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Two strongly convergent algorithms with momentum for the split common fixed point problems without prior knowledge of operator norms and an application to signal processing

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Abstract. Two novel iterative algorithms are proposed for solving the split common fixed point problem with demicontractive operators. Strong convergence theorems of the proposed algorithms are proved. The step-sizes of our algorithms are chosen such that they do not depend on operator norms. The main results proven in this paper extend and improve some results in the literature. Finally, an application to the signal processing problem is presented to demonstrate the efficiency of our proposed algorithms.

1. Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is formulated as finding a point x satisfying the property

$$x \in C \text{ such that } Ax \in Q,$$

where $A : H_1 \to H_2$ is a bounded linear operator. The split feasibility problem (SFP) has been shown to have broad applicability across various fields, including computer tomography, image restoration, radiation therapy treatment, and numerous other impactful real-world applications, for instance, see [5, 6, 9, 11, 25]. Due to it has applications across various fields, recently, the SFP has been widely studied by many authors (see [1, 5, 12, 13, 20, 22–24, 26]). Due to application in signal processing, Byrne [5] introduced the so-called CQ algorithm. For any $x_0 \in H_1$ and define $\{x_n\}$ as

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \tag{1}$$

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where $0 < \gamma < \frac{2}{\rho(A^*A)}$ and where P_C denotes the projection onto C and $\rho(A^*A)$ is the spectral radius of the operator A^*A. It is known that the CQ algorithm converges weakly to a solution of the SFP if such a solution exists.

In the case where both C and Q consist of fixed point sets of some nonlinear operators, the SFP is known as the split common fixed point problem (SCFP). More specifically, the SCFP is to find

$$x \in Fix(S) \text{ such that } Ax \in Fix(T),$$

where $Fix(S)$ and $Fix(T)$ are the fixed point sets of $S : H_1 \to H_1$ and $T : H_2 \to H_2$, respectively. We denote the solution set of the SCFP by

$$\Omega := \{x \in H_1 : x \in Fix(S) \text{ and } Ax \in Fix(T)\}. \tag{2}$$

When S and T are directed operators, Censor and Segal [8] proposed and proved the convergence of the following algorithm in the setting of the finite dimensional spaces:

$$x_{n+1} = S(I - \gamma A^*(I - T)A)x_n.$$

Note that a class of directed operators include the metric projection. So the results of Censor and Segal recover Byrne's CQ algorithm.

In the case that S and T are demicontractive mappings with constants $\beta \in [0, 1)$, $\mu \in [0, 1)$ respectively, Moudafi [14] introduced the following algorithm for solving the SCFP (2) as follows:

$$\begin{cases} x_0 \in H_1 \\ u_n = x_n - \gamma A^*(I - T)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n S u_n, \end{cases} \tag{3}$$

where $\alpha_n \in (\delta, 1 - \beta - \delta)$ for a small enough $\delta > 0$ and $\gamma \in (0, \frac{1-\mu}{\rho})$ with ρ being the spectral radius of A^*A and he presented the weak convergence of the sequence generated by algorithm (3). It is obvious that to solve the SCFP (2) for demicontractive operators by the sequence generated by (3) requires the norm of the linear mapping A. This is quite challenging in practice. To overcome this difficulty recently, some authors considered alternative ways of constructing variable step sizes.

In [10], Cui and Wang combined algorithm (3) with a self-adaptive step size and introduced the following algorithm for solving the SCFP (2):

$$\begin{cases} x_0 \in H_1 \\ u_n = x_n - \tau_n A^*(I - T)Ax_n, \\ x_{n+1} = (1 - \lambda)u_n + \lambda S u_n, \end{cases}$$

where $\lambda \in (0, 1 - \beta)$ and

$$\tau_n = \begin{cases} \frac{(1-\mu)\|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|^2}, & \text{if } Ax_n \neq T(Ax_n), \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

They obtained a weak convergence result for demicontractive operators provided that $\Omega \neq \emptyset$.

The split common fixed point problem (SCFP) (2) for demicontractive operators with weak convergent results has since garnered significant attention and has been extensively investigated by numerous researchers (see [21, 22, 27]).

A natural question that arises in the case of infinite dimensional Hilbert spaces is how to design an algorithm which provides strong convergence for solving SCFP (2). Based on the algorithm of Cui and Wang, Boikanyo [3] developed the following Halpern-type algorithm for demicontractive operators, which generates sequences that consistently converge strongly to a solution of the SCFPP (2) with step sizes that do not depend on the norm of the operator A.

$$\begin{cases} x_0, u \in H_1 \\ u_n = x_n - \tau_n A^*(I - T)Ax_n, \\ y_n = (1 - \lambda)u_n + \lambda S u_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \end{cases}$$

and the step size τ_n is chosen as in (4).

Now, let us mention an inertial type algorithm. We know that the problem of finding a zero of a maximal monotone operator A on a real Hilbert H is formulated as

$$\text{find } x \in H \text{ such that } 0 \in A(x). \tag{5}$$

One of the fundamental approaches to solving it is the proximal method, which generates the next iteration x_{n+1} by solving the subproblem

$$0 \in \lambda_n A(x) + (x - x_n), \tag{6}$$

where x_n is the current iteration and λ_n is a regularization parameter, see [4, 16, 17]. In 2001, Attouch and Alvarez [2] applied an inertial technique to the algorithm (6) to contruct an inertial proximal method for solving (5). It works as follows: given $x_{n-1},\ x_n \in H$ and two parameters $\theta_n \in [0, 1), \lambda_n > 0$, find $x_{n+1} \in H$ such that

$$0 \in \lambda_n A(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1}),$$

which can be written equivalently to the following

$$x_{n+1} = J^A_{\lambda_n}(x_n + \theta_n(x_n - x_{n-1})),$$

where $J^A_{\lambda_n}$ is the resolvent of A with parameter λ_n and the inertia is induced by the term $\theta_n(x_n - x_{n-1})$ and it can be regarded as procedure of seeding up the convergence properties (see, e.g., [2, 15]).

