Filomat 39:33 (2025), 11701–11716
https://doi.org/10.2298/FIL2533701D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Normal structure and generalized von Neumann-Jordan type constant

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Abstract. We introduce the generalized von Neumann-Jordan type constant of a Banach space and give some geometric properties concerning this new coefficient. Moreover, we establish several sufficient conditions for normal structure of a Banach space in terms of the generalized von Neumann-Jordan type constant, the generalized James constant, the coefficient of weak orthogonality and the generalized García-Falset coefficient. Our main results in this paper significantly improve and generalize many known results in the literature.

1. Introduction and preliminaries

Many investigations have been devoted to the study of geometric constants of Banach spaces, which enables us to make precise descriptions of various geometric properties of Banach spaces. In 1965, Kirk [24] proved that every reflexive Banach space with normal structure has the fixed point property, that is, for every nonempty closed bounded convex subset C of X and for every nonexpansive mapping $T : C \to C$ there exists a point $x \in C$ such that $x = Tx$. Whether or not a Banach space has normal structure depends on the geometry of its unit ball and its unit sphere. To check that a given Banach space has normal structure is not an easy task. However, one can do this with the help of certain geometric constants. To be more precise, motivated by Kirk's celebrated work, many studies have focused on finding sufficient conditions involving different geometric constants for a Banach space to have either normal structure or uniform normal structure, and therefore the fixed point property. For a great deal of the results related to this topic, we refer the interested readers to [4, 6–13, 15–17, 22, 26–29] and the references therein.

Throughout this paper, let X be a real Banach space with the dual space X^*. As usual, we will denote by $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \leq 1\}$ the unit sphere and the closed unit ball of X, respectively.

Recall that a Banach space X is said to have (weak) normal structure [1] if for every (weakly compact) closed bounded convex subset K of X that contains more than one point, there exists a point $x_0 \in K$ such that

$$\sup\left\{\|x_0 - y\| : y \in K\right\} < \operatorname{diam}(K),$$

2020 *Mathematics Subject Classification*. Primary 46B20; Secondary 47H09, 47H10.

Keywords. uniform normal structure, generalized von Neumann-Jordan type constant, generalized James constant, coefficient of weak orthogonality, generalized García-Falset coefficient.

Received: 01 April 2022; Accepted: 30 September 2025

Communicated by Dragan S. Djordjević

The research of the first-named author (Mina Dinarvand) was in part supported by a grant from IPM (No. 1404460032).

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where $\operatorname{diam}(K) = \sup\{\|x - y\| : x, y \in K\}$ denotes the diameter of K.

For a reflexive Banach space X, normal structure and weak normal structure coincide. It is worth noticing that if X fails to have weak normal structure, then there exist a weakly compact convex subset $C \subset X$ and a sequence $\{x_n\} \subset C$ such that $\operatorname{dist}(x_{n+1}, \operatorname{co}\{x_k\}_{k=1}^n) \to \operatorname{diam}(C) = 1$ [19].

A Banach space X is said to have uniform normal structure if there exists $0 < c < 1$ such that for every closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\Big\{\|x_0 - y\| : y \in K\Big\} < c \operatorname{diam}(K).$$

A Banach space X is called uniformly non-square provided that there exists $\delta > 0$ such that either $\dfrac{\|x + y\|}{2} \le 1 - \delta$ or $\dfrac{\|x - y\|}{2} \le 1 - \delta$ whenever $x, y \in S_X$. In [20] it was proved that uniformly non-square Banach spaces are super-reflexive and therefore, reflexive.

We next recall some basic facts about the ultrapowers of Banach spaces which are the main ingredient of our results. For a widespread discussion on the Banach space ultrapower construction, the reader is directed to [19, 23, 30].

Let $\mathcal{F}$ be a filter on $\mathbb{N}$ and let X be a Banach space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in \mathbb{N} : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on $\mathbb{N}$ is called an ultrafilter if it is maximal with respect to set inclusion. An ultrafilter is called trivial if it is of the form $\{A \subset \mathbb{N} : i_0 \in A\}$ for some fixed $i_0 \in \mathbb{N}$, otherwise, it is called nontrivial.

Let $\ell_\infty(X)$ denote the linear subspace of the product space $\prod_{n\in\mathbb{N}} X$ as

$$\ell_\infty(X) = \Big\{\{x_n\}_{n\in\mathbb{N}} : x_n \in X \text{ for all } n \in \mathbb{N} \text{ and } \sup_{n\in\mathbb{N}} \|x_n\| < \infty\Big\}$$

equipped with the norm

$$\|\{x_n\}_{n\in\mathbb{N}}\| := \sup_{n\in\mathbb{N}} \|x_n\|.$$

Furthermore, let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let $N_{\mathcal{U}}(X)$ denote the normed subspace of $\ell_\infty(X)$ as

$$N_{\mathcal{U}}(X) = \Big\{\{x_n\}_{n\in\mathbb{N}} \subset \ell_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\Big\}.$$

The ultrapower of Banach space X with respect to ultrafilter $\mathcal{U}$ is the quotient space

$$\widetilde{X} = \{X\}_{\mathcal{U}} = \frac{\ell_\infty(X)}{N_{\mathcal{U}}(X)},$$

equipped with the norm

$$\|\{x_n\}_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_n\|, \tag{1}$$

where $\{x_n\}_{\mathcal{U}}$ is the equivalence class in $\{X\}_{\mathcal{U}}$ generated by $\{x_n\}_{n\in\mathbb{N}} \in \ell_\infty(X)$. Note that $N_{\mathcal{U}}(X)$ is actually a closed subspace of $\ell_\infty(X)$ and also the limit (1) always exists and equals to the standard quotient norm.

If $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically. It should be pointed out that if X is super-reflexive, that is $\widetilde{X^*} = (\widetilde{X})^*$, then X has uniform normal structure if and only if $\widetilde{X}$ has normal structure [23].

The modulus of convexity of X [2] is the function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\Big\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| = \varepsilon\Big\}, \qquad (0 \le \varepsilon \le 2).$$

The function δ_X is continuous on $[0, 2)$, nondecreasing on $[0, \varepsilon_0(X)]$ and strictly increasing on $[\varepsilon_0(X), 2]$. Here $\varepsilon_0(X) = \sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\}$ is the characteristic of convexity of X. A Banach space X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2)$, a property which in its turn implies that X is uniformly non-square. It is worthwhile to mention that $\delta_X(1) > 0$ implies X has uniform normal structure [18].

The James and von Neumann-Jordan constants are two most widely studied constants, due to their connections with various geometric structures of Banach spaces. The following two constants of a Banach space X,

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \ : \ x, y \in X, \ \|x\| + \|y\| > 0\right\}$$

and

$$J(X) = \sup\left\{\min\left\{\|x+y\|, \|x-y\|\right\} \ : \ x, y \in B_X\right\}$$

are called the von Neumann-Jordan [3] and James constants [16], respectively. These constants are generalized in the following ways [6, 7]:

$$C_{NJ}(a, X) = \sup\left\{\frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \ : \ x, y, z \in X, \ \|x\| + \|y\| + \|z\| > 0, \ \|y-z\| \le a\|x\|\right\}$$

and

$$J(a, X) = \sup\left\{\min\left\{\|x+y\|, \|x-z\|\right\} \ : \ x, y, z \in B_X, \ \|y-z\| \le a\|x\|\right\},$$

where $0 \le a \le 2$. It is clear that $C_{NJ}(0, X) = C_{NJ}(X)$ and $J(0, X) = J(X)$.

