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# Weighted g-MPD inverse

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**Abstract.** The significance of weighted generalized inverses inspired us to extend the concepts of the g-MPD and g-DMP inverses. In particular, we define weighted versions of the g-MPD and g-DMP inverses solving two systems of operator equations. Characterizations and representations of weighted g-MPD and g-DMP inverses are presented. The operator matrix forms of the weighted g-MPD inverse are given. Consequently, we get new results for g-MPD and g-DMP inverses and extend some known properties of g-MPD and g-DMP inverses. Applying the weighted g-MPD and g-DMP inverses, we verify solvability of some systems of linear equations and minimizations problems.

## 1. Introduction

For arbitrary Hilbert spaces $H$ and $K$, let $\mathcal{B}(H, K)$ be the set of bounded linear operators from $H$ to $K$, and let $\mathcal{B}(H) = \mathcal{B}(H, H)$. If $A \in \mathcal{B}(H, K)$, we stand $A^*$, $N(A)$ and $R(A)$ for the adjoint, null space and range of $A$, respectively. Denote by $P_M$ the orthogonal projector onto $M$ and by $P_{M,S}$ the projector on $M$ along $S$, where $M$ and $S$ are closed subspaces.

Solvability of various systems of linear equations is a problem in many areas: physics, economy, statistics, operational research, and others. As solutions of systems of linear equations, the concepts of numerous generalized inverses were presented. The Moore–Penrose inverse of $A \in \mathcal{B}(H, K)$ is the unique solution $X = A^\dagger \in \mathcal{B}(K, H)$ to the system

$$(1)\ AXA = A, \qquad (2)\ XAX = X, \qquad (3)\ (AX)^* = AX, \qquad (4)\ (XA)^* = XA.$$

It is known that $A^\dagger$ exists if and only if $R(A)$ is closed in $K$ [8]. The Moore-Penrose inverse is applicable in linear estimation, image restoration, Markov chains, differential and difference equations, graphics, coding

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theory, cryptography, robotics [3, 5]. The relation between the Moore–Penrose inverse and the least-squares solution problem, as a special case of nonlinear optimization problems, was given in [21].

If $M$ and $L$ are subspaces of $H$ and $K$, respectively, the outer inverse of $A \in \mathcal{B}(H,K)$ with the fixed range $M$ and null space $L$ is the unique operator (if it exists) $X = A^{(2)}_{M,L} \in \mathcal{B}(K,H)$ satisfying $XAX = X$, $R(X) = M$ and $N(X) = L$ [3]. For $\alpha \subseteq \{1,2,3,4\}$, the $\alpha$-inverse $X = A^{(\alpha)}_{M,L}$ of $A \in \mathcal{B}(H,K)$ with the fixed range $M$ and null space $L$ is the unique operator (if it exists) for which equations from the definition of the Moore–Penrose inverse contained in $\alpha$ hold.

For a fixed $W \in \mathcal{B}(K,H)\backslash\{0\}$, the $Wg$–Drazin inverse of $A \in \mathcal{B}(H,K)$ [7] is the uniquely determined solution $X = A^{d,W} \in \mathcal{B}(H,K)$ to

$$AWX = XWA, \quad XWAWX = X \quad \text{and} \quad A - AWXWA \text{ is quasinilpotent.}$$

If $A - AWXWA$ is nilpotent, $A^{d,W}$ reduces to the $W$-weighted Drazin inverse $A^{D,W}$ [6]. When $H = K$ and $W = I$, $A^d = A^{d,I}$ is the generalized Drazin inverse of $A$ [12] and $A^D = A^{D,I}$ is the Drazin inverse of $A$ [9]. For the Drazin inverse, the statement that $A - AXA$ is nilpotent is equivalent to $A^{k+1}B = A^k$, for some non-negative integer $k$; and the smallest such $k$ is the index $ind(A)$ of $A$. If $ind(A) \le 1$, $A^\# = A^D$ is the group inverse of $A$. Set $\mathcal{B}(H,K)^{d,W}$, $\mathcal{B}(H,K)^{D,W}$, $\mathcal{B}(H)^d$ and $B(H)^\#$, respectively, for the sets of $Wg$–Drazin invertible and $W$-weighted Drazin invertible operators of $\mathcal{B}(H,K)$, and generalized Drazin invertible and group invertible operators of $\mathcal{B}(H)$. We know that $A \in \mathcal{B}(H,K)^{d,W}$ if and only if $AW \in \mathcal{B}(K)^d$ if and only if $WA \in \mathcal{B}(H)^d$ [7]. In addition, $A^{d,W} = ((AW)^d)^2A = A((WA)^d)^2$, $(AW)^d = A^{d,W}W$ and $(WA)^d = WA^{d,W}$.

The notion of the weighted core–EP inverse was defined in [10] for a rectangular matrix and extended to a $Wg$-Drazin invertible bounded linear operator between two Hilbert spaces in [17]. If $W \in \mathcal{B}(K,H)\backslash\{0\}$ and $A \in \mathcal{B}(H,K)^{d,W}$, there exists the unique $W$–weighted core–EP inverse $X = A^{\textcircled{d},W} \in \mathcal{B}(H,K)$ of $A$ such that

$$WAWX = P_{R((WA)^d)} \quad \text{and} \quad R(X) \subseteq R((AW)^d).$$

We observe that $A^{\textcircled{d},W}WAWA^{\textcircled{d},W} = A^{\textcircled{d},W}$, $R(A^{\textcircled{d},W}) = R((AW)^d)$ and $R((A^{\textcircled{d},W})^*) = R((WA)^d)$. Dually, the $W$–weighted $*$core–EP inverse of $A$ is the unique operator $X = A_{\textcircled{d},W} \in \mathcal{B}(H,K)$ satisfying

$$XWAW = P_{R(((AW)^d)^*)} \quad \text{and} \quad R(X^*) \subseteq R(((WA)^d)^*).$$

When $H = K$ and $W = I$, $A^{\textcircled{d}} = A^{\textcircled{d},I}$ (or $A_{\textcircled{d}} = A_{\textcircled{d},I}$) is the core–EP (or $*$core–EP) inverse of $A$ [17, 20]. For a Drazin invertible operator $A \in \mathcal{B}(H)$ and $ind(A) \le 1$, $A^{\textcircled{\#}} = A^{\textcircled{d}}$ (or $A_{\textcircled{\#}} = A_{\textcircled{d}}$) is the (dual) core inverse of $A$ [1]. According to [17] and [14], $A^{\textcircled{d},W} = A[(WA)^{\textcircled{d}}]^2$ and $A_{\textcircled{d},W} = [(AW)_{\textcircled{d}}]^2A$. Properties of the weighted core–EP inverse can be found in [2, 11, 13].

As generalizations of the Moore–Penrose inverse, the generalized Moore–Penrose (or gMP) inverse and its dual were introduced in [22] for a generalized Drazin invertible Hilbert space operator. Extending the concepts of the gMP inverse and its dual, the weighted generalized Moore–Penrose inverse and its dual were presented in [19] for operators between two Hilbert spaces. For $W \in \mathcal{B}(K,H)\backslash\{0\}$ and $A \in \mathcal{B}(H,K)^{d,W}$, the $W$-weighted generalized Moore–Penorse (or $W$-gMP) inverse $A^{\diamond,W}$ of $A$ is defined as the unique solution to

$$XWAWX = X, \qquad WAWX = WAW(A^{\textcircled{d},W}WAW)^\dagger A^{\textcircled{d},W} \qquad \text{and}$$

$$XWAW = (A^{\textcircled{d},W}WAW)^\dagger A^{\textcircled{d},W}WAW,$$

which is expressed by

$$A^{\diamond,W} = (A^{\textcircled{d},W}WAW)^\dagger A^{\textcircled{d},W}.$$

The dual $W$-weighted generalized Moore–Penorse (or dual $W$-gMP) inverse $A_{\diamond,W}$ of $A$ is the unique solution to

$$XWAWX = X, \qquad WAWX = WAWA_{\textcircled{d},W}(WAWA_{\textcircled{d},W})^\dagger \qquad \text{and}$$

$$XWAW = A_{\textcircled{d},W}(WAWA_{\textcircled{d},W})^\dagger WAW,$$

which is represented as

$$A_{\diamond,W} = A_{\textcircled{\dagger},W}(WAWA_{\textcircled{\dagger},W})^{\dagger}.$$

In the case that $H = K$ and $W = I$, $A^{\diamond} = A^{\diamond,I}$ is the gMP inverse and $A_{\diamond} = A_{\diamond,I}$ is the dual gMP inverse [22]. If $A \in \mathcal{B}(H)^{\#}$, $A^{\diamond}$ becomes $A^{\dagger}$. Various expressions for the gMP inverse were developed in [4, 16, 22–24].