Motivated by Boikanyo's work [3], utilizing the inertial technique in this examined direction, we developed two new algorithms for demicontractive operators that converge strongly to a solution of the problem (2). The aim of our work in this study is as follows:

1. First, we introduce two new iterative algorithms that combine the algorithm (3) the inertial technique. The motivation behind this combination is to improve the convergence rate of the algorithms for solving SCFP.
2. Second, under the appropriate conditions, we prove strong convergence results of the iterative sequences generated by our algorithms without prior knowledge of operator norm A.
3. Third, we apply our algorithms to the signal processing problem to illustrate the performance of our algorithms.

This paper is organized as follows: In Sect. 2, we recall some definitions and preliminary results for further use. Sect. 3 deals with analyzing the convergence of the proposed algorithms. Finally, in Sect. 4 we perform several numerical examples to support the convergence of our algorithms.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$.

For each $x, y \in H$, we have the following:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{7}$$

Definition 2.1. *Assume that $T : H \to H$ is a nonlinear operator with $Fix(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in H, the following implication holds:*

$$x_n \rightharpoonup x \text{ and } (I - T)x_n \to 0 \Longrightarrow x \in Fix(T).$$

Definition 2.2. *Let $T : H \to H$ be an operator with $Fix(T) \neq \emptyset$. Then*

- *$T : H \to H$ is called directed if*

$$\langle z - Tx, x - Tx\rangle \leq 0 \quad \forall z \in Fix(T), x \in H, \tag{8}$$

or equivalently

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \|x - Tx\|^2 \quad \forall z \in Fix(T), x \in H,$$

- *$T : H \to H$ is called quasi-nonexpansive if*

$$\|Tx - z\| \leq \|x - z\| \quad \forall z \in Fix(T), x \in H;$$

- *$T : H \to H$ is called β-demicontractive with $0 \leq \beta < 1$ if*

$$\|Tx - z\|^2 \leq \|x - z\|^2 + \beta\|(I - T)x\|^2 \quad \forall z \in Fix(T), x \in H,$$

or equivalently

$$\langle Tx - x, x - z\rangle \leq \frac{\beta - 1}{2}\|x - Tx\|^2 \quad \forall z \in Fix(T), x \in H,$$

or equivalently

$$\langle Tx - z, x - z\rangle \leq \|x - z\|^2 + \frac{\beta - 1}{2}\|x - Tx\|^2 \quad \forall z \in Fix(T), x \in H.$$

Lemma 2.3. *Let $U : H \to H$ is β- demicontractive with $F(U) \neq \emptyset$ and set $U_\lambda = (1 - \lambda)I + \lambda U$, $\lambda \in (0, 1 - \beta)$ then:*

$$\|U_\lambda x - z\|^2 \leq \|x - z\|^2 - \lambda(1 - \beta - \lambda)\|(I - U)x\|^2 \quad \forall x \in H, z \in Fix(U).$$

Proof. We have

$$\begin{aligned}
\|U_\lambda x - z\|^2 &= \|(1 - \lambda)x + \lambda Ux - z\|^2 \\
&= \|(x - z) + \lambda(Ux - x)\|^2 \\
&= \|x - z\|^2 + 2\lambda\langle x - z, Ux - x\rangle + \lambda^2\|Ux - x\|^2 \\
&\leq \|x - z\|^2 + \lambda(\beta - 1)\|Ux - x\|^2 + \lambda^2\|Ux - x\|^2 \\
&= \|x - z\|^2 - \lambda(1 - \beta - \lambda)\|(I - U)x\|^2 \\
&= \|x - z\|^2 - \frac{1 - \beta - \lambda}{\lambda}\|(I - U_\lambda)x\|^2.
\end{aligned}$$

□

More information on quasi-nonexpansive mappings and demicontractive mappings can be found, for example [7, 19].

Lemma 2.4. ([18]) *Let $\{a_n\}$ be sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n b_n, \quad \forall n \ge 1.$$

If $\limsup_{k\to\infty} b_{n_k} \le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0$ then $\lim_{n\to\infty} a_n = 0$.

3. Strong convergence results

Our strong convergence theorems are established under the following conditions:

Condition 3.1. *The solution set $\Omega \neq \emptyset$.*

Condition 3.2. *$S : H_1 \to H_1$ and $T : H_2 \to H_2$ are two demicontractive operators with constants $\beta \in [0,1)$ and $\mu \in [0,1)$, respectively such that $I - S$ and $I - T$ are demiclosed at zero.*

Condition 3.3. *$A : H_1 \to H_2$ is a bounded linear operator with its adjoint operator A^*.*

Now, we introduce the first algorithm:

Algorithm 3.4.
Initialization: *Let $\lambda \in (0, 1-\beta)$, $\gamma \in (0,1)$, $\alpha > 0$, and $v_0, v_1 \in H$ be arbitrary. We assume that $\{\rho_n\}$, $\{\xi_n\}$, $\{\epsilon_n\}$ are three positive sequences such that $\{\xi_n\} \subset [a, \frac{1}{2}]$, for some $a > 0$ and $\epsilon_n = 0(1-\rho_n)$, that is $\lim_{n\to\infty} \frac{\epsilon_n}{1-\rho_n} = 0$, where $\{\rho_n\} \subset (0,1)$ satisfies the following conditions:*

$$\lim_{n\to\infty}(1-\rho_n) = 0, \quad \sum_{n=1}^{\infty}(1-\rho_n) = \infty.$$

