



A sharp upper bound on the spectral radius of $\theta(1, 3, 3)$ -free graphs with given size

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Abstract. A graph G is F -free if G does not contain F as a subgraph. Let $\rho(G)$ be the spectral radius of a graph G . Let $\theta(1, p, q)$ denote the theta graph, which is obtained by connecting two distinct vertices with three internally disjoint paths with lengths $1, p, q$, where $p \leq q$. Let $S_{n,k}$ denote the graph obtained by joining every vertex of K_k to $n - k$ isolated vertices and $S_{n,k}^-$ denote the graph obtained from $S_{n,k}$ by deleting an edge incident to a vertex of degree k , respectively. In this paper, we show that if $\rho(G) \geq \rho(S_{\frac{m+4}{2}, 2}^-)$ for a graph G with even size $m \geq 92$, then G contains a $\theta(1, 3, 3)$ unless $G \cong S_{\frac{m+4}{2}, 2}^-$.

1. Introduction

Throughout this paper, we consider all graphs are always undirected and simple. We follow the traditional notation and terminology [1]. Let G be a graph of order n with vertex set $V(G) = \{u_1, u_2, \dots, u_n\}$ and size m with edge set $E(G)$. For a vertex $u \in V(G)$, let $N_G(u)$ be the neighborhood set of a vertex u , $N_G[u] = N_G(u) \cup \{u\}$ and $N_G^2(u)$ be the set of vertices of distance two to u in G . In particular, $N_S(u) = N(u) \cap S$ and $d_S(u) = |N_S(u)|$ for a subset $S \subseteq V(G)$. Let $d_G(u) = |N_G(u)|$ be the degree of a vertex u . For the sake of simplicity, we omit all the subscripts if G is clear from the context, for example, $N(u)$, $N[u]$, $N^2(u)$ and $d(u)$. For a graph G and a subset $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S . For two vertex subsets S and T of $V(G)$ (where $S \cap T$ may not be empty), let $e(S, T)$ denote the number of edges with one endpoint in S and the other in T . $e(S, S)$ is simplified by $e(S)$. Given two vertex-disjoint graphs G_1 and G_2 , we denote by $G_1 \cup G_2$ the disjoint union of the two graphs, and by $G_1 \vee G_2$ the joint graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with each vertex of G_2 . The adjacency matrix of a graph G is an $n \times n$ matrix $A(G)$ whose (i, j) -entry is 1 if u_i is adjacent to u_j and 0 otherwise. The spectral radius $\rho(G)$ of a graph G is the largest eigenvalue of its adjacency matrix $A(G)$.

Let $P_n, C_n, K_{1,n-1}$ and $K_{a,b}$ be the path of order n , the cycle of order n , the star graph of order n and the complete bipartite graph with two parts of sizes a, b , respectively. Let S_n^k be the graph obtained from $K_{1,n-1}$ by adding k disjoint edges within its independent sets. Let $S_{n,k}$ be the graph obtained by joining every

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vertex of K_k to $n - k$ isolated vertices. Let $S_{n,k}^-$ be the graph obtained from $S_{n,k}$ by deleting an edge incident to a vertex of degree k . Let $\theta(1, p, q)$ denote the theta graph, which is obtained by connecting two distinct vertices with three internally disjoint paths with lengths $1, p, q$, where $q \leq p$.

Given a graph F , a graph G is F -free if it does not contain F as a subgraph. Let $\mathcal{G}(m, F)$ denote the family of F -free graphs with m edges and without isolated vertices. The classic Turán type problem asks what is the maximum number of edges in an F -free graph of order n . In spectral graph theory, Nikiforov [16] posed a spectral Turán type problem which asks to determine the maximum spectral radius of an F -free graphs of n vertices, which is known as the Brualdi-Solheid-Turán type problem. In the past decades, this problem has received much attention. For more details, we suggest the reader to see surveys [3, 6, 10, 17], and references therein. In addition, Brualdi and Hoffman raised another spectral Turán type problem: What is the maximal spectral radius of an F -free graph with given size m ? This problem is called the Brualdi-Hoffman-Turán type problem. This problem has been studied for various families of graphs. For example, K_3 [18], K_{r+1} [14, 15], $K_{2,r+1}$ [20], F_{2k+2} [8] (where $F_k = K_1 \vee P_{k-1}$), $F_{k,3}$ [8] (where $F_{k,3}$ is the friendship graph obtained from k triangles by sharing a common vertex).

For theta graphs, Sun, Li and Wei [19] characterized the extremal graph with maximum spectral radius of $\theta(1, 2, 3)$ -free and $\theta(1, 2, 4)$ -free graphs with odd size. Fang and You [5] characterized the extremal graph with maximum spectral radius of $\theta(1, 2, 3)$ -free graphs with even size. Liu and Wang [11] characterized the extremal graph with maximum spectral radius of $\theta(1, 2, 4)$ -free graphs with even size. Lu, Lu and Li [12] characterized the extremal graph with maximum spectral radius of $\theta(1, 2, 5)$ -free graphs with given size. Li, Zhai and Shu [9] characterized the extremal graph with maximum spectral radius of $\theta(1, 2, 2k - 1)$ -free or $\theta(1, 2, 2k)$ -free graphs with given size.

Recently, Li, Zhao and Zou [8] characterized the extremal graph with maximum spectral radius of $\theta(1, p, q)$ -free graphs with size m for $q \geq p \geq 3$ and $p + q \geq 7$.

Theorem 1.1. [8] Let $k \geq 3$ be fixed and $m \geq \frac{9}{4}k^6 + 6k^5 + 46k^4 + 56k^3 + 196k^2$. If $G \in \mathcal{G}(m, \theta(1, p, q)) \cup \mathcal{G}(m, \theta(1, r, s))$ with $q \geq p \geq 3$, $s \geq r \geq 3$, $p + q = 2k + 1$ and $r + s = 2k + 2$, then

$$\rho(G) \leq \frac{k - 1 + \sqrt{4m - k^2 + 1}}{2},$$

and equality holds if and only if $G \cong K_k \vee (\frac{m}{k} - \frac{k-1}{2})K_1$.

At the same time, they [8] proposed the following problem.

Problem 1.2. [8] How can we characterize the graphs among $\mathcal{G}(m, \theta(1, 3, 3))$ having the largest spectral radius?

Theorem 1.3. [7] Let $G \in \mathcal{G}(m, \theta(1, 3, 3))$ be a graph of size $m \geq 43$. Then

$$\rho(G) \leq \frac{1 + \sqrt{4m - 3}}{2},$$

and equality holds if and only if $G \cong S_{\frac{m+3}{2}, 2}$.

