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Riemann integration of Banach space-valued functions on time scale

Vikuozonuo Sekhose[a], Hemanta Kalita[b,*], Hemen Bharali[a]

[a] *Assam Don Bosco University, Sonapur, Assam 782402, India*
[b] *Mathematics Division, VIT Bhopal University, Kothrikalan, Bhopal-Indore Highway, Madhya Pradesh 466114, India*

Abstract. In this article, we explore the Riemann integration of Banach-valued functions on time scale. We attempt to give an alternate definition of the previously defined Riemann Δ Banach-valued integral. Basic properties and results such as the criterion of integrability of the function, uniqueness of integral, linearity property, additivity property and results related to boundedness and continuity of functions are formulated and established. Finally, we state and proof the Fundamental Theorem of Calculus for the integral.

1. Introduction

S. Hilger, under the supervision of B. Aulbach, introduced the theory of measure chain calculus (commonly known as time scale calculus) as part of his doctorial degree titled- "*Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*" (in German), University of Würzburg, Germany, 1988, later published as [11].

Hilger's main motivation was the analogy between discrete and continuous analysis and the aim to unify them, a direct quote from his paper- "*These analogies which are described in the relevant literature lead to the idea to develop some higher ranging calculus which in special cases covers those two concepts*"[11] beautifully portrays this objective. As theoretical framework (see [11, 12]), Hilger introduced the forward jump operator (σ) and backward jump operator (ρ); and for the notion of measure chains Hilger presented three axioms in an attempt to construct an abstract structure, and concluded that any set that satisfied these three axioms were to be called a measure chain (or time scale). Using the forward jump operator, Hilger introduced the delta derivative (Δ-derivative) and also a descriptive sense of the integral (named the Cauchy integral).

For Banach space-valued functions (which we will simply refer to as Banach-valued functions), the Δ-derivative was defined by S. Hilger [11]; and the ∇-derivative was defined by B. Satco [14] in 2011.

The Δ-derivative was formulated using the σ operator, hence it was just a matter of time before another derivative using ρ was formulated. This was done by C. D. Ahlbrandt et al. [1] in 2000. They formulated a so-called alpha derivative (α-derivative): when $\alpha = \sigma$ the Δ-derivative was deduced, and when $\alpha = \rho$ the derivative formulated, they called the ρ-derivative. This 'ρ-derivative' came to be known as the nabla

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* Corresponding author: Hemanta Kalita
Email addresses: vikuosekhose4@gmail.com (Vikuozonuo Sekhose), hemanta30kalita@gmail.com (Hemanta Kalita), hemen.bharali@gmail.com (Hemen Bharali)
ORCID iDs: https://orcid.org/0009-0007-1465-1604 (Vikuozonuo Sekhose), https://orcid.org/0000-0002-9798-6608 (Hemanta Kalita), https://orcid.org/0000-0002-4800-9536 (Hemen Bharali)

derivative and denoted by ∇-derivative by F. M. Atici et al. in [3]. For more insight into the theory of time scale calculus one may refer [4–6].

Various integration notions, in their constructive sense, are discussed in literature including the Riemann integral for real-valued functions on time scales [9, 10] and Riemann integral for Banach-valued functions on time scales [2].

The Riemann integral for real-valued functions on time scale was formulated by S. Sailer [4, 9], using the concept of Darboux sum definition of the integral; and by G. Sh. Guseinov et al. [9] (also see [10]), using the concept of Riemann sum definition of the integral. The latter also proved that the two different approaches of the Riemann integral for real-valued functions on time scales are in essence equal [9]. While the Riemann integral for Banach-valued functions on time scale was formulated by B. Aulbach et al. [2] (also see [15]), the theory is not nearly as developed as it's real-valued counterpart.

Thus in this paper, we attempt to give an alternate definition of the previously defined Riemann integral by B. Aulbach et al. [2], and formulate and establish some basic properties and results for Banach-valued functions on time scale.

For all definitions, properties and results, there are corresponding ones for the nabla derivatives and ∇-integral, which we will not bother to consider in this paper.

2. Preliminary

In this section, we recall a few definitions and results on the theory of time scale calculus [4–6], on Riemann integration for real-valued functions [9, 10] and on classical Banach space theory [7, 8, 13].

A time scale $\mathcal{T}$ is any non-empty closed subset of $\mathbb{R}$.

Definition 2.1. *[11, pp. 20] Forward Jump Operator. The forward jump operator denoted by σ is a mapping, $\sigma : \mathcal{T} \to \mathcal{T}$ defined by,*

$$\sigma(t) = \inf\left\{r \in \mathcal{T} : r > t\right\}.$$

Definition 2.2. *[11, pp. 20] Backward Jump Operator. The backward jump operator denoted by ρ is a mapping, $\rho : \mathcal{T} \to \mathcal{T}$ defined by,*

$$\rho(t) = \sup\left\{r \in \mathcal{T} : r < t\right\}.$$

Elements of $\mathcal{T}$, $(t \in \mathcal{T})$, can be classified using the jump operators as [11]:

- right-dense (rd) if $\sigma(t) = t < \sup \mathcal{T}$; right-scattered (rd) if $\sigma(t) > t$;
- left-dense (ld) if $\rho(t) = t > \inf \mathcal{T}$; right-scattered (rd) if $\sigma(t) > t$;
- dense if $\sigma(t) = t = \rho(t)$; isolated if $\sigma(t) < t < \rho(t)$.

Definition 2.3. *[11, pp. 27] $\mathcal{T}^k$ is defined as,*

$$\mathcal{T}^k = \begin{cases} \mathcal{T} \setminus \left(\rho(\sup \mathcal{T}), \sup \mathcal{T}\right] & \text{if} \quad \sup \mathcal{T} < \infty \\ \mathcal{T} & \text{otherwise.} \end{cases}$$

Assuming $p \leq q$, intervals in $\mathcal{T}$ are defined as [4]-
$[p, q]_{\mathcal{T}} = \left\{t \in \mathcal{T} : p \leq t \leq q\right\}$; $(p, q)_{\mathcal{T}} = \left\{t \in \mathcal{T} : p < t < q\right\}$
$[p, q)_{\mathcal{T}} = \left\{t \in \mathcal{T} : p \leq t < q\right\}$, $(p, q]_{\mathcal{T}} = \left\{t \in \mathcal{T} : p < t \leq q\right\}$.

Definition 2.4. *[9, pp. 1002] Given $[p, q]_{\mathcal{T}}$ where $p < q$. A partition $\mathcal{V}$ is any finite ordered subset $\mathcal{V} = \left\{t_h\right\}_{h=0}^{n} \subseteq [p, q]_{\mathcal{T}}$ where,*

$$\mathcal{V} = \left\{p = t_0 < t_1 < \ldots < t_n = q\right\}.$$

Definition 2.5. *[9] Riemann Δ-integral (using Riemann sum definition). Let f be a real-valued bounded function on $[p,q]_{\mathcal{T}}$ and $\mathfrak{P}$ be the collection of all possible partitions of $[p,q]_{\mathcal{T}}$, then f is said to be Riemann Δ-integrable if and only if there exists a number $\bar{I} \in \mathbb{R}$ with the property that for all $\varepsilon > 0$ there exists a $\delta > 0$, and for any partition $\mathcal{V}_\delta \in \mathfrak{P}$ (i.e., partitions whose subintervals have length less than δ) such that,*

$$\left|\overline{R} - \bar{I}\right| < \varepsilon.$$

Here $\bar{I} = \overline{R}\int_p^q f(t)\Delta t$ and $\overline{R} = \sum_{h=1}^{n} f(\xi_h) \cdot (t_h - t_{h-1})$, where $\xi_h \in [t_{h-1}, t_h)_{\mathcal{T}}$.

We now provide some definitions on classical Banach space theory.

Definition 2.6. *[13, pp. 5] Metric Space. A metric space is a set S along with a metric or distance function $\mathrm{d} : S \times S \to \mathbb{R}$ such that the following three conditions are satisfied, $\forall\, x, y, z \in S$:*

1. $\mathrm{d}(x,y) \geq 0$, *and* $\mathrm{d}(x,y) = 0$ *if and only if* $x = y$;
2. $\mathrm{d}(x,y) = \mathrm{d}(y,x)$;
3. $\mathrm{d}(x,z) \leq \mathrm{d}(x,y) + \mathrm{d}(y,z)$ *(the triangle inequality).*

Definition 2.7. *[13, pp. 9] Normed Space. Let $\mathfrak{X}$ be a vector space. A norm on $\mathfrak{X}$ is a real-valued function $\|\cdot\|$ on $\mathfrak{X}$ such that the following conditions are satisfied by all members x and y of $\mathfrak{X}$ and each scalar α:*

1. $\|x\| \geq 0$, *and* $\|x\| = 0$ *if and only if* $x = 0$;
2. $\|\alpha x\| = |\alpha|\|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$ *(the triangle inequality).*

The ordered pair $\left(\mathfrak{X}, \|\cdot\|\right)$ is called a normed space/ normed vector space/ normed linear space.

Definition 2.8. *[13, pp. 13] Banach Space. A Banach space $\mathfrak{X}$ is a vector space over $\mathbb{R}$ equipped with the norm $\|\cdot\|$ such that $\mathfrak{X}$ is a complete metric space with respect to the metric-*

$$\mathrm{d}(x_1, x_2) = \left\|x_1 - x_2\right\|, \quad x_1, x_2 \in \mathfrak{X}.$$

Throughout this paper $\mathfrak{X}$ will denote a Banach space and $\mathfrak{X}^*$ its dual.

