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Correction to: Fixed point results under nonlinear Suzuki $(F, \mathcal{R}^{\neq})$-contractions with an application

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Abstract. Correction to: M. Arif and M. Imdad, Fixed point results under nonlinear Suzuki $(F, \mathcal{R}^{\neq})$-contractions with an application, Filomat 36:9 (2022), 3155–3165. https://doi.org/10.2298/FIL2209155A.
Here, we have noticed an obvious flaw in the proof of the alternative part of the main results, especially in Theorems 4.1 and 4.2, where the contraction condition can be applied if the Suzuki condition is met out in the presence of amorphous binary relation. We made an error in applying the contraction condition (denoted as (e)) as noted on page 3160, line 18 (from the top) of reference [1]. In our analysis, we used the condition $\frac{1}{2}d(u, u_{n_k}) < d(u, u_{n_k})$ to employ condition (e) on the pair (u, u_{n_k}). But, condition (e) is applicable only for those pairs $(u, v) \in \mathcal{R}$ for which either $\frac{1}{2}d(u, Tu) < d(u, v)$ or $\frac{1}{2}d(v, Tv) < d(u, v)$ holds. Therefore, in order to apply hypothesis (e) to the pair (u, u_{n_k}), we need to establish either $\frac{1}{2}d(u_{n_k}, u_{n_k+1}) < d(u_{n_k}, u)$ or $\frac{1}{2}d(u_{n_k+1}, u_{n_k+2}) < d(u_{n_k+1}, u)$. However, the main results of [1] corresponding to $\mathcal{R}$-continuity of T never demands any modification.

Before rectifying an obvious flaw in the proof of [1, Theorem 4.1] corresponding to d-self-closedness of $\mathcal{R}$, we record [1, Theorem 4.1], which is as under:

Theorem 0.1. *Suppose T be a self-mapping on a metric space $(\mathcal{M}, d)$ and $\mathcal{R}$ a binary relation defined on $\mathcal{M}$. Assume that the following conditions hold:*

- (a) *$(\mathcal{M}, d)$ is $\mathcal{R}$-complete;*
- (b) *$\mathcal{M}(T, \mathcal{R})$ is non-empty;*
- (c) *$\mathcal{R}$ is T-closed;*
- (d) *T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed;*
- (e) *there exist $\phi \in \Phi$ and $F \in \mathcal{F}$ such that T is nonlinear Suzuki $(F, \mathcal{R}^{\neq})$-contraction.*

Then T has a fixed point.

Proof. As $\mathcal{R}$ is d-self-closed, therefore, for every $\mathcal{R}$-preserving sequence $\{u_n\}$ with $u_n \xrightarrow{d} u$, the d-self-closedness of $\mathcal{R}$ implies that the existence of a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $[u_{n_k}, u] \in \mathcal{R} \quad (\forall\ k \in \mathbb{N}_0)$. We assert that (for all $k \in \mathbb{N}_0$)

$$\frac{1}{2}d(u_{n_k}, u_{n_k+1}) < d(u_{n_k}, u) \text{ or } \frac{1}{2}d(u_{n_k+1}, u_{n_k+2}) < d(u_{n_k+1}, u). \tag{1}$$

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Let on contrary that if it is not so (for some $k_o \in \mathbb{N}_0$)

$$\frac{1}{2}d(u_{n_{k_o}}, u_{n_{k_o}+1}) \geq d(u_{n_{k_o}}, u) \text{ and } \frac{1}{2}d(u_{n_{k_o}+1}, u_{n_0+2}) \geq d(u_{n_{k_o}+1}, u)$$