For $A \in \mathcal{B}(H)^{d}$, the product $AA^{\textcircled{\dagger}}A$ is very significant for decomposing the operator $A$, for representing well-known generalized inverses and for investigating partial orders. Notice that $A^{\textcircled{\dagger}} = (AA^{\textcircled{\dagger}}A)^{\textcircled{\#}}$ and $A^{\diamond} = (AA^{\textcircled{\dagger}}A)^{\dagger}$. Solving some system of operator equations, a new generalized inverse of $AA^{\textcircled{\dagger}}A$ was defined in [15], represented by the generalized Moore–Penrose inverse and the generalized Drazin inverse of $A$ and called a generalized MPD inverse of $AA^{\textcircled{\dagger}}A$. Precisely, if $A \in \mathcal{B}(H)^{d}$, the generalized MPD (or g-MPD) inverse of $AA^{\textcircled{\dagger}}A$ is the uniquely determined solution to

$$XAA^{\textcircled{\dagger}}AX = X, \qquad AA^{\textcircled{\dagger}}AX = AA^{d} \qquad \text{and} \qquad XAA^{\textcircled{\dagger}}A = (A^{\textcircled{\dagger}}A)^{\dagger}A^{\textcircled{\dagger}}A$$

and given by

$$A^{\diamond,d} = (A^{\textcircled{\dagger}}A)^{\dagger}A^{d} = A^{\diamond}AA^{d}.$$

The generalized DMP (or g-DMP) inverse of $AA_{\textcircled{\dagger}}A$ is the unique solution to

$$XAA_{\textcircled{\dagger}}AX = X, \qquad AA_{\textcircled{\dagger}}AX = AA_{\textcircled{\dagger}}(AA_{\textcircled{\dagger}})^{\dagger} \qquad \text{and} \qquad XAA_{\textcircled{\dagger}}A = A^{d}A,$$

which is expressed as

$$A_{d,\diamond} = A^{d}(AA_{\textcircled{\dagger}})^{\dagger} = A^{d}AA_{\diamond}.$$

If $A \in \mathcal{B}(H)^{\#}$, $A^{\diamond,d} = A_{\textcircled{\#}}$ and $A_{d,\diamond} = A^{\textcircled{\#}}$. Applying the g-MPD and g-DMP inverses, some systems of linear equations and minimization problems were solved in [15].

The weighted generalized inverses, defined as extensions of corresponding generalized inverses, have many applications in the weighted linear least-square problems, statistics, neural networks and numerical analysis [3]. The great importance of weighted generalized inverses motivated us to further investigate this area. Our aim is to generalize the notions of the g-MPD and g-DMP inverses of $AA^{\textcircled{\dagger}}A$ when the operator $A$ is generalized Drazin invertible Hilbert space operator, to the case that $A$ is $Wg$–Drazin invertible operator between two Hilbert spaces. Precisely, solving two systems of operator equations, we introduce the weighted versions of g-MPD and g-DMP inverses as unique solutions of these systems. Many characterizations and expressions of weighted g-MPD and g-DMP inverses are established. We give operator matrix representations of the weighted g-MPD inverse. The weighted g-MPD and g-DMP inverses can be applied in solving systems of linear equations and minimizations problems. Thus, several known results for g-MPD and g-DMP inverses are extended and some new properties for g-MPD and g-DMP inverses are presented as consequences.

This paper is organized in the next manner. In Section 2, we define and characterize the weighted g-MPD and g-DMP inverses. The operator matrix forms and some expressions of the weighted g-MPD inverse are parts of Section 3. Section 4 involves applications of weighted g-MPD and g-DMP inverses in solving systems of linear equations and minimization problems.

## 2. Weighted g-MPD inverse

Generalizing the concepts of g-MPD and g-DMP inverses, we present their weighted versions as solutions of the following systems of operator equations.

**Theorem 2.1.** *If $W \in \mathcal{B}(K,H)\backslash\{0\}$ and $A \in \mathcal{B}(H,K)^{d,W}$, then*

(a) $X = (A^{\textcircled{\dagger},W}WAW)^{\dagger}A^{d,W}$ *is the uniquely determined solution to*

$$XWAWA^{\textcircled{\dagger},W}WAWX = X, \quad XWAWA^{\textcircled{\dagger},W}WAW = (A^{\textcircled{\dagger},W}WAW)^{\dagger}A^{\textcircled{\dagger},W}WAW$$
$$\text{and} \quad WAWA^{\textcircled{\dagger},W}WAWX = WAWA^{d,W}; \tag{1}$$

(b) $X = A^{d,W}(WAWA_{\textcircled{d},W})^\dagger$ *is the uniquely determined solution to*

$$XWAWA_{\textcircled{d},W}WAWX = X, \quad WAWA_{\textcircled{d},W}WAWX = WAWA_{\textcircled{d},W}(WAWA_{\textcircled{d},W})^\dagger$$
$$\text{and} \quad XWAWA_{\textcircled{d},W}WAW = A^{d,W}WAW.$$

*Proof.* (a) The projector $A^{\textcircled{d},W}WAW$ has a closed range, which implies the existence of $(A^{\textcircled{d},W}WAW)^\dagger$. Using $A^{d,W}WAWA^{\textcircled{d},W} = A^{\textcircled{d},W}$, we check that $X = (A^{\textcircled{d},W}WAW)^\dagger A^{d,W}$ satisfies the first two equations in (1). The equality $A^{d,W} = A^{\textcircled{d},W}WAWA^{d,W}$ yields

$$\begin{aligned} WAWA^{\textcircled{d},W}WAWX &= WAWA^{\textcircled{d},W}WAW(A^{\textcircled{d},W}WAW)^\dagger A^{d,W} \\ &= WAW(A^{\textcircled{d},W}WAW(A^{\textcircled{d},W}WAW)^\dagger A^{\textcircled{d},W}WAW)A^{d,W} \\ &= WAW(A^{\textcircled{d},W}WAWA^{d,W}) = WAWA^{d,W}. \end{aligned}$$

Hence, $X = (A^{\textcircled{d},W}WAW)^\dagger A^{d,W}$ is a solution to (1).

For a solution $X$ of the system (1), we have

$$\begin{aligned} X &= (XWAWA^{\textcircled{d},W}WAW)X = (A^{\textcircled{d},W}WAW)^\dagger A^{\textcircled{d},W}WAWX \\ &= (A^{\textcircled{d},W}WAW)^\dagger A^{\textcircled{d},W}(WAWA^{\textcircled{d},W}WAWX) \\ &= (A^{\textcircled{d},W}WAW)^\dagger(A^{\textcircled{d},W}WAWA^{d,W}) \\ &= (A^{\textcircled{d},W}WAW)^\dagger A^{d,W}. \end{aligned}$$

So, $X = (A^{\textcircled{d},W}WAW)^\dagger A^{d,W}$ represents the unique solution to (1).

Analogously, we verify part (b). □

**Definition 2.2.** *For $W \in \mathcal{B}(K,H)\backslash\{0\}$ and $A \in \mathcal{B}(H,K)^{d,W}$,*

(a) *the W-weighted g-MPD (or W-g-MPD) inverse of $WAWA^{\textcircled{d},W}WAW$ is introduced by*

$$A^{\diamond,d,W} = (A^{\textcircled{d},W}WAW)^\dagger A^{d,W}.$$

(b) *the W-weighted g-DMP (or W-g-DMP) inverse of $WAWA_{\textcircled{d},W}WAW$ is introduced by*

$$A_{d,\diamond,W} = A^{d,W}(WAWA_{\textcircled{d},W})^\dagger.$$

In the case that $H = K$ and $W = I$, the $W$-g-MPD inverse $A^{\diamond,d,W}$ and the $W$-g-DMP inverse $A_{d,\diamond,W}$, respectively, become the g-MPD inverse $A^{\diamond,d}$ and the g-DMP inverse $A_{d,\diamond}$.