In [17], Gao and Saejung defined the generalized Zbăganu constant for $a \ge 0$ as

$$C_Z(a, X) = \sup\left\{\frac{2\|x+y\|\,\|x-z\|}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \ : \ x, y, z \in X, \ \|x\| + \|y\| + \|z\| > 0, \ \|y-z\| \le a\|x\|\right\},$$

which is a generalization of the Zbăganu constant [34] defined by

$$C_Z(X) = C_Z(0, X) = \sup\left\{\frac{\|x+y\|\,\|x-y\|}{\|x\|^2 + \|y\|^2} \ : \ x, y \in X, \ \|x\| + \|y\| > 0\right\}.$$

Recently, Yin et al. [25] introduced a new constant based on the Zbăganu constant as

$$C_Z^{(p)}(a, X) = \sup\left\{\frac{\|x+y\|^{\frac{p}{2}}\|x-z\|^{\frac{p}{2}}}{2^{p-2}\left(\|x\|^p + \|y\|^p + \|z\|^p\right)} \ : \ x, y, z \in X, \ \|x\| + \|y\| + \|z\| > 0, \ \|y-z\| \le a\|x\|\right\}$$

for all $1 \le p < \infty$ and all $a \ge 0$.

In [4], Cui et al. defined the generalized von Neumann-Jordan constant by

$$C_{NJ}^{(p)}(X) = \sup\left\{\frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}\left(\|x\|^p + \|y\|^p\right)} \ : \ x, y \in X, \ (x, y) \ne (0, 0)\right\},$$

where $1 \le p < \infty$. From the definitions, it is obvious that $C_{NJ}^{(2)}(X) = C_{NJ}(X)$ and $C_Z^{(2)}(X) = C_Z(X)$.

Afterwards, Dinarvand [9] introduced a new constant based on the generalized von Neumann-Jordan constant as

$$C_{NJ}^{(p)}(a, X) = \sup\left\{\frac{\|x+y\|^p + \|x-z\|^p}{2^{p-1}\|x\|^p + 2^{p-2}\left(\|y\|^p + \|z\|^p\right)} \ : \ x, y, z \in X, \ \|x\| + \|y\| + \|z\| > 0, \ \|y-z\| \le a\|x\|\right\}$$

for all $1 \le p < \infty$ and all $a \ge 0$.

In 2008, Takahashi [33] has introduced the von Neumann-Jordan type constant by

$$C_t(X) = \sup\left\{\frac{J_{X,t}^2(\tau)}{1+\tau^2} \,:\, 0 \le \tau \le 1\right\}$$

for $-\infty \le t < \infty$, where the James type constant $J_{X,t}(\tau)$ is defined as

$$J_{X,t}(\tau) = \begin{cases} \sup\left\{\left(\frac{\|x+\tau y\|^t + \|x-\tau y\|^t}{2}\right)^{\frac{1}{t}} \,:\, x, y \in S_X\right\}, & -\infty < t < \infty, t \neq 0, \\ \sup\left\{\sqrt{\|x+\tau y\|\,\|x-\tau y\|} \,:\, x, y \in S_X\right\}, & t = 0, \\ \sup\left\{\min\left\{\|x+\tau y\|, \|x-\tau y\|\right\} \,:\, x, y \in S_X\right\}, & t = -\infty. \end{cases}$$

Here, we remark that $J(X) = J_{X,-\infty}(1)$. By taking $t = -\infty$ in the definition of $C_t(X)$, we get the constant

$$C_{-\infty}(X) = \sup\left\{\frac{\min\left\{\|x+y\|^2, \|x-y\|^2\right\}}{\|x\|^2 + \|y\|^2} \,:\, x, y \in X,\ \|x\| + \|y\| > 0\right\}.$$

In this paper, we introduce the generalized von Neumann-Jordan type constant $C_{-\infty}^{(p)}(a, X)$ of a Banach space X and investigate some properties and results related to this novel geometric constant. Furthermore by applying the ultraproduct techniques, we obtain some new sufficient conditions for normal structure and uniform normal structure in terms of this new constant and the generalized James constant, the coefficient of weak orthogonality and the generalized García-Falset coefficient. Our results improve and generalize many comparable results in the existing literature.

2. The generalized von Neumann-Jordan type constant

We define the generalized von Neumann-Jordan type constant $C_{-\infty}^{(p)}(a, X)$ of a Banach space X to be

$$C_{-\infty}^{(p)}(a, X) = \sup\left\{\frac{\min\left\{\|x+y\|^p, \|x-z\|^p\right\}}{2^{p-2}\|x\|^p + 2^{p-3}\left(\|y\|^p + \|z\|^p\right)} \,:\, x, y, z \in X,\ \|x\| + \|y\| + \|z\| > 0,\ \|y-z\| \le a\|x\|\right\}$$

for all $1 \le p < \infty$ and all $a \ge 0$. It is apparent that

$$C_{-\infty}^{(p)}(0, X) = C_{-\infty}^{(p)}(X) = \sup\left\{\frac{\min\left\{\|x+y\|^p, \|x-y\|^p\right\}}{2^{p-2}\left(\|x\|^p + \|y\|^p\right)} \,:\, x, y \in X,\ \|x\| + \|y\| > 0\right\},$$

where $1 \le p < \infty$. It should be noted that $C_{-\infty}^{(2)}(a, X) = C_{-\infty}(a, X)$ and $C_{-\infty}^{(2)}(0, X) = C_{-\infty}(X)$.

We immediately have the following proposition.

Proposition 2.1. *Let X be a Banach space and $1 \le p < \infty$. Then, for all $a \ge 0$,*

$$\begin{aligned} C_{-\infty}^{(p)}(a, X) &= \sup\left\{\frac{\min\left\{\|x+y\|^p, \|x-z\|^p\right\}}{2^{p-2}\|x\|^p + 2^{p-3}\left(\|y\|^p + \|z\|^p\right)} \,:\, x, y, z \in X,\ \|x\| + \|y\| + \|z\| > 0,\ \|y-z\| \le a\|x\|\right\} \\ &= \sup\left\{\frac{\min\left\{\|x+y\|^p, \|x-z\|^p\right\}}{2^{p-2}\|x\|^p + 2^{p-3}\left(\|y\|^p + \|z\|^p\right)} \,:\, x, y, z \in B_X,\ \|x\| + \|y\| + \|z\| > 0,\ \|y-z\| \le a\|x\|\right\} \\ &= \sup\left\{\frac{\min\left\{\|x+y\|^p, \|x-z\|^p\right\}}{2^{p-2}\|x\|^p + 2^{p-3}\left(\|y\|^p + \|z\|^p\right)} \,:\, x, y, z \in B_X \text{ of which at least one belongs to } S_X,\right. \\ &\qquad \left. \|y-z\| \le a\|x\|\right\}. \end{aligned}$$

Now let us collect some important properties concerning this constant.