Applying the triangle inequality, we obtain

$$\begin{aligned} d(u_{n_{k_o}}, u_{n_{k_o}+1}) &\leq d(u_{n_{k_o}}, u) + d(u_{n_{k_o}+1}, u) \\ &\leq \frac{1}{2}d(u_{n_{k_o}}, u_{n_{k_o}+1}) + \frac{1}{2}d(u_{n_{k_o}+1}, u_{n_{k_o}+2}) \\ &< \frac{1}{2}\{d(u_{n_{k_o}}, u_{n_{k_o}+1}) + d(u_{n_{k_o}}, u_{n_{k_{o+1}}})\} = d(u_{n_{k_o}}, u_{n_{k_o}+1}) \text{ (becouse } d(u_{n_{k_o}+1}, u_{n_{k_o}+2}) < d(u_{n_{k_o}}, u_{n_{k_o}+1})), \end{aligned}$$

which is a contradiction. Therefore, (1) holds (for all $k \in \mathbb{N}_0$). One can choose a subsequence $\{n_{k_l}\}$ of sequence $\{n_k\}$ such that $\frac{1}{2}d(u_{n_{kl}}u_{n_{kl}+1}) < d(u_{n_{kl}}, u)$ (for all $l \in \mathbb{N}$), which on employing assumption (e), $[u_{n_k}, u] \in \mathcal{R}$ and henceforth using Proposition 3.13 of [1], we get

$$\begin{aligned} F(d(u_{n_{kl}+1}, Tu)) &\leq F(d(u_{n_{kl}}, u)) - \phi(d(u_{n_{kl}}, u)) \\ \Rightarrow F(du_{n_{kl}+1}, Tu)) &< F(d(u_{n_{kl}}, u)) \end{aligned} \quad (2)$$

In view of (F_1), we deduce that $d(u_{n_{kl}+1}, Tu) < d(u_{n_{kl}}, u)$ for all $l \in \mathbb{N}$. Letting $l \to \infty$ in above inequality and using the fact that $u_{n_{kl}} \xrightarrow{d} u$ as $l \to \infty$, yields that $u_{n_{kl}+1} \xrightarrow{d} T(u)$. Owing to the uniqueness of limit, we have $T(u) = u$ so that u is a fixed point of T, which concludes the proof.
□

To correct the similar flaw in the corresponding uniqueness namely [1, Theorem 4.2], we first opt to state this theorem, which is as under:

Theorem 0.2. *Suppose all the hypotheses of Theorem 0.1 (i.e.,* [1, Theorem 4.1]*) together with the following condition holds:*

(v) *:* $T(\mathcal{M})$ *is* $\mathcal{R}^s$*-connected.*

Then T has a unique fixed point.

Proof. By Theorem 0.1, $F(T) \neq \emptyset$. If $F(T)$ is singleton, then there is nothing to prove. Otherwise, let us choose $u, v \in F(T)$, then for all $n \in \mathbb{N}_0$, we have

$$T^n(u) = u \text{ and } T^n(v) = v. \quad (3)$$

In lieu of assumption (v), there exists a path (say $\{\xi_0, \xi_1, \xi_2, ..., \xi_l\}$) of some finite length l in $\mathcal{R}^s$ from u to v so that

$$\xi_0 = u,\ \xi_l = v \text{ and } [\xi_i, \xi_{i+1}] \in \mathcal{R} \text{ for each } i\ (0 \leq i \leq l-1). \quad (4)$$

Since $\mathcal{R}$ is T-closed, utilizing **Propositions 3.4-3.5 of [1]** and (4), we obtain

$$[T^n\xi_i, T^n\xi_{i+1}] \in \mathcal{R} \text{ for each } i\ (0 \leq i \leq l-1) \text{ and for each } n \in \mathbb{N}_0. \quad (5)$$

Now, for each $n \in \mathbb{N}_0$ and for each i $(0 \leq i \leq l-1)$, write $\delta_n^i := d(T^n\xi_i, T^n\xi_{i+1})$.
We assert that

$$\lim_{n\to\infty} \delta_n^i = 0 \quad \text{for each } i\ (0 \leq i \leq l-1). \quad (6)$$

For fix i, we distinguish two cases. Firstly, consider that $\delta_{n_0}^i = d(T^{n_0}\xi_i, T^{n_0}\xi_{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, *i.e.*, $T^{n_0}(\xi_i) = T^{n_0}(\xi_{i+1})$, which implies that $T^{n_0+1}(\xi_i) = T^{n_0+1}(\xi_{i+1})$. Consequently, we get $\delta_{n_0+1}^i = d(T^{n_0+1}\xi_i, T^{n_0+1}\xi_{i+1}) = 0$. Thus by induction on n, we obtain $\delta_n^i = 0\ \forall\ n \geq n_0$ so that $\lim\limits_{n\to\infty} \delta_n^i = 0$.