Iterative Steps: *Calculate v_{n+1} as follows:*
Step 1. *Given the current iterates v_{n-1} and v_n $(n \ge 1)$, compute*

$$\begin{cases} q_n = v_n + \alpha_n(v_n - v_{n-1}), \\ u_n = q_n - \tau_n A^*(I-T)Aq_n, \end{cases}$$

where

$$\tau_n = (1-\mu)\gamma \frac{\|(I-T)Aq_n\|^2}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2},$$

and

$$\alpha_n = \begin{cases} \min\{\alpha, \dfrac{\epsilon_n}{\|v_n - v_{n-1}\|}\} & \text{if } v_n \neq v_{n-1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{9}$$

If $\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2 = 0$ then stop and q_n is a solution of problem (2). Otherwise, go to **Step 2**.
Step 2. *Compute*

$$v_{n+1} = (1-\xi_n)(\rho_n v_n) + \xi_n U_\lambda u_n,$$

where $U_\lambda := (1-\lambda)I + \lambda S$.
Let $n := n+1$ and return to **Step 1**.

Remark 3.5. *1. We prove that if* $||q_n - Sq_n||^2 + ||A^*(I-T)Aq_n||^2 = 0$, $q_n \in \Omega$. *Indeed, since* $||q_n - Sq_n||^2 + ||A^*(I-T)Aq_n||^2 = 0$, *it implies* $Sq_n = q_n$, *thus* $q_n \in Fix(S)$ *and* $A^*(I-T)Aq_n = 0$. *Since* $\Omega \neq \emptyset$, *there exists a point* $z \in Fix(S)$ *such that* $Az \in Fix(T)$. *Since the operator* T *is* μ*-demicontractive, we have*

$$\begin{aligned}\frac{1-\mu}{2}||(I-T)Aq_n||^2 &\leq \langle (I-T)Aq_n, Aq_n - Az\rangle \\ &= \langle A^*(I-T)Aq_n, q_n - z\rangle = 0.\end{aligned}$$

This implies that $(I-T)Aq_n = 0$. *Thus* $Aq_n = T(Aq_n)$. *Therefore* $q_n \in \Omega$, *as asserted, that is* q_n *is a solution of problem* (2).

2. From the definition of $\{\alpha_n\}$ *and* $\lim_{n\to\infty} \dfrac{\epsilon_n}{1-\rho_n} = 0$ *we have* $\lim_{n\to+\infty} \dfrac{\alpha_n}{1-\rho_n}||v_n - v_{n-1}|| = 0$.

Theorem 3.6. *Assume that the Conditions* 3.1 − 3.3 *hold. Then the sequence* $\{v_n\}$ *generated by Algorithm 3.4 converges strongly to an element* $x^* \in \Omega$, *where* $||x^*|| = \min\{||u|| : u \in \Omega\}$.

Proof. **Claim 1.** The sequence $\{v_n\}$ is bounded. Indeed, from $x^* \in \Omega$ and using inequality (8) we have

$$\begin{aligned}||u_n - x^*||^2 &= ||q_n - \tau_n A^*(I-T)Aq_n - x^*||^2 \\ &= ||q_n - x^*||^2 + \tau_n^2||A^*(I-T)Aq_n||^2 - 2\tau_n\langle A^*(I-T)Aq_n, q_n - x^*\rangle \\ &= ||q_n - x^*||^2 + \tau_n^2||A^*(I-T)At_n||^2 - 2\tau_n\langle (I-T)Aq_n, Aq_n - Ax^*\rangle \\ &\leq ||q_n - x^*||^2 + \tau_n^2||A^*(I-T)Aq_n||^2 - (1-\mu)\tau_n||(I-T)Aq_n||^2 \\ &= ||q_n - x^*||^2 + (1-\mu)^2\gamma^2\frac{||(I-T)Aq_n||^4}{(||q_n - Sq_n||^2 + ||A^*(I-T)Aq_n||^2)^2}||A^*(I-T)Aq_n||^2 \\ &\quad - (1-\mu)^2\gamma\frac{||(I-T)Aq_n||^4}{||q_n - Sq_n||^2 + ||A^*(I-T)Aq_n||^2} \\ &\leq ||q_n - x^*||^2 + (1-\mu)^2\gamma^2\frac{||(I-T)Aq_n||^4}{||q_n - Sq_n||^2 + ||A^*(I-T)Aq_n||^2} \\ &\quad - (1-\mu)^2\gamma\frac{||(I-T)Aq_n||^4}{||q_n - Sq_n||^2 + ||A^*(I-T)Aq_n||^2} \\ &= ||q_n - x^*||^2 - (1-\mu)^2\gamma(1-\gamma)\frac{||(I-T)Aq_n||^4}{||q_n - Sq_n||^2 + ||A^*(I-T)Aq_n||^2}. \end{aligned} \tag{10}$$

This implies that

$$||u_n - x^*|| \leq ||q_n - x^*||. \tag{11}$$

Let $z_n = U_\lambda u_n$, we get

$$\begin{aligned}||z_n - x^*||^2 &= ||(1-\lambda)u_n + \lambda Su_n - x^*||^2 \\ &= ||u_n - x^* + \lambda(Su_n - u_n)||^2 \\ &= ||u_n - x^*||^2 + 2\lambda\langle Su_n - u_n, u_n - x^*\rangle + \lambda^2||Su_n - u_n||^2 \\ &\leq ||u_n - x^*||^2 + \lambda(\beta - 1)||Su_n - u_n||^2 + \lambda^2||Su_n - u_n||^2 \\ &= ||u_n - x^*||^2 - \lambda(1-\beta-\lambda)||Su_n - u_n||^2.\end{aligned} \tag{12}$$