The Δ-derivative on $\mathfrak{X}$ was defined by S. Hilger [11]-

Definition 2.9. *[11, pp. 27] Delta Derivative (on $\mathfrak{X}$). Let function F be a mapping, $F : \mathcal{T} \to \mathbb{R}$ and $t \in \mathcal{T}^k$. $F^\Delta(t)$, provided it exists is called the delta derivative of F at t, if for any $\varepsilon > 0$, there exists a neighbourhood $\mathcal{W} = (t-\delta, t+\delta) \cap \mathcal{T}$ of t for $\delta > 0$ such that,*

$$\left\|F(\sigma(t)) - F(r) - (\sigma(t) - r) \cdot F^\Delta(t)\right\| \leq \varepsilon \cdot \left|\sigma(t) - r\right| \text{ for all } r \in \mathcal{W}.$$

Given the Δ-derivative (on $\mathfrak{X}$), Hilger also presented an appropriate notion of integral via antiderivatives in [11] (also view [12]). The antiderivative of the Δ-derivative we will call the Δ-antiderivative. The integral defined by means of the Δ-antiderivative was called the Cauchy integral by Hilger [11], we will call this descriptive integral of the Δ-derivative as Δ-integral.

Definition 2.10. *[11] Δ-Integral (on $\mathfrak{X}$). A function $F : \mathcal{T} \to \mathfrak{X}$ is called a Δ-antiderivative of $f : \mathcal{T}^k \to \mathfrak{X}$ provided $F^\Delta(t) = f(t)$ for all $t \in \mathcal{T}^k$. In this case the Δ-integral from p to q of f is defined by*

$$\int_p^q f(t)\Delta t = F(q) - F(p) \text{ for all } p, q \in \mathcal{T}.$$

We now proceed to define Riemann Δ-integral for Banach-valued functions in the following section and establish a few basic results.

3. Riemann Δ Banach-Valued Integral

Let $\mathcal{T}$ be a time scale, $[p,q]_{\mathcal{T}}$ be a closed interval on $\mathcal{T}$ such that $p < q$. Let $\mathfrak{P}$ be the collection of all possible partitions of $[p,q]_{\mathcal{T}}$.

Let $\mathcal{V} \in \mathfrak{P}$, $\mathcal{V} = \{p = t_0 < t_1 < \ldots < t_n = q\}$, with $t_0, t_1, \ldots, t_n$ being the points of division. We consider subintervals of the form $[t_{h-1}, t_h)_{\mathcal{T}}$, for $1 \leq h \leq n$, and from each subinterval we choose ϑ_h arbitrarily, defined as $\vartheta_h \in [t_{h-1}, t_h)_{\mathcal{T}}$, and call it the tag point of the respective subinterval. For $\mathcal{V} \in \mathfrak{P}$, we define a point-interval collection as $\check{\mathcal{V}} = \left\{\left(\vartheta_h, [t_{h-1}, t_h)_{\mathcal{T}}\right)\right\}_{h=1}^{n}$, and call it the tagged partition. We define the mesh of $\mathcal{V}$ as,

$$\mathrm{mesh}(\mathcal{V}) = \left|\mathcal{V}\right| = \max_{1\leq h\leq n}(t_h - t_{h-1}) > 0.$$

For some $\delta > 0$, $\mathcal{V}_\delta$ will represent a partition of $[p,q]_{\mathcal{T}}$ with mesh δ satisfying the property: For each $h = 1, 2, \ldots, n$ we have either-

$$(t_h - t_{h-1}) \leq \delta \quad \text{or} \quad (t_h - t_{h-1}) > \delta \wedge \rho(t_h) = t_{h-1},$$

here $\wedge$ stands for "and". Hence, $\check{\mathcal{V}}_\delta$ will mean a tagged partition with mesh δ satisfying the above property.

We now form the Riemann Δ-sum, $(\overline{R})(f; \check{\mathcal{V}}_\delta)$, of Banach-valued function f evaluated at the tags as,

$$(\overline{R})(f; \check{\mathcal{V}}_\delta) = \sum_{h=1}^{n}(t_h - t_{h-1}) \cdot f(\vartheta_h).$$

Proceeding, we first give the definition of the Riemann Δ-integral for Banach-valued function on $[p,q]_{\mathcal{T}}$, according to B. Aulbach et. al [2], and consequently attempt to give an alternate definition of the same, similar to it's classical counterpart [see [7]].

Definition 3.1. *[2] A function $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ is Riemann Δ-integrable, denoted by $(\overline{R})$, if there exists an $\overline{I} \in \mathfrak{X}$ such that, for all $\varepsilon > 0$ there exists a $\delta > 0$ so that if $\check{\mathcal{V}}_\delta$ is any tagged partition of $[p,q]_{\mathcal{T}}$ with mesh δ, then*

$$\left\|(\overline{R})(f; \check{\mathcal{V}}_\delta) - \overline{I}\right\| < \varepsilon.$$

Here $\overline{I} = (\overline{R}) \int_p^q f(t)\Delta t = (\overline{R})$.

The set of all Riemann Δ Banach-valued integrable function on $[p,q]_{\mathcal{T}}$ will be denoted by $(\mathfrak{R})_\Delta\left([p,q]_{\mathcal{T}}, \mathfrak{X}\right)$.

We will also agree to the following convention:

$$(\overline{R}) \int_p^q f(t)\Delta t = -(\overline{R}) \int_q^p f(t)\Delta t.$$

Finally, tagged partition $\check{\mathcal{V}}_1$ is a refinement of the tagged partition $\check{\mathcal{V}}_2$ if the points of $\mathcal{V}_2$ form a subset of the points of $\mathcal{V}_1$, i.e. $\mathcal{V}_1$ refines $\mathcal{V}_2$.

We proceed to give the alternate definition of the Riemann Δ-integral for Banach-valued function on $[p,q]_{\mathcal{T}}$ and consequently prove the equivalence of the two above definitions in Theorem 3.3.

Definition 3.2. *A function $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ is Riemann Δ-integrable, denoted by $(\overline{R}_\varepsilon)$, if there exists a number $\overline{I} \in \mathfrak{X}$ such that for all $\varepsilon > 0$ there exists a partition $\mathcal{V}_\varepsilon \in \mathfrak{P}([p,q]_{\mathcal{T}})$ such that,*

$$\left\|(\overline{R}_\varepsilon)(f; \check{\mathcal{V}}) - \overline{I}\right\| < \varepsilon,$$

whenever $\check{\mathcal{V}}$ is a tagged partition of $[p,q]_{\mathcal{T}}$ that refines $\check{\mathcal{V}}_\varepsilon$.

We now construct a theorem to prove that $(\overline{R})$ integrable function is necessarily $(\overline{R}_\epsilon)$ integrable and vice versa.

Theorem 3.3. *A function $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ is $(\overline{R}_\varepsilon)$ integrable on $[p,q]_{\mathcal{T}}$ if, and only if it is $(\overline{R})$ integrable on $[p,q]_{\mathcal{T}}$.*

Proof. Suppose that f is $(\overline{R}_\varepsilon)$ integrable on $[p,q]_{\mathcal{T}}$. Let $\bar{I}$ be the $(\overline{R}_\varepsilon)$ integral of f on $[p,q]_{\mathcal{T}}$ and let D be a bound for f on $[p,q]_{\mathcal{T}}$.

Let $\varepsilon > 0$ and choose a partition $\mathcal{V}_\varepsilon = \left\{p = t_0 < \ldots < t_n = q\right\} \in \mathfrak{P}$ such that,

$$\left\|(\overline{R}_\varepsilon)(f;\check{\mathcal{V}}) - \bar{I}\right\| < \frac{\varepsilon}{2},$$

whenever $\check{\mathcal{V}}$ is a tagged partition of $[p,q]_{\mathcal{T}}$ that refines $(\mathcal{V}_\varepsilon)$. Let $\delta = \frac{\varepsilon}{4Dn}$. We will show that,

$$\left\|(\overline{R})(f;\check{\mathcal{V}}) - \bar{I}\right\| < \varepsilon,$$

whenever $\check{\mathcal{V}}$ is a tagged partition of $[p,q]_{\mathcal{T}}$ with mesh δ.

It then follows that f is $(\overline{R})$ integrable on $[p,q]_{\mathcal{T}}$.

Let $\check{\mathcal{V}}$ be such a tagged partition. Form a tagged partition $\check{\mathcal{V}}_1$ such that points of both $\check{\mathcal{V}}$ and $\check{\mathcal{V}}_\varepsilon$ is a subset of $\check{\mathcal{V}}_1$. Let the tag of each interval of $\check{\mathcal{V}}_1$ coincides with those of $\check{\mathcal{V}}$ and arbitrary for the remaining intervals. Let $\left\{[c_k, d_k)_{\mathcal{T}} : 1 \le k \le N\right\}$ be the intervals of $\check{\mathcal{V}}$ that contains points of $\check{\mathcal{V}}_\varepsilon$ in their interior, note that $N \le n-1$. In the interval $[c_k, d_k)_{\mathcal{T}}$ let $c_k = u_0^k < u_1^k < \ldots < u_{n_k-1}^k < u_{n_k}^k = d_k$ where the points $\left\{u_h^k : 1 \le h \le n_k - 1\right\}$ are the points of $\check{\mathcal{V}}_\varepsilon$ in $(c_k, d_k)_{\mathcal{T}}$. Let ϑ_h be the tag of $\check{\mathcal{V}}$ for $[c_k, d_k)_{\mathcal{T}}$ and let ξ_h^k be the tag of $\check{\mathcal{V}}_1$ for $[u_{h-1}^k, u_h^k)_{\mathcal{T}}$. Then,

$$\begin{aligned}\left\|(\overline{R})(f;\check{\mathcal{V}}) - (\overline{R}_\varepsilon)(f;\check{\mathcal{V}}_1)\right\| &= \left\|\sum_{k=1}^{N}\{(d_k - c_k)\cdot f(\vartheta_k) - \sum_{h=1}^{n_k}(u_h^k - u_{h-1}^k)\cdot f(\xi_h^k)\}\right\| \\ &\le \sum_{k=1}^{N}\sum_{h=1}^{n_k}(u_h^k - u_{h-1}^k)\cdot\left\|f(\vartheta_k) - f(\xi_h^k)\right\| \\ &\le 2D\sum_{k=1}^{N}(d_k - c_k) \le 2D(n-1)\delta < \frac{\varepsilon}{2}.\end{aligned}$$