**Corollary 2.3.** *For $W \in \mathcal{B}(K,H)\backslash\{0\}$ and $A \in \mathcal{B}(H,K)^{d,W}$, we have*

$$A^{\diamond,d,W} = A^{\diamond,W}WAWA^{d,W}$$

*and*

$$A_{d,\diamond,W} = A^{d,W}WAWA_{\diamond,W}.$$

*Proof.* We only prove the first equality, because the second equality follows in a similar manner. Note that

$$A^{\diamond,d,W} = (A^{\textcircled{d},W}WAW)^\dagger A^{d,W} = (A^{\textcircled{d},W}WAW)^\dagger A^{\textcircled{d},W}WAWA^{d,W} = A^{\diamond,W}WAWA^{d,W}.$$

□

Remark that, for $A \in \mathcal{B}(H,K)^{D,W}$ in Definition 2.2 and Corollary 2.3,

$$A^{\diamond,D,W} = (A^{\textcircled{D},W}WAW)^\dagger A^{D,W} = A^{\diamond,W}WAWA^{D,W}$$

and

$$A_{D,\diamond,W} = A^{D,W}(WAWA_{\textcircled{D},W})^\dagger = A^{D,W}WAWA_{\diamond,W}.$$

**Example 2.4.** *For the complex matrices*

$$A=\begin{bmatrix} 3&0&0&0&1\\ 0&1&0&1&1\\ 0&0&1&1&1\\ 0&0&0&0&1 \end{bmatrix} \quad\text{and}\quad W=\begin{bmatrix} 1&0&0&0\\ 0&3&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix},$$

*we have*

$$(WA)^{\textcircled{D}}=\begin{bmatrix} \frac{1}{3}&0&0&0&0\\ 0&\frac{1}{3}&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix},\quad A^{\textcircled{D},W}=A[(WA)^{\textcircled{D}}]^2=\begin{bmatrix} \frac{1}{3}&0&0&0&0\\ 0&\frac{1}{9}&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0 \end{bmatrix},$$

$$(WA)^{D}=\begin{bmatrix} \frac{1}{3}&0&0&0&\frac{1}{9}\\ 0&\frac{1}{3}&0&\frac{1}{3}&\frac{4}{9}\\ 0&0&1&1&2\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix},\quad A^{D,W}=A[(WA)^{D}]^2=\begin{bmatrix} \frac{1}{3}&0&0&0&\frac{1}{9}\\ 0&\frac{1}{9}&0&\frac{1}{9}&\frac{4}{27}\\ 0&0&1&1&2\\ 0&0&0&0&0 \end{bmatrix},$$

$$(A^{\textcircled{D},W}WAW)^{\dagger}=\begin{bmatrix} 1&0&0&0\\ 0&\frac{18}{19}&-\frac{3}{19}&0\\ 0&-\frac{3}{19}&\frac{10}{19}&0\\ 0&\frac{3}{19}&\frac{9}{19}&0 \end{bmatrix},\quad A^{\diamond,W}=\begin{bmatrix} \frac{1}{3}&0&0&0&0\\ 0&\frac{2}{19}&-\frac{3}{19}&0&0\\ 0&-\frac{1}{57}&\frac{10}{19}&0&0\\ 0&\frac{1}{57}&\frac{9}{19}&0&0 \end{bmatrix}$$

*and*

$$A^{\diamond,D,W}=\begin{bmatrix} \frac{1}{3}&0&0&0&\frac{1}{9}\\ 0&\frac{2}{19}&-\frac{3}{19}&-\frac{1}{19}&-\frac{10}{57}\\ 0&-\frac{1}{57}&\frac{10}{19}&\frac{29}{57}&\frac{176}{171}\\ 0&\frac{1}{57}&\frac{9}{19}&\frac{28}{57}&\frac{166}{171} \end{bmatrix}.$$

New expressions of $A^{\diamond,d,W}$ and $A_{d,\diamond,W}$ are given.

**Lemma 2.5.** *If $W\in\mathcal{B}(K,H)\backslash\{0\}$ and $A\in\mathcal{B}(H,K)^{d,W}$, then*

$$A^{\diamond,d,W}=P_{R((A^{\textcircled{d},W}WAW)^*)}A^{d,W}=(WA^{\textcircled{d},W}WAW)^{\dagger}WA^{d,W}$$

*and*

$$A_{d,\diamond,W}=A^{d,W}P_{R(WAWA_{\textcircled{d},W})}=A^{d,W}W(WAWA_{\textcircled{d},W}W)^{\dagger}.$$

*Proof.* Since $R(WA^{\textcircled{d},W}WAW)=R(WA^{\textcircled{d},W}WA)$ is closed, $(WA^{\textcircled{d},W}WAW)^{\dagger}$ exists. Now,

$$\begin{aligned} A^{\diamond,d,W} &= (A^{\textcircled{d},W}WAW)^{\dagger}A^{d,W}=(A^{\textcircled{d},W}WAW)^{\dagger}A^{\textcircled{d},W}WAWA^{d,W}\\ &= P_{R((A^{\textcircled{d},W}WAW)^*)}A^{d,W}=P_{R((WA^{\textcircled{d},W}WAW)^*)}A^{d,W}\\ &= (WA^{\textcircled{d},W}WAW)^{\dagger}WA^{\textcircled{d},W}WAWA^{d,W}\\ &= (WA^{\textcircled{d},W}WAW)^{\dagger}WA^{d,W}. \end{aligned}$$

□

We establish necessary and sufficient conditions for an operator to coincide with $A^{\diamond,d,W}$.

**Theorem 2.6.** *Let $W\in\mathcal{B}(K,H)\backslash\{0\}$, $A\in\mathcal{B}(H,K)^{d,W}$ and $X\in\mathcal{B}(H,K)$. Then the following statements are equivalent:*

(i) $X = A^{\diamond,d,W}$;

(ii) $XWAWA^{\oplus,W}WAWX = X$, $WAWA^{\oplus,W}WAWXWAWA^{\oplus,W}WAW = WAWA^{\oplus,W}WAW$, $XWAWA^{\oplus,W}WAW = (A^{\oplus,W}WAW)^{\dagger}A^{\oplus,W}WAW$ and $WAWA^{\oplus,W}WAWX = WAWA^{d,W}$;

(iii) $XWAWA^{\oplus,W}WAWX = X$, $WAWXWAW = WAW(A^{\oplus,W}WAW)^{\dagger}A^{d,W}WAW$, $XWAW = (A^{\oplus,W}WAW)^{\dagger}A^{d,W}WAW$ and $WAWX = WAW(A^{\oplus,W}WAW)^{\dagger}A^{d,W}$;

(iv) $XWAWA^{\oplus,W}WAWX = X$, $XWAW = (A^{\oplus,W}WAW)^{\dagger}A^{d,W}WAW$ and $WAWX = WAW(A^{\oplus,W}WAW)^{\dagger}A^{d,W}$;

(v) $XWAWA^{d,W} = X$ and $XWAW = (A^{\oplus,W}WAW)^{\dagger}A^{d,W}WAW$;

(vi) $XWAWA^{d,W} = X$ and $XWAWA^{\oplus,W} = (A^{\oplus,W}WAW)^{\dagger}A^{\oplus,W}(= A^{\diamond,W})$;

(vii) $XWAWA^{d,W} = X$ and $XWAWA^{\oplus,W}WAW = (A^{\oplus,W}WAW)^{\dagger}A^{\oplus,W}WAW$;

(viii) $XWAWA^{d,W} = X$ and $XWAWA^{\oplus,W}WAW(A^{\oplus,W}WAW)^{\dagger} = (A^{\oplus,W}WAW)^{\dagger}$;

(ix) $XWAWA^{d,W} = X$ and $XWAWA^{\oplus,W}WAW(A^{\oplus,W}WAW)^{*} = (A^{\oplus,W}WAW)^{*}$;

(x) $(A^{\oplus,W}WAW)^{\dagger}A^{\oplus,W}WAWX = X$ and $WAWX = WAW(A^{\oplus,W}WAW)^{\dagger}A^{d,W}$;

(xi) $(A^{\oplus,W}WAW)^{\dagger}A^{\oplus,W}WAWX = X$ and $A^{\oplus,W}WAWX = A^{d,W}$;

(xii) $(A^{\oplus,W}WAW)^{\dagger}A^{\oplus,W}WAWX = X$ and $WAWA^{\oplus,W}WAWX = WAWA^{d,W}$.

*Proof.* (i) ⇒ (ii)–(xii): Using $X = A^{\diamond,d,W} = (A^{\oplus,W}WAW)^{\dagger}A^{d,W}$, we show these implications.

(ii) ⇒ (i): It is evident by Theorem 2.1.

(iii) ⇒ (iv): Obviously.

(iv) ⇒ (i): We observe that

$$\begin{aligned} X &= (XWAW)A^{\oplus,W}(WAWX) \\ &= (A^{\oplus,W}WAW)^{\dagger}(A^{d,W}WAWA^{\oplus,W})WAW(A^{\oplus,W}WAW)^{\dagger}A^{d,W} \\ &= (A^{\oplus,W}WAW)^{\dagger}A^{\oplus,W}WAW(A^{\oplus,W}WAW)^{\dagger}A^{d,W} \\ &= (A^{\oplus,W}WAW)^{\dagger}A^{d,W}. \end{aligned}$$

(v) ⇒ (i): The assumptions $XWAWA^{d,W} = X$ and $XWAW = (A^{\oplus,W}WAW)^{\dagger}A^{d,W}WAW$ give

$$\begin{aligned} X &= (XWAW)A^{d,W} = (A^{\oplus,W}WAW)^{\dagger}(A^{d,W}WAWA^{d,W}) \\ &= (A^{\oplus,W}WAW)^{\dagger}A^{d,W}. \end{aligned}$$

(vi) ⇒ (i): Recall that $A^{d,W} = A^{\oplus,W}WAWA^{d,W}$. Note that $XWAWA^{d,W} = X$ and $XWAWA^{\oplus,W} = (A^{\oplus,W}WAW)^{\dagger}A^{\oplus,W}(= A^{\diamond,W})$ yield

$$\begin{aligned} X &= XWAWA^{d,W} = (XWAWA^{\oplus,W})WAWA^{d,W} \\ &= (A^{\oplus,W}WAW)^{\dagger}(A^{\oplus,W}WAWA^{d,W}) \\ &= (A^{\oplus,W}WAW)^{\dagger}A^{d,W}. \end{aligned}$$

(ix) ⇒ (viii) ⇒ (vii) ⇒ (vi): These implications are clear by the properties of the Moore-Penrose inverse and the $W$-weighted core–EP inverse.