Hence, we get

$$||z_n - x^*|| \leq ||u_n - x^*||. \tag{13}$$

Combining (11) and (13) we get

$$\|z_n - x^*\| \le \|u_n - x^*\| \le \|q_n - x^*\|. \tag{14}$$

On the other hand, from the definition of q_n, we get

$$\begin{aligned}\|q_n - x^*\| &= \|v_n + \alpha_n(v_n - v_{n-1}) - x^*\| \\ &\le \|v_n - x^*\| + \alpha_n\|v_n - v_{n-1}\| \\ &= \|v_n - x^*\| + (1-\rho_n)\frac{\alpha_n}{1-\rho_n}\|v_n - v_{n-1}\|. \end{aligned} \tag{15}$$

By (9), we get $\frac{\alpha_n}{1-\rho_n}\|v_n - v_{n-1}\| \to 0$, it follows that there exists a constant $A_1 > 0$ such that

$$\frac{\alpha_n}{1-\rho_n}\|v_n - v_{n-1}\| \le A_1, \quad \forall n \ge 1. \tag{16}$$

Combining (14), (15) and (16), we obtain

$$\|z_n - x^*\| \le \|q_n - x^*\| \le \|v_n - x^*\| + (1-\rho_n)A_1. \tag{17}$$

Now, from the definition of $\{v_n\}$, we get

$$\begin{aligned}\|v_{n+1} - x^*\| &= \|(1-\xi_n)(\rho_n v_n) + \xi_n z_n - x^*\| \\ &= \|(1-\xi_n)\rho_n(v_n - x^*) + \xi_n(z_n - x^*) - (1-\rho_n)(1-\xi_n)x^*\| \\ &\le \|(1-\xi_n)\rho_n(v_n - x^*) + \xi_n(z_n - x^*)\| + (1-\rho_n)(1-\xi_n)\|x^*\|. \end{aligned} \tag{18}$$

Now, we estimate $\|(1-\xi_n)\rho_n(v_n - x^*) + \xi_n(z_n - x^*)\|$

$$\begin{aligned}&\|(1-\xi_n)\rho_n(v_n - x^*) + \xi_n(z_n - x^*)\|^2 \\ &= (1-\xi_n)^2\rho_n^2\|v_n - x^*\|^2 + 2(1-\xi_n)\rho_n\xi_n\langle v_n - x^*, z_n - x^*\rangle + \xi_n^2\|z_n - x^*\|^2 \\ &\le (1-\xi_n)^2\rho_n^2\|v_n - x^*\|^2 + 2(1-\xi_n)\rho_n\xi_n\|v_n - x^*\|\|z_n - x^*\| + \xi_n^2\|z_n - x^*\|^2 \\ &= [(1-\xi_n)\rho_n\|v_n - x^*\| + \xi_n\|z_n - x^*\|]^2. \end{aligned}$$

Thus

$$\|(1-\xi_n)\rho_n(v_n - x^*) + \xi_n(z_n - x^*)\| \le (1-\xi_n)\rho_n\|v_n - x^*\| + \xi_n\|z_n - x^*\|. \tag{19}$$

Combining (17) and (19), we deduce that

$$\begin{aligned}&\|(1-\xi_n)\rho_n(v_n - x^*) + \xi_n(z_n - x^*)\| \\ &\qquad \le (1-\xi_n)\rho_n\|v_n - x^*\| + \xi_n\|v_n - x^*\| + (1-\rho_n)\xi_n A_1 \\ &\qquad = (1-(1-\rho_n)(1-\xi_n))\|v_n - x^*\| + (1-\rho_n)\xi_n A_1 \\ &\qquad \le (1-(1-\rho_n)(1-\xi_n))\|v_n - x^*\| + (1-\rho_n)(1-\xi_n)A_1 \\ &\qquad\quad (\text{ by } 0 < a \le \xi_n \le \frac{1}{2}, \text{ thus } \xi_n \le 1-\xi_n). \end{aligned} \tag{20}$$

Substituting (20) into (18), we get

$$\begin{aligned}\|v_{n+1} - x^*\| &\le (1-(1-\rho_n)(1-\xi_n))\|v_n - x^*\| + (1-\rho_n)(1-\xi_n)(\|x^*\| + A_1) \\ &\le \max\{\|v_n - x^*\|, \|x^*\| + A_1\} \\ &\le \dots \le \max\{\|v_0 - x^*\|, \|x^*\| + A_1\}. \end{aligned}$$

Therefore, the sequence $\{v_n\}$ is bounded.
Claim 2.

$$a[1-(1-\xi_n)(1-\rho_n)](1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n-Sq_n\|^2+\|A^*(I-T)Aq_n\|^2}$$
$$+a[1-(1-\xi_n)(1-\rho_n)]\lambda(1-\beta-\lambda)\|Su_n-u_n\|^2$$
$$\leq \|v_n-x^*\|^2-\|v_{n+1}-x^*\|^2+(1-\rho_n)[\xi_n(1-(1-\xi_n)(1-\rho_n))A_1+(1-\xi_n)A_2].$$

for some $A_1, A_2 > 0$.