However, for a $\theta(1, 3, 3)$ -free graph G with size m , the bound $\rho(G) \leq \frac{1 + \sqrt{4m - 3}}{2}$ is sharp only for odd m . Motivated by this, we want to obtain a sharp upper bound of $\rho(G)$ for even m . Our result is presented as follows.

Theorem 1.4. Let $G \in \mathcal{G}(m, \theta(1, 3, 3))$ be a graph of even size $m \geq 92$, then $\rho(G) \leq \rho(S_{\frac{m+4}{2}, 2}^-)$, and equality holds if and only if $G \cong S_{\frac{m+4}{2}, 2}^-$.

2. Preliminary

In this section, we introduce some lemmas which are used to prove our result.

Lemma 2.1. [21] Let u, v be two distinct vertices of the connected graph G , $\{v_i | i = 1, 2, \dots, s\} \subseteq N(v) \setminus N(u)$, and $X = (x_1, x_2, \dots, x_n)^T$ be the Perron vector of G . Let $G' = G - \sum_{i=1}^s v_i v + \sum_{i=1}^s v_i u$. If $x_u \geq x_v$, then $\rho(G) < \rho(G')$.

Definition 2.2. [4] Given a graph G , the vertex partition $\Pi: V(G) = V_1 \cup V_2 \cup \dots \cup V_k$ is said to be an equitable partition if, for each $u \in V_i$, $|V_j \cap N(u)| = b_{ij}$ is a constant depending only on i, j ($1 \leq i, j \leq k$). The matrix $B_\Pi = (b_{ij})$ is called the quotient matrix of G with respect to Π .

Lemma 2.3. [4] Let $\Pi: V(G) = V_1 \cup V_2 \cup \dots \cup V_k$ be an equitable partition of a graph G with quotient matrix B_Π . Then $\det(xI - B_\Pi) \mid \det(xI - A(G))$. Furthermore, the largest eigenvalue of B_Π is just the spectral radius of G .

Lemma 2.4. [20] Let G^* be the extremal graph with the maximum spectral radius in $\mathcal{G}(m, F)$. Let $X = (x_1, x_2, \dots, x_n)^T$ be the Perron vector of the graph G^* . If F is a 2-connected graph and $x_{u^*} = \max\{x_v \mid v \in V(G^*)\}$, then the following statements hold.

- (i) G^* is connected.
- (ii) There exists no cut vertex in $V(G^*) \setminus \{u^*\}$, and hence $d(u) \geq 2$ for any $u \in V(G^*) \setminus N[u^*]$.

Lemma 2.5. [2] Let G be a bipartite graph of size m . Then $\rho(G) \leq \sqrt{m}$, and equality holds if and only if G is a disjoint union of a complete bipartite graph and isolated vertices.

Lemma 2.6. [13] $\rho(S_{\frac{m+4}{2}, 2}^-) > \frac{1+\sqrt{4m-5}}{2}$ for $m \geq 6$.

Lemma 2.7. [13] Let $X = (x_1, x_2, \dots, x_n)^T$ be the Perron vector of a connected graph G of size m and let $x_{u^*} = \max\{x_v \mid v \in V(G)\}$ and $W = V(G) \setminus N[u^*]$. If $\rho(G) > \frac{1+\sqrt{4m-5}}{2}$ and there exists a vertex v of G such that $x_v < (1-\beta)x_{u^*}$, where $0 < \beta < 1$, then

$$e(W) < e(N(u^*)) - |N(u^*) \setminus N_0(u^*)| + \frac{3}{2} - \beta d_{N(u^*)}(v),$$

for $v \in N^2(u^*)$.

3. Proof of Theorem 1.4.

Let G^* be the extremal graph with maximum spectral radius in $\mathcal{G}(m, \theta(1, 3, 3))$. Let $\rho = \rho(G^*)$ and $X^* = (x_1, x_2, \dots, x_n)^T$ be the Perron vector of G^* with coordinate x_v corresponding to the vertex $v \in V(G^*)$. A vertex u^* in G^* is said to be an extremal vertex if $x_{u^*} = \max\{x_v \mid v \in V(G^*)\}$. Let $W = V(G^*) \setminus N[u^*]$ and $N_+(u^*) = N(u^*) \setminus N_0(u^*)$, where $N_0(u^*)$ denotes the set of isolated vertices of $G^*[N(u^*)]$. Let $W_H = \cup_{u \in V(H)} N_W(u)$ for any component H of $G^*[N(u^*)]$.

Lemma 3.1. Let $G^* \in \mathcal{G}(m, \theta(1, 3, 3))$. Then $G^*[N(u^*)]$ is P_5 -free, that is, each component H of $G^*[N(u^*)]$ is one of the following.

- (i) a graph with C_4 as its spanning subgraph, that is, C_4 , $\theta(1, 2, 2)$ or K_4 ;
- (ii) a copy of S_{r+1}^1 for $r \geq 2$, where S_3^1 is a triangle for $r = 2$;
- (iii) a double star $D_{a,b}$ for $a, b \geq 1$, which is obtained from two stars $K_{1,a}$ and $K_{1,b}$ by joining a new edge between their centers;
- (iv) a star $K_{1,r}$ for $r \geq 0$, where $K_{1,0}$ is a singleton component.

Lemma 3.2. [7] For any non-trivial component H in $G^*[N(u^*)]$, if H contains a cycle of length four, then $N_W(u) \cap N_W(v) = \emptyset$ for any two vertices u and v in the cycle of length four.

Since $A(G^*)X^* = \rho X^*$, we have

$$\rho x_{u^*} = \sum_{u \in N_+(u^*)} x_u + \sum_{u \in N_0(u^*)} x_u.$$

Furthermore, we have

$$\begin{aligned} \rho^2 x_{u^*} &= \sum_{u \in N_+(u^*)} \rho x_u + \sum_{u \in N_0(u^*)} \rho x_u \\ &= |N_+(u^*)| x_{u^*} + \sum_{u \in N_+(u^*)} d_{N(u^*)}(u) x_u + \sum_{w \in N_W(u)} d_W(u) x_w + |N_0(u^*)| x_{u^*} \\ &= d(u^*) x_{u^*} + \sum_{u \in N_+(u^*)} d_{N(u^*)}(u) x_u + \sum_{w \in N^2(u^*)} d_{N(u^*)}(w) x_w. \end{aligned}$$