We finish this proof in a similar way. □

Using Theorem 2.6, we obtain the next consequence for $W$-weighted Drazin invertible operator $A$.

**Corollary 2.7.** *Let $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{D,W}$, $\max\{ind(AW), ind(WA)\} = k$ and $X \in \mathcal{B}(H,K)$. Then the following statements are equivalent:*

(i) $X = A^{\diamond,D,W}$;

(ii) $XWAWA^{D,W} = X$ *and* $XW(AW)^{k+1} = (A^{\textcircled{d},W}WAW)^{\dagger}(AW)^{k}$;

(iii) $(A^{\mathbb{D},W}WAW)^{\dagger}A^{\mathbb{D},W}WAWX = X$ *and* $[(WA)^{k}]^{\dagger}WAWX = [(WA)^{k}]^{\dagger}WAWA^{D,W}$;

(iv) $(A^{\mathbb{D},W}WAW)^{\dagger}A^{\mathbb{D},W}WAWX = X$ *and* $[(WA)^{k}]^{*}WAWX = [(WA)^{k}]^{*}WAWA^{D,W}$.

*Proof.* (i) ⇒ (ii): It is clear by Theorem 2.6(v).
(ii) ⇒ (i): Applying $AWA^{\mathbb{D},W}WA^{\mathbb{D},W} = A^{\mathbb{D},W}$ and Theorem 2.6(vi), we get this implication.
(i) ⇔ (iii): This part is evident by Theorem 2.6(xii) and $WAWA^{\mathbb{D},W} = (WA)^{k}[(WA)^{k}]^{\dagger}$.
(i) ⇔ (iv): It is evident by properties of the Moore-Penrose inverse. □

As with Theorem 2.6, we show the following equivalent conditions for $X = A_{d,\diamond,W}$.

**Theorem 2.8.** *Let* $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{d,W}$ *and* $X \in \mathcal{B}(H,K)$. *Then the following statements are equivalent:*

(i) $X = A^{d,W}(WAWA_{\textcircled{d},W})^{\dagger}$;

(ii) $XWAWA_{\textcircled{d},W}WAWX = X$, $WAWA_{\textcircled{d},W}WAWXWAWA_{\textcircled{d},W}WAW = WAWA_{\textcircled{d},W}WAW$, $WAWA_{\textcircled{d},W}WAWX = WAWA_{\textcircled{d},W}(WAWA_{\textcircled{d},W})^{\dagger}$ *and* $XWAWA_{\textcircled{d},W}WAW = A^{d,W}WAW$;

(iii) $XWAWA_{\textcircled{d},W}WAWX = X$, $WAWXWAW = WAWA^{d,W}(WAWA_{\textcircled{d},W})^{\dagger}WAW$, $WAWX = WAWA^{d,W}(WAWA_{\textcircled{d},W})^{\dagger}$ *and* $XWAW = A^{d,W}(WAWA_{\textcircled{d},W})^{\dagger}WAW$;

(iv) $XWAWA_{\textcircled{d},W}WAWX = X$, $WAWX = WAWA^{d,W}(WAWA_{\textcircled{d},W})^{\dagger}$ *and* $XWAW = A^{d,W}(WAWA_{\textcircled{d},W})^{\dagger}WAW$;

(v) $A^{d,W}WAWX = X$ *and* $WAWX = WAWA^{d,W}(WAWA_{\textcircled{d},W})^{\dagger}$;

(vi) $A^{d,W}WAWX = X$ *and* $A_{\textcircled{d},W}WAWX = A_{\textcircled{d},W}(WAWA_{\textcircled{d},W})^{\dagger}(= A_{\diamond,W})$;

(vii) $A^{d,W}WAWX = X$ *and* $WAWA_{\textcircled{d},W}WAWX = WAWA_{\textcircled{d},W}(WAWA_{\textcircled{d},W})^{\dagger}$;

(viii) $A^{d,W}WAWX = X$ *and* $(WAWA_{\textcircled{d},W})^{\dagger}WAWA_{\textcircled{d},W}WAWX = (WAWA_{\textcircled{d},W})^{\dagger}$;

(ix) $A^{d,W}WAWX = X$ *and* $(WAWA_{\textcircled{d},W})^{*}WAWA_{\textcircled{d},W}WAWX = (WAWA_{\textcircled{d},W})^{*}$;

(x) $XWAWA_{\textcircled{d},W}(WAWA_{\textcircled{d},W})^{\dagger} = X$ *and* $XWAW = A^{d,W}(WAWA_{\textcircled{d},W})^{\dagger}WAW$;

(xi) $XWAWA_{\textcircled{d},W}(WAWA_{\textcircled{d},W})^{\dagger} = X$ *and* $XWAWA_{\textcircled{d},W} = A^{d,W}$;

(xii) $XWAWA_{\textcircled{d},W}(WAWA_{\textcircled{d},W})^{\dagger} = X$ *and* $XWAWA_{\textcircled{d},W}WAW = A^{d,W}WAW$.

Theorem 2.8 gives more characterizations for $X = A^{D,\diamond,W}$.

**Corollary 2.9.** *Let* $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{D,W}$, $\max\{ind(AW), ind(WA)\} = k$ *and* $X \in \mathcal{B}(H,K)$. *Then the following statements are equivalent:*

(i) $X = A^{D,\diamond,W}$;

(ii) $A^{D,W}WAWX = X$ *and* $(WA)^{k+1}WX = (WA)^{k}(WAWA_{\mathbb{D},W})^{\dagger}$;

(iii) $XWAWA_{\mathbb{D},W}(WAWA_{\mathbb{D},W})^{\dagger} = X$ *and* $XWAW[(AW)^{k}]^{\dagger} = A^{D,W}WAW[(AW)^{k}]^{\dagger}$;

(iv) $XWAWA_{\mathbb{D},W}(WAWA_{\mathbb{D},W})^{\dagger} = X$ *and* $XWAW[(AW)^{k}]^{*} = A^{D,W}WAW[(AW)^{k}]^{*}$.

By Theorem 2.6 and Theorem 2.8, we deduce that $A^{\diamond,d,W}$ and $A_{d,\diamond,W}$, respectively, are both outer and inner inverses of $WAWA^{\textcircled{d},W}WAW$ and $WAWA_{\textcircled{d},W}WAW$. We find their ranges and kernels in the following result.

**Lemma 2.10.** *If* $W \in \mathcal{B}(K,H)\backslash\{0\}$ *and* $A \in \mathcal{B}(H,K)^{d,W}$, *then*

(i) $A^{\diamond,d,W}WAWA^{\textcircled{\dagger},W}WAW$ *is the orthogonal projection onto* $R((A^{\textcircled{\dagger},W}WAW)^*)$;

(ii) $WAWA^{\textcircled{\dagger},W}WAWA^{\diamond,d,W}$ *is the projection onto* $R(WA^{d,W})$ *along* $N(A^{d,W})$;

(iii) $A^{\diamond,d,W} = (WAWA^{\textcircled{\dagger},W}WAW)^{(1,2,4)}_{R((A^{\textcircled{\dagger},W}WAW)^*),N(A^{d,W})}$;

(iv) $A_{d,\diamond,W}WAWA_{\textcircled{\dagger},W}WAW$ *is a projection onto* $R(A^{d,W})$ *along* $N(A^{d,W}W)$;

(v) $WAWA_{\textcircled{\dagger},W}WAWA_{d,\diamond,W}$ *is the orthogonal projection onto* $R(WAWA_{\textcircled{\dagger},W})$;

(vi) $A_{d,\diamond,W} = (WAWA_{\textcircled{\dagger},W}WAW)^{(1,2,3)}_{R(A^{d,W}),N((WAWA_{\textcircled{\dagger},W})^*)}$.

*Proof.* (i) Theorem 2.1 gives $A^{\diamond,d,W}WAWA^{\textcircled{\dagger},W}WAW = (A^{\textcircled{\dagger},W}WAW)^\dagger A^{\textcircled{\dagger},W}WAW$ is the orthogonal projection onto $R((A^{\textcircled{\dagger},W}WAW)^\dagger) = R((A^{\textcircled{\dagger},W}WAW)^*)$.

(ii) $WAWA^{\textcircled{\dagger},W}WAWA^{\diamond,d,W} = WAWA^{d,W}$ is the projection onto $R(WAWA^{d,W}) = R(WA^{d,W})$ along $N(WAWA^{d,W}) = N(A^{d,W})$.

(iii) It follows by (i) and (ii).

Similarly, we show the rest. □

Lemma 2.10 yields the next consequences.