Indeed, we have

$$\begin{aligned}
&\|v_{n+1}-x^*\|^2=\|(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*)-(1-\rho_n)(1-\xi_n)x^*\|^2\\
&=\|(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*)\|^2+(1-\rho_n)^2(1-\xi_n)^2\|x^*\|^2\\
&-2(1-\rho_n)(1-\xi_n)\langle(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*),x^*\rangle\\
&\leq\|(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*)\|^2+(1-\rho_n)(1-\xi_n)\Big[(1-\rho_n)(1-\xi_n)\|x^*\|\\
&+2\|(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*)\|\|x^*\|\Big]\\
&\leq\|(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*)\|^2+(1-\rho_n)(1-\xi_n)A_2.
\end{aligned}\tag{21}$$

The last inequality obtains by the boundedness of $\{v_n\}$, $\{z_n\}$, $\{\xi_n\}$, and $\{\rho_n\}$, implies there exists $A_2 > 0$ such that

$$(1-\rho_n)(1-\xi_n)\|x^*\|+2\|(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*)\|\|x^*\|\leq A_2,\ \forall.$$

Now, we estimate $\|(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*)\|^2$. We have

$$\begin{aligned}
&\|(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*)\|^2\\
&\quad=(1-\xi_n)^2\rho_n^2\|v_n-x^*\|^2+2(1-\xi_n)\rho_n\xi_n\langle v_n-x^*,z_n-x^*\rangle+\xi_n^2\|z_n-x^*\|^2\\
&\quad\leq(1-\xi_n)^2\rho_n^2\|v_n-x^*\|^2+2(1-\xi_n)\rho_n\xi_n\|v_n-x^*\|\|z_n-x^*\|+\xi_n^2\|z_n-x^*\|^2\\
&\quad\leq(1-\xi_n)^2\rho_n^2\|v_n-x^*\|^2+(1-\xi_n)\rho_n\xi_n\|v_n-x^*\|^2+(1-\xi_n)\rho_n\xi_n\|z_n-x^*\|^2+\xi_n^2\|z_n-x^*\|^2\\
&\quad=(1-\xi_n)\rho_n(1-(1-\xi_n)(1-\rho_n))\|v_n-x^*\|^2+\xi_n[1-(1-\xi_n)(1-\rho_n)]\|z_n-x^*\|^2.
\end{aligned}\tag{22}$$

On the other hand using the inequalities (12) and (10) , we have

$$\begin{aligned}
\|z_n-x^*\|^2&\leq\|u_n-x^*\|^2-\lambda(1-\beta-\lambda)\|Su_n-u_n\|^2\\
&\leq\|q_n-x^*\|^2-(1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n-Sq_n\|^2+\|A^*(I-T)Aq_n\|^2}-\lambda(1-\beta-\lambda)\|Su_n-u_n\|^2.
\end{aligned}\tag{23}$$

Combining (23) and (22), we get

$$\begin{aligned}
&\|(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*)\|^2\\
&\leq(1-\xi_n)\rho_n(1-(1-\xi_n)(1-\rho_n))\|v_n-x^*\|^2+\xi_n[1-(1-\xi_n)(1-\rho_n)]\|q_n-x^*\|^2]\\
&\quad-\xi_n[1-(1-\xi_n)(1-\rho_n)](1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n-Sq_n\|^2+\|A^*(I-T)Aq_n\|^2}\\
&\quad-\xi_n[1-(1-\xi_n)(1-\rho_n)]\lambda(1-\beta-\lambda)\|Su_n-u_n\|^2
\end{aligned}\tag{24}$$

Substituting (24) into (21), we get

$$\begin{aligned}
\|v_{n+1}-x^*\|^2 &\le (1-(1-\xi_n)(1-\rho_n))^2\|v_n-x^*\|^2\\
&\quad+(1-\rho_n)(\xi_n(1-(1-\xi_n)(1-\rho_n))A_1+(1-\xi_n)A_2)\\
&\quad-\xi_n[1-(1-\xi_n)(1-\rho_n)](1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n-Sq_n\|^2+\|A^*(I-T)Aq_n\|^2}\\
&\quad-\xi_n[1-(1-\xi_n)(1-\rho_n)]\lambda(1-\beta-\lambda)\|Su_n-u_n\|^2\\
&\le \|v_n-x^*\|^2 \text{ (by } (1-(1-\xi_n)(1-\rho_n))^2\le 1 \text{ and } \ \xi_n\ge a)\\
&\quad+(1-\rho_n)[\xi_n(1-(1-\xi_n)(1-\rho_n))A_1+(1-\xi_n)A_2]\\
&\quad-a[1-(1-\xi_n)(1-\rho_n)](1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n-Sq_n\|^2+\|A^*(I-T)Aq_n\|^2}\\
&\quad-a[1-(1-\xi_n)(1-\rho_n)]\lambda(1-\beta-\lambda)\|Su_n-u_n\|^2\\
&\le (1-\xi_n)\rho_n(1-(1-\xi_n)(1-\rho_n))\|v_n-x^*\|^2+\xi_n[1-(1-\xi_n)(1-\rho_n)]\|v_n-x^*\|^2\\
&\quad+\xi_n(1-(1-\xi_n)(1-\rho_n))(1-\rho_n)A_1\\
&\quad-\xi_n[1-(1-\xi_n)(1-\rho_n)](1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n-Sq_n\|^2+\|A^*(I-T)Aq_n\|^2}\\
&\quad-a[1-(1-\xi_n)(1-\rho_n)]\lambda(1-\beta-\lambda)\|Su_n-u_n\|^2\\
&= (1-(1-\xi_n)(1-\rho_n))^2\|v_n-x^*\|^2+\xi_n(1-(1-\xi_n)(1-\rho_n))(1-\rho_n)A_1\\
&\quad-\xi_n[1-(1-\xi_n)(1-\rho_n)](1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n-Sq_n\|^2+\|A^*(I-T)Aq_n\|^2}\\
&\quad-a[1-(1-\xi_n)(1-\rho_n)]\lambda(1-\beta-\lambda)\|Su_n-u_n\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
&a[1-(1-\xi_n)(1-\rho_n)](1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n-Sq_n\|^2+\|A^*(I-T)Aq_n\|^2}\\
&+a[1-(1-\xi_n)(1-\rho_n)]\lambda(1-\beta-\lambda)\|Su_n-u_n\|^2\\
&\le \|v_n-x^*\|^2-\|v_{n+1}-x^*\|^2+(1-\rho_n)[\xi_n(1-(1-\xi_n)(1-\rho_n))A_1+(1-\xi_n)A_2].
\end{aligned}$$