Since $\check{\mathcal{V}}_1$ is a refinement of $\check{\mathcal{V}}_\varepsilon$, we obtain,

$$\begin{aligned}\left\|(\overline{R})(f;\check{\mathcal{V}}) - \bar{I}\right\| &\le \left\|(\overline{R})(f;\check{\mathcal{V}}_1) - (\overline{R}_\varepsilon)(f;\check{\mathcal{V}}_1)\right\| + \left\|(\overline{R}_\varepsilon)(f;\check{\mathcal{V}}_1) - \bar{I}\right\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Converse of the proof follows directly from Definition 3.1.
This completes the proof. □

Remark 3.4. *The function $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ is Riemann Δ- integrable on $[p,q]_{\mathcal{T}}$ if f is either $(\overline{R})$ or $(\overline{R}_\varepsilon)$ integrable on $[p,q]_{\mathcal{T}}$.*

Theorem 3.5. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be Riemann Δ-integrable, then the value of the integral, $\bar{I}$, is unique.*

Proof. Let us assume that the Banach-valued function f has two integral values, say $\bar{I}'$ and $\bar{I}''$ both satisfying the definition and let $\varepsilon > 0$.

Then, there exists $\delta'_{\frac{\varepsilon}{2}} > 0$ such that for any tagged partition $\check{\mathcal{V}}_{\delta'_{\frac{\varepsilon}{2}}}$ we have,

$$\left\|(\overline{R})\left(f;\check{\mathcal{V}}_{\delta'_{\frac{\varepsilon}{2}}}\right)-\overline{I}'\right\|<\frac{\varepsilon}{2}.$$

Also, there exists $\delta''_{\frac{\varepsilon}{2}} > 0$ such that for any tagged partition $\check{\mathcal{V}}_{\delta''_{\frac{\varepsilon}{2}}}$ we have,

$$\left\|(\overline{R})\left(f;\check{\mathcal{V}}_{\delta''_{\frac{\varepsilon}{2}}}\right)-\overline{I}''\right\|<\frac{\varepsilon}{2}.$$

Now, let $\delta_\varepsilon = \min\left\{\delta'_{\frac{\varepsilon}{2}}, \delta''_{\frac{\varepsilon}{2}}\right\} > 0$ and let $\check{\mathcal{V}}_{\delta_\varepsilon}$ be the corresponding tagged partition. Since mesh of $\check{\mathcal{V}}_{\delta_\varepsilon}$ is lesser or equal to the mesh of $\check{\mathcal{V}}_{\delta'_{\frac{\varepsilon}{2}}}$ and $\check{\mathcal{V}}_{\delta''_{\frac{\varepsilon}{2}}}$, thus we have by definition-

$$\left\|(\overline{R})\left(f;\check{\mathcal{V}}_{\delta_{\frac{\varepsilon}{2}}}\right)-\overline{I}'\right\|<\frac{\varepsilon}{2} \quad \text{and} \quad \left\|(\overline{R})\left(f;\check{\mathcal{V}}_{\delta_{\frac{\varepsilon}{2}}}\right)-\overline{I}''\right\|<\frac{\varepsilon}{2},$$

whence it follows from triangle inequality that,

$$\begin{aligned}\left\|\overline{I}'-\overline{I}''\right\| &= \left\|\overline{I}'-(\overline{R})\left(f;\check{\mathcal{V}}_{\delta_\varepsilon}\right)+(\overline{R})\left(f;\check{\mathcal{V}}_{\delta_\varepsilon}\right)-\overline{I}''\right\| \\ &\le \left\|\overline{I}'-(\overline{R})\left(f;\check{\mathcal{V}}_{\delta_\varepsilon}\right)\right\|+\left\|(\overline{R})\left(f;\check{\mathcal{V}}_{\delta_\varepsilon}\right)-\overline{I}''\right\|<\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, thus we conclude that $\overline{I}' = \overline{I}''$. □

We now state and proof the Cauchy criterion of integrability in terms of the Riemann Δ-integral for Banach-valued functions on time scale.

Theorem 3.6. *Cauchy Criterion. Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be Riemann Δ-integrable on $[p,q]_{\mathcal{T}}$ if, and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for partition $\mathcal{V}_1, \mathcal{V}_2 \in [p,q]_{\mathcal{T}}$ having mesh δ for both partitions $\mathcal{V}_1$ and $\mathcal{V}_2$ with respective tagged partition $\check{\mathcal{V}}_1$ and $\check{\mathcal{V}}_2$ implies,*

$$\left\|(\overline{R})\left(f;\check{\mathcal{V}}_1\right)-(\overline{R})\left(f;\check{\mathcal{V}}_2\right)\right\|<\varepsilon. \tag{1}$$

Proof. Suppose first that $f \in (\mathfrak{R})_\Delta\left([p,q]_{\mathcal{T}}, \mathfrak{X}\right)$ and consider $\varepsilon > 0$ there exists $\delta > 0$ such that for partition $\mathcal{V}_1$ and $\mathcal{V}_2$ of $[p,q]_{\mathcal{T}}$ having mesh$(\mathcal{V}_1) < \delta$ and mesh$(\mathcal{V}_2) < \delta$ with respective tagged partition $\check{\mathcal{V}}_1$ and $\check{\mathcal{V}}_2$ satisfying,

$$\left\|(\overline{R})\left(f;\check{\mathcal{V}}_1\right)-\overline{I}\right\|<\frac{\varepsilon}{2} \quad \text{and} \quad \left\|(\overline{R})\left(f;\check{\mathcal{V}}_2\right)-\overline{I}\right\|<\frac{\varepsilon}{2}.$$

Therefore we have,

$$\begin{aligned}\left\|(\overline{R})\left(f;\check{\mathcal{V}}_1\right)-(\overline{R})\left(f;\check{\mathcal{V}}_2\right)\right\| &= \left\|(\overline{R})\left(f;\check{\mathcal{V}}_1\right)-\overline{I}+\overline{I}-(\overline{R})\left(f;\check{\mathcal{V}}_2\right)\right\| \\ &\le \left\|(\overline{R})\left(f;\check{\mathcal{V}}_1\right)-\overline{I}\right\|+\left\|(\overline{R})\left(f;\check{\mathcal{V}}_2\right)-\overline{I}\right\| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Hence, given f is Riemann Δ Banach-valued function integrable Eq. 1 holds.

Conversely, for each $n \in \mathbb{N}$, let $\delta_n > 0$ be such that if $\check{\mathcal{V}}_1$ and $\check{\mathcal{V}}_2$ are tagged partition with mesh$< \delta_n$, then

$$\left\|(\overline{R})\left(f;\check{\mathcal{V}}_1\right)-(\overline{R})\left(f;\check{\mathcal{V}}_2\right)\right\|<\frac{1}{n}.$$

Evidently, we may assume that $\delta_n \geq \delta_{n+1}$ for $n \in \mathbb{N}$; otherwise, we replace δ_n by $\delta'_n = \min\{\delta_1, \ldots, \delta_n\}$.
For each $n \in \mathbb{N}$, let $\check{\mathcal{V}}_{1_n}$ be a tagged partition with mesh$(\check{\mathcal{V}}_{1_n}) < \delta_n$. Clearly if $m > n$ then both $\check{\mathcal{V}}_{1_m}$ and $\check{\mathcal{V}}_{1_n}$ have mesh$< \delta_n$, so that,

$$\left\|(\overline{R})\left(f;\check{\mathcal{V}}_{1_n}\right) - (\overline{R})\left(f;\check{\mathcal{V}}_{1_m}\right)\right\| < \frac{1}{n} \quad \text{for} \quad m > n. \tag{2}$$

Consequently, the sequence $\left\{(\overline{R})\left(f;\check{\mathcal{V}}_{1_m}\right)\right\}_{m=1}^{\infty}$ is Cauchy in $\mathfrak{X}$. Since $\mathfrak{X}$ is a Banach space, hence there exists $A \in \mathfrak{X}$ such that,

$$\lim_{m}(\overline{R})\left(f;\check{\mathcal{V}}_{1_n}\right) = A.$$

Passing to the limit in Eq. 2 as $m \to \infty$, we have,

$$\left\|(\overline{R})\left(f;\check{\mathcal{V}}_{1_n}\right) - A\right\| \leq \frac{1}{n} \quad \forall\, n \in \mathbb{N}.$$

To see that A is the Riemann Δ-integral of f, given $\varepsilon > 0$, let $K \in \mathbb{N}$ satisfy $K > \frac{2}{\varepsilon}$. If $\check{\mathcal{V}}_2$ is any tagged partition with mesh-$(\check{\mathcal{V}}_2) < \delta_K$, then

$$\begin{aligned}\left\|(\overline{R})\left(f;\check{\mathcal{V}}_2\right) - A\right\| &\leq \left\|(\overline{R})\left(f;\check{\mathcal{V}}_2\right) - (\overline{R})\left(f;\check{\mathcal{V}}_{1_K}\right) - A\right\| + \left\|(\overline{R})\left(f;\check{\mathcal{V}}_{1_K}\right) - A\right\| \\ &\leq \frac{1}{K} + \frac{1}{K} < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary then $f \in (\mathfrak{R})_{\Delta}([p,q]_{\mathcal{T}}, \mathfrak{X})$ with integral A. □

Corollary 3.7. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be Riemann Δ-integrable on $[p,q]_{\mathcal{T}}$ and if for each $\varepsilon > 0$ there exists partition $\mathcal{V}_{\varepsilon} \in \mathcal{P}([p,q]_{\mathcal{T}})$ such that for all tagged partition $\check{\mathcal{V}}_1$ and $\check{\mathcal{V}}_2$ of $[p,q]_{\mathcal{T}}$ that refines $\check{\mathcal{V}}_{\varepsilon}$, we have*

$$\left\|(\overline{R})\left(f;\check{\mathcal{V}}_1\right) - (\overline{R})\left(f;\check{\mathcal{V}}_2\right)\right\| < \varepsilon.$$

We will now establish a few basic results which will hold true, given a Banach-valued function is Riemann Δ-integrable.