Therefore,

$$\begin{aligned} (\rho^2 - \rho) x_{u^*} &= d(u^*) x_{u^*} + \sum_{v \in N_+(u^*)} (d_{N(u^*)}(v) - 1) x_v + \sum_{w \in N^2(u^*)} d_{N(u^*)}(w) x_w - \sum_{v \in N_0(u^*)} x_v \\ &\leq |N(u^*)| x_{u^*} + \sum_{u \in N_+(u^*)} (d_{N(u^*)}(u) - 1) x_u + e(N(u^*), W) x_{u^*} - \sum_{u \in N_0(u^*)} x_u. \end{aligned} \quad (1)$$

Note that $S_{\frac{m+4}{2}, 2}^-$ is $\theta(1, 3, 3)$ -free, we have $\rho \geq \rho(S_{\frac{m+4}{2}, 2}^-) > \frac{1+\sqrt{4m-5}}{2} > 10$ for $m \geq 92$. By Lemma 2.6, we have

$$(\rho^2 - \rho) x_{u^*} > (m - \frac{3}{2}) x_{u^*} = (|N(u^*)| + e(N_+(u^*)) + e(N(u^*), W) + e(W) - \frac{3}{2}) x_{u^*}. \quad (2)$$

Combining with (1) and (2), we get

$$\sum_{u \in N_+(u^*)} (d_{N(u^*)}(u) - 1) x_u > \left(e(N_+(u^*)) + e(W) + \sum_{u \in N_0(u^*)} \frac{x_u}{x_{u^*}} - \frac{3}{2} \right) x_{u^*}.$$

Let \mathcal{H} be the set of all non-trivial components in $G^*[N(u^*)]$. For each non-trivial connected component H of \mathcal{H} , we denote $\zeta(H) = \sum_{u \in V(H)} (d_H(u) - 1) x_u$. Obviously,

$$\sum_{H \in \mathcal{H}} \zeta(H) > \left(e(N_+(u^*)) + e(W) + \sum_{u \in N_0(u^*)} \frac{x_u}{x_{u^*}} - \frac{3}{2} \right) x_{u^*}. \quad (3)$$

Lemma 3.3. $G^*[N(u^*)]$ contains no any cycle of length four.

Proof. Let \mathcal{H}' be the family of components of $G^*[N(u^*)]$ each of which contains C_4 as a spanning subgraph and $\mathcal{H} \setminus \mathcal{H}'$ be the family of other non-trivial components of $G^*[N(u^*)]$ each of which contains no C_4 as a spanning subgraph. By Lemma 3.1 (ii)-(iv), for each $H \in \mathcal{H} \setminus \mathcal{H}'$, we have

$$\zeta(H) = \sum_{v \in V(H)} (d_H(v) - 1) x_v \leq (2e(H) - |V(H)|) x_{u^*} \leq e(H) x_{u^*}.$$

Next we show that

$$\zeta(H) < (e(H) - \frac{3}{2}) x_{u^*} + \frac{2 \sum_{w \in W_H} x_w}{\rho - 3}$$

for each $H \in \mathcal{H}'$. Let $H^* \in \mathcal{H}'$ with $V(H^*) = \{u_1, u_2, u_3, u_4\}$ and the cycle of length four be $u_1 u_2 u_3 u_4 u_1$.

First, we consider $W_{H^*} = \emptyset$. Assume that $x_{u_1} = \max\{x_{u_i} | 1 \leq i \leq 4\}$. Then

$$\rho x_{u_1} = \sum_{u \in N(u_1)} x_u \leq x_{u^*} + x_{u_2} + x_{u_3} + x_{u_4} \leq x_{u^*} + 3x_{u_1}.$$

Hence, $x_{u_1} \leq \frac{1}{\rho-3}x_{u^*} < \frac{1}{4}x_{u^*}$ for $\rho \geq \rho(S_{\frac{m+4}{2}, 2}^-) > \frac{1+\sqrt{4m-5}}{2} > 10$ due to $m \geq 92$. Furthermore,

$$\zeta(H^*) \leq (2e(H^*) - |V(H^*)|)x_{u_1} < (e(H^*) - \frac{3}{2})x_{u^*} + \frac{2 \sum_{w \in W_{H^*}} x_w}{\rho - 3},$$

as desired.

In the following, we assume that $W_{H^*} \neq \emptyset$, we consider the following two cases.

Case 1. All vertices in W_{H^*} have a unique common neighbor in $V(H^*)$.

Without loss of generality, let $u_1 \in V(H^*)$ be the unique common neighbor. Thus, $N_W(u_i) = \emptyset$ for $i \in \{2, 3, 4\}$. Assume that $x_{u_2} = \max\{x_{u_i} | 2 \leq i \leq 4\}$, we have

$$\rho x_{u_2} \leq x_{u^*} + x_{u_1} + x_{u_3} + x_{u_4} \leq 2x_{u_2} + 2x_{u^*}.$$

Thus, $x_{u_2} \leq \frac{2}{\rho-2}x_{u^*} < \frac{1}{4}x_{u^*}$ due to $\rho > 10$. Thus,

$$\begin{aligned} \zeta(H^*) &= \sum_{u \in V(H^*)} (d_{H^*}(u) - 1)x_u \\ &\leq (d_{H^*}(u_1) - 1)x_{u_1} + (2e(H^*) - d_{H^*}(u_1) - 3)x_{u_2} \\ &< \left(d_{H^*}(u_1) - 1 + \frac{1}{2}e(H^*) - \frac{1}{4}d_{H^*}(u_1) - \frac{3}{4}\right)x_{u^*} \\ &\leq \left(\frac{3}{4}d_{H^*}(u_1) + \frac{1}{2}e(H^*) - \frac{7}{4}\right)x_{u^*} \\ &\leq \left(\frac{1}{2}e(H^*) + \frac{1}{2}\right)x_{u^*} \\ &< (e(H^*) - \frac{3}{2})x_{u^*} + \frac{2 \sum_{w \in W_{H^*}} x_w}{\rho - 3} \end{aligned}$$

due to $d_{H^*}(u_1) \leq 3$, as desired.