Claim 3.

$$\begin{aligned}
\|v_{n+1}-x^*\|^2 &\le (1-(1-\xi_n)(1-\rho_n))\|v_n-x^*\|^2\\
&\quad+(1-\xi_n)(1-\rho_n)\Big[\frac{\alpha_n}{1-\rho_n}\|v_n-v_{n-1}\|(1-(1-\xi_n)(1-\rho_n))\frac{\xi_n}{1-\xi_n}A_3-2\langle x^*,v_{n+1}-x^*\rangle\Big],
\end{aligned}$$

for some $A_3>0$. Indeed, using the inequalities (7), (22) and (14) we have

$$\begin{aligned}
\|v_{n+1}-x^*\|^2 &= \|(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*)-(1-\rho_n)(1-\xi_n)x^*\|^2\\
&\le \|(1-\xi_n)\rho_n(v_n-x^*)+\xi_n(z_n-x^*)\|^2-2(1-\rho_n)(1-\xi_n)\langle x^*,v_{n+1}-x^*\rangle\\
&\le (1-\xi_n)\rho_n(1-(1-\xi_n)(1-\rho_n))\|v_n-x^*\|^2\\
&\quad+\xi_n(1-(1-\xi_n)(1-\rho_n))\|z_n-x^*\|^2-2(1-\rho_n)(1-\xi_n)\langle x^*,v_{n+1}-x^*\rangle\\
&\le (1-\xi_n)\rho_n(1-(1-\xi_n)(1-\rho_n))\|v_n-x^*\|^2\\
&\quad+\xi_n(1-(1-\xi_n)(1-\rho_n))\|q_n-x^*\|^2-2(1-\rho_n)(1-\xi_n)\langle x^*,v_{n+1}-x^*\rangle.
\end{aligned}\tag{25}$$

On the other hand, by the definition of $\{q_n\}$, we have

$$\begin{aligned}\|q_n - x^*\|^2 &= \|v_n - x^* + \alpha_n(v_n - v_{n-1})\|^2\\ &= \|v_n - x^*\|^2 + \alpha_n^2\|v_n - v_{n-1}\|^2 + 2\alpha_n\langle v_n - x^*, v_n - v_{n-1}\rangle\\ &\le \|v_n - x^*\|^2 + \alpha_n^2\|v_n - v_{n-1}\|^2 + 2\alpha_n\|v_n - x^*\|\|v_n - v_{n-1}\|\\ &\le \|v_n - x^*\|^2 + \alpha_n\|v_n - v_{n-1}\|(\alpha_n\|v_n - v_{n-1}\| + 2\|v_n - x^*\|)\\ &\le \|v_n - x^*\|^2 + \alpha_n\|v_n - v_{n-1}\|A_3, \end{aligned} \tag{26}$$

for some $A_3 > 0$.
Substituting (26) into (25) we deduce that

$$\begin{aligned}\|v_{n+1} - x^*\|^2 &\le (1-\xi_n)\rho_n(1-(1-\xi_n)(1-\rho_n))\|v_n - x^*\|^2 + \xi_n(1-(1-\xi_n)(1-\rho_n))\|v_n - x^*\|^2\\ &\quad + \xi_n(1-(1-\xi_n)(1-\rho_n)\alpha_n\|v_n - v_{n-1}\|A_3 - 2(1-\rho_n)(1-\xi_n)\langle x^*, v_{n+1} - x^*\rangle\\ &= (1-(1-\xi_n)(1-\rho_n))^2\|v_n - x^*\|^2\\ &\quad + (1-\xi_n)(1-\rho_n)\Big[\frac{\alpha_n}{1-\rho_n}\|v_n - v_{n-1}\|(1-(1-\xi_n)(1-\rho_n))\frac{\xi_n}{1-\xi_n}A_3 - 2\langle x^*, v_{n+1} - x^*\rangle\Big]\\ &\le (1-(1-\xi_n)(1-\rho_n))\|v_n - x^*\|^2\\ &\quad + (1-\xi_n)(1-\rho_n)\Big[\frac{\alpha_n}{1-\rho_n}\|v_n - v_{n-1}\|(1-(1-\xi_n)(1-\rho_n))\frac{\xi_n}{1-\xi_n}A_3 - 2\langle x^*, v_{n+1} - x^*\rangle\Big].\end{aligned}$$