Theorem 3.8. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be Riemann Δ-integrable on $[p,q]_{\mathcal{T}}$ and if for each $\varepsilon > 0$ there exist functions β_{ε} and γ_{ε} in $(\mathfrak{R})_{\Delta}\left([p,q]_{\mathcal{T}}, \mathfrak{X}\right)$ with*

$$\beta_{\epsilon}(t) \leq f(t) \leq \gamma_{\varepsilon}(t) \quad \forall \;\; t \in [p,q]_{\mathcal{T}}, \tag{3}$$

and such that

$$(\overline{R})\int_p^q (\gamma_{\varepsilon} - \beta_{\varepsilon})(t)\Delta t < \varepsilon. \tag{4}$$

Proof. ($\Rightarrow$) Take $\beta_{\varepsilon} = \gamma_{\varepsilon} = f$ for all $\varepsilon > 0$.
($\Leftarrow$) Let $\varepsilon > 0$. Since β_{ε} and γ_{ε} belong to $(\mathfrak{R})_{\Delta}\left([p,q]_{\mathcal{T}}, \mathfrak{X}\right)$, there exists $\delta_{\varepsilon} > 0$ such that if $\check{\mathcal{V}}$ is any tagged partition with mesh δ_{ε} then,

$$\left\|(\overline{R})(\beta_{\varepsilon};\check{\mathcal{V}}) - (\overline{R})\int_p^q \beta_{\varepsilon}(t)\Delta t\right\| < \varepsilon \quad \text{and} \quad \left\|(\overline{R})(\gamma_{\varepsilon};\check{\mathcal{V}}) - (\overline{R})\int_p^q \gamma_{\varepsilon}(t)\Delta t\right\| < \varepsilon.$$

It follows from these inequalities that,

$$(\overline{R})\int_p^q \beta_{\varepsilon}(t)\Delta t - \varepsilon < (\overline{R})(\beta_{\varepsilon};\check{\mathcal{V}}) \quad \text{and} \quad (\overline{R})(\gamma_{\varepsilon};\check{\mathcal{V}}) < (\overline{R})\int_p^q \gamma_{\varepsilon}(t)\Delta t + \varepsilon.$$

In view of inequality Eq. 3, we have $(\overline{R})(\beta_\varepsilon;\check{\mathcal{V}}) \leq (\overline{R})(f;\check{\mathcal{V}}) \leq (\overline{R})(\gamma_\varepsilon;\check{\mathcal{V}})$, whence

$$(\overline{R})\int_p^q \beta_\varepsilon(t)\Delta t - \varepsilon < (\overline{R})(f;\check{\mathcal{V}}) < (\overline{R})\int_p^q \gamma_\varepsilon(t)\Delta t + \varepsilon.$$

If $\check{\mathcal{H}}$ is another tagged partition with mesh δ_ε, then we also have

$$(\overline{R})\int_p^q \beta_\varepsilon(t)\Delta t - \varepsilon < (\overline{R})(f;\check{\mathcal{H}}) < (\overline{R})\int_p^q \gamma_\varepsilon(t)\Delta t + \varepsilon.$$

Subtracting the above two inequalities and using Eq. 4, we conclude that,

$$\begin{aligned}\left\|(\overline{R})(f;\check{\mathcal{V}}) - (\overline{R})(f;\check{\mathcal{H}})\right\| &< (\overline{R})\int_p^q \gamma_\varepsilon(t)\Delta t - (\overline{R})\int_p^q \beta_\varepsilon(t)\Delta t + 2\varepsilon \\ &= (\overline{R})\int_p^q (\gamma_\varepsilon - \beta_\varepsilon)(t)\Delta t + 2\varepsilon < 3\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the Cauchy criterion implies that $f \in (\Re)_\Delta([p,q]_{\mathcal{T}}, \mathfrak{X})$. □

Theorem 3.9. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$, $f_1 : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ and $f_2 : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be Riemann Δ-integrable functions, and let $r \in \mathbb{R}$. Then,*

1. *$r \cdot f$ is Riemann Δ-integrable on $[p,q]_{\mathcal{T}}$ and*

$$(\overline{R})\int_p^q (r\cdot f)(t)\Delta t = r \cdot (\overline{R})\int_p^q f(t)\Delta t;$$

2. *$f_1 + f_2$ is Riemann Δ-integrable on $[p,q]_{\mathcal{T}}$ and*

$$(\overline{R})\int_p^q (f_1 + f_2)(t)\Delta t = (\overline{R})\int_p^q f_1(t)\Delta t + (\overline{R})\int_p^q f_2(t)\Delta t.$$

Proof. Given $f \in (\Re)_\Delta\big([p,q]_{\mathcal{T}}, \mathfrak{X}\big)$, $r \in \mathbb{R}$ and let $\varepsilon > 0$, we have

$$\left\|(\overline{R})(f;\check{\mathcal{V}}) - (\overline{R})\int_p^q f(t)\Delta t\right\| < \varepsilon.$$

Now,

$$\begin{aligned}(\overline{R})(r\cdot f;\check{\mathcal{V}}) &= \sum_{h=1}^{n}(t_h - t_{h-1})\cdot(r\cdot f)(\vartheta_h) \\ &= r\cdot\sum_{h=1}^{n}(t_h - t_{h-1})\cdot f(\vartheta_h) = r\cdot(\overline{R})(f;\check{\mathcal{V}}).\end{aligned}$$

And

$$\begin{aligned}\left\|(\overline{R})(r\cdot f;\check{\mathcal{V}}) - r\cdot(\overline{R})\int_p^q f(t)\Delta t\right\| &= \left\|r\cdot(\overline{R})(f;\check{\mathcal{V}}) - r\cdot(\overline{R})\int_p^q f(t)\Delta t\right\| \\ &= |r|\cdot\left\|(\overline{R})(f;\check{\mathcal{V}}) - (\overline{R})\int_p^q f(t)\Delta t\right\| < |r|\cdot\varepsilon,\end{aligned}$$

since $\varepsilon > 0$ is arbitrary, hence

$$(\overline{R})\int_p^q (r\cdot f)(t)\Delta t = r\cdot\left[(\overline{R})\int_p^q f(t)\Delta t\right].$$

□

Proof. Given $f_1, f_2 \in (\Re)_\Delta\Big([p,q]_{\mathcal{T}}, \mathfrak{X}\Big)$ and let $\varepsilon > 0$ we have,

$$\left\|(\overline{R})(f_1;\check{\mathcal{V}}) - (\overline{R})\int_p^q f_1(t)\Delta t\right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\|(\overline{R})(f_2;\check{\mathcal{V}}) - (\overline{R})\int_p^q f_2(t)\Delta t\right\| < \frac{\varepsilon}{2}.$$

Also

$$\begin{aligned}(\overline{R})(f_1+f_2;\check{\mathcal{V}}) &= \sum_{h=0}^{n}(t_h - t_{h-1})\cdot(f_1+f_2)(\vartheta_h)\\ &= \sum_{h=0}^{n}(t_h - t_{h-1})\cdot\{f_1(\vartheta_h)+f_2(\vartheta_h)\}\\ &= \sum_{h=0}^{n}(t_h - t_{h-1})\cdot f_1(\vartheta_h) + \sum_{h=0}^{n}(t_h - t_{h-1})\cdot f_2(\vartheta_h)\\ &= (\overline{R})(f_1;\check{\mathcal{V}}) + (\overline{R})(f_2;\check{\mathcal{V}}).\end{aligned}$$

Now

$$\begin{aligned}\left\|(\overline{R})(f_1+f_2;\check{\mathcal{V}}) - \left[(\overline{R})\int_p^q f_1(t)\Delta t + (\overline{R})\int_p^q f_2(t)\Delta t\right]\right\| &= \Big\|(\overline{R})(f_1;\check{\mathcal{V}})\\ &\quad + (\overline{R})(f_2;\check{\mathcal{V}}) - (\overline{R})\int_p^q f_1(t)\Delta t - (\overline{R})\int_p^q f_2(t)\Delta t\Big\|\end{aligned}$$

$$\begin{aligned}&\le \left\|(\overline{R})(f_1;\check{\mathcal{V}}) - (\overline{R})\int_p^q f_1(t)\Delta t\right\| + \left\|(\overline{R})(f_2;\check{\mathcal{V}}) - (\overline{R})\int_p^q f_2(t)\Delta t\right\|\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

since $\varepsilon > 0$ is arbitrary, hence

$$(\overline{R})\int_p^q (f_1+f_2)(t)\Delta t = (\overline{R})\int_p^q f_1(t)\Delta t + (\overline{R})\int_p^q f_2(t)\Delta t.$$

□

Remark 3.10. *Taking* $r = -1$

$$(\overline{R})\int_p^q (-f)(t)\Delta t = -(\overline{R})\int_p^q f(t)\Delta t.$$

Remark 3.11. *In general, if* $f_1, f_2, \ldots, f_n \in (\Re)_\Delta\Big([p,q]_{\mathcal{T}}, \mathfrak{X}\Big)$ *implies* $(f_1 + f_2 + \ldots + f_n) \in (\Re)_\Delta\Big([p,q]_{\mathcal{T}}, \mathfrak{X}\Big)$.