Case 2. There are at least two distinct vertices of W_{H^*} such that they have distinct neighbors in $V(H^*)$. Since

$$\begin{cases} \rho x_{u_1} \leq x_{u^*} + x_{u_2} + x_{u_3} + x_{u_4} + \sum_{w \in N_{W_{H^*}}(u_1)} x_w, \\ \rho x_{u_2} \leq x_{u^*} + x_{u_1} + x_{u_3} + x_{u_4} + \sum_{w \in N_{W_{H^*}}(u_2)} x_w, \\ \rho x_{u_3} \leq x_{u^*} + x_{u_1} + x_{u_2} + x_{u_4} + \sum_{w \in N_{W_{H^*}}(u_3)} x_w, \\ \rho x_{u_4} \leq x_{u^*} + x_{u_1} + x_{u_2} + x_{u_3} + \sum_{w \in N_{W_{H^*}}(u_4)} x_w, \end{cases}$$

we obtain

$$\rho(x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4}) \leq 3(x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4}) + 4x_{u^*} + \sum_{i=1}^4 \sum_{w \in N_{W_{H^*}}(u_i)} x_w.$$

By Lemma 3.2, we get $N_{W_{H^*}}(u_i) \cap N_{W_{H^*}}(u_j) = \emptyset$ for arbitrary two distinct vertices $u_i, u_j \in V(H^*)$. Thus, $\sum_{w \in W_{H^*}} x_w = \sum_{w \in N_W(V(H^*))} x_w = \sum_{i=1}^4 \sum_{w \in N_{W_{H^*}}(u_i)} x_w$. Combining with $\rho > 10$, we obtain

$$\begin{aligned} x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4} &\leq \frac{4x_{u^*}}{\rho - 3} + \frac{\sum_{w \in W_{H^*}} x_w}{\rho - 3} \\ &< \frac{4}{7}x_{u^*} + \frac{\sum_{w \in W_{H^*}} x_w}{\rho - 3}. \end{aligned}$$

Hence, by the definition of $\zeta(H^*)$,

$$\begin{aligned}\zeta(H^*) &\leq 2(x_{u_1} + x_{u_2} + x_{u_3} + x_{u_4}) \\ &< \frac{8}{7}x_{u^*} + \frac{2\sum_{w \in W_{H^*}} x_w}{\rho - 3} \\ &< (e(H^*) - \frac{3}{2})x_{u^*} + \frac{2\sum_{w \in W_{H^*}} x_w}{\rho - 3}.\end{aligned}$$

Thus, we obtain $\zeta(H) < (e(H) - \frac{3}{2})x_{u^*} + \frac{2\sum_{w \in W_H} x_w}{\rho - 3}$ for each $H \in \mathcal{H}'$. Recall that $\zeta(H) \leq e(H)x_{u^*}$ for each $H \in \mathcal{H} \setminus \mathcal{H}'$. Furthermore,

$$\begin{aligned}\sum_{H \in \mathcal{H}} \zeta(H) &= \sum_{H \in \mathcal{H}'} \zeta(H) + \sum_{H \in \mathcal{H} \setminus \mathcal{H}'} \zeta(H) \\ &< \sum_{H \in \mathcal{H}'} (e(H) - \frac{3}{2})x_{u^*} + \sum_{H \in \mathcal{H}'} \frac{2\sum_{w \in W_H} x_w}{\rho - 3} + \sum_{H \in \mathcal{H} \setminus \mathcal{H}'} e(H)x_{u^*} \\ &= e(N_+(u^*))x_{u^*} - \frac{3}{2} \sum_{H \in \mathcal{H}'} x_{u^*} + \sum_{H \in \mathcal{H}'} \frac{2\sum_{w \in W_H} x_w}{\rho - 3}.\end{aligned}$$

For any $H \in \mathcal{H}'$ satisfying $W_H = \emptyset$, we have $\sum_{w \in W_H} x_w = 0$. For any $H \in \mathcal{H}'$ satisfying $W_H \neq \emptyset$ and any $w \in W_H$, since G^* is $\theta(1, 3, 3)$ -free, we get $W_H \cap W_{N(u^*) \setminus H} = \emptyset$. Then $d_{N(u^*) \setminus H}(w) = 0$. By Lemma 3.2, we have $d_H(w) = 1$. Furthermore, $d_{N(u^*)}(w) = 1$ and $d_W(w) \geq 1$ from Lemma 2.4. Thus, $\sum_{H \in \mathcal{H}'} \sum_{w \in W_H} x_w \leq \sum_{H \in \mathcal{H}'} \sum_{w \in W_H} d_W(w)x_w \leq \sum_{H \in \mathcal{H}'} \sum_{w \in W_H} d_W(w)x_{u^*} \leq 2e(W)x_{u^*}$. Note that $\rho > 10$. Thus,

$$\begin{aligned}\sum_{H \in \mathcal{H}} \zeta(H) &< e(N_+(u^*))x_{u^*} - \frac{3}{2} \sum_{H \in \mathcal{H}'} x_{u^*} + \sum_{H \in \mathcal{H}'} \frac{2\sum_{w \in W_H} x_w}{\rho - 3} \\ &< (e(N_+(u^*)) + \frac{4}{7}e(W) - \sum_{H \in \mathcal{H}'} \frac{3}{2})x_{u^*} \\ &< (e(N_+(u^*)) + e(W) - \sum_{H \in \mathcal{H}'} \frac{3}{2})x_{u^*},\end{aligned}$$

which contradicts (3). Furthermore, $G^*[N(u^*)]$ contains no C_4 . This completes the proof of Lemma 3.3. \square

By Lemma 3.1, we know that each non-trivial component of $G^*[N(u^*)]$ is either a tree or a unicyclic graph S_{r+1}^1 with $r \geq 2$. Let c be the number of non-trivial tree components of $G^*[N(u^*)]$. Then

$$\sum_{H \in \mathcal{H}} \zeta(H) \leq \sum_{H \in \mathcal{H}} \sum_{u \in V(H)} (d_H(u) - 1)x_{u^*} = \sum_{H \in \mathcal{H}} (2e(H) - |H|)x_{u^*} = (e(N_+(u^*)) - c)x_{u^*}.$$

Combining with (3), we get

$$e(W) < \frac{3}{2} - c - \sum_{u \in N_0(u^*)} \frac{x_u}{x_{u^*}}. \quad (4)$$

Thus, $e(W) \leq 1$ and $c \leq 1$. In addition, if $e(W) = 1$, then $c = 0$ and $\sum_{u \in N_0(u^*)} \frac{x_u}{x_{u^*}} < \frac{1}{2}$.

Lemma 3.4. $e(W) = 0$.