Remark 2.2. Let X be a Banach space and $1 \leq p < \infty$. Then,

(1) $2^{2-p} \leq C^{(p)}_{-\infty}(a, X) \leq C^{(p)}_{Z}(a, X) \leq C^{(p)}_{NJ}(a, X) \leq 2$ for $a \geq 0$ and also the inequalities are strict in some Banach spaces;

(2) $C^{(p)}_{-\infty}(a, X)$ is a non-decreasing function for $a \geq 0$;

(3) if $C^{(p)}_{-\infty}(a, X) < 2$ for some $a \geq 0$, then $C^{(p)}_{-\infty}(X) < 2$ and consequently, X is uniformly non-square;

(4) $C^{(p)}_{-\infty}(a, X)$ is a continuous function for $a \geq 0$.

Proposition 2.3. *Let X be a Banach space and $1 \leq p < \infty$. Then,*

(i) $\dfrac{(2+a)^p}{2^{p-2}(2^p + a^p)} \leq C^{(p)}_{-\infty}(a, X) \leq 2$ *for all $a \in [0, 2]$.*

(ii) $C^{(p)}_{-\infty}(a, X) = 2$ *for all $a \geq 2$.*

Proof. (i) Suppose that $a \in [0, 2]$ and $x \in S_X$. Take $y = \left(\frac{a}{2}\right)x = -z$. Thus, $y - z = ax$. Hence, we get

$$\begin{aligned} C^{(p)}_{-\infty}(a, X) &\geq \frac{\min\left\{\|x+y\|^p, \|x-z\|^p\right\}}{2^{p-2}\|x\|^p + 2^{p-3}\left(\|y\|^p + \|z\|^p\right)} = \frac{\min\left\{(1+\frac{a}{2})^p\|x\|^p, (1+\frac{a}{2})^p\|x\|^p\right\}}{2^{p-2}\|x\|^p + 2^{p-2}(\frac{a^p}{2^p})\|x\|^p} \\ &= \frac{(1+\frac{a}{2})^p}{2^{p-2}(1+\frac{a^p}{2^p})} = \frac{(2+a)^p}{2^{p-2}(2^p + a^p)}, \end{aligned}$$

which implies the left inequality. To prove the right inequality, we have

$$\begin{aligned} 2\min\left\{\|x+y\|^p, \|x-z\|^p\right\} &\leq \|x+y\|^p + \|x-z\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p) + 2^{p-1}(\|x\|^p + \|z\|^p) \\ &= 2^p\|x\|^p + 2^{p-1}(\|y\|^p + \|z\|^p), \end{aligned}$$

from which it follows that

$$\min\left\{\|x+y\|^p, \|x-z\|^p\right\} \leq 2^{p-1}\|x\|^p + 2^{p-2}(\|y\|^p + \|z\|^p) = 2\left(2^{p-2}\|x\|^p + 2^{p-3}(\|y\|^p + \|z\|^p)\right).$$

Therefore, $C^{(p)}_{-\infty}(a, X) \leq 2$.

(ii) It is to be noted that the function $a \mapsto \dfrac{(2+a)^p}{2^{p-2}(2^p + a^p)}$ is strictly increasing on $[0, 2]$ and attains its maximum of 2 at $a = 2$. This fact concludes that $C^{(p)}_{-\infty}(a, X) = 2$ for all $a \geq 2$. □

Example 2.4. (Day-James $\ell_\infty - \ell_1$ space) Let X be $\mathbb{R}^2$ endowed with the $\ell_\infty - \ell_1$ norm defined by

$$\|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_\infty, & \text{if } x_1x_2 \geq 0, \\ \|(x_1, x_2)\|_1, & \text{if } x_1x_2 \leq 0. \end{cases}$$

Put $x = (1, 1)$, $y = (0, 1)$ and $z = (-1, 0)$. Hence, we have $y - z = (1, 1) = x$ and $\|x + y\| = \|(1, 2)\|_\infty = 2$, $\|x - z\| = \|(2, 1)\|_\infty = 2$ and $\|z\| = 1$. Thus, we obtain

$$2 = \frac{2^p}{2^{p-1}} = \frac{\min\left\{\|x+y\|^p, \|x-z\|^p\right\}}{2^{p-2}\|x\|^p + 2^{p-3}(\|y\|^p + \|z\|^p)} \leq C^{(p)}_{-\infty}(a, X) \leq 2.$$

Therefore, $C^{(p)}_{-\infty}(a, X) = 2$ for all $a \geq 1$.

Example 2.5. (Day-James $\ell_2 - \ell_1$ space) Let X be $\mathbb{R}^2$ endowed with the $\ell_2 - \ell_1$ norm defined by

$$\|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_2, & \text{if } x_1x_2 \geq 0, \\ \|(x_1, x_2)\|_1, & \text{if } x_1x_2 < 0. \end{cases}$$

Put $x = (-\frac{1}{2}, \frac{1}{2})$, $y = (0, 1)$ and $z = (1, 0)$. Thus, we get $y - z = (-1, 1) = 2(-\frac{1}{2}, \frac{1}{2}) = 2x$ and $\|x + y\| = \|(-\frac{1}{2}, \frac{3}{2})\|_1 = 2$, $\|x - z\| = \|(-\frac{3}{2}, \frac{1}{2})\|_1 = 2$ and $\|z\| = \|(1, 0)\|_2 = 1$. Hence, we have

$$2 = \frac{2^p}{2^{p-1}} = \frac{\min\left\{\|x + y\|^p, \|x - z\|^p\right\}}{2^{p-2}\|x\|^p + 2^{p-3}(\|y\|^p + \|z\|^p)} \leq C^{(p)}_{-\infty}(a, X) \leq 2.$$

Therefore, $C^{(p)}_{-\infty}(a, X) = 2$ for all $a \geq 2$.

Remark 2.6. Examples 2.4 and 2.5 show that sometimes it is easy to compute $C^{(p)}_{-\infty}(a, X)$ for all $a \geq 1$, but not at $a = 0$.

Now, we present a relation between the constant $C^{(p)}_{-\infty}(\cdot, X)$ and the modulus of convexity $\delta_X(\cdot)$.

Theorem 2.7. *Let X be a Banach space, $\varepsilon \in [0, 2]$ and $\beta \geq 0$. If $C^{(p)}_{-\infty}(\beta, X) < \dfrac{|\varepsilon - \beta|^p}{2^{p-2}\left(3 + (1 + \beta)^p\right)}$ for $1 \leq p < \infty$, then $\delta_X(\varepsilon) > 0$.*