Claim 4. $\{\|v_n - x^*\|^2\}$ converges to zero. Indeed, by Lemma 2.4 it suffices to show that $\limsup_{k\to\infty}\langle x^*, v_{n_k+1} - x^*\rangle \le 0$ for every subsequence $\{\|v_{n_k} - x^*\|\}$ of $\{\|v_n - x^*\|\}$ satisfying

$$\liminf_{k\to\infty}(\|v_{n_k+1} - x^*\| - \|v_{n_k} - x^*\|) \ge 0.$$

For this, suppose that $\{\|v_{n_k} - x^*\|\}$ is a subsequence of $\{\|v_n - x^*\|\}$ such that $\liminf_{k\to\infty}(\|v_{n_k+1} - x^*\| - \|v_{n_k} - x^*\|) \ge 0$. Then

$$\liminf_{k\to\infty}(\|v_{n_k+1} - x^*\|^2 - \|v_{n_k} - x^*\|^2) = \liminf_{k\to\infty}[(\|v_{n_k+1} - x^*\| - \|v_{n_k} - x^*\|)(\|v_{n_k+1} - x^*\| + \|v_{n_k} - x^*\|)] \ge 0.$$

By Claim 2 we obtain

$$\begin{aligned}&\limsup_{k\to\infty}\Big(a[1-(1-\xi_n)(1-\rho_n)](1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_{n_k}\|^4}{\|q_{n_k} - Sq_{n_k}\|^2 + \|A^*(I-T)Aq_{n_k}\|^2}\\ &\quad + a[1-(1-\xi_{n_k})(1-\rho_{n_k})]\lambda(1-\beta-\lambda)\|Su_{n_k} - u_{n_k}\|^2\Big)\\ &\le \limsup_{k\to\infty}[\|v_{n_k} - x^*\|^2 - \|v_{n_k+1} - x^*\|^2 + (1-\rho_{n_k})[\xi_{n_k}(1-(1-\xi_{n_k})(1-\rho_{n_k}))A_1 + (1-\xi_{n_k})A_2]]\\ &\le \limsup_{k\to\infty}[\|v_{n_k} - x^*\|^2 - \|v_{n_k+1} - x^*\|^2]\\ &\quad + \limsup_{k\to\infty}(1-\rho_{n_k})[\xi_{n_k}(1-(1-\xi_{n_k})(1-\rho_{n_k}))A_1 + (1-\xi_{n_k})A_2]\\ &= \limsup_{k\to\infty}[\|v_{n_k} - x^*\|^2 - \|v_{n_k+1} - x^*\|^2]\\ &= -\liminf_{k\to\infty}[\|v_{n_k+1} - x^*\|^2 - \|v_{n_k} - x^*\|^2]\\ &\le 0.\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\frac{\|(I-T)Aq_{n_k}\|^4}{\|q_{n_k} - Sq_{n_k}\|^2 + \|A^*(I-T)Aq_{n_k}\|^2} = 0 \text{ and } \lim_{k\to\infty}\|Su_{n_k} - u_{n_k}\|^2 = 0. \tag{27}$$

Moreover, since $\{v_n\}$ is bounded, using the inequality (26) we also obtain $\{q_n\}$ is bounded. By Lemma 2.3 we get $\{q_{n_k} - Sq_{n_k}\}$ is bounded and $\{A^*(I-T)Aq_{n_k}\}$ is bounded. Therefore it follows from (27) that

$$\lim_{n\to\infty} \|(I-T)Aq_{n_k}\| = 0 \text{ and } \lim_{k\to\infty} \|u_{n_k} - Su_{n_k}\| = 0. \tag{28}$$

On the other hand, using the definition of $\{u_{n_k}\}$, see that

$$\begin{aligned}\|u_{n_k} - q_{n_k}\| &= \tau_{n_k}\|A^*(I-T)Aq_{n_k}\| \\ &\le \frac{\|(I-T)Aq_{n_k}\|^2}{\|q_{n_k} - Sq_{n_k}\|^2 + \|A^*(I-T)Aq_{n_k}\|^2}\|A^*\|\|(I-T)Aq_{n_k}\| \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{29}$$

Using (28) and the definition of $\{z_n\}$, we get

$$\|u_{n_k} - z_{n_k}\| = \lambda\|Su_{n_k} - u_{n_k}\| \to 0 \text{ as } n \to \infty. \tag{30}$$

Now, we show that

$$\|v_{n_k+1} - v_{n_k}\| \to 0 \text{ as } n \to \infty. \tag{31}$$

Indeed, combining (29) and (30) we have

$$\|z_{n_k} - q_{n_k}\| \to 0. \tag{32}$$

and

$$\|v_{n_k} - q_{n_k}\| = \alpha_{n_k}\|v_{n_k} - v_{n_k-1}\| = (1-\rho_{n_k})\frac{\alpha_{n_k}}{1-\rho_{n_k}}\|v_{n_k} - v_{n_k-1}\| \to 0. \tag{33}$$

Combining (32) and (33), we deduce that

$$\lim_{k\to+\infty} \|z_{n_k} - v_{n_k}\| = 0.$$

Therefore, we have

$$\begin{aligned}\|v_{n_k+1} - v_{n_k}\| &= \|(1-\xi_{n_k})\rho_{n_k}v_{n_k} + \xi_{n_k}z_{n_k} - v_{n_k}\| \\ &= \|\xi_{n_k}(z_{n_k} - v_{n_k}) - (1-\rho_{n_k})(1-\xi_{n_k})v_{n_k}\| \\ &\le \xi_{n_k}\|z_{n_k} - v_{n_k}\| + (1-\rho_{n_k})(1-\xi_{n_k})\|v_{n_k}\| \to 0.\end{aligned}$$

Since the sequence $\{v_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{v_{n_{k_j}}\}$ of $\{v_{n_k}\}$, which converges weakly to some $z \in H$, such that

$$\limsup_{k\to\infty}\langle x^*, v_{n_k} - x^*\rangle = \lim_{j\to\infty}\langle x^*, v_{n_{k_j}} - x^*\rangle = \langle x^*, z - x^*\rangle. \tag{34}$$