Result related to boundedness of a function on $\mathfrak{X}$ is presented below.

Theorem 3.12. *Let* $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ *be Riemann* Δ*-integrable, then* f *is bounded on* $[p,q]_{\mathcal{T}}$.

Proof. By contradiction, suppose there exists a Riemann Δ-integrable function f on $[p,q]_{\mathcal{T}}$ that is unbounded. Let $\bar{I}$ denote the value $(\overline{R})\int_p^q f(t)\Delta t$.
Taking $\varepsilon := 1$, there exists $\delta > 0$ such that,

$$\left\|(\overline{R})(f;\check{\mathcal{V}}) - \bar{I}\right\| < 1,$$

for all tagged partitions $\check{\mathcal{V}}$ of $[p,q]_{\mathcal{T}}$ having mesh strictly less than δ. The reverse triangle inequality then yields,

$$\left\|(\overline{R})(f;\check{\mathcal{V}})\right\| < 1 + \left\|\overline{I}\right\| \tag{5}$$

for all such $\check{\mathcal{V}}$. We now construct a special tagged partition $\check{\mathcal{H}}$ which will contradict the above.
Let $\mathcal{H} = \left\{[t_{h-1}, t_h)_{\mathcal{T}}\right\}_{h=1}^{n}$ be any partition of $[p,q]_{\mathcal{T}}$ whose mesh is strictly less than δ. Since f is unbounded on $[p,q]_{\mathcal{T}}$, it is unbounded on atleast one interval of $\mathcal{H}$. Hence, there is an index $h = 1, 2, \ldots, n$ and $\vartheta_h \in [t_{h-1}, t_h)_{\mathcal{T}}$ such that,

$$\left\|(t_h - t_{h-1}) \cdot f(\vartheta_h)\right\| > 1 + \left\|\overline{I}\right\| + \left\|\sum_{j \neq h}(t_j - t_{j-1}) \cdot f(\vartheta_h)\right\|.$$

Hence, we have chose $\vartheta_j := t_j$ for all $j \neq h$. But then, if we give $\mathcal{H}$ these tags,

$$\begin{aligned}
\left\|(\overline{R})(f;\check{\mathcal{H}})\right\| &= \left\|(t_h - t_{h-1}) \cdot f(\vartheta_h) + \sum_{j \neq h}(t_j - t_{j-1}) \cdot f(\vartheta_j)\right\| \\
&\geq \left\|(t_h - t_{h-1}) \cdot f(\vartheta_h)\right\| - \left\|\sum_{j \neq h}(t_j - t_{j-1}) \cdot f(\vartheta_h)\right\| \\
&> 1 + \left\|\overline{I}\right\|,
\end{aligned}$$

which contradicts Eq. 5, hence if f is Riemann Δ-integrable then f is bounded on $[p,q]_{\mathcal{T}}$. □

Theorem 3.13. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be Riemann Δ-integrable, then the following holds,*

$$\left\|(\overline{R})\int_p^q f(t)\Delta t\right\| \leq (q-p) \cdot \sup_{t \in \mathfrak{X}} \left\|f(t)\right\|. \tag{6}$$

Proof. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that,

$$\left\|(\overline{R})(f;\check{\mathcal{V}}) - (\overline{R})\int_p^q f(t)\Delta t\right\| < \varepsilon,$$

for all tagged partition $\check{\mathcal{V}}$ of $[p,q]_{\mathcal{T}}$ whose mesh is strictly less than δ. Now, fix any such partition $\check{\mathcal{V}}$ and write,

$$\begin{aligned}
\left\|(\overline{R})\int_p^q f(t)\Delta t\right\| &= \left\|(\overline{R})\int_p^q f(t)\Delta t - (\overline{R})(f;\check{\mathcal{V}}) + (\overline{R})(f;\check{\mathcal{V}})\right\| \\
&\leq \left\|(\overline{R})\int_p^q f(t)\Delta t - (\overline{R})(f;\check{\mathcal{V}})\right\| + \left\|(\overline{R})(f;\check{\mathcal{V}})\right\| \\
&< \varepsilon + \left\|(\overline{R})(f;\check{\mathcal{V}})\right\| \\
&\leq \varepsilon + \sum_{h=1}^{n}\left|(t_h - t_{h-1})\right| \cdot \left\|f(\vartheta_h)\right\| \\
&\leq \varepsilon + \left(\sum_{h=1}^{n}(t_h - t_{h-1})\right) \cdot \sup_{t \in \mathcal{X}} \left\|f(t)\right\| \\
&= \varepsilon + (q-p) \cdot \sup_{t \in \mathfrak{X}} \left\|f(t)\right\|,
\end{aligned}$$

since $\varepsilon > 0$ is arbitrary, hence Eq. 6 holds. □

Theorem 3.14. *Let $f_1 : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ and $f_2 : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be Riemann Δ-integrable functions, if $f_1 \leq f_2$ for all $t \in [p,q]_{\mathcal{T}}$, then*

$$(\overline{R})\int_p^q f_1(t)\Delta t \leq (\overline{R})\int_p^q f_2(t)\Delta t.$$

Proof. Given $f_1, f_2 \in (\Re)_\Delta\Big([p,q]_{\mathcal{T}}, \mathfrak{X}\Big)$ and let $\varepsilon > 0$ we have,

$$\left\|(\overline{R})(f_1;\check{\mathcal{V}}) - (\overline{R})\int_p^q f_1(t)\Delta t\right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\|(\overline{R})(f_2;\check{\mathcal{V}}) - (\overline{R})\int_p^q f_2(t)\Delta t\right\| < \frac{\varepsilon}{2}.$$

Also, given $f_1 \leq f_2$ for all $t \in [p,q]_{\mathcal{T}}$ thus $(\overline{R})(f_1;\check{\mathcal{V}}) \leq (\overline{R})(f_2;\check{\mathcal{V}})$.

Applying triangle inequality, we get,

$$(\overline{R})\int_p^q f_1(t)\Delta t - \frac{\varepsilon}{2} < (\overline{R})(f_1;\check{\mathcal{V}}) \quad \text{and} \quad (\overline{R})(f_2;\check{\mathcal{V}}) < (\overline{R})\int_p^q f_2(t)\Delta t + \frac{\varepsilon}{2}.$$

Hence,

$$(\overline{R})\int_p^q f_1(t)\Delta t \leq (\overline{R})\int_p^q f_2(t)\Delta t + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$(\overline{R})\int_p^q f_1\Delta t \leq (\overline{R})\int_p^q f_2\Delta t.$$

□

Theorem 3.15. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be Riemann Δ-integrable, and let $\mathcal{V}_1$ and $\mathcal{V}_2$ be partition on $[p,q]_{\mathcal{T}}$ such that $\mathcal{V}_1 \subseteq \mathcal{V}_2$. Then,*

$$(\overline{R})(f;\check{\mathcal{V}}_1) \leq (\overline{R})(f;\check{\mathcal{V}}_2).$$

Proof. Given $\mathcal{V}_1$ and $\mathcal{V}_2$ be partition on $[p,q]_{\mathcal{T}}$ such that $\mathcal{V}_1 \subseteq \mathcal{V}_2$.

An induction argument, let us assume that $\mathcal{V}_2$ has only one more point, say r, than $\mathcal{V}_1$. If

$$\mathcal{V}_1 = \Big\{p = t_0 < t_1 < \ldots < t_{h-1} < t_h < \ldots < t_n = q\Big\},$$

then,

$$\mathcal{V}_2 = \Big\{p = t_0 < t_1 < \ldots < t_{h-1} < r < t_h < \ldots < t_n = q\Big\},$$

for some $h = \Big\{1, 2, \ldots, n\Big\}$.

Given $\mathcal{V}_1$ and $\mathcal{V}_2$ let the respective tagged partition be $\check{\mathcal{V}}_1$ and $\check{\mathcal{V}}_2$. We find the difference between $(\overline{R})(f;\check{\mathcal{V}}_1)$ and $(\overline{R})(f;\check{\mathcal{V}}_2)$.

$$(\overline{R})(f;\check{\mathcal{V}}_2) - (\overline{R})(f;\check{\mathcal{V}}_1) = (r - t_{h-1}) \cdot f(\vartheta_{r_1}) + (t_h - r) \cdot f(\vartheta_{r_2}) - (t_h - t_{h-1}) \cdot f(\vartheta_h),$$

where $\vartheta_{r_1} \in [t_{h-1}, r)$, $\vartheta_{r_2} \in [r, t_h)_{\mathcal{T}}$ and $\vartheta_h \in [t_{h-1}, t_h)_{\mathcal{T}}$. The above is non-negative because,

$$\begin{aligned}(t_h - t_{h-1}) \cdot f(\vartheta_h) &= (t_h - r + r - t_{h-1}) \cdot f(\vartheta_h) \\ &\leq (t_h - r) \cdot f(\vartheta_{r_2}) + (r - t_{h-1}) \cdot f(\vartheta_{r_1})\end{aligned}$$

Thus $(\overline{R})(f;\check{\mathcal{V}}_1) \leq (\overline{R})(f;\check{\mathcal{V}}_2)$. □

Results related to continuity of a function on $\mathfrak{X}$ are presented below.