Proof. Suppose on the contrary that $\delta_X(\varepsilon) = 0$. Thus, there exist $x_n, y_n \in S_X$ such that $\|x_n - y_n\| = \varepsilon$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$. Put $z_n = y_n - \beta x_n$. So, for all $n \in \mathbb{N}$, we get $y_n - z_n = \beta x_n$, $\|z_n\| = \|y_n - \beta x_n\| \leq 1 + \beta$ and $\|x_n - z_n\| \geq \left|\|x_n - y_n\| - \|\beta x_n\|\right| = |\varepsilon - \beta|$. Hence, we obtain

$$\begin{aligned} \frac{|\varepsilon - \beta|^p}{2^{p-2}\left(3 + (1 + \beta)^p\right)} &= \frac{\min\left\{2^p, |\varepsilon - \beta|^p\right\}}{2^{p-3}\left(3 + (1 + \beta)^p\right)} \leq \liminf_{n\to\infty} \frac{\min\left\{\|x_n + y_n\|^p, \|x_n - z_n\|^p\right\}}{2^{p-2}\|x_n\|^p + 2^{p-3}\left(\|y_n\|^p + \|z_n\|^p\right)} \\ &\leq C^{(p)}_{-\infty}(\beta, X) < \frac{|\varepsilon - \beta|^p}{2^{p-2}\left(3 + (1 + \beta)^p\right)}, \end{aligned}$$

which is a contradiction and the proof is complete. □

Corollary 2.8. *Let X be a Banach space and $1 \leq p < \infty$. If $C^{(p)}_{-\infty}(0, X) < \left(\frac{\varepsilon}{2}\right)^p$ for all $\varepsilon \in [0, 2]$, then $\delta_X(\varepsilon) > 0$.*

Remark 2.9. It is worthwhile to mention that for $1 \leq p < \infty$,

(1) Corollary 2.8 shows that if $C^{(p)}_{-\infty}(X) < 2^{-p}$, then $\delta_X(1) > 0$, which in turn implies that X has uniform normal structure.

(2) $C^{(p)}_{-\infty}(0, X) < 2$ if and only if $C^{(p)}_{-\infty}(0, X) < \left(\frac{\varepsilon}{2}\right)^p$ for some $\varepsilon \in (0, 2)$, which is a simple proof of this fact that "$C^{(p)}_{-\infty}(0, X) < 2$ if and only if X is uniformly non-square".

Now let us present the following proposition indicating the relation between constants $J(a, X)$ and $C^{(p)}_{-\infty}(a, X)$.

Proposition 2.10. *Let X be a Banach space and $1 < p < \infty$. Then, for all $a \geq 0$,*

$$J(a, X) \leq 2^{\frac{p-1}{p}} \sqrt[p]{C^{(p)}_{-\infty}(a, X)}.$$

We give a proposition that will constitute a main tool in proving our results.

Proposition 2.11. *Let X be a Banach space and $1 \le p < \infty$. Then $C^{(p)}_{-\infty}(a, X) = C^{(p)}_{-\infty}(a, \widetilde{X})$.*

Proof. Take $1 \le p < \infty$. Obviously, $C^{(p)}_{-\infty}(a, X) \le C^{(p)}_{-\infty}(a, \widetilde{X})$. We only have to prove that $C^{(p)}_{-\infty}(a, \widetilde{X}) \le C^{(p)}_{-\infty}(a, X)$. To this end, suppose that $\delta > 0$, $\alpha \in [0, a]$ and $\widetilde{x}, \widetilde{y}, \widetilde{z} \in \widetilde{X}$ not all of which are zero and for which the relation $\|\widetilde{y} - \widetilde{z}\| = \alpha\|\widetilde{x}\|$ holds. If $\widetilde{x} = 0$, then

$$\frac{\min\left\{\|\widetilde{x} + \widetilde{y}\|^p, \|\widetilde{x} - \widetilde{z}\|^p\right\}}{2^{p-2}\|\widetilde{x}\|^p + 2^{p-3}\left(\|\widetilde{y}\|^p + \|\widetilde{z}\|^p\right)} = \frac{1}{2^{p-2}} \le C^{(p)}_{-\infty}(a, X).$$

If $\widetilde{x} \neq 0$, then choose $\varepsilon > 0$ such that $\varepsilon < \delta\|\widetilde{x}\|$. Since $\|\widetilde{x}\| = \lim_{\mathcal{U}} \|x_n\|$ and

$$c := \frac{\min\left\{\|\widetilde{x} + \widetilde{y}\|^p, \|\widetilde{x} - \widetilde{z}\|^p\right\}}{2^{p-2}\|\widetilde{x}\|^p + 2^{p-3}\left(\|\widetilde{y}\|^p + \|\widetilde{z}\|^p\right)} = \lim_{\mathcal{U}} \frac{\min\left\{\|x_n + y_n\|^p + \|x_n - z_n\|^p\right\}}{2^{p-2}\|x_n\|^p + 2^{p-3}\left(\|y_n\|^p + \|z_n\|^p\right)} := \lim_{\mathcal{U}} c_n,$$

it follows that the set $E := \left\{n \in \mathbb{N} \ : \ |c_n - c| < \delta, \|y_n - z_n\| \le \alpha\|x_n\| + \varepsilon < (\alpha + \delta)\|x_n\|\right\}$ belongs to $\mathcal{U}$. In particular, noticing that $x_n \neq 0$ for every $n \in E$, there exists n such that

$$c < \frac{\min\left\{\|x_n + y_n\|^p + \|x_n - z_n\|^p\right\}}{2^{p-2}\|x_n\|^p + 2^{p-3}\left(\|y_n\|^p + \|z_n\|^p\right)} + \delta \le C^{(p)}_{-\infty}(a + \delta, X) + \delta.$$

Hence, the conclusion follows from the arbitrariness of δ and the continuity of $C^{(p)}_{-\infty}(\cdot, X)$. This completes the proof. □

Saejung proved the following lemma in [28].

Lemma 2.12. ([28]) *If X is a super-reflexive Banach space and fails to have normal structure, then for $r \in (0, 1]$ there are $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in S_{\widetilde{X}}$ and $\widetilde{f_1}, \widetilde{f_2}, \widetilde{f_3} \in S_{(\widetilde{X})^*}$ such that*

(a) $\|\widetilde{x}_i - \widetilde{x}_j\| = 1$ *and* $\widetilde{f_i}(\widetilde{x}_j) = 0$ *for all* $i \neq j$;

(b) $\widetilde{f_i}(\widetilde{x}_i) = 1$ *for* $i = 1, 2, 3$;

(c) $\|\widetilde{x}_3 - (\widetilde{x}_2 + r\widetilde{x}_1)\| \ge \|\widetilde{x}_2 + r\widetilde{x}_1\|$.

Theorem 2.13. *A Banach space X has uniform normal structure if there exists $0 \le a \le 1$ such that*

$$C^{(p)}_{-\infty}(a, X) < \frac{1}{2^{p-2}}\left((1 - a)\left(1 + \left(\frac{1 + a}{J(a, X) + 2a}\right)^p\right) + 2a\right)^{p-1},$$

where $1 < p < \infty$.

