and ϵ enough small that we want, by (15), we obtain

$$\tilde{J}(tu) = c_{16}\frac{1}{q_-}\max(K_2, 1)\max(\|tu\|^{p_-}, \|tu\|^{p_+}) - \int_{\partial\Omega} G(x,tu)d\sigma,$$

from (14), the above inequality implies

$$\tilde{J}(tu) \leqslant c_{17}\|tu\|^{p_-} - |t|^\lambda c_0\int_{\partial\Omega}\left[\frac{\epsilon}{\lambda}|u|^{p_+} + C_\epsilon|u|^{s(x)}\right]d\sigma \leqslant c_{17}|t|^{p_-}\|u\|^{p_-} - c_{18}|t|^\lambda,$$

where $c_{17} = c_{16}\frac{1}{q_-}\max(K_2, 1) > 0$ and $c_{18} = c_0\int_{\partial\Omega}\left[\frac{\epsilon}{\lambda}|u|^{p_+} + C_\epsilon|u|^{s(x)}\right]d\sigma > 0$.
From (G3), we obtain

$$\lim_{t\to+\infty}\tilde{J}(tu) \leqslant \lim_{t\to+\infty}\left[|t|^{p_-}(c_{17}\|u\|^{p_-} - c_{18}|t|^{\lambda-p_-})\right] = -\infty.$$

Hence, $\lim_{t\to+\infty}\tilde{J}(tu) = -\infty$.. Consequently, we deduce the existence of a subspace W of E and a positive constant M_m independent of μ, where $\dim W = m$ and $\max_{u\in W} J_\mu(u) < M_m$. Thus, the proof of Proposition 3.13 is concluded. □

From (H5), we deduce J_μ is even, moreover, we have $J_\mu \in C^1(E, \mathbb{R})$ satisfying (PS)-condition and $J_\mu(0) = 0$. According to the propositions 3.6, 3.12, and 3.13, Theorem 3.10 allows us to infer that the problem (1) possesses infinitely many nontrivial solutions.

References

[1] Aberqi, A., Bennouna, J., Benslimane, O. and Ragusa, M. A. *Existence results for double phase problem in Sobolev–Orlicz spaces with variable exponents in complete manifold.* Mediterranean Journal of Mathematics. 19 (4), (2022) 158.

[2] Aberqi, A., Benslimane, B., Elmassoudi, M. and Ragusa, M. A. *Nonnegative solution of a class of double phase problems with logarithmic nonlinearity*. Boundary Value Problems. 1, (2022) 57.

[3] Aberqi,A., Benslimane, B. and Knifda, M. *On a class of double phase problem involving potentials terms*. Journal of Elliptic and Parabolic Equations. 8(2), (2022) 791-811.

[4] Allaoui, M. , ElAmrouss, A. and Ourraoui, A. *Existence of infinitely many solutions for a Steklov problem involving the p(x)-Laplace operator*. Electronic Journal of Qualitative Theory of Differential Equations. 20, (2014) 1–10.

[5] Antontsev, S. and Rodrigues, J. F. *On stationary thermorheological viscous flows*. Annali-Universita Di Ferrara Sezione 7. 52(1), (2006) 19.

[6] Antontsev, S. and Shmarev, S. *Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions*. In Handbook of differential equations: stationary partial differential equations. North-Holland. 3, (2006) 1–100.

[7] Belaouidel, H., Ourraoui, A. and Tsouli, N. *General quasilinear problems involving p (x)-Laplacian with Robin boundary condition*. Ural Mathematical Journal. 6.1 (10), (2020) 30–41.

[8] Benslimane, O., Aberqi, A. and Bennouna, J. *Existence results for double phase obstacle problems with variable exponents*. Journal of Elliptic and Parabolic Equations. 7, (2021) 875-890.

[9] Benslimane, O. and Aberqi, A. *Singular two-phase problem on a complete manifold: analysis and insights*. Arabian Journal of Mathematics, 13 (1), (2024) 45-62.

[10] Bocea, M. and Mihailescu, M. *Γ-convergence of power-law functional with variable exponents*. Nonlinear Analysis: Theory, Methods and Applications. 73(1), (2010) 110–121.

[11] Bocea, M. Mihailescu, M. and Popovici, C. *On the asymptotic behavior of variable exponent power-law functional and applications*. Ricerche di Matematica. 59(2), (2010) 207–238.

[12] Chen, Y., Levine, S. and Rao, M. *Variable exponent, linear growth functionals in image restoration*. Siam journal on Applied Mathematics. 66(4), (2006) 1383-1406.

[13] Conca, C., Diaz, J. I., Linan, A., and Timofte, C. *Homogenization in chemical reactive flows*. Electronic Journal of Differential Equations. 40, (2004) 1-22.