Definition 3.16. *[11] Continuous (on $\mathfrak{X}$). A function $f : \mathcal{T} \to \mathfrak{X}$ is continuous at $\tilde{t} \in \mathcal{T}$ if for each $\varepsilon > 0$, there exists $\delta > 0$ (depending on ε and $\tilde{t}$) such that $\|f(t) - f(\tilde{t})\| < \varepsilon$ whenever $\tilde{t} \in (t-\delta, t+\delta) \cap \mathcal{T}$, and write $\lim_{t\to\tilde{t}} f(t) = f(\tilde{t})$.*

Definition 3.17. *Uniform Continuous Function (on $\mathfrak{X}$). A function $f : \mathcal{T} \to \mathfrak{X}$ is uniformly continuous on $\tilde{t} \in [p,q]_{\mathcal{T}}$, if for each $\varepsilon > 0$ there exists $\delta > 0$ (depending on ε) such that for all $\tilde{t}, t \in [p,q]_{\mathcal{T}}$ we get $\left\|f(\tilde{t}) - f(t)\right\| < \varepsilon$ whenever $\tilde{t} \in (t-\delta, t+\delta) \cap \mathcal{T}$.*

Theorem 3.18. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be continuous. For each $\varepsilon > 0$ there exists $\delta > 0$ such that, for any tagged partition $\check{\mathcal{V}}$ of $[p,q]_{\mathcal{T}}$ whose mesh is strictly less than δ, and any refinement $\check{\mathcal{H}}$ of $\check{\mathcal{V}}$, the following holds,*

$$\left\|(\overline{R})(f;\check{\mathcal{V}}) - (\overline{R})(f;\check{\mathcal{H}})\right\| < \varepsilon.$$

Proof. Since f is continuous and $[p,q]_{\mathcal{T}}$ is compact, the function f is uniformly continuous. Therefore given $\varepsilon > 0$, we may find $\delta > 0$ such that

$$\left\|f(r) - f(u)\right\| < \frac{\varepsilon}{q-p},$$

for all $r, u \in [p,q]_{\mathcal{T}}$ satisfying $|r-u| < \delta$. Let $\check{\mathcal{V}}$ be a tagged partition of $[p,q]_{\mathcal{T}}$ with mesh$(\mathcal{V}) < \delta$.
Suppose that $\check{\mathcal{H}}$ is any refinement of $\check{\mathcal{V}}$ with its own set of tags. Any subinterval $[t_{h-1}, t_h)_{\mathcal{T}}$ of $\check{\mathcal{V}}$ may then be decomposed into a finite union of subintervals from $\check{\mathcal{H}}$,

$$[t_{h-1}, t_h)_{\mathcal{T}} = \bigcup_{j=1}^{i_h} \left[\tilde{t}_{h_j-1}, \tilde{t}_{h_j}\right)_{\mathcal{T}}.$$

Therefore,

$$\begin{aligned}
\left\|(\overline{R})(f;\check{\mathcal{V}}) - (\overline{R})(f;\check{H})\right\| &= \left\|\sum_{h=1}^{n}(t_h - t_{h-1}) \cdot f(\vartheta_h) - \sum_{h=1}^{n}\sum_{j=1}^{i_h}(\tilde{t}_{h_j} - \tilde{t}_{h_j-1}) \cdot f(\tilde{\vartheta}_{h_j})\right\| \\
&= \left\|\sum_{h=1}^{n}\sum_{j=1}^{i_h}(\tilde{t}_{h_j} - \tilde{t}_{h_j-1}) \cdot [f(\vartheta_h) - f(\tilde{\vartheta}_{h_j})]\right\| \\
&\le \sum_{h=1}^{n}\sum_{j=1}^{i_h}(\tilde{t}_{h_j} - \tilde{t}_{h_j-1})\left\|f(\vartheta_h) - f(\tilde{\vartheta}_{h_j})\right\|,
\end{aligned}$$

where ϑ_h are tags from $\check{\mathcal{V}}$ and $\tilde{\vartheta}_h$ are tags from $\check{\mathcal{H}}$. However, since the mesh of $\check{\mathcal{V}}$ is strictly less than δ, the above implies that,

$$\left\|(\overline{R})(f;\check{\mathcal{V}}) - (\overline{R})(f;\check{\mathcal{H}})\right\| < \frac{\varepsilon}{q-p}\sum_{h=1}^{n}\sum_{j=1}^{i_h}(\tilde{t}_{h_j} - \tilde{t}_{h_j-1}) = \varepsilon.$$

□

Theorem 3.19. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be a continuous function. Then f is Riemann Δ-integrable on $[p,q]_{\mathcal{T}}$.*

Proof. This proof relies upon Theorem 3.18. Let $\varepsilon > 0$ be given, and there exists $\delta > 0$. Let $\check{\mathcal{V}}$ and $\check{\mathcal{H}}$ be two tagged partitions of $[p,q]_{\mathcal{T}}$, both having mesh strictly less than δ. Let $\check{\mathcal{F}} = \check{\mathcal{V}} \cup \check{\mathcal{H}}$ denote their common refinement, then following Theorem 3.18 we get,

$$\left\|(\overline{R})(f;\check{\mathcal{V}}) - (\overline{R})(f;\check{\mathcal{F}})\right\| < \varepsilon \quad \text{and} \quad \left\|(\overline{R})(f;\check{\mathcal{H}}) - (\overline{R})(f;\check{\mathcal{F}})\right\| < \varepsilon$$

$$\begin{aligned}\left\|(\overline{R})(f;\check{\mathcal{V}})-(\overline{R})(f;\check{\mathcal{H}})\right\| &= \left\|(\overline{R})(f;\check{\mathcal{V}})-(\overline{R})(f;\check{\mathcal{F}})+(\overline{R})(f;\check{\mathcal{F}})-(\overline{R})(f;\check{\mathcal{H}})\right\| \\ &\leq \left\|(\overline{R})(f;\check{\mathcal{V}})-(\overline{R})(f;\check{\mathcal{F}})\right\|+\left\|(\overline{R})(f;\check{\mathcal{F}})-(\overline{R})(f;\check{\mathcal{H}})\right\|\end{aligned}$$

whence,

$$\begin{aligned}\left\|(\overline{R})(f;\check{\mathcal{V}})-(\overline{R})(f;\check{\mathcal{H}})\right\| &< \varepsilon+\varepsilon \\ &= 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, hence from Corollary 3.7 gives that f is Riemann Δ-integrable. □

Theorem 3.20. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be continuous. Then,*

$$\left\|(\overline{R})\int_p^q f(t)\Delta t\right\| \leq \overline{R}\int_p^q \left\|f(t)\right\|\Delta t.$$

Proof. Given f is continuous thus f and $\|f\|$ are Riemann Δ-integrable on $[p,q]_{\mathcal{T}}$.
Let $\left\{\check{\mathcal{V}}_m\right\}_{m=1}^{\infty}$ be a sequence of partitions of $[p,q]_{\mathcal{T}}$ whose mesh tends to zero as $m \to \infty$. Then since $f \in (\mathfrak{R})_\Delta\left([p,q]_{\mathcal{T}},\mathfrak{X}\right)$ we have,

$$(\overline{R})\int_p^q f(t)\Delta t = \lim_{m\to\infty}(\overline{R})(f;\check{\mathcal{V}}_m).$$

By continuity of the norm, this extends to,

$$\left\|(\overline{R})\int_p^q f(t)\Delta t\right\| = \lim_{m\to\infty}\left\|(\overline{R})(f;\check{\mathcal{V}}_m)\right\|.$$

Fix an index $m \in \mathbb{N}$ and let,

$$p = t_0 < \ldots < t_n = q$$

denote the end points of the sub-interval in $\check{\mathcal{V}}_m$, and let ϑ_h be the tag from the h^{th} interval. Clearly,

$$\left\|(\overline{R})(f;\check{\mathcal{V}}_m)\right\| \leq \sum_{h=1}^{n}(t_h - t_{h-1})\left\|f(\vartheta_h)\right\|,$$

which is itself a Riemann Δ-sum for the function $\left\|f(t)\right\|$ on $[p,q]_{\mathcal{T}}$. Since $\left\|f\right\|$ is Riemann Δ-integrable on $[p,q]_{\mathcal{T}}$, putting all the pieces together gives,

$$\begin{aligned}\left\|(\overline{R})\int_p^q f(t)\Delta t\right\| &= \lim_{m\to\infty}\left\|(\overline{R})(f;\check{\mathcal{V}}_m)\right\| \\ &\leq \lim_{m\to\infty}(\overline{R})\left(\|f\|;\check{\mathcal{V}}_m\right) \\ &= \overline{R}\int_p^q \left\|f(t)\right\|\Delta t.\end{aligned}$$

This completes the proof. □

Theorem 3.21. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ and if $p < r < q$ and f is Riemann Δ-integrable on $[p,r]_{\mathcal{T}}$ and on $[r,q]_{\mathcal{T}}$, then f is Riemann Δ-integrable on $[p,q]_{\mathcal{T}}$ and,*

$$(\overline{R})\int_p^q f(t)\Delta t = (\overline{R})\int_p^r f(t)\Delta t + (\overline{R})\int_r^q f(t)\Delta t. \tag{7}$$

The converse also holds.

Proof. Since f is bounded on both $[p,r]_{\mathcal{T}}$ and $[r,q]_{\mathcal{T}}$, f is bounded on $[p,q]_{\mathcal{T}}$.