Proof. Take $1 < p < \infty$. Since $(1 + \frac{a}{2})^p \le (J(a, X))^p \le 2^{p-1}C^{(p)}_{-\infty}(a, X)$, it follows by the above inequality that X is uniformly non-square, and consequently, X is super-reflexive. We will prove that X has normal structure. Arguing by contradiction, we suppose that X fails to have normal structure and the inequality above holds for some $0 \le a \le 1$. Then there are elements $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in S_{\widetilde{X}}$ and $\widetilde{f_1}, \widetilde{f_2}, \widetilde{f_3} \in S_{(\widetilde{X})^*}$ satisfying all the conditions in Lemma 2.12. We now use these elements to estimate the constants $C^{(p)}_{-\infty}(a, \widetilde{X})$ and $J(a, \widetilde{X})$ of the Banach space ultrapower $\widetilde{X}$ which are the same as $C^{(p)}_{-\infty}(a, X)$ and $J(a, X)$ of the space X, respectively. Consider

$$\beta := \|\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1\|, \qquad \gamma := \|\widetilde{x}_3 - \widetilde{x}_2 - \widetilde{x}_1\|.$$

On the one hand, if we put

$$\widetilde{x} = \widetilde{x}_3 - \widetilde{x}_1, \qquad \widetilde{y} = \frac{1-a}{\beta^p}(\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1) + a\widetilde{x}_3, \qquad \widetilde{z} = \frac{1-a}{\beta^p}(\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1) + a\widetilde{x}_1,$$

then

$$\|\widetilde{x}\| = 1, \qquad \|\widetilde{y}\| \le \frac{1-a}{\beta^{p-1}} + a, \qquad \|\widetilde{z}\| \le \frac{1-a}{\beta^{p-1}} + a, \qquad \|\widetilde{y} - \widetilde{z}\| = a\|\widetilde{x}\|.$$

Thus, we have

$$\begin{aligned}
C^{(p)}_{-\infty}(a, X) &= C^{(p)}_{-\infty}(a, \widetilde{X}) \\
&\ge \frac{\min\left\{\|\widetilde{x} + \widetilde{y}\|^p, \|\widetilde{x} - \widetilde{z}\|^p\right\}}{2^{p-2}\|\widetilde{x}\|^p + 2^{p-3}\left(\|\widetilde{y}\|^p + \|\widetilde{z}\|^p\right)} \\
&= \frac{1}{2^{p-2}\left(1 + \left(\frac{1-a}{\beta^{p-1}} + a\right)^p\right)}\Bigg[\min\Bigg\{\left\|\widetilde{x}_3 - \widetilde{x}_1 + a\widetilde{x}_3 + \frac{1-a}{\beta^p}(\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1)\right\|^p, \\
&\quad \left\|\widetilde{x}_3 - \widetilde{x}_1 - a\widetilde{x}_1 - \frac{1-a}{\beta^p}(\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1)\right\|^p\Bigg\}\Bigg] \\
&\ge \frac{1}{2^{p-2}\left(1 + \left(\frac{1-a}{\beta^{p-1}} + a\right)^p\right)}\Bigg[\min\Bigg\{\left(\widetilde{f_3}\left(\widetilde{x}_3 - \widetilde{x}_1 + a\widetilde{x}_3 + \frac{1-a}{\beta^p}(\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1)\right)\right)^p, \\
&\quad \left((-\widetilde{f_1})\left(\widetilde{x}_3 - \widetilde{x}_1 - a\widetilde{x}_1 - \frac{1-a}{\beta^p}(\widetilde{x}_3 - \widetilde{x}_2 + \widetilde{x}_1)\right)\right)^p\Bigg\}\Bigg] \\
&\ge \frac{\left(1 + \frac{1-a}{\beta^p} + a\right)^p}{2^{p-2}\left(1 + \left(\frac{1-a}{\beta^{p-1}} + a\right)^p\right)} \ge \frac{\left(1 + \frac{1-a}{\beta^p} + a\right)^p}{2^{p-2}\left(1 + \frac{1-a}{\beta^p} + a\right)} = \frac{1}{2^{p-2}}\left(1 + \frac{1-a}{\beta^p} + a\right)^{p-1}.
\end{aligned}$$

On the other hand, if we put

$$\widetilde{x} = \widetilde{x}_2 - \widetilde{x}_1, \qquad \widetilde{y} = \frac{1-a}{\gamma^p}(-\widetilde{x}_3 + \widetilde{x}_2 + \widetilde{x}_1) + a\widetilde{x}_2, \qquad \widetilde{z} = \frac{1-a}{\gamma^p}(-\widetilde{x}_3 + \widetilde{x}_2 + \widetilde{x}_1) + a\widetilde{x}_1,$$

then

$$\|\widetilde{x}\| = 1, \qquad \|\widetilde{y}\| \le \frac{1-a}{\gamma^{p-1}} + a, \qquad \|\widetilde{z}\| \le \frac{1-a}{\gamma^{p-1}} + a, \qquad \|\widetilde{y} - \widetilde{z}\| = a\|\widetilde{x}\|.$$

Hence, we have

$$\begin{aligned}
C^{(p)}_{-\infty}(a,X) &= C^{(p)}_{-\infty}(a,\widetilde{X}) \\
&\geq \frac{\min\left\{\|\widetilde{x}+\widetilde{y}\|^p, \|\widetilde{x}-\widetilde{z}\|^p\right\}}{2^{p-2}\|\widetilde{x}\|^p + 2^{p-3}\left(\|\widetilde{y}\|^p + \|\widetilde{z}\|^p\right)} \\
&= \frac{1}{2^{p-2}\left(1+\left(\frac{1-a}{\gamma^{p-1}}+a\right)^p\right)}\Bigg[\min\Bigg\{\left\|\widetilde{x}_2-\widetilde{x}_1+a\widetilde{x}_2+\frac{1-a}{\gamma^p}(-\widetilde{x}_3+\widetilde{x}_2+\widetilde{x}_1)\right\|^p, \\
&\quad \left\|\widetilde{x}_2-\widetilde{x}_1-a\widetilde{x}_1-\frac{1-a}{\gamma^p}(-\widetilde{x}_3+\widetilde{x}_2+\widetilde{x}_1)\right\|^p\Bigg\}\Bigg] \\
&\geq \frac{1}{2^{p-2}\left(1+\left(\frac{1-a}{\gamma^{p-1}}+a\right)^p\right)}\Bigg[\min\Bigg\{\left(\widetilde{f}_2\left(\widetilde{x}_2-\widetilde{x}_1+a\widetilde{x}_2+\frac{1-a}{\gamma^p}(-\widetilde{x}_3+\widetilde{x}_2+\widetilde{x}_1)\right)\right)^p, \\
&\quad \left((-\widetilde{f}_1)\left(\widetilde{x}_2-\widetilde{x}_1-a\widetilde{x}_1-\frac{1-a}{\gamma^p}(-\widetilde{x}_3+\widetilde{x}_2+\widetilde{x}_1)\right)\right)^p\Bigg\}\Bigg] \\
&\geq \frac{\left(1+\frac{1-a}{\gamma^p}+a\right)^p}{2^{p-2}\left(1+\left(\frac{1-a}{\gamma^{p-1}}+a\right)^p\right)} \geq \frac{\left(1+\frac{1-a}{\gamma^p}+a\right)^p}{2^{p-2}\left(1+\frac{1-a}{\gamma^p}+a\right)} = \frac{1}{2^{p-2}}\left(1+\frac{1-a}{\gamma^p}+a\right)^{p-1}.
\end{aligned}$$

Furthermore, we put

$$\widetilde{x} = \widetilde{x}_3 - \widetilde{x}_2, \qquad \widetilde{y} = a\widetilde{x}_3 + (1-a)\widetilde{x}_1, \qquad \widetilde{z} = a\widetilde{x}_2 + (1-a)\widetilde{x}_1.$$

It is easy to see that

$$\|\widetilde{y}\| \leq 1, \qquad \|\widetilde{z}\| \leq 1, \qquad \widetilde{y}-\widetilde{z} = a\widetilde{x}.$$