**Corollary 2.11.** *If* $W \in \mathcal{B}(K,H)\backslash\{0\}$ *and* $A \in \mathcal{B}(H,K)^{d,W}$, *then*

(i) $A^{\diamond,d,W} = (WAWA^{\textcircled{\dagger},W}WAW)^\dagger$ *if and only if* $WAWA^{d,W} = (WAWA^{d,W})^*$;

(ii) $A_{d,\diamond,W} = (WAWA_{\textcircled{\dagger},W}WAW)^\dagger$ *if and only if* $A^{d,W}WAW = (A^{d,W}WAW)^*$.

*Proof.* (i) It is clear by Lemma 2.10(iii) and $WAWA^{\textcircled{\dagger},W}WAWA^{\diamond,d,W} = WAWA^{d,W}$.

The part (ii) follows in a same manner. □

**Corollary 2.12.** *If* $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{D,W}$ *and* $\max\{ind(AW), ind(WA)\} = k$, *then*

(i) $A^{\diamond,D,W}WAWA^{\textcircled{D},W}WAW$ *is the orthogonal projection onto* $R((WAW)^*(WA)^k)$;

(ii) $WAWA^{\textcircled{D},W}WAWA^{\diamond,D,W}$ *is the projection onto* $R((WA)^k)$ *along* $N((WA)^k)$;

(iii) $A^{\diamond,D,W} = (WAWA^{\textcircled{D},W}WAW)^{(1,2,4)}_{R((WAW)^*(WA)^k),N((WA)^k)}$;

(iv) $A_{D,\diamond,W}WAWA_{\textcircled{D},W}WAW$ *is a projection onto* $R((AW)^k)$ *along* $N((AW)^k)$;

(v) $WAWA_{\textcircled{D},W}WAWA_{D,\diamond,W}$ *is the orthogonal projection onto* $R(WAW[(AW)^k]^*)$;

(vi) $A_{D,\diamond,W} = (WAWA_{\textcircled{D},W}WAW)^{(1,2,3)}_{R((AW)^k),N((AW)^k(WAW)^*)}$.

Some expressions for $A^{\diamond,W}$ and $A_{\diamond,W}$ presented in [19] imply the following representations of $A^{\diamond,d,W}$ and $A_{d,\diamond,W}$.

**Corollary 2.13.** *If* $W \in \mathcal{B}(K,H)\backslash\{0\}$ *and* $A \in \mathcal{B}(H,K)^{d,W}$, *we have*

$$A^{\diamond,d,W} = (WAWA^{\textcircled{\dagger},W}WAW)^\dagger WAWA^{d,W}$$

*and*

$$A_{d,\diamond,W} = A^{d,W}WAW(WAWA_{\textcircled{\dagger},W}WAW)^\dagger.$$

*Proof.* According to [19], $A^{\diamond,W} = (WAWA^{\textcircled{\dagger},W}WAW)^\dagger$ and $A_{\diamond,W} = (WAWA_{\textcircled{\dagger},W}WAW)^\dagger$. The rest follows by Corollary 2.3. □

Consequently, we obtain the next result for $W$-weighted Drazin invertible operator.

**Corollary 2.14.** *Let $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{D,W}$ and $\max\{ind(AW), ind(WA)\} = k$. Then*

$$A^{\diamond,D,W} = ((WA)^k[(WA)^k]^\dagger WAW)^\dagger WAWA^{D,W}$$

*and*

$$A_{D,\diamond,W} = A^{D,W}WAW(WAW[(AW)^k]^\dagger (AW)^k)^\dagger.$$

*Proof.* By [19], $A^{\diamond,W} = ((WA)^k[(WA)^k]^\dagger WAW)^\dagger$ and $A_{\diamond,W} = (WAW[(AW)^k]^\dagger (AW)^k)^\dagger$. □

More characterizations for $A^{\diamond,d,W}$ and $A_{d,\diamond,W}$ follow.

**Theorem 2.15.** *If $W \in \mathcal{B}(K,H)\backslash\{0\}$ and $A \in \mathcal{B}(H,K)^{d,W}$, then*

(a) *$X = A^{\diamond,d,W}$ is the uniquely determined solution to*

$$WAWA^{\textcircled{d},W}WAWX = P_{R(WA^{d,W}),N(A^{d,W})} \quad \text{and} \quad R(X) \subseteq R((A^{\textcircled{d},W}WAW)^*); \tag{2}$$

(b) *$X = A^{\diamond,d,W}$ is the uniquely determined solution to*

$$XWAWA^{\textcircled{d},W}WAW = P_{R((A^{\textcircled{d},W}WAW)^*)} \quad \text{and} \quad R(X^*) \subseteq R((A^{d,W})^*); \tag{3}$$

(c) *$X = A_{d,\diamond,W}$ is the uniquely determined solution to*

$$WAWA_{\textcircled{d},W}WAWX = P_{R(WAWA_{\textcircled{d},W})} \quad \text{and} \quad R(X) \subseteq R(A^{d,W});$$

(d) *$X = A_{d,\diamond,W}$ is the uniquely determined solution to*

$$XWAWA_{\textcircled{d},W}WAW = P_{R(A^{d,W}),N(A^{d,W}W)} \quad \text{and} \quad R(X^*) \subseteq R(WAWA_{\textcircled{d},W}).$$

*Proof.* (a) Using Lemma 2.10, we deduce that $X = (A^{\textcircled{d},W}WAW)^\dagger A^{d,W}$ satisfies conditions in (2).
Let (2) hold for two operators $Z$ and $X$. Then

$$WAWA^{\textcircled{d},W}WAW(Z - X) = P_{R(WA^{d,W}),N(A^{d,W})} - P_{R(WA^{d,W}),N(A^{d,W})} = 0$$

gives $R(Z - X) \subseteq N(WAWA^{\textcircled{d},W}WAW) \subseteq N(A^{\textcircled{d},W}WAW)$. Because

$$R(Z - X) \subseteq R((A^{\textcircled{d},W}WAW)^*) \cap N(A^{\textcircled{d},W}WAW) \subseteq N(A^{\textcircled{d},W}WAW)^\perp \cap N(A^{\textcircled{d},W}WAW) = \{0\},$$

we have $Z = X = A^{\diamond,d,W}$ is the unique solution to (2).

(b) Note that $X = (A^{\textcircled{d},W}WAW)^\dagger A^{d,W}$ is a solution to (3) by Lemma 2.10 and $R(X^*) \subseteq R((A^{d,W})^*) = R([(WA)^d]^*)$.

If two operators $Z$ and $X$ satisfy (3), then, by $(WAWA^{\textcircled{d},W})^* = WAWA^{\textcircled{d},W}$,

$$\begin{aligned} R(Z^* - X^*) &\subseteq R((A^{d,W})^*) \cap N((WAWA^{\textcircled{d},W}WAW)^*) \\ &= R([(WA)^d]^*) \cap N(WAWA^{\textcircled{d},W}) = R([(WA)^d]^*) \cap N(A^{\textcircled{d},W}) \\ &= R([(WA)^d]^*) \cap R((A^{\textcircled{d},W})^*) = R([(WA)^d]^*) \cap R((WA)^d) = \{0\}, \end{aligned}$$

i.e. $Z = X$.

The parts (c) and (d) follow similarly. □

**Corollary 2.16.** *If $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{D,W}$ and $\max\{ind(AW), ind(WA)\} = k$, then*

(a) $X = A^{\diamond,D,W}$ *is the uniquely determined solution to*

$$WAWA^{\text{Ⓓ},W}WAWX = P_{R((WA)^k),N((WA)^k)} \quad \textit{and} \quad R(X) \subseteq R((WAW)^*(WA)^k);$$

(b) $X = A^{\diamond,d,W}$ *is the uniquely determined solution to*

$$XWAWA^{\text{Ⓓ},W}WAW = P_{R((WAW)^*(WA)^k)} \quad \textit{and} \quad R(X^*) \subseteq R([(WA)^k]^*);$$

(c) $X = A_{D,\diamond,W}$ *is the uniquely determined solution to*

$$WAWA_{\text{Ⓓ},W}WAWX = P_{R(WAW[(AW)^k]^*)} \quad \textit{and} \quad R(X) \subseteq R((AW)^k);$$

(d) $X = A_{D,\diamond,W}$ *is the uniquely determined solution to*

$$XWAWA_{\text{Ⓓ},W}WAW = P_{R((AW)^k),N((AW)^k)} \quad \textit{and} \quad R(X^*) \subseteq R(WAW[(AW)^k]^*).$$

We can characterize the $W$-g-MPD inverse in the next way.