Suppose that the restriction f_1 of f on $[p,r]_{\mathcal{T}}$, and the restriction f_2 of f to $[r,q]_{\mathcal{T}}$ are Riemann Δ-integrable to $\bar{I}_1$ and $\bar{I}_2$ respectively. Then, given $\varepsilon > 0$ there exists $\delta' > 0$ such that for tagged partition $\check{\mathcal{V}}_1$ with mesh δ', we have

$$\left\|(\overline{R})(f_1;\check{\mathcal{V}}_1) - \bar{I}_1\right\| < \frac{\varepsilon}{3}.$$

Similarly there exists $\delta'' > 0$ such that for tagged partition $\check{\mathcal{V}}_2$ with mesh δ'' we have,

$$\left\|(\overline{R})(f_2;\check{\mathcal{V}}_2) - \bar{I}_2\right\| < \frac{\varepsilon}{3}.$$

If D is a bound for $\|f\|$, we define $\delta := \min\left\{\delta', \delta'', \frac{\varepsilon}{6D}\right\}$ such that for tagged partition $\check{\mathcal{V}}$ with mesh δ we have

$$\left\|(\overline{R})(f;\check{\mathcal{V}}) - (\bar{I}_1 + \bar{I}_2)\right\| < \varepsilon. \tag{8}$$

1. If r is a partition point of $\check{\mathcal{V}}$, we split $\check{\mathcal{V}}$ into $\check{\mathcal{V}}_1$ of $[p,r]_{\mathcal{T}}$ and $\check{\mathcal{V}}_2$ of $[r,q]_{\mathcal{T}}$. Since $(\overline{R})(f;\check{\mathcal{V}}) = (\overline{R})(f_1;\check{\mathcal{V}}_1) + (\overline{R})(f_2;\check{\mathcal{V}}_2)$, with mesh δ' for $\check{\mathcal{V}}_1$ and mesh δ'' for $\check{\mathcal{V}}_2$, the inequality in Eq. 8 is clear.
2. If r is not a partition point in $\check{\mathcal{V}} = \left\{\left([t_{k-1},t_k),\vartheta_k\right)\right\}_{k=1}^{n}$, $\exists k \le n$ such that $r \in (t_{k-1},t_k)$. We let $\check{\mathcal{V}}_1$ be the tagged partition of $[p,r]_{\mathcal{T}}$ defined by,

$$\check{\mathcal{V}}_1 := \left\{\left([t_0,t_1)_{\mathcal{T}},\vartheta_1\right),\dots,\left([t_{k-1},t_k)_{\mathcal{T}},\vartheta_k\right),\left([t_{k-1},r)_{\mathcal{T}},r\right)\right\},$$

$$\check{\mathcal{V}}_2 := \left\{\left([r,t_k)_{\mathcal{T}},r\right),\left([t_k,t_{k-1})_{\mathcal{T}},\vartheta_{k+1}\right),\dots,\left([t_{n-1},t_n)_{\mathcal{T}},\vartheta_n\right)\right\}.$$

A straightforward calculation shows that,

$$\begin{aligned}(\overline{R})(f;\check{\mathcal{V}}) - (\overline{R})(f;\check{\mathcal{V}}_1) - (\overline{R})(f;\check{\mathcal{V}}_2) &= (t_k - t_{k-1})\cdot f(\vartheta_k) - (t_k - t_{k-1})\cdot f(r)\\ &= (t_k - t_{k-1})\cdot\left(f(\vartheta_k) - f(r)\right),\end{aligned}$$

whence it follows that,

$$\left\|(\overline{R})(f;\check{\mathcal{V}}) - (\overline{R})(f;\check{\mathcal{V}}_1) - (\overline{R})(f;\check{\mathcal{V}}_2)\right\| \le 2D(t_k - t_{k-1}) < \frac{\varepsilon}{3}.$$

But since $\left\|\check{\mathcal{V}}_1\right\| < \delta \le \delta'$ and $\left\|\check{\mathcal{V}}_2\right\| < \delta \le \delta''$, it follows that,

$$\left\|(\overline{R})(f;\check{\mathcal{V}}_1) - \bar{I}_1\right\| < \frac{\varepsilon}{3} \quad \text{and} \quad \left\|(\overline{R})(f;\check{\mathcal{V}}_2) - \bar{I}_2\right\| < \frac{\varepsilon}{3},$$

from which we obtain Eq. 8. Since $\varepsilon > 0$ is arbitrary, we infer that $f \in (\Re)_\Delta\left([p,q]_{\mathcal{T}},\mathfrak{X}\right)$ and that Eq, 7 holds.

Conversely, we suppose that $f \in (\Re)_\Delta\left([p,q]_{\mathcal{T}},\mathfrak{X}\right)$ and given $\varepsilon > 0$ we let $\delta > 0$ satisfy the Cauchy criterion (Theorem 3.6). Let f_1 be the restriction of f to $[p,r]_{\mathcal{T}}$ with tagged partition $\check{\mathcal{V}}_1, \check{\mathcal{H}}_1$ and mesh δ for both partitions. By adding additional partition points and tags from $[r,q]_{\mathcal{T}}$, we can extend $\check{\mathcal{V}}_1$ and $\check{\mathcal{H}}_1$ to tagged partition $\check{\mathcal{V}}$ and $\check{\mathcal{H}}$ of $[p,q]_{\mathcal{T}}$ and mesh δ for both partitions.

If we use the same additional points and tags in $[r,q]_{\mathcal{T}}$ for both $\check{\mathcal{V}}$ and $\check{\mathcal{H}}$, then

$$(\overline{R})(f_1;\check{\mathcal{V}}_1) - (\overline{R})(f_1;\check{\mathcal{H}}_1) = (\overline{R})(f;\check{\mathcal{V}}) - (\overline{R})(f;\check{\mathcal{H}}).$$

Since both $\check{\mathcal{V}}$ and $\check{\mathcal{H}}$ have mesh δ, thus

$$\left\|(\overline{R})(f_1;\check{\mathcal{V}}_1) - (\overline{R})(f_1;\check{\mathcal{H}}_1)\right\| < \varepsilon.$$

Therefore the Cauchy condition shows that the restriction f_1 and f to $[p,r]_{\mathcal{T}}$ is in $(\Re)_\Delta\left([p,q]_{\mathcal{T}},\mathfrak{X}\right)$.

In a similar way we see that restriction f_2 of f to $[r,q]_{\mathcal{T}}$ is in $(\Re)_\Delta\left([p,q]_{\mathcal{T}},\mathfrak{X}\right)$. Hence equality Eq. 7 holds. □

Theorem 3.22. *Let* $f \in (\Re)_\Delta\Big([\tilde{t}, \sigma(\tilde{t})]_{\mathcal{T}}, \mathfrak{X}\Big)$*, we have*

$$(\overline{R})\int_{\tilde{t}}^{\sigma(\tilde{t})} f(t)\Delta t = \Big(\sigma(\tilde{t}) - \tilde{t}\Big) \cdot f(\tilde{t}).$$

Proof. There are two possible cases:
1. When $\sigma(\tilde{t}) = \tilde{t}$ which is 0, and
2. When $\sigma(\tilde{t}) > \tilde{t}$ which we proof below-
Consider the tagged partition $\mathcal{V} = \Big\{\tilde{t} = t_0 < t_1 = \sigma(\tilde{t})\Big\}$, with tag point $\tilde{t}$. Hence for $\varepsilon > 0$ there exists $\delta > 0$ and

$$(\overline{R})\int_{\tilde{t}}^{\sigma(\tilde{t})} f(t)\Delta t = \Big(\sigma(\tilde{t}) - \tilde{t}\Big) \cdot f(\tilde{t}).$$

□

We delve into a few examples-

1. If $\mathbb{T} = \mathbb{R}$, Definition 3.1 coincides with the usual definition of Riemann integration for Banach-valued functions on which domain is $\mathbb{R}$.
2. If $\mathcal{T} = r\mathbb{Z}$ with $r > 0$ and $p, q \in r\mathbb{Z}$, and $f \in (\Re)_\Delta\Big([p, q]_{\mathcal{T}}, \mathfrak{X}\Big)$ then

$$(\overline{R})\int_p^q f(t)\Delta t = \sum_{u=\frac{p}{r}}^{\frac{q}{r}-1} f(ru) \cdot r.$$

3. If $\mathbb{T} = \mathbb{Z}$ and $p, q \in \mathbb{Z}$, and $f \in (\Re)_\Delta\Big([p, q]_{\mathcal{T}}, \mathfrak{X}\Big)$ then

$$(\overline{R})\int_p^q f(t)\Delta t = \sum_{u=p}^{q-1} f(u).$$

4. If $f \in (\Re)_\Delta\Big([p, q]_{\mathcal{T}}, \mathfrak{X}\Big)$ such that $f(t) = C$, a constant set, for all $t \in [p, q]_{\mathcal{T}}$, then

$$(\overline{R})\int_p^q f(t)\Delta t = C \cdot (q - p).$$

In the following section we establish some basic results on the Fundamental Theorem of Calculus for Riemann Δ Banach-valued integrable functions on time scale.

3.1. Fundamental Theorem of Calculus for Riemann Δ Banach-valued integrable functions.

Theorem 3.23. *Let* $f : [p, q]_{\mathcal{T}} \to \mathfrak{X}$ *be continuous. Define-*

$$F : [p, q]_{\mathcal{T}} \to \mathfrak{X}, \qquad F(\tilde{t}) := (\overline{R})\int_p^{\tilde{t}} f(t)\Delta t.$$

Then F is continuously differentiable on $[p, q]_{\mathcal{T}}$ *and satisfy* $F^\Delta \equiv f$ *on* $[p, q]_{\mathcal{T}}$.