Secondly, suppose that $\delta_n^i > 0 \ \forall \ n \in \mathbb{N}_0$, then we show either $\frac{1}{2}d(T^n\xi_{i+1}, T^{n+1}\xi_{i+1}) < d(T^n\xi_i, T^n\xi_{i+1})$ or $\frac{1}{2}d(T^n\xi_{i+1}, T^{n+1}\xi_{i+1}) < d(T^n\xi_i, T^n\xi_{i+1})$. Due to (3), we have $T^n\xi_0 \to u$. By induction hypothesis assume that $T^n\xi_k \to u$ for some $k < l$. We prove for $T^n\xi_{k+1} \to u$. Let on contrary $T^n\xi_{k+1} \neq u$ $(\forall n \in \mathbb{N}_0)$, then there exists positive integer $N \in \mathbb{N}$ such that $\frac{1}{2}d(T^n\xi_k, T^{n+1}\xi_k) < d(T^n\xi_k, T^n\xi_{k+1})$ with $\forall n \geq N$. On applying (5), the contractivity condition (e), and Proposition 3.13 (contained in [1]) we deduce, (for all $n \in \mathbb{N}$ with $\forall n \geq N$)

$$\begin{aligned} \phi(d(T^n\xi_k, T^n\xi_{k+1})) + F(d(T^{n+1}\xi_k, T^{n+1}\xi_{k+1})) &\leq F(d(T^n\xi_k, T^n\xi_{k+1})) \ or, \\ F(d(T^{n+1}\xi_k, T^{n+1}\xi_{k+1})) &\leq F(d(T^n\xi_k, T^n\xi_{k+1})) - \phi(d(T^n\xi_k, T^n\xi_{k+1})) \end{aligned} \tag{7}$$

Due to (7) and property (F_1), the sequence $\{\delta_n^k\}$ (for each k, $0 \leq k \leq l-1$) is decreasing . Therefore, there exists $\delta^k \geq 0$ such that $\lim_{n\to\infty} \delta_n^k = \delta^k$ (for each k, $0 \leq k \leq l-1$). Suppose that $\delta^k > 0$. Proceeding on the lines of pattern (3) (contained in [1, Theorem 4.1]), the following holds:

$$\begin{aligned} F(d(T^{n+1}\xi_k, T^{n+1}\xi_{k+1})) &\leq F(d(\xi_k, \xi_{k+1})) - \phi(d(T^n\xi_k, T^n\xi_{k+1})) - \phi(d(T^{n-1}\xi_k, T^{n-1}\xi_{k+1})) - \cdots - \phi(d(\xi_k, \xi_{k+1})) \\ &= F(d(\xi_k, \xi_{k+1})) - \phi(\delta_n^k) - \phi(\delta_{n-1}^k) - \cdots - \phi(\delta_0^k) \end{aligned} \tag{8}$$

On setting $\phi(\delta_{p_n}^k) := \min\{\phi(\delta_n^k), \phi(\delta_{n-1}^k), \cdots, \phi(\delta_0^k)\}$ for all $n \in \mathbb{N}_0$, inequality (8), reduces to

$$F(d(T^{n+1}\xi_k, T^{n+1}\xi_{k+1})) \leq F(d(\xi_k, \xi_{k+1})) - n\phi(\delta_{p_n}^k). \tag{9}$$

Proceeding on the lines of pattern (4) (contained in [1, Theorem 4.1]), we deduce that $\lim\limits_{n\to\infty} \delta_n^k = 0$ for each k $(0 \leq k \leq l-1)$. Making use of (3), (5), (6) and the triangular inequality, we have

$$d(u, v) = d(T^n\xi_0, T^n\xi_l) \leq \delta_n^0 + \delta_n^1 + \cdots + \delta_n^{l-1} \to 0 \ \ as \ \ n \to \infty$$

so that $u = v$. Hence T has a unique fixed point, which concludes the proof. □

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References

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