**Theorem 2.17.** *Let $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{d,W}$ and $X \in \mathcal{B}(H,K)$. Then the following statements are equivalent:*

(i) $X = A^{\diamond,d,W}$;

(ii) $R(X) = R((A^{\text{ⓓ},W}WAW)^*)$ *and* $WAWA^{\text{ⓓ},W}WAWX = WAWA^{d,W}$;

(iii) $R(X) = R((A^{\text{ⓓ},W}WAW)^*)$ *and* $A^{\text{ⓓ},W}WAWX = A^{d,W}$;

(iv) $N(X) = N(A^{d,W})$ *and* $XWAWA^{\text{ⓓ},W}WAW = (A^{\text{ⓓ},W}WAW)^\dagger A^{\text{ⓓ},W}WAW$;

(v) $N(X) = N(A^{d,W})$ *and* $XWAWA^{d,W} = (A^{\text{ⓓ},W}WAW)^\dagger A^{d,W}$.

*Proof.* (i) ⇒ (ii): This implication is evident using Theorem 2.1 and Lemma 2.10.

(ii) ⇒ (iii): Since $WAWA^{\text{ⓓ},W}WAWX = WAWA^{d,W}$, we have

$$\begin{aligned} A^{\text{ⓓ},W}WAWX &= A^{\text{ⓓ},W}(WAWA^{\text{ⓓ},W}WAWX) \\ &= A^{\text{ⓓ},W}WAWA^{d,W} \\ &= A^{d,W}. \end{aligned}$$

(iii) ⇒ (i): Becuase $R(X) = R((A^{\text{ⓓ},W}WAW)^*)$, $X = (A^{\text{ⓓ},W}WAW)^*U$, for some $U \in \mathcal{B}(H,K)$. Now,

$$\begin{aligned} X &= (A^{\text{ⓓ},W}WAW)^\dagger A^{\text{ⓓ},W}WAW(A^{\text{ⓓ},W}WAW)^*U \\ &= (A^{\text{ⓓ},W}WAW)^\dagger A^{\text{ⓓ},W}WAWX \\ &= (A^{\text{ⓓ},W}WAW)^\dagger A^{d,W} \\ &= A^{\diamond,d,W}. \end{aligned}$$

The rest can be completed in a similar manner. □

We verify the next result related to the $W$-g-DMP inverse as Theorem 2.17.

**Theorem 2.18.** *Let $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{d,W}$ and $X \in \mathcal{B}(H,K)$. Then the following statements are equivalent:*

(i) $X = A_{d,\diamond,W}$;

(ii) $N(X) = N((WAWA_{\text{ⓓ},W})^*)$ *and* $XWAWA_{\text{ⓓ},W}WAW = A^{d,W}WAW$;

(iii) $N(X) = N((WAWA_{㉪,W})^*)$ *and* $XWAWA_{㉪,W} = A^{d,W}$;

(iv) $R(X) = R(A^{d,W})$ *and* $WAWA_{㉪,W}WAWX = WAWA_{㉪,W}(WAWA_{㉪,W})^\dagger$;

(v) $R(X) = R(A^{d,W})$ *and* $A^{d,W}WAWX = A^{d,W}(WAWA_{㉪,W})^\dagger$.

It is interesting to study equivalent conditions for $A^{\diamond,d,W} = A^{\diamond,W}$.

**Theorem 2.19.** *If* $W \in \mathcal{B}(K,H)\backslash\{0\}$ *and* $A \in \mathcal{B}(H,K)^{d,W}$, *the following statements are equivalent:*

(i) $A^{\diamond,d,W} = A^{\diamond,W}$;

(ii) $A^{d,W} = A^{㉪,W}$;

(iii) $A_{d,\diamond,W} = A_{\diamond,W}$.

*Proof.* (i) ⇒ (ii): Notice that $A^{\diamond,d,W} = A^{\diamond,W}$, i.e. $(A^{㉪,W}WAW)^\dagger A^{d,W} = (A^{㉪,W}WAW)^\dagger A^{㉪,W}$ gives

$$\begin{aligned} A^{d,W} &= A^{㉪,W}WAWA^{d,W} = A^{㉪,W}WAW(A^{㉪,W}WAW)^\dagger(A^{㉪,W}WAWA^{d,W}) \\ &= A^{㉪,W}WAW((A^{㉪,W}WAW)^\dagger A^{d,W}) = A^{㉪,W}WAW(A^{㉪,W}WAW)^\dagger A^{㉪,W} \\ &= A^{㉪,W}WAW(A^{㉪,W}WAW)^\dagger A^{㉪,W}WAWA^{㉪,W} \\ &= A^{㉪,W}WAWA^{㉪,W} = A^{㉪,W} \end{aligned}$$

(ii) ⇒ (i): It is clear.
(i) ⇔ (iii): This equivalence follows similarly as (i) ⇔ (ii). □

Consequently, we characterize the equality $A^{\diamond,d} = A^{\diamond}$.

**Corollary 2.20.** *If* $A \in \mathcal{B}(H)^d$, *the following statements are equivalent:*

(i) $A^{\diamond,d} = A^{\diamond}$;

(ii) $A^d = A^{㉪}$;

(iii) $A_{d,\diamond} = A_{\diamond}$.

## 3. Expressions for $A^{\diamond,d,W}$

The following operator matrix forms of $A$ and $W$ with the corresponding expression for $A^{d,W}$ were developed in [17], and the adequate representation for the $W$-gMP inverse was established with [19, Theorem 3.2].

**Lemma 3.1.** [17, Theorem 2.1] *Let* $W \in \mathcal{B}(K,H)\backslash\{0\}$ *and* $A \in \mathcal{B}(H,K)^{d,W}$. *Then*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \to \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix} \tag{4}$$

*and*

$$W = \begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix} : \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix} \to \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix}, \tag{5}$$

*where* $A_1 \in \mathcal{B}(R((WA)^d), R((AW)^d))^{-1}$, $W_1 \in \mathcal{B}(R((AW)^d), R((WA)^d))^{-1}$, $A_3W_3 \in \mathcal{B}(N[((AW)^d)^*])^{qnil}$ *and* $W_3A_3 \in \mathcal{B}(N[((WA)^d)^*])^{qnil}$. *Furthermore,*

$$A^{d,W} = \begin{bmatrix} (W_1A_1W_1)^{-1} & W_1^{-1}U \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \to \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix} \tag{6}$$

*and*

$$A^{\diamond,W} = \begin{bmatrix} (I+EE^*)^{-1}(W_1A_1W_1)^{-1} & 0 \\ E^*(I+EE^*)^{-1}(W_1A_1W_1)^{-1} & 0 \end{bmatrix} : \begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \to \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix}, \tag{7}$$

*where*

$$U = \sum_{n=0}^{\infty} (W_1A_1)^{-(n+2)}(W_1A_2 + W_2A_3)(W_3A_3)^n$$

*and*

$$E = W_1^{-1}W_2 + W_1^{-1}A_1^{-1}(A_2W_3 + W_1^{-1}W_2A_3W_3).$$

We now present the operator matrix form of $A^{\diamond,d,W}$.

**Theorem 3.2.** *If $W \in \mathcal{B}(K,H)\backslash\{0\}$ and $A \in \mathcal{B}(H,K)^{d,W}$ are given as in* (4) *and* (5)*, respectively, then*

$$A^{\diamond,d,W} = \begin{bmatrix} (I+EE^*)^{-1}(W_1A_1W_1)^{-1} & (I+EE^*)^{-1}W_1^{-1}U \\ E^*(I+EE^*)^{-1}(W_1A_1W_1)^{-1} & E^*(I+EE^*)^{-1}W_1^{-1}U \end{bmatrix} :$$

$$\begin{bmatrix} R((WA)^d) \\ N[((WA)^d)^*] \end{bmatrix} \to \begin{bmatrix} R((AW)^d) \\ N[((AW)^d)^*] \end{bmatrix}, \tag{8}$$

*where $E$ and $U$ are represented in Lemma 3.1.*

*Proof.* Let $A$, $W$, $A^{d,W}$ and $A^{\diamond,W}$, respectively, be represented by (4), (5), (6) and (7). We complete this proof using the expression $A^{\diamond,d,W} = A^{\diamond,W}WAWA^{d,W}$ presented in Corollary 2.3. □

With respect to the orthogonal sums $H = \overline{R(W)} \oplus N(W^*)$ and $K = \overline{R(A)} \oplus N(A^*)$, the operators $A \in \mathcal{B}(H,K)$ and $W \in \mathcal{B}(K,H)\backslash\{0\}$ can be represented by [18] as follows:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(A)} \\ N(A^*) \end{bmatrix} \tag{9}$$

and

$$W = \begin{bmatrix} W_1 & W_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(A)} \\ N(A^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix}, \tag{10}$$

where $A_1A_1^* + A_2A_2^* \in \mathcal{B}(\overline{R(A)})$.