Proof. Suppose on the contrary that $e(W) = 1$. In this case, we have $c = 0$ and $\sum_{u \in N_0(u^*)} \frac{x_u}{x_{u^*}} < \frac{1}{2}$. It follows that each component of $G^*[N(u^*)]$ is isomorphic to a unicyclic graph S_{r+1}^1 with $r \geq 2$. That is, each component

of $G^*[N(u^*)]$ contains a triangle. Let H^* be a component of $G^*[N(u^*)]$ and $u_1u_2u_3$ is a C_3 of H^* . Let w_1w_2 be the unique edge of $e(W)$. By Lemma 2.4, we obtain that $d_{N(u^*)}(w_i) \geq 1$ for each $i \in \{1, 2\}$. If $H^* \cong S_3^1$, then we obtain $d_{S_3^1}(w_i) \leq 3$ for each $i \in \{1, 2\}$. If $H^* \cong S_{r+1}^1$ with $r \geq 3$, then let $d_{H^*}(u_2) = d_{H^*}(u_3) = 2$ and u_4, u_5, \dots, u_{r+1} be the neighbors of u_1 . For $w \in W_{H^*}$, we claim $d_{H^*}(w) \leq 1$. Otherwise, we consider the following five cases in the sense of symmetry. If $\{u_1, u_2\} \subseteq N_{H^*}(w)$, then $u_1u_3, u_1wu_2u_3, u_1u_4u^*u_3$ are three internally disjoint paths of lengths 1, 3, 3 between u_1 and u_3 , a contradiction. If $\{u_2, u_3\} \subseteq N_{H^*}(w)$, then $u_1u_3, u_1u_2wu_3, u_1u_4u^*u_3$ are three internally disjoint paths of lengths 1, 3, 3 between u_1 and u_3 , a contradiction. If $\{u_1, u_4\} \subseteq N_{H^*}(w)$, then $u_1u^*, u_1u_2u_3u^*, u_1wu_4u^*$ are three internally disjoint paths of lengths 1, 3, 3 between u_1 and u^* , a contradiction. If $\{u_2, u_4\} \subseteq N_{H^*}(w)$, then $u^*u_2, u_2wu_4u^*, u_2u_1u_3u^*$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and u_2 , a contradiction. If $\{u_4, u_5\} \subseteq N_{H^*}(w)$, then $u^*u_4, u_4wu_5u^*, u_4u_1u_3u^*$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and u_4 , a contradiction. Thus, $d_{H^*}(w) \leq 1$ for $w \in W_{H^*}$ and $H^* \cong S_{r+1}^1$ with $r \geq 3$. In addition, we can check that $W_{H^*} \cap W_{N(u^*) \setminus H^*} = \emptyset$ for $H^* \cong S_{r+1}^1$ with $r \geq 2$. Consequently, $d(w) = d_{H^*}(w)$ for $w \in W_{H^*}$. Furthermore, we consider the following two cases.

Case 1. At least a vertex $w_i \in \cup_{j=1}^3 N_W(u_j)$ for some $i \in \{1, 2\}$.

Subcase 1.1. $w_1, w_2 \in \cup_{j=1}^3 N_W(u_j)$.

In this case, we obtain $(\cup_{j=1}^3 N_W(u_j)) \cap (W_{N(u^*)} \setminus \{\cup_{j=1}^3 N_W(u_j)\}) = \emptyset$. Thus, $N_{N(u^*)}(w) \subseteq \cup_{j=1}^3 N_W(u_j)$ for each $w \in \cup_{j=1}^3 N_W(u_j)$. If $N_{C_3}(w_1) \cap N_{C_3}(w_2) = \emptyset$, then there exists a $\theta(1, 3, 3)$, a contradiction. If $|N_{C_3}(w_1) \cap N_{C_3}(w_2)| \geq 2$, then there exists a $\theta(1, 3, 3)$, a contradiction. Thus, $|N_{C_3}(w_1) \cap N_{C_3}(w_2)| = 1$ and hence there exists a cut vertex, which contradicts Lemma 2.4 (ii).

Subcase 1.2. $w_1 \in \cup_{j=1}^3 N_W(u_j)$ and $w_2 \notin \cup_{j=1}^3 N_W(u_j)$.

In this case, suppose that $w_2 \in W_{H^*}$, where $H^* \cong S_{r+1}^1$ with $r \geq 3$. Then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u_1} + x_{u_2} + x_{u_3} \leq x_{w_2} + 3x_{u^*}, \\ \rho x_{w_2} = x_{w_1} + x_{u_k} \leq x_{w_1} + x_{u^*}, \end{cases}$$

where $u_k \in V(H^*) \setminus \{u_1, u_2, u_3\}$. Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{3\rho+1}{\rho^2-1}x_{u^*}$. Since $f(x) = \frac{3x+1}{x^2-1}$ is decreasing with $x > 10$, we have $x_{w_1} \leq \frac{3\rho+1}{\rho^2-1}x_{u^*} < \frac{31}{99}x_{u^*}$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{68}{99} = \frac{161}{99}$, a contradiction. Suppose that $w_2 \in W_{H^*}$, where $H^* \subseteq N_+(u^*) \setminus H^*$ and $H^* \cong S_{r'+1}^1$ with $r' \geq 2$. Furthermore, if $H^* \cong S_3^1 : u_1u_2u_3$ and $H^* \cong S_3^1 : u'_1u'_2u'_3$, then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u_1} + x_{u_2} + x_{u_3} \leq x_{w_2} + 3x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + x_{u'_1} + x_{u'_2} + x_{u'_3} \leq x_{w_1} + 3x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{3}{\rho-1}x_{u^*} < \frac{1}{3}x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$, a contradiction. If $H^* \cong S_3^1$ and $H^* \cong S_{r'+1}^1$ with $r' \geq 3$, then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u_1} + x_{u_2} + x_{u_3} \leq x_{w_2} + 3x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{3\rho+1}{\rho^2-1}x_{u^*}$. Since $f(x) = \frac{3x+1}{x^2-1}$ is decreasing with $x > 10$, we have $x_{w_1} \leq \frac{3\rho+1}{\rho^2-1}x_{u^*} < \frac{31}{99}x_{u^*}$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{68}{99} = \frac{161}{99}$, a contradiction. If $H^* \cong S_{r+1}^1$ with $r \geq 3$ and $H^* \cong S_{r'+1}^1$ with $r' \geq 3$, then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{1}{\rho-1}x_{u^*} < \frac{1}{9}x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{8}{9} = \frac{11}{18}$, a contradiction. Suppose that $w_2 \in W_{N_0(u^*)}$ and $H^* \cong S_3^1$, then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u_1} + x_{u_2} + x_{u_3} \leq x_{w_2} + 3x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + \sum_{u \in N_0(u^*)} x_u < x_{w_1} + \frac{1}{2}x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} < \frac{3\rho+\frac{1}{2}}{\rho^2-1}x_{u^*} < \frac{30\frac{1}{2}}{99}x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{68\frac{1}{2}}{99} = \frac{80}{99}$, a contradiction. Suppose that $w_2 \in W_{N_0(u^*)}$ and $H^* \cong S_{r+1}^1$ with $r \geq 3$, then

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + \sum_{u \in N_0(u^*)} x_u < x_{w_1} + \frac{1}{2}x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} < \frac{\rho+\frac{1}{2}}{\rho^2-1}x_{u^*} < \frac{10\frac{1}{2}}{99}x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{88\frac{1}{2}}{99} = \frac{20}{33}$, a contradiction.