Therefore, we have

$$\begin{aligned}
J(a,X) &= J(a,\widetilde{X}) \\
&\geq \min\left\{\|\widetilde{x}+\widetilde{y}\|, \|\widetilde{x}-\widetilde{z}\|\right\} \\
&= \min\left\{\|(1+a)\widetilde{x}_3-\widetilde{x}_2+(1-a)\widetilde{x}_1\|, \|\widetilde{x}_3-(1+a)\widetilde{x}_2-(1-a)\widetilde{x}_1\|\right\} \\
&\geq \min\left\{(1+a)\|\widetilde{x}_3-\widetilde{x}_2+\widetilde{x}_1\| - a\|\widetilde{x}_2-2\widetilde{x}_1\|, (1+a)\|\widetilde{x}_3-\widetilde{x}_2-\widetilde{x}_1\| - a\|\widetilde{x}_3-2\widetilde{x}_1\|\right\}.
\end{aligned}$$

Notice that $\|\widetilde{x}_2-2\widetilde{x}_1\| \leq \|\widetilde{x}_2-\widetilde{x}_1\| + \|\widetilde{x}_1\| = 2$. Similarly, $\|\widetilde{x}_3-2\widetilde{x}_1\| \leq 2$. Hence, we obtain

$$J(a,X) = J(a,\widetilde{X}) \geq (1+a)\min\left\{\|\widetilde{x}_3-\widetilde{x}_2+\widetilde{x}_1\|, \|\widetilde{x}_3-\widetilde{x}_2-\widetilde{x}_1\|\right\} - 2a = (1+a)\min\{\beta,\gamma\} - 2a,$$

from which it follows that

$$\frac{1}{\min\{\beta,\gamma\}} \geq \frac{1+a}{J(a,X)+2a}.$$

Consequently, we obtain

$$C^{(p)}_{-\infty}(a,X) \geq \frac{1}{2^{p-2}}\left((1-a)\left(1+\frac{1}{\min\left\{\beta^p,\gamma^p\right\}}\right)+2a\right)^{p-1} \geq \frac{1}{2^{p-2}}\left((1-a)\left(1+\left(\frac{1+a}{J(a,X)+2a}\right)^p\right)+2a\right)^{p-1},$$

which contradicts the hypothesis. Hence, X has normal structure if the inequality is satisfied. Since $C^{(p)}_{-\infty}(a,X) = C^{(p)}_{-\infty}(a,\widetilde{X})$ and $J(a,X) = J(a,\widetilde{X})$, it follows from the above arguments that $\widetilde{X}$ has normal structure. Therefore, X has uniform normal structure if the inequality is satisfied. This finishes the proof. □

Remark 2.14. Take $1 < p < \infty$. Since $C^{(p)}_{-\infty}(a,X) \le C^{(p)}_{NJ}(a,X)$, it follows from Theorem 2.13 that a Banach space X has uniform normal structure if there exists $0 \le a \le 1$ such that

$$C^{(p)}_{NJ}(a,X) < \frac{1}{2^{p-2}}\left((1-a)\left(1+\left(\frac{1+a}{J(a,X)+2a}\right)^{p}\right)+2a\right)^{p-1}.$$

Moreover, in [9, Theorem 3.12] it was proved that a Banach space X has uniform normal structure if there exists $0 \le a \le 1$ such that

$$C^{(p)}_{NJ}(a,X) < \frac{1}{2^{p-1}}\left(1+a+\frac{1-a^2}{J(a,X)+2a}\right)^{p}.$$

It is easy to check that

$$\frac{1}{2^{p-1}}\left(1+a+\frac{1-a^2}{J(a,X)+2a}\right)^{p} < \frac{1}{2^{p-2}}\left((1-a)\left(1+\left(\frac{1+a}{J(a,X)+2a}\right)^{p}\right)+2a\right)^{p-1}$$

for all $0 \le a < 1$. Hence, Theorem 2.13 improves Theorem 3.12 in [9].

Remark 2.15. Take $1 < p < \infty$. Since $C^{(p)}_{-\infty}(a,X) \le C^{(p)}_{Z}(a,X)$, it follows from Theorem 2.13 that a Banach space X has uniform normal structure if there exists $0 \le a \le 1$ such that

$$C^{(p)}_{Z}(a,X) < \frac{1}{2^{p-2}}\left((1-a)\left(1+\left(\frac{1+a}{J(a,X)+2a}\right)^{p}\right)+2a\right)^{p-1}.$$

Furthermore, in [25, Theorem 2.2] it was proved that a Banach space X has normal structure if there exists $0 \le a \le 1$ such that

$$C^{(p)}_{Z}(a,X) < \frac{1}{3\cdot 2^{p-2}}\left(1+a+\frac{1-a^2}{J(a,X)+2a}\right)^{p}.$$

It is easy to see that

$$\frac{1}{3\cdot 2^{p-2}}\left(1+a+\frac{1-a^2}{J(a,X)+2a}\right)^{p} < \frac{1}{2^{p-2}}\left((1-a)\left(1+\left(\frac{1+a}{J(a,X)+2a}\right)^{p}\right)+2a\right)^{p-1}$$

for all $0 \le a < 1$. Thus, Theorem 2.13 improves Theorem 2.2 in [25].

By letting $p = 2$ in Theorem 2.13, we obtain immediately the following result.

Corollary 2.16. *A Banach space X has uniform normal structure if there exists $0 \le a \le 1$ such that*

$$C_{-\infty}(a,X) < (1-a)\left(1+\left(\frac{1+a}{J(a,X)+2a}\right)^{2}\right)+2a.$$

Remark 2.17. By applying the inequality $C_{-\infty}(a,X) \le C_Z(a,X)$, we conclude that Corollary 2.16 improves Theorem 6 in [17].

Corollary 2.18. *A Banach space X has uniform normal structure if there exists $1 < p < \infty$ for which*

$$C^{(p)}_{-\infty}(X) < \frac{1}{2^{p-2}}\left(1+\frac{1}{(J(X))^p}\right)^{p-1} \qquad \text{or} \qquad C^{(p)}_{-\infty}(1,X) < 2.$$

Corollary 2.19. *If $C_{-\infty}(X) < 1 + \frac{1}{(J(X))^2}$ or $C_{-\infty}(1,X) < 2$, then X has uniform normal structure.*

Corollary 2.20. *If $C_{-\infty}(X) < \frac{1+\sqrt{3}}{2} \approx 1.366025$, then X has uniform normal structure.*

Remark 2.21. Corollaries 2.19 and 2.20 are improvements of the following results.
A Banach space X has uniform normal structure if one of the following conditions is satisfied:

(1) $C_Z(X) < 1 + \frac{1}{(J(X))^2}$ [17, Corollary 7];

(2) $C_Z(1, X) < 2$ [17, Corollary 7];

(3) $C_Z(X) < \frac{1+\sqrt{3}}{2}$ [17, Corollary 8];

(4) $C_{NJ}(X) < 1 + \frac{1}{(J(X))^2}$ [5, Corollary 3.17];

(5) $C_{NJ}(1, X) < 2$ [7, Theorem 3.6];

(6) $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$ [27, Theorem 2] and [5, Theorem 3.16].