**Theorem 3.3.** *If $W \in \mathcal{B}(K,H)\backslash\{0\}$ and $A \in \mathcal{B}(H,K)^{d,W}$ are given as in* (9) *and* (10)*, respectively, then*

$$A^{\diamond,d,W} = \begin{bmatrix} (A_1^{\textcircled{a},W_1}W_1A_1W_1)^*D^\dagger A_1^{d,W_1} & (A_1^{\textcircled{a},W_1}W_1A_1W_1)^*D^\dagger A_1^{\textcircled{a},W_1}W_1A_1^{d,W_1}W_1A_2 \\ (A_1^{\textcircled{a},W_1}W_1A_1W_2)^*D^\dagger A_1^{d,W_1} & (A_1^{\textcircled{a},W_1}W_1A_1W_2)^*D^\dagger A_1^{\textcircled{a},W_1}W_1A_1^{d,W_1}W_1A_2 \end{bmatrix} :$$

$$\begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(A)} \\ N(A^*) \end{bmatrix},$$

*where*

$$D = A_1^{\textcircled{a},W_1}W_1A_1W_1(A_1^{\textcircled{a},W_1}W_1A_1W_1)^* + A_1^{\textcircled{a},W_1}W_1A_1W_2(A_1^{\textcircled{a},W_1}W_1A_1W_2)^*.$$

*Proof.* For $A$ and $W$ expressed as in (9) and (10), respectively, we have by [18] and [19] that

$$A^{d,W} = \begin{bmatrix} A_1^{d,W_1} & (A_1^{d,W_1}W_1)^2A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(A)} \\ N(A^*) \end{bmatrix}$$

and

$$A^{\diamond,W} = \begin{bmatrix} (A_1^{\textcircled{a},W_1}W_1A_1W_1)^*D^\dagger A_1^{\textcircled{a},W_1} & 0 \\ (A_1^{\textcircled{a},W_1}W_1A_1W_2)^*D^\dagger A_1^{\textcircled{a},W_1} & 0 \end{bmatrix} : \begin{bmatrix} \overline{R(W)} \\ N(W^*) \end{bmatrix} \to \begin{bmatrix} \overline{R(A)} \\ N(A^*) \end{bmatrix}.$$

Applying $A^{\diamond,d,W} = A^{\diamond,W}WAWA^{d,W}$, we finish the proof. □

The maximal classes of operators $G$ and $H$ such that $A^{\diamond,d,W} = (GWAW)^{\dagger}H$ are investigated next.

**Theorem 3.4.** *Let $W \in \mathcal{B}(K,H)\backslash\{0\}$, $G, E \in \mathcal{B}(H,K)$ and $A \in \mathcal{B}(H,K)^{d,W}$ such that $R(GWAW)$ is closed. Then the following statements are equivalent:*

(i) $A^{\diamond,d,W} = (GWAW)^{\dagger}E$;

(ii) $(GWAW)^{\dagger}EWAW = (A^{\circledast,W}WAW)^{\dagger}A^{d,W}WAW$ *and* $(GWAW)^{\dagger}E(I - WAWA^{d,W}) = 0$.

*Proof.* (i) ⇒ (ii): Because $A^{\diamond,d,W} = (A^{\circledast,W}WAW)^{\dagger}A^{d,W} = (GWAW)^{\dagger}E$, it follows

$$(A^{\circledast,W}WAW)^{\dagger}A^{d,W}WAW = (GWAW)^{\dagger}EWAW$$

and

$$(GWAW)^{\dagger}E(I - WAWA^{d,W}) = (A^{\circledast,W}WAW)^{\dagger}A^{d,W}(I - WAWA^{d,W}) = 0.$$

(ii) ⇒ (i): Using $(GWAW)^{\dagger}EWAW = (A^{\circledast,W}WAW)^{\dagger}A^{d,W}WAW$ and $(GWAW)^{\dagger}E(I - WAWA^{d,W}) = 0$, we obtain

$$\begin{aligned} A^{\diamond,d,W} &= (A^{\circledast,W}WAW)^{\dagger}A^{d,W} = ((A^{\circledast,W}WAW)^{\dagger}A^{d,W}WAW)A^{d,W} \\ &= (GWAW)^{\dagger}EWAWA^{d,W} = (GWAW)^{\dagger}E. \end{aligned}$$

□

We consequently get new properties of the g-MPD inverse.

**Corollary 3.5.** *Let $G, E \in \mathcal{B}(H)$ and $A \in \mathcal{B}(H)^{d}$ such that $R(GA)$ is closed. Then the following statements are equivalent:*

(i) $A^{\diamond,d} = (GA)^{\dagger}E$;

(ii) $(GA)^{\dagger}EA = (A^{\circledast}A)^{\dagger}A^{d}A$ *and* $(GA)^{\dagger}E(I - AA^{d,W}) = 0$.

One more expression for $A^{\diamond,d,W}$ is given.

**Theorem 3.6.** *If $W \in \mathcal{B}(K,H)\backslash\{0\}$ and $A \in \mathcal{B}(H,K)^{d,W}$, we have*

$$A^{\diamond,d,W} = ((A^{\circledast,W})^{\dagger}A^{\circledast,W}WAW)^{\dagger}WAWA^{d,W}.$$

*Proof.* According to [19], $A^{\diamond,W} = ((A^{\circledast,W})^{\dagger}A^{\circledast,W}WAW)^{\dagger}$. The rest follows by $A^{\diamond,d,W} = A^{\diamond,W}WAWA^{d,W}$. □

As a consequence of Theorem 3.6, we obtain new representation for $A^{\diamond,d}$.

**Corollary 3.7.** *If $A \in \mathcal{B}(H)^{d}$, we have*

$$A^{\diamond,d} = ((A^{\circledast})^{\dagger}A^{\circledast}A)^{\dagger}AA^{d}.$$

## 4. Applications of $W$-g-MPD inverse

We firstly apply the $W$-g-MPD inverse to solve certain systems of linear equations and represent their solutions.

**Theorem 4.1.** *If $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{d,W}$ and $b \in H$, the general solution to*

$$A^{\circledast,W}WAWx = A^{d,W}b \tag{11}$$

*is*

$$x = A^{\diamond,D,W}b + (I - A^{\diamond,W}WAW)c, \tag{12}$$

*for arbitrary $c \in H$.*

*Proof.* Notice that

$$\begin{aligned} A^{\circledast,W}WAWA^{\diamond,d,W} &= A^{\circledast,W}WAW(A^{\circledast,W}WAW)^{\dagger}A^{d,W} \\ &= (A^{\circledast,W}WAW(A^{\circledast,W}WAW)^{\dagger}A^{\circledast,W}WAW)A^{d,W} \\ &= A^{\circledast,W}WAWA^{d,W} \\ &= A^{d,W}. \end{aligned}$$

Now, for $x$ given by (12), we have

$$A^{\circledast,W}WAWx = A^{d,W}b + (A^{\circledast,W}WAW - A^{\circledast,W}WAW)c = A^{d,W}b.$$

We deduce that $x$ is a solution to the equation (11).

If $x$ is a solution to (11), then

$$(A^{\circledast,W}WAW)^{\dagger}A^{\circledast,W}WAWx = (A^{\circledast,W}WAW)^{\dagger}A^{d,W}b = A^{\diamond,d,W}b,$$

which yields

$$x = A^{\diamond,d,W}b + x - (A^{\circledast,W}WAW)^{\dagger}A^{\circledast,W}WAWx = A^{\diamond,d,W}b + (I - A^{\diamond,W}WAW)x.$$

So, (12) is the form of $x$. □

Theorem 4.1 implies the next result about $W$-weighted Drazin invertible operators.

**Corollary 4.2.** *Let $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{D,W}$ and $\max\{ind(AW), ind(WA)\} = k$. If $b \in H$, the general solution to*

$$[(WA)^k]^*WAWx = [(WA)^k]^*WA(WA)^Db \tag{13}$$

*is expressed by* (12).

*Proof.* Using $(WA)^{\circledast} = (WA)^D(WA)^k[(WA)^k]^{\dagger}$, $A^{\circledast,W} = A[(WA)^D]^2(WA)^k[(WA)^k]^{\dagger}$. Since

$$\begin{aligned} A^{\circledast,W}WAWx = A^{d,W}b \Leftrightarrow \;&\Leftrightarrow\; A[(WA)^D]^2(WA)^k[(WA)^k]^{\dagger}WAWx = A[(WA)^D]^2b \\ &\Leftrightarrow\; (WA)^D(WA)^k[(WA)^k]^{\dagger}WAWx = (WA)^Db \\ &\Leftrightarrow\; (WA)^k[(WA)^k]^{\dagger}WAWx = WA(WA)^Db \\ &\Leftrightarrow\; [(WA)^k]^*WAWx = [(WA)^k]^*WA(WA)^Db, \end{aligned}$$

by Theorem 4.1, the rest follows. □

Taking $b \in R((WA)^k)$ in Corollary 4.2, we solve the following equation.