Case 2. $w_i \notin \cup_{j=1}^3 N_W(u_j)$ for any $i \in \{1, 2\}$.

Suppose that $w_1, w_2 \in W_{H^*}$, where $H^* \cong S_{r+1}^1$ for $r \geq 3$. Then $N_{H^*}(w_1) \cap N_{H^*}(w_2) = \emptyset$. Otherwise there exists a cut vertex, which contradicts Lemma 2.4 (ii). By lemma 2.4, we get $|N_{H^*}(w_i)| = 1$ for each $i \in \{1, 2\}$. Moreover,

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{1}{\rho-1}x_{u^*} < \frac{1}{9}x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{8}{9} = \frac{11}{18}$, a contradiction. Suppose that $w_1 \in W_{H^*}$, where $H^* \cong S_{r+1}^1$ for $r \geq 3$ and $w_2 \in W_{H^*}$, where $H^* \cong S_{r'+1}^1$ for $r' \geq 3$ and $H^* \in N_+(u^*) \setminus H^*$. Then $|N_{H^*}(w_2)| = 1$. Similar with above, we get a contradiction. Suppose that $w_1 \in W_{H^*}$, where $H^* \cong S_{r+1}^1$ for $r \geq 3$ and $w_2 \in W_{N_0(u^*)}$. Then $|N_{H^*}(w_1)| = 1$ and $|N_{N_0(u^*)}(w_2)| \leq |N_0(u^*)|$. Thus, we get

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + \sum_{u \in N_0(u^*)} x_u < x_{w_1} + \frac{1}{2}x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} \leq \frac{\rho+\frac{1}{2}}{\rho^2-1}x_{u^*} < \frac{10\frac{1}{2}}{99}x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{88\frac{1}{2}}{99} = \frac{60}{99}$, a contradiction. Suppose that $w_1, w_2 \in W_{N_0(u^*)}$. Then $|N_{N_0(u^*)}(w_i)| \leq |N_0(u^*)|$ for each $i \in \{1, 2\}$. Thus, we get

$$\begin{cases} \rho x_{w_1} \leq x_{w_2} + \sum_{u \in N_0(u^*)} x_u < x_{w_2} + \frac{1}{2}x_{u^*}, \\ \rho x_{w_2} \leq x_{w_1} + \sum_{u \in N_0(u^*)} x_u < x_{w_1} + \frac{1}{2}x_{u^*}. \end{cases}$$

Combining with the above system of inequalities, we get $x_{w_1} < \frac{1}{2(\rho-1)}x_{u^*} < \frac{1}{18}x_{u^*}$. By Lemma 2.7, we get $1 = e(W) < 0 + \frac{3}{2} - \frac{17}{18} = \frac{5}{9}$, a contradiction. This completes the proof of Lemma 3.4. \square

Lemma 3.5. $G^*[N(u^*)]$ contains no triangle.

Proof. Suppose on the contrary that $G^*[N(u^*)]$ contains triangles. Then $G^*[N(u^*)]$ contains a component which is isomorphic to S_{r+1}^1 for $r \geq 2$. Let $H^* \cong S_{r+1}^1$ be a component of $G^*[N(u^*)]$. Then $e(H^*) = r + 1$. Let $u_1 u_2 u_3 u_1$ be the triangle of H^* and $d_{H^*}(u_1) = d_{H^*}(u_2)$. If $W_{H^*} = \emptyset$, then $x_{u_1} = x_{u_2}$. Furthermore,

$$\rho x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} \leq x_{u_1} + 2x_{u^*}.$$

Thus, $x_{u_1} \leq \frac{2}{\rho-1}x_{u^*} < \frac{2}{9}x_{u^*}$ due to $\rho > 10$.

$$\begin{aligned}\zeta(H^*) &= x_{u_1} + x_{u_2} + (r-1)x_{u_3} \\ &< \left(\frac{4}{9} + r - 1\right)x_{u^*} \\ &= (e(H^*) - \frac{14}{9})x_{u^*}.\end{aligned}$$

Recall that $\zeta(H) \leq e(H)x_{u^*}$ for $H \in \mathcal{H} \setminus H^*$. Thus,

$$\begin{aligned}\sum_{H \in \mathcal{H}} \zeta(H) &= \zeta(H^*) + \sum_{H \in \mathcal{H} \setminus H^*} \zeta(H) \\ &< (e(H^*) - \frac{14}{9})x_{u^*} + \sum_{H \in \mathcal{H} \setminus H^*} e(H)x_{u^*}, \\ &= (e(N_+(u^*)) - \frac{14}{9})x_{u^*}\end{aligned}$$

which contradicts (3). Thus, $W_{H^*} \neq \emptyset$.

Since $e(W) = 0$, combining with Lemma 2.4, we have $d_{N(u^*)}(w) \geq 2$. If $r \geq 3$, then $W_{H^*} \cap W_{N(u^*) \setminus H^*} = \emptyset$, and hence $d(w) = d_{H^*}(w) \leq 1$ for $w \in W_{H^*}$, a contradiction. Thus, $r = 2$, that is H^* is a triangle $u_1u_2u_3$. Since G^* is $\theta(1, 3, 3)$ -free, we obtain that $W_{H^*} \cap W_{N(u^*) \setminus H^*} = \emptyset$, and hence $d(w) = d_{H^*}(w) \leq 3$. First, we assume that $|W_{H^*}| = 1$. Let $W_{H^*} = \{w\}$. As H^* is a triangle, we obtain $2 \leq d(w) = d_{H^*}(w) \leq 3$. We consider two cases as follows.

Case 1. $d_{H^*}(w) = 2$.