3. The coefficient of weak orthogonality

The WORTH-property was introduced by Sims in [31]. Recall that a Banach space X has the WORTH-property if

$$\lim_{n\to\infty} \Big| \|x_n + x\| - \|x_n - x\| \Big| = 0$$

for all $x \in X$ and all weakly null sequences $\{x_n\}$. In [32], Sims defined the coefficient of weak orthogonality, which measures the "degree of WORTHwhileness". As in [21], we prefer to use its reciprocal, i.e. $\mu(X) \in [1, 3]$, defined by

$$\mu(X) = \inf\Big\{\lambda > 0 \ : \ \limsup_{n\to\infty} \|x_n + x\| \le \lambda \limsup_{n\to\infty} \|x_n - x\|\Big\},$$

where the infimum is taken over all $x \in X$ and all weakly null sequences $\{x_n\}$ in X. It is proved that $\mu(X) = \mu(X^*)$ in a reflexive Banach space X [22].
The following lemma can be found in [29].

Lemma 3.1. ([29]) *If a super-reflexive Banach space X fails to have normal structure, then there are $\widetilde{x}_1, \widetilde{x}_2 \in S_{\widetilde{X}}$ and $\widetilde{f}_1, \widetilde{f}_2 \in S_{(\widetilde{X})^*}$ such that*

(a) $\|\widetilde{x}_1 - \widetilde{x}_2\| = 1$ *and* $\widetilde{f}_i(\widetilde{x}_j) = 0$ *for all* $i \neq j$;

(b) $\widetilde{f}_i(\widetilde{x}_i) = 1$ *for* $i = 1, 2$;

(c) $\|\widetilde{x}_1 + \widetilde{x}_2\| \le \mu(X)$.

Theorem 3.2. *A Banach space X has normal structure if there exists $0 \le a \le 1$ such that*

$$C^{(p)}_{-\infty}(a, X) < \frac{\Big(1 + \frac{1-a}{(\mu(X))^p} + a\Big)^p}{2^{p-2}\Big(1 + \Big(\frac{1-a}{(\mu(X))^{p-1}} + a\Big)^p\Big)},$$

where $1 < p < \infty$.

Proof. Take $1 < p < \infty$. By applying Corollary 2.18, the condition $C^{(p)}_{-\infty}(1, X) < 2$ implies that X has normal structure. Since $\mu(X) \geq 1$, it follows by the above inequality that X is uniformly non-square, and consequently, X is super-reflexive. Arguing by contradiction, we suppose that X fails to have normal structure and there exists $0 \leq a < 1$ such that the above inequality holds. Then there are elements $\widetilde{x}_1, \widetilde{x}_2 \in S_{\widetilde{X}}$ and $\widetilde{f}_1, \widetilde{f}_2 \in S_{(\widetilde{X})^*}$ satisfying all the conditions in Lemma 3.1. We now use these elements to estimate the generalized von Neumann-Jordan type constant of the Banach space ultrapower $\widetilde{X}$ which is the same as that of the space X itself. We will estimate $\eta := \|\widetilde{x}_1 + \widetilde{x}_2\|$ in terms of the generalized von Neumann-Jordan type constant of X. Now, we put

$$\widetilde{x} = \widetilde{x}_2 - \widetilde{x}_1, \qquad \widetilde{y} = \frac{1-a}{\eta^p}(\widetilde{x}_1 + \widetilde{x}_2) + a\widetilde{x}_2, \qquad \widetilde{z} = \frac{1-a}{\eta^p}(\widetilde{x}_1 + \widetilde{x}_2) + a\widetilde{x}_1.$$

We note here that $\|\widetilde{x}_2 + \widetilde{x}_1\| \geq \widetilde{f}_2(\widetilde{x}_2 + \widetilde{x}_1) = 1$. It is easy to see that

$$\|\widetilde{x}\| = 1, \qquad \|\widetilde{y}\| \leq \frac{1-a}{\eta^{p-1}} + a, \qquad \|\widetilde{z}\| \leq \frac{1-a}{\eta^{p-1}} + a, \qquad \|\widetilde{y} - \widetilde{z}\| = a\|\widetilde{x}\|.$$

Therefore, we have

$$\begin{aligned}
C^{(p)}_{-\infty}(a, X) &= C^{(p)}_{-\infty}(a, \widetilde{X}) \\
&\geq \frac{\min\left\{\|\widetilde{x} + \widetilde{y}\|^p, \|\widetilde{x} - \widetilde{z}\|^p\right\}}{2^{p-2}\|\widetilde{x}\|^p + 2^{p-3}\left(\|\widetilde{y}\|^p + \|\widetilde{z}\|^p\right)} \\
&= \frac{1}{2^{p-2}\left(1 + \left(\frac{1-a}{\eta^{p-1}} + a\right)^p\right)}\Bigg[\min\Bigg\{\left\|\widetilde{x}_2 - \widetilde{x}_1 + a\widetilde{x}_2 + \frac{1-a}{\eta^p}(\widetilde{x}_1 + \widetilde{x}_2)\right\|^p, \\
&\qquad \left\|\widetilde{x}_2 - \widetilde{x}_1 - a\widetilde{x}_1 - \frac{1-a}{\eta^p}(\widetilde{x}_1 + \widetilde{x}_2)\right\|^p\Bigg\}\Bigg] \\
&\geq \frac{1}{2^{p-2}\left(1 + \left(\frac{1-a}{\eta^{p-1}} + a\right)^p\right)}\Bigg[\min\Bigg\{\left(\widetilde{f}_2\left(\widetilde{x}_2 - \widetilde{x}_1 + a\widetilde{x}_2 + \frac{1-a}{\eta^p}(\widetilde{x}_1 + \widetilde{x}_2)\right)\right)^p, \\
&\qquad \left((-\widetilde{f}_1)\left(\widetilde{x}_2 - \widetilde{x}_1 - a\widetilde{x}_1 - \frac{1-a}{\eta^p}(\widetilde{x}_1 + \widetilde{x}_2)\right)\right)^p\Bigg\}\Bigg] \\
&\geq \frac{\left(1 + \frac{1-a}{\eta^p} + a\right)^p}{2^{p-2}\left(1 + \left(\frac{1-a}{\eta^{p-1}} + a\right)^p\right)} \geq \frac{\left(1 + \frac{1-a}{(\mu(X))^p} + a\right)^p}{2^{p-2}\left(1 + \left(\frac{1-a}{(\mu(X))^{p-1}} + a\right)^p\right)},
\end{aligned}$$

which is a contradiction. This completes the proof. □

Remark 3.3. Take $1 < p < \infty$. Since $C^{(p)}_{-\infty}(a, X) \leq C^{(p)}_{NJ}(a, X)$, it follows from Theorem 3.2 that a Banach space X has normal structure if there exists $0 \leq a \leq 1$ such that

$$C^{(p)}_{NJ}(a, X) < \frac{\left(1 + \frac{1-a}{(\mu(X))^p} + a\right)^p}{2^{p-2}\left(1 + \left(\frac{1-a}{(\mu(X))^{p-1}} + a\right)^p\right)}.$$

Moreover, in [9, Theorem 5.2] it was proved that a Banach space X has normal structure if there exists $0 \leq a \leq 1$ such that

$$C^{(p)}_{NJ}(a, X) < \frac{1}{2^{p-1}}\left(1 + \frac{1-a}{\mu(X)} + a\right)^p.$$

It is easy to see that

$$\frac{1}{2^{p-1}}\Big(1+\frac{1-a}{\mu(X)}+a\Big)^p < \frac{\Big(1+\frac{1-a}{(\mu(X))^p}+a\Big)^p}{2^{p-2}\Big(1+\big(\frac{1-a}{(\mu(X))^{p-1}}+a\big)^p\Big)}$$

for all $0 \le a < 1$. Hence, Theorem 3.2 improves Theorem 5.2 in [9]. Furthermore, Theorem 3.2 improves Theorem 3.7 in [4] in the case $a = 0$.