**Corollary 4.3.** *If $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{D,W}$ and $\max\{ind(AW), ind(WA)\} = k$, the general solution to*

$$[(WA)^k]^*WAWx = [(WA)^k]^*b, \quad b \in R((WA)^k),$$

*is expressed by*

$$x = A^{\diamond,W}b + (I - A^{\diamond,W}WAW)c,$$

*for arbitrary $c \in H$.*

*Proof.* The fact $b \in R((WA)^k) = R(WA(WA)^D)$ gives $b = WA(WA)^Db$ and so, by Corollary 2.3, $A^{\diamond,D,W}b = A^{\diamond,W}WAWA^{D,W}b = A^{\diamond,W}WA(WA)^Db = A^{\diamond,W}$. The rest is clear by Corollary 4.2. □

We study when the equation (11) has the unique solution.

**Theorem 4.4.** *If $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{d,W}$ and $b \in H$, $x = A^{\diamond,d,W}b$ is the uniquely determined solution in $R((A^{\circledast,W}WAW)^*)$ of* (11).

*Proof.* The equation (11) has a solution $x = A^{\diamond,d,W}b$ by Theorem 4.1. Lemma 2.10 gives that $x = A^{\diamond,d,W}b \in R(A^{\diamond,d,W}) = R((A^{\textcircled{d},W}WAW)^*)$.

Assume that $x, z \in R((A^{\textcircled{d},W}WAW)^*)$ are two solutions to (11). From

$$x - z \in R((A^{\textcircled{d},W}WAW)^*) \cap N(A^{\textcircled{d},W}WAW) = N(A^{\textcircled{d},W}WAW)^{\perp} \cap N(A^{\textcircled{d},W}WAW) = \{0\},$$

(11) has in $R((A^{\textcircled{d},W}WAW)^*)$ the unique solution $z = x = A^{\diamond,d,W}b$. □

Using $W$-g-DMP inverse, we solve certain systems of linear equations in an analogue way.

**Theorem 4.5.** *If $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{d,W}$ and $b \in H$, the general solution to*

$$A_{\textcircled{d},W}WAWx = A_{\diamond,W}b \tag{14}$$

*is*

$$x = A_{d,\diamond,W}b + (I - A^{d.W}WAW)c,$$

*for arbitrary $c \in K$.*

**Corollary 4.6.** *Let $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{D,W}$ and $\max\{ind(AW), ind(WA)\} = k$. If $b \in H$, the general solution to*

$$(AW)^k x = (AW)^k A_{\diamond,W}b \quad (\textit{or equivalently} \;\; [(AW)^k]^{\dagger}(AW)^k x = A_{\diamond,W}b)$$

*is*

$$x = A_{D,\diamond,W}b + (I - A^{D,W}WAW)c,$$

*for arbitrary $c \in K$.*

**Theorem 4.7.** *If $W \in \mathcal{B}(K,H)\backslash\{0\}$, $A \in \mathcal{B}(H,K)^{d,W}$ and $b \in H$, $x = A_{d,\diamond,W}b$ is the uniquely determined solution in $R(A^{d,W})$ of* (14).

Applying the $W$-g-MPD and $W$-g-DMP inverses, solvability of certain minimization problems are verified.

**Theorem 4.8.** *If $B \in \mathbb{C}^{m\times m}$, $W \in \mathbb{C}^{n\times m}\backslash\{0\}$, $A \in \mathbb{C}^{m\times n}$ and $\max\{ind(AW), ind(WA)\} = k$, $X = BA^{\diamond,D,W}$ is the uniquely determined solution to*

$$\min \|XWAWA^{\mathbb{D},W}WAW - B\|_F \quad \text{subject to} \quad X \in \mathbb{C}^{m\times n}(WA)^k. \tag{15}$$

*Proof.* Using Theorem 2.1 and Corollary 2.12, we know that $A^{\diamond,D,W}WAWA^{\mathbb{D},W}WAW = (A^{\mathbb{D},W}WAW)^{\dagger}A^{\mathbb{D},W}WAW$ is the orthogonal projection onto $R((WAW)^*(WA)^k)$. Since

$$\begin{aligned} B &= BA^{\diamond,D,W}WAWA^{\mathbb{D},W}WAW + B(I - A^{\diamond,D,W}WAWA^{\mathbb{D},W}WAW) \\ &= B(A^{\mathbb{D},W}WAW)^{\dagger}A^{\mathbb{D},W}WAW + B(I - (A^{\mathbb{D},W}WAW)^{\dagger}A^{\mathbb{D},W}WAW), \end{aligned}$$

Pythagorean theorem implies

$$\begin{aligned} \|XWAWA^{\mathbb{D},W}WAW - B\|_F^2 &= \|XWAWA^{\mathbb{D},W}WAW - B(A^{\mathbb{D},W}WAW)^{\dagger}A^{\mathbb{D},W}WAW\|_F^2 \\ &+ \|B(I - (A^{\mathbb{D},W}WAW)^{\dagger}A^{\mathbb{D},W}WAW)\|_F^2 \end{aligned}$$

is minimal for $XWAWA^{\mathbb{D},W}WAW = B(A^{\mathbb{D},W}WAW)^{\dagger}A^{\mathbb{D},W}WAW = BA^{\diamond,D,W}WAWA^{\mathbb{D},W}WAW$, attained at $X = BA^{\diamond,D,W} \in \mathbb{C}^{m\times n}(WA)^k$.

For a solution $X$ to minimization problem (15) and for some $Y \in \mathbb{C}^{m\times n}$, we get $X = Y(WA)^k = (Y(WA)^k)WA(WA)^D = XWA(WA)^D$. So,

$$\begin{aligned} X &= XWA(WA)^D = (XWAWA^{\mathbb{D},W}WAW)A^{D,W} \\ &= BA^{\diamond,D,W}WAW(A^{\mathbb{D},W}WAWA^{D,W}) = BA^{\diamond,D,W}WAWA^{D,W} \\ &= BA^{\diamond,D,W} \end{aligned}$$

is the unique solution to (15). □

Similarly as Theorem 4.8, we obtain solvability of the next minimization problem.

**Theorem 4.9.** *If $B \in \mathbb{C}^{n\times n}$, $W \in \mathbb{C}^{n\times m}\backslash\{0\}$, $A \in \mathbb{C}^{m\times n}$ and $\max\{ind(AW), ind(WA)\} = k$, $X = A_{D,\diamond,W}B$ is the uniquely determined solution to*

$$\min \|WAWA_{\mathbb{D},W}WAWX - B\|_F \quad \text{subject to} \quad X \in (AW)^k\mathbb{C}^{m\times n}.$$

To illustrate the previous results, we present the next example.

**Example 4.10.** *Let A and W be given as in Example 2.4,* $b = \begin{bmatrix} 3 \\ 171 \\ 171 \\ 0 \end{bmatrix}$ *and* $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$. *Since*

$$\begin{aligned} x &= A^{\diamond,D,W}b + (I - A^{\diamond,W}WAW)c \\ &= \begin{bmatrix} 1 \\ -9 + \frac{1}{19}c_2 + \frac{3}{19}c_3 - \frac{3}{19}c_4 \\ 87 + \frac{9}{57}c_2 + \frac{9}{19}c_3 - \frac{9}{19}c_4 \\ 84 - \frac{9}{57}c_2 - \frac{9}{19}c_3 + \frac{9}{19}c_4 \end{bmatrix}, \end{aligned}$$

*we confirm Theorem 4.1 by*

$$A^{\mathbb{D},W}WAWx = \begin{bmatrix} 1 \\ 19 \\ 171 \\ 0 \end{bmatrix} = A^{D,W}b.$$

*Theorem 4.4 implies that* $A^{\diamond,D,W}b = \begin{bmatrix} 1 \\ -9 \\ 87 \\ 84 \end{bmatrix}$ *is the unique solution to* $A^{\mathbb{D},W}WAWx = A^{D,W}b$ *in*

$$R((A^{\mathbb{D},W}WAW)^*) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \frac{1}{3}y_2 + y_3 \end{bmatrix} : y_1, y_2, y_3 \in \mathbb{C} \right\}.$$

*Set* $B = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 57 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Applying Theorem 4.8, it follows that*

$$X = BA^{\diamond,D,W} = \begin{bmatrix} 3 & 0 & 0 & 0 & 1 \\ 0 & 6 & -9 & -3 & -10 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

*is the uniquely determined solution to minimization problem (15), where* $ind(WA) = ind(AW) = 2$ *and*

$$\begin{aligned} X &\in \mathbb{C}^{4\times 5}(WA)^2 \\ &= \left\{ \begin{bmatrix} 9u_1 & 9u_2 & u_3 & 9u_2 + u_3 & 3u_1 + 12u_2 + 2u_3 \\ 9u_4 & 9u_5 & u_6 & 9u_5 + u_6 & 3u_4 + 12u_5 + 2u_6 \\ 9u_7 & 9u_8 & u_9 & 9u_8 + u_9 & 3u_7 + 12u_8 + 2u_9 \\ 9u_{10} & 9u_{11} & u_{12} & 9u_{11} + u_{12} & 3u_{10} + 12u_{11} + 2u_{12} \end{bmatrix} : u_i \in \mathbb{C},\ i = \overline{1, 12} \right\}. \end{aligned}$$

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