In this case, without loss of generality, we suppose $N(w) = \{u_1, u_2\}$. Then $x_{u_1} = x_{u_2}$. Since

$$\rho x_{u_3} = x_{u_1} + x_{u_2} + x_{u^*} \leq 3x_{u^*}.$$

Furthermore, $x_{u_3} \leq \frac{3}{\rho}x_{u^*}$. Similarly,

$$\rho x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} + x_w \leq x_{u_1} + \frac{3}{\rho}x_{u^*} + 2x_{u^*},$$

we obtain that $x_{u_1} \leq \frac{2\rho+3}{\rho(\rho-1)}x_{u^*}$. Thus,

$$\zeta(H^*) = x_{u_1} + x_{u_2} + x_{u_3} \leq \frac{7\rho+3}{\rho(\rho-1)}x_{u^*}.$$

Since $\frac{7x+3}{x(x-1)}$ is decreasing in variable $x > 10$, we get

$$\zeta(H^*) < \frac{73}{90}x_{u^*} < (e(H^*) - \frac{3}{2})x_{u^*}.$$

Recall that $\zeta(H) \leq e(H)x_{u^*}$ for $H \in \mathcal{H} \setminus H^*$. Thus,

$$\begin{aligned}\sum_{H \in \mathcal{H}} \zeta(H) &= \zeta(H^*) + \sum_{H \in \mathcal{H} \setminus H^*} \zeta(H) \\ &< (e(H^*) - \frac{3}{2})x_{u^*} + \sum_{H \in \mathcal{H} \setminus H^*} e(H)x_{u^*}, \\ &= (e(N_+(u^*)) - \frac{3}{2})x_{u^*}\end{aligned}$$

which contradicts (3).

Case 2. $d_{H^*}(w) = 3$.

In this case, let $N_{H^*}(w) = \{u_1, u_2, u_3\}$. Then $x_{u_1} = x_{u_2} = x_{u_3}$. Since

$$\rho x_{u_1} = x_{u_2} + x_{u_3} + x_{u^*} + x_w \leq 2x_{u_1} + 2x_{u^*},$$

we obtain $x_{u_1} \leq \frac{2}{\rho-2}x_{u^*}$. Furthermore,

$$\zeta(H^*) = x_{u_1} + x_{u_2} + x_{u_3} \leq \frac{6}{\rho-2}x_{u^*} < \frac{3}{4}x_{u^*} < (e(H^*) - \frac{3}{2})x_{u^*}.$$

Thus,

$$\sum_{H \in \mathcal{H}} \zeta(H) < (e(N_+(u^*)) - \frac{3}{2})x_{u^*},$$

a contradiction. Thus, $|W_{H^*}| \geq 2$.

Recall that $W_{H^*} \cap W_{N(u^*) \setminus H^*} = \emptyset$ and $2 \leq d(w) = d_{H^*}(w) \leq 3$. Let $w_1 \in W_{H^*}$ such that $N(w_1) = \{u_1, u_2, u_3\}$ and $w_2 \neq w_1$ in W_{H^*} satisfying $d(w_2) \geq 2$. Suppose that $u_1, u_2 \in N(w_2)$. Then $u^*u_1, u^*u_3w_1u_1$ and $u^*u_2w_2u_1$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and u_1 , a contradiction. Hence, $d(w) = d_{H^*}(w) = 2$ for any $w \in W_{H^*}$. This implies $1 \leq |N(w_1) \cap N(w_2)| \leq 2$ for any two vertices $w_1, w_2 \in W_{H^*}$. If $|N(w_1) \cap N(w_2)| = 1$, without loss of generality, let $N(w_1) = \{u_1, u_2\}$ and $N(w_2) = \{u_1, u_3\}$, then $u^*u_1, u^*u_3w_2u_1$ and $u^*u_2w_1u_1$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and u_1 , a contradiction. Thus, $|N(w_1) \cap N(w_2)| = 2$. That is $N(w_1) = N(w_2)$ for any two vertices $w_1, w_2 \in W_{H^*}$. Without loss of generality, suppose that $N(w) = \{u_1, u_2\}$ for any $w \in W_{H^*}$. Let $G_1 = G^* - \{u_1w | w \in N_W(u_1)\} + \{u^*w | w \in N_W(u_1)\}$. Obviously, G_1 is $\theta(1, 3, 3)$ -free. By Lemma 2.1, we get $\rho(G_1) > \rho$, a contradiction. This completes the proof of Lemma 3.5. \square

Proof of Theorem 1.4. By Lemmas 3.1, 3.3 and 3.5, we obtain that each component of $G^*[N_+(u^*)]$ is a non-trivial tree. If $c = 0$, then G^* is bipartite. By Lemma 2.5, we have $\rho \leq \sqrt{m} < \frac{1+\sqrt{4m-5}}{2}$ for $m \geq 92$, a contradiction. Thus $c = 1$ and $\sum_{u \in N_0(u^*)} \frac{x_u}{x_{u^*}} < \frac{1}{2}$ from Inequality (4). Let H be the unique component of $G^*[N_+(u^*)]$, where H is a non-trivial tree. By Lemma 3.1 (iii) and (iv), we have $\text{diam}(H) \leq 3$.

If $\text{diam}(H) = 3$, then H is a double star. Let u_1 and u_2 be the two center vertices of H . Let $\{v_1, v_2, \dots, v_a\} \in N_H(u_1) \setminus u_2$ and let $\{z_1, z_2, \dots, z_b\} \in N_H(u_2) \setminus u_1$ for $a, b \geq 1$. If $W_H = \emptyset$, without loss of generality, assume that $x_{u_1} \geq x_{u_2}$, then let $G_2 = G^* - \{u_2v | v \in N_H(u_2) \setminus \{u_1\}\} + \{u_1v | v \in N_H(u_2) \setminus \{u_2\}\}$. We can verify that G_2 is $\theta(1, 3, 3)$ -free. By Lemma 2.1, we get $\rho(G_2) > \rho$, a contradiction. Thus, $W_H \neq \emptyset$. It is easily checked $W_H \cap W_{N(u^*) \setminus H} = \emptyset$. Hence, $N(w) \subseteq V(H)$ and by Lemma 2.4, $d(w) = d_H(w) \geq 2$. We claim $d(w) = d_H(w) = 2$. Suppose on the contrary that $d_H(w) \geq 3$. If $\{u_1, u_2, v_1\} \in N_H(w)$, then $u_1u_2, u_1v_1wu_2, u_1u^*z_1u_2$ are three internally disjoint paths of lengths 1, 3, 3 between u_1 and u_2 , a contradiction. If $\{u_1, v_1, v_2\} \in N_H(w)$, then $u^*v_2, u^*v_1wu_2, u^*u_2u_1v_2$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and u_2 , a contradiction. If $\{u_2, v_1, v_2\} \in N_H(w)$, then $u^*u_2, u^*v_1wu_2, u^*v_2u_1u_2$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and u_2 , a contradiction. If $\{v_1, v_2, z_1\} \in N_H(w)$, then $v_1u^*, v_1wu_2u^*, v_1u_1u_2u^*$ are three internally disjoint paths of lengths 1, 3, 3 between v_1 and u^* , a contradiction. If $\{v_1, v_2, v_3\} \in W_H(w)$, then $u^*v_1, u^*v_3wu_1, u^*u_2u_1v_1$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and v_1 , a contradiction. Thus, $d(w) = d_H(w) = 2$. We claim $N(w) = \{u_1, u_2\}$. Otherwise, we consider five cases in the sense of symmetry as follows. If $N(w) = \{u_1, v_1\}$, then $u^*u_1, u^*v_1wu_1, u^*z_1u_2u_1$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and u_1 , a contradiction. If $N(w) = \{u_1, z_1\}$, then $u_1u_2, u_1wz_1u_2, u_1v_1u^*u_2$ are three internally disjoint paths of lengths 1, 3, 3 between u_1 and u_2 , a contradiction. If $N(w) = \{v_1, v_2\}$, then $u^*v_1, u^*v_2wu_1, u^*u_2u_1v_1$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and v_1 , a contradiction. If $N(w) = \{v_1, z_1\}$, then $u^*v_1, u^*z_1wu_1, u^*u_2u_1v_1$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and v_1 , a contradiction. Thus, $N(w) = \{u_1, u_2\}$. Let $G_3 = G^* - \{u_2w | w \in N_W(u_2)\} + \{u^*w | w \in N_W(u_2)\}$. We can verify that G_3 is $\theta(1, 3, 3)$ -free. By Lemma 2.1, we get $\rho(G_3) > \rho$, a contradiction.