By letting $p = 2$ in Theorem 3.2, we obtain immediately the following result.

Corollary 3.4. *A Banach space X has normal structure if there exists $0 \le a \le 1$ such that*

$$C_{-\infty}(a,X) < \frac{\Big(1+\frac{1-a}{(\mu(X))^2}+a\Big)^2}{1+\big(\frac{1-a}{\mu(X)}+a\big)^2}.$$

Remark 3.5. By applying the inequality $C_{-\infty}(a,X) \le C_Z(a,X)$, we conclude that Corollary 3.4 improves Theorem 12 in [17].

Corollary 3.6. *A Banach space X has normal structure if there exists $1 < p < \infty$ for which*

$$C^{(p)}_{-\infty}(X) < \frac{\Big(1+\frac{1}{(\mu(X))^p}\Big)^p}{2^{p-2}\Big(1+\big(\frac{1}{(\mu(X))^{p-1}}\big)^p\Big)}.$$

Corollary 3.7. *If $C_{-\infty}(X) < 1+\frac{1}{(\mu(X))^2}$, then X has normal structure.*

Remark 3.8. Corollary 3.7 is an improvement of the following results.

A Banach space X has normal structure if one of the following conditions is satisfied:

(1) $C_Z(X) < 1+\frac{1}{(\mu(X))^2}$ [26, Theorem 4];

(2) $C_{NJ}(X) < 1+\frac{1}{(\mu(X))^2}$ [22, Theorem 1].

4. The generalized García-Falset coefficient

The generalized García-Falset coefficient $R(\alpha,X)$ of a Banach space X for a given $\alpha \ge 0$, was introduced in 1996 by Domínguez-Benavides [14] as

$$R(\alpha,X) = \sup\Big\{\liminf_{n\to\infty}\|x_n + x\|\Big\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \le \alpha$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$D(\{x_n\}) = \limsup_{n\to\infty}\Big(\limsup_{m\to\infty}\|x_n - x_m\|\Big) \le 1.$$

It is noteworthy that $1 \le R(1,X) \le 2$.

The following key lemma was proved by Dinarvand in [8].

Lemma 4.1. ([8]) *If a super-reflexive Banach space X fails to have normal structure, then there are $\widetilde{x}_1, \widetilde{x}_2 \in S_{\widetilde{X}}$ and $\widetilde{f_1}, \widetilde{f_2} \in S_{(\widetilde{X})^*}$ such that*

(a) $\|\widetilde{x}_1 - \widetilde{x}_2\| = 1$ *and* $\widetilde{f}_i(\widetilde{x}_j) = 0$ *for all* $i \neq j$;

(b) $\widetilde{f}_i(\widetilde{x}_i) = 1$ *for* $i = 1, 2$;

(c) $\|\widetilde{x}_1 + \widetilde{x}_2\| \leq R(1, X)$.

Theorem 4.2. *A Banach space X has normal structure if there exists* $0 \leq a \leq 1$ *such that*

$$C^{(p)}_{-\infty}(a, X) < \frac{\left(1 + \frac{1-a}{(R(1,X))^p} + a\right)^p}{2^{p-2}\left(1 + \left(\frac{1-a}{(R(1,X))^{p-1}} + a\right)^p\right)},$$

where $1 < p < \infty$.

Proof. The proof of this theorem can obtained by using similar arguments as those given in the proof of Theorem 3.2 and applying Lemma 4.1. □

Remark 4.3. By using similar arguments as those given in Remark 3.3, we conclude that Theorem 4.2 improves Theorem 4.2 in [9].

Remark 4.4. Take $1 < p < \infty$. Since $C^{(p)}_{-\infty}(a, X) \leq C^{(p)}_{Z}(a, X)$, it follows from Theorem 4.2 that a Banach space X has normal structure if there exists $0 \leq a \leq 1$ such that

$$C^{(p)}_{Z}(a, X) < \frac{\left(1 + \frac{1-a}{(R(1,X))^p} + a\right)^p}{2^{p-2}\left(1 + \left(\frac{1-a}{(R(1,X))^{p-1}} + a\right)^p\right)}.$$

Moreover, in [25, Theorem 3.1] it was proved that a Banach space X has normal structure if there exists $0 \leq a \leq 1$ such that

$$C^{(p)}_{Z}(a, X) < \frac{1}{3 \cdot 2^{p-2}}\left(1 + \frac{1-a}{R(1, X)} + a\right)^p.$$

It is easy to see that

$$\frac{1}{3 \cdot 2^{p-2}}\left(1 + \frac{1-a}{R(1, X)} + a\right)^p < \frac{\left(1 + \frac{1-a}{(R(1,X))^p} + a\right)^p}{2^{p-2}\left(1 + \left(\frac{1-a}{(R(1,X))^{p-1}} + a\right)^p\right)}$$

for all $0 \leq a < 1$. Hence, Theorem 4.2 improves Theorem 3.1 in [25].

By letting $p = 2$ in Theorem 4.2, we obtain immediately the following result.

Corollary 4.5. *A Banach space X has normal structure if there exists* $0 \leq a \leq 1$ *such that*

$$C_{-\infty}(a, X) < \frac{\left(1 + \frac{1-a}{(R(1,X))^2} + a\right)^2}{1 + \left(\frac{1-a}{R(1,X)} + a\right)^2}.$$

Remark 4.6. By applying the inequalities $C_{-\infty}(a, X) \leq C_Z(a, X) \leq C_{NJ}(a, X)$, we conclude that Corollary 4.5 improves Theorems 3.8 and 3.13 in [8].

Corollary 4.7. *A Banach space X has normal structure if there exists $1 < p < \infty$ for which*

$$C_{-\infty}^{(p)}(X) < \frac{\left(1 + \frac{1}{(R(1,X))^p}\right)^p}{2^{p-2}\left(1 + \left(\frac{1}{(R(1,X))^{p-1}}\right)^p\right)}.$$

Corollary 4.8. *If $C_{-\infty}(X) < 1 + \frac{1}{(R(1,X))^2}$, then X has normal structure.*

Remark 4.9. Corollary 4.8 is an improvement of the following results.

A Banach space X has normal structure if one of the following conditions is satisfied:

(1) $C_Z(X) < 1 + \frac{1}{(R(1,X))^2}$ [8, Corollary 3.14];

(2) $C_{NJ}(X) < 1 + \frac{1}{(R(1,X))^2}$ [8, Corollary 3.9].

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