[14] Cruz-Uribe, D. V. and Fiorenza, A. *Variable Lebesgue Spaces*. Foundations and harmonic analysis. Springer Science and Business Media. (2013).

[15] Deng, S. G. *Positive solutions for Robin problem involving the p(x)-Laplacian*. Journal of Mathematical Analysis and Applications. 360(2), (2009) 548–560.

[16] Diening, L. Harjulehto, P. H*ästö*, P. and *Ružička*, M. *Lebesgue and Sobolev spaces with variable exponents*. Springer. 2011.

[17] ElAmrouss, A., Moradi, F. and Ourraoui, A. *Neumann problem in divergence form modeled on the p(x)-Laplace equation*. Boletim da Sociedade Paranaense de Matemática. 32(2), (2014) 109–117.

[18] Fabian, M., Habala, P., Hajék, P., Montesinos, V. and Zizler, V. *Banach space theory: the basis for linear and nonlinear analysis*. Springer, New York. (2011).

[19] Fan, X. *Boundary trace embedding theorems for variable exponent Sobolev spaces*. Journal of Mathematical Analysis and Applications. 339 (2), (2008) 1395–1412.

[20] Fan, X. L. and Zhao, D. *On the generalized Orlicz-Sobolev space* $W^{k,p(x)}(\Omega)$. Gansu Educ. College. 12 (1), (1998) 1-6.

[21] Fan, X. and Zhao, D. *On the spaces* $L^{p(x)}(\Omega)$ *and* $W^{m,p(x)}(\Omega)$. Journal of Mathematical Analysis and Applications. 263 (2), (2001) 424-446.

[22] Fu, Y. and Zhang, X. *A multiplicity result for* $p(x)-$*Laplacian problem in* $\mathbb{R}^{\mathbb{N}}$. Nonlinear Analysis: Theory, Methods and Applications. 70 (6), (2009) 2261–2269.

[23] Ge, B. and Zhou, Q. M. *Multiple solutions for a Robin-type differential inclusion problem involving the* $p(x)-$*Laplacian*. Mathematical Methods in the Applied Sciences. 40 (18), (2017) 6229–6238.

[24] Ge, B., Wang, L. Y., and Lu, L. F. *On a class of double-phase problem without Ambrosetti–Rabinowitz-type conditions*. Applicable Analysis. 100 (10), (2021) 2147–2162.

[25] Hurtado, E. J., Miyagaki, O. H. and Rodrigues, R. S. *Existence and multiplicity of solutions for a class of elliptic equations without Ambrosetti–Rabinowitz type conditions*. Journal of Dynamics and Differential Equations. 30, (2018) 405–432.

[26] Mihailescu, M. *On a class of nonlinear problems involving a* $p(x)-$*Laplace type operator*. Czechoslovak Mathematical Journal. 58 (1), (2008) 155–172.

[27] Ni, W. M. and Serrin, J. *Nonexistence theorems for quasilinear partial differential equations*. Rend. Circ. Mat. Palermo (2) Suppl. 8, (1985) 171-185.

[28] Ourraoui, A. *Some results for Robin type problem involving* $p(x)-$ *Laplacian*. Filomat. 36 (6), (2022) 2105-2117.

[29] Rabinowitz, P.H (Ed). *Minimax methods in critical point theory with applications to differential equations*. American Mathematical Soc. 65, (1986).

[30] Ragusa, M. A. and Tachikawa, A. *Regularity for minimizers for functionals of double phase with variable exponents*. Advances in Nonlinear Analysis. 9 (1), (2019) 710–728.

[31] Rajagopal, K. R. and Ruzicka, M. *Mathematical modeling of electrorheological materials*. Continuum Mechanics and Thermodynamics. 13 (1), (2001) 59-78.

[32] Silva, E. A. and Xavier, M. S. *Multiplicity of solutions for quasilinear elliptic problems involving critical Sobolev exponents*. Annales de l'Institut Henri Poincaré C, Analyse non linéaire. 20 (2), (2003) 341–358.

[33] Struwe, M. *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. Ergebnisse der

Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer-Verlag Berlin Heidelberg, 2008.
[34] Timofte, C. *On the homogenization of a climatization problem*. Stud. Univ. Babes-Bolyai Math. 2, (2007) 117–125.
[35] Vetro, F., and Winkert, P. *Constant sign solutions for double phase problems with variable exponents*. Applied Mathematics Letters. 135, (2023) 108404.
[36] Wang, L. L. Fan, Y. H. and Ge, W. G. *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)-$ Laplace operator.* Nonlinear Analysis: Theory, Methods and Applications. 71(9), (2009) 4259-4270.
[37] Zhao, D., Qiang, W. J. and Fan, X. L. *On generalized Orlicz spaces $L^{p(x)}(\Omega)$*. J. Gansu Sci. 9 (2), (1997) 1-7.