If $\text{diam}(H) \leq 2$, then $H \cong K_{1,r}$ with $r \geq 1$. Let $V(H) = \{u_0, u_1, \dots, u_r\}$ and u_0 be the center vertex of H with $r \geq 1$. We claim $r \geq 9$. By $\rho > 10$ and Inequality (4), we have

$$10x_{u^*} < \rho x_{u^*} = x_{u_0} + x_{u_1} + \dots + x_{u_r} + \sum_{v \in N_0(u^*)} x_v < (r+1 + \frac{1}{2})x_{u^*}.$$

Thus, $r \geq 9$. We claim $W_H = \emptyset$. Suppose on the contrary that $W_H \neq \emptyset$.

First, we assume $d_H(w) \geq 3$ for any vertex $w \in W_H$. If $\{u_0, u_1, u_2\} \in N_H(w)$, then $u^*u_1, u^*u_2wu_1$ and $u^*u_3u_0u_1$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and u_1 , a contradiction. If $\{u_1, u_2, u_3\} \in N_H(w)$, then $u^*u_3, u^*u_4u_0u_3$ and $u^*u_1wu_3$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and u_3 , a contradiction. Thus, $d_H(w) \leq 2$. If $d_H(w) = 1$, then we obtain that w is only adjacent to the center vertex u_0 . Otherwise, let $N_H(w) = u_1$. By Lemma 2.4 and $e(W) = 0$, we obtain $|N_{N_0(u^*)}(w)| \geq 1$. Then $u^*u_1, u^*vwu_1, u^*u_2u_0u_1$ are three internally disjoint paths of lengths 1, 3, 3 between u^* and u_1 , where $v \in N_{N_0(u^*)}(w)$, a contradiction. Thus, $N_H(w) = \{u_0\}$. By Inequality (4), we have

$$\rho x_w \leq x_{u_0} + \sum_{v \in N_0(u^*)} x_v < \frac{3}{2} x_{u^*}.$$

Thus, $x_w < \frac{3}{2\rho} x_{u^*} < \frac{3}{20} x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $0 = e(W) < -1 + \frac{3}{2} - \frac{17}{20} = -\frac{7}{20}$, a contradiction. If $d_H(w) = 2$, then we have $N_{N_0(u^*)}(w) = \emptyset$. Otherwise, G^* contains a $\theta(1, 3, 3)$. Furthermore, we get

$$\rho x_w \leq x_{u_0} + x_{u_1} \leq 2x_{u^*}.$$

This implies that $x_w \leq \frac{1}{2\rho} x_{u^*} < \frac{1}{20} x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $0 = e(W) < -1 + \frac{3}{2} - \frac{19}{20} < 0$, a contradiction. Thus, $W_H = \emptyset$. If $W \neq \emptyset$, then by Lemma 2.1, we obtain $d(w) = d_{N_0(u^*)}(w)$ for any vertex $w \in W$. Furthermore, $N_0(u^*) \neq \emptyset$ and

$$\rho x_w \leq \sum_{v \in N_0(u^*)} x_v < \frac{1}{2} x_{u^*}.$$

This implies that $x_w < \frac{1}{2\rho} x_{u^*} < \frac{1}{20} x_{u^*}$ due to $\rho > 10$. By Lemma 2.7, we get $0 = e(W) < -1 + \frac{3}{2} - \frac{19}{20} = -\frac{9}{20}$, a contradiction. Thus, $W = \emptyset$ and hence $G^* \cong G_4$ (see Figure. 1). Let $|N_0(u^*)| = t$. Since m is even, we obtain that t is odd and $t \geq 1$. By Lemma 2.3, we obtain that ρ is the largest root of the equation $f(x, t) = 0$ where

$$f(x, t) = x^4 - mx^2 - (m - t - 1)x + \frac{t(m - t - 1)}{2}$$

for $m = t + 1 + 2r \geq 92$. Since

$$f(x, t) - f(x, 1) = (t - 1)x + \frac{(t - 1)(m - t - 2)}{2} > 0$$

for $x > 0$ and $t \geq 3$, which implies that $t = 1$ for the extremal graph G^* . By Lemma 2.6, we have $\rho(S_{\frac{m+4}{2}, 2}^-) > \frac{1 + \sqrt{4m-5}}{2}$ for $m \geq 92$ and $G^* \cong S_{\frac{m+4}{2}, 2}^-$, as desired. This completes the proof of Theorem 1.4. ■

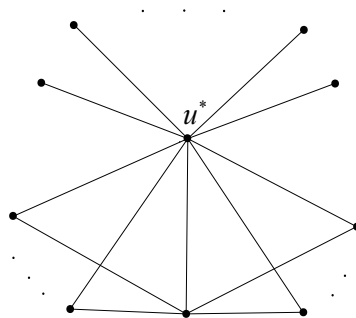


Figure 1: The graph G_4 .

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no conflict of interest.

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