Proof. Let us fix u, v with $p \le u < v \le q$. Then, using additivity of integral we get,

$$\begin{aligned} F(v) = (\overline{R})\int_p^v f\Delta t &= (\overline{R})\int_p^u f\Delta t + (\overline{R})\int_u^v f\Delta t \\ &= F(u) + (\overline{R})\int_u^v f\Delta t, \end{aligned}$$

$$F(v) - F(u) = (\overline{R})\int_u^v f\Delta t.$$

Now,

$$(\overline{R})\int_u^v f(u)\Delta t = (v-u)\cdot f(u).$$

Therefore,

$$\begin{aligned}\Big\|F(v)-F(u)-(v-u)\cdot f(u)\Big\| &= \Big\|(\overline{R})\int_u^v f(t)\Delta t-(\overline{R})\int_u^v f(u)\Delta t\Big\|\\ &= \Big\|(\overline{R})\int_u^v \big(f(t)-f(u)\big)\Delta t\Big\|\\ &\leq \Big\|v-u\Big\|\sup_{u\leq t\leq v}\Big\|f(t)-f(u)\Big\|.\end{aligned}$$

Given f is continuous on $[p,q]_{\mathcal{T}}$, implies that f is also uniformly continuous on this interval, hence the above implies

$$\lim_{v\to u}\frac{F(v)-F(u)}{v-u} = f(u).$$

The above covers the case $u = p$ thus we may assume that $u > p$ and proceed to proof similarly.

Let v be such that $p \leq v < u$. As above, we get

$$F(v) = (\overline{R})\int_p^v f\Delta t \quad\text{and}\quad F(u) = (\overline{R})\int_p^u f\Delta t,$$

so that,

$$F(u)-F(v) = (\overline{R})\int_v^u f\Delta t \quad\text{or}\quad F(v)-F(u) = -(\overline{R})\int_v^u f\Delta t.$$

Therefore,

$$\begin{aligned}\Big\|(F(v)-F(u))-(v-u)\cdot f(u)\Big\| &= \Big\|-(\overline{R})\int_v^u f(t)\Delta t-(v-u)\cdot f(u)\Big\|\\ &= \Big\|(\overline{R})\int_u^v f(t)\Delta t-(u-v)\cdot f(u)\Big\|\\ &< (u-v)\sup_{v\leq t\leq u}\Big\|f(t)-f(u)\Big\|.\end{aligned}$$

Given f is continuous on $[p,q]_{\mathcal{T}}$, implies that f is also uniformly continuous on this interval, hence the above implies

$$\lim_{v\to u}\frac{F(v)-F(u)}{v-u} = f(u).$$

This proves the claim. □

Theorem 3.24. *Suppose that $F:\mathcal{T}\to\mathfrak{X}$ is continuous on $\mathcal{T}$ and has a Δ-derivative at each point of $\mathcal{T}^k$. Then there exists $\tilde{t}_1,\tilde{t}_2\in\mathcal{T}^k$ such that*

$$F^{\Delta}(\tilde{t}_1)\cdot(q-p)\geq F(q)-F(p)\geq F^{\Delta}(\tilde{t}_2)\cdot(q-p)\ \text{ for all }\ p,q\in\mathcal{T}.$$

Theorem 3.25. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be Riemann Δ-integrable on $[p,q]_{\mathcal{T}}$. If f has a Δ-antiderivative $F : [p,q]_{\mathcal{T}} \to \mathfrak{X}$, then*

$$(\overline{R}) \int_p^q f(t)\Delta t = F(q) - F(p).$$

Proof. Let $\varepsilon > 0$. By Theorem 3.6 there is a $\delta > 0$ such that for tagged partitions $\check{\mathcal{V}}'$, $\check{\mathcal{V}}'' \in \mathfrak{P}$ both having mesh δ implies

$$\left\|(\overline{R})(f;\check{\mathcal{V}}') - (\overline{R})(f;\check{\mathcal{V}}'')\right\| < \varepsilon. \tag{9}$$

Applying Theorem 3.24 to $F : [t_{h-1}, t_h]_{\mathcal{T}} \to \mathfrak{X}$ for each $h = 1, 2, \ldots, n$ we obtain $\tilde{t}_1, \tilde{t}_2 \in [t_{h-1}, t_h)_{\mathcal{T}}$ such that

$$f(\tilde{t}'_h) \cdot (t_h - t_{h-1}) \geq F(t_h) - F(t_{h-1}) \geq f(\tilde{t}''_h) \cdot (t_h - t_{h-1}).$$

Hence summing we have,

$$\sum_{h=1}^{n} f(\tilde{t}'_h) \cdot (t_h - t_{h-1}) \geq F(q) - F(p) \geq \sum_{h=1}^{n} f(\tilde{t}''_h) \cdot (t_h - t_{h-1})$$
$$(\overline{R})\left(f;\check{\mathcal{V}}'\right) \geq F(q) - F(p) \geq (\overline{R})\left(f;\check{\mathcal{V}}''\right)$$

using Eq. 9, and the fact that f is Riemann Δ-integrable, we conclude that

$$\left\|(\overline{R}) \int_p^q f(t)\Delta t - (F(q) - F(p))\right\| < \varepsilon.$$

Since ε is arbitrary, hence

$$(\overline{R}) \int_p^q f(t)\Delta t = F(q) - F(p).$$

□

Theorem 3.26. *Let $f : [p,q]_{\mathcal{T}} \to \mathfrak{X}$ be Riemann Δ-integrable on $[p,q]_{\mathcal{T}}$. For $\tilde{t} \in [p,q]_{\mathcal{T}}$, let $F(\tilde{t}) = (\overline{R}) \int_p^{\tilde{t}} f(t)\Delta t$. Then F is continuous on $[p,q]_{\mathcal{T}}$. Further, let $t_u \in [p,q]_{\mathcal{T}}$ and let f be arbitrary at t_u if t_u is right-scattered, and let f be continuous at t_u if t_u is right-dense. Then F is Δ-differentiable at t_u and $F^{\Delta}(t_u) = f(t_u)$.*

Proof. Let $D > 0$ such that $\left\|F(\tilde{t})\right\| \leq D$ for all $\tilde{t} \in [p,q)_{\mathcal{T}}$. If $\tilde{t}, \tilde{t}_v \in [p,q]_{\mathcal{T}}$ and $\tilde{t} - \tilde{t}_v < \frac{\varepsilon}{D}$ where $\tilde{t} < \tilde{t}_v$, say, then

$$\left\|F(\tilde{t}_v) - F(\tilde{t})\right\| = \left\|(\overline{R}) \int_{\tilde{t}}^{\tilde{t}_v} f(t)\Delta t\right\| \leq \overline{R} \int_{\tilde{t}}^{\tilde{t}_v} \left\|f(t)\right\|\Delta t$$
$$\leq \overline{R} \int_{\tilde{t}}^{\tilde{t}_v} D\Delta t = D \cdot (\tilde{t}_v - \tilde{t}) < \varepsilon.$$

This shows that F is uniformly continuous on $[p,q]_{\mathcal{T}}$. Let $t_u \in [p,q)_{\mathcal{T}}$ be right-scattered. Then, since F is continuous, it is Δ-differentiable at t_u and we have by Theorem 3.21 and Theorem 3.22,

$$F^{\Delta}(t_u) = \frac{F(\sigma(t_u)) - F(t_u)}{\sigma(t_u) - t_u} = \frac{1}{\sigma(t_u) - t_u} \cdot \left\|(\overline{R}) \int_p^{\sigma(t_u)} f(t)\Delta t - (\overline{R}) \int_p^{t_u} f(t)\Delta t\right\|$$
$$= \frac{1}{\sigma(t_u) - t_u} \cdot (\overline{R}) \int_{t_u}^{\sigma(t_u)} f(t)\Delta t = f(t_u),$$

which is the desired result.

Suppose now that t_u is right-dense and f is continuous at t_u. In this case

$$\begin{aligned} F^{\Delta}(t_u) &= \lim_{t\to t_u} \frac{F(t)-F(t_u)}{t-t_u} \\ &= \lim_{t\to t_u} \frac{1}{t-t_u} \cdot \left\| (\overline{R}) \int_a^{\tilde{t}} f(t)\Delta t - (\overline{R}) \int_p^{t_u} f(t)\Delta t \right\| \\ &= \lim_{t\to t_u} \frac{1}{t-t_u} \cdot (\overline{R}) \int_{t_u}^{\tilde{t}} f(t)\Delta t. \end{aligned}$$

Let $\varepsilon > 0$. Since f is continuous at t_u, there exists $\delta > 0$ such that $t \in [p,q)_{\mathcal{T}}$ and $|t - t_u| < \delta$ imply $\left\| f(t) - f(t_u) \right\| < \varepsilon$. Then

$$\left\| \frac{1}{\tilde{t}-t_u} \cdot (\overline{R}) \int_{t_u}^{\tilde{t}} f(t)\Delta t - f(t_u) \right\| = \frac{1}{|\tilde{t}-t_u|} \cdot \left\| (\overline{R}) \int_{t_u}^{\tilde{t}} f(t)\Delta t - (\overline{R}) \int_{t_u}^{\tilde{t}} f(t_u)\Delta t \right\|$$

$$\frac{1}{|\tilde{t}-t_u|} \cdot \overline{R} \int_{t_u}^{\tilde{t}} \left\| f(t) - f(t_u) \right\| \Delta t \leq \frac{\varepsilon}{|\tilde{t}-t_u|} \cdot |\tilde{t}-t_u| = \varepsilon,$$

for all $t \in [p,q]_{\mathcal{T}}$ such that $|\tilde{t} - t_u| < \delta$ and $\tilde{t} = t_u$. Hence the desired result follows. □

Conclusion

This paper explores the theory of Riemann integration for Banach-valued functions on time scale and discuss a few fascinating results.

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