From (33) and (29) we get

$$q_{n_k} \rightharpoonup z \text{ and } u_{n_k} \rightharpoonup z,$$

this together with (28) and the demiclosedness of $I-T$ and $I-S$ at zero, we obtain $z \in \Omega$ and, from (34) and the definition of $x^* = P_\Omega(0)$, we have

$$\limsup_{k\to\infty}\langle x^*, v_{n_k} - x^*\rangle = \langle x^*, z - x^*\rangle \le 0. \tag{35}$$

Combining (31) and (35), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle x^*, v_{n_k+1} - x^*\rangle &\le \limsup_{k\to\infty}\langle x^*, v_{n_k} - x^*\rangle \\ &= \langle x^*, z - x^*\rangle \\ &\le 0.\end{aligned} \tag{36}$$

Hence, by (36), $\lim_{n\to\infty} \frac{\alpha_n}{1-\rho_n}\|v_n - v_{n-1}\| = 0$, Claim 3 and Lemma 2.4, we have $\lim_{n\to\infty} \|v_n - x^*\| = 0$. This is the desired result.

□

Now, we introduce the second algorithm:

Algorithm 3.7.
Initialization: *Let $\lambda \in (0, 1-\beta)$, $\gamma \in (0,1)$, $\alpha > 0$, and $v_0, v_1 \in H$ be arbitrary. We assume that $\{\rho_n\}$, $\{\xi_n\}, \{\epsilon_n\}$ are three positive sequences such that $\{\xi_n\} \subset (a,b) \subset (0, 1-\rho_n)$ and $\epsilon_n = 0(\rho_n)$, that is $\lim_{n\to\infty} \frac{\epsilon_n}{\rho_n} = 0$, where $\{\rho_n\} \subset (0,1)$ satisfies the following conditions:*

$$\lim_{n\to\infty} \rho_n = 0, \quad \sum_{n=1}^{\infty} \rho_n = \infty.$$

Iterative Steps: *Calculate v_{n+1} as follows:*
Step 1. *Given the current iterates v_{n-1} and v_n $(n \geq 1)$, compute*

$$\begin{cases} q_n = v_n + \alpha_n(v_n - v_{n-1}), \\ u_n = q_n - \tau_n A^*(I-T)Aq_n, \end{cases}$$

where

$$\tau_n = (1-\mu)\gamma \frac{\|(I-T)Aq_n\|^2}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2},$$

and

$$\alpha_n = \begin{cases} \min\{\alpha, \dfrac{\epsilon_n}{\|v_n - v_{n-1}\|}\} & \text{if } v_n \neq v_{n-1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{37}$$

If $u_n = 0$ then stop and q_n is a solution of problem (2). Otherwise, go to **Step 2**.
Step 2. *Compute*

$$v_{n+1} = (1 - \xi_n - \rho_n)v_n + \xi_n U_\lambda u_n,$$

where $U_\lambda := (1-\lambda)I + \lambda S$.
Let $n := n+1$ and return to **Step 1**.

Theorem 3.8. *Assume that the Conditions 3.1 − 3.3 hold. Then the sequence $\{v_n\}$ generated by Algorithm 3.7 converges strongly to an element $x^* \in \Omega$, where $\|x^*\| = \min\{\|u\| : u \in \Omega\}$.*

Proof. **Claim 1.** The sequence $\{v_n\}$ is bounded. Indeed, let $z_n = U_\lambda u_n$, from $x^* \in \Omega$ using (23), we get

$$\begin{aligned} \|z_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \lambda(1-\beta-\lambda)\|Su_n - u_n\|^2 \\ &\leq \|q_n - x^*\|^2 - (1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} - \lambda(1-\beta-\lambda)\|Su_n - u_n\|^2. \end{aligned} \tag{38}$$

This follows that

$$\|z_n - x^*\| \leq \|q_n - x^*\|. \tag{39}$$

On the other hand, from the definition of q_n, we get

$$\begin{aligned}\|q_n - x^*\| &= \|v_n + \alpha_n(v_n - v_{n-1}) - x^*\| \\ &\le \|v_n - x^*\| + \alpha_n\|v_n - v_{n-1}\| \\ &= \|v_n - x^*\| + \rho_n \frac{\alpha_n}{\rho_n}\|v_n - v_{n-1}\|. \end{aligned} \tag{40}$$

By (37), we get $\frac{\alpha_n}{\rho_n}\|v_n - v_{n-1}\| \to 0$, it follows that there exists a constant $M_1 > 0$ such that

$$\frac{\alpha_n}{\rho_n}\|v_n - v_{n-1}\| \le M_1, \quad \forall n \ge 1. \tag{41}$$

Combining (39), (40) and (41), we obtain

$$\|z_n - x^*\| \le \|q_n - x^*\| \le \|v_n - x^*\| + \rho_n M_1.$$

Thus

$$\|z_n - x^*\| \le \|q_n - x^*\| \le \|v_n - x^*\| + \rho_n M_1, \tag{42}$$

for some $M_1 > 0$. We also have

$$\begin{aligned}\|v_{n+1} - x^*\| &= \|(1 - \xi_n - \rho_n)v_n + \xi_n z_n - x^*\| \\ &= \|(1 - \xi_n - \rho_n)(v_n - x^*) + \xi_n(z_n - x^*) - \rho_n x^*\| \\ &\le \|(1 - \xi_n - \rho_n)(v_n - x^*) + \xi_n(z_n - x^*)\| + \rho_n\|x^*\|. \end{aligned} \tag{43}$$

Moreover

$$\begin{aligned}&\|(1 - \xi_n - \rho_n)(v_n - x^*) + \xi_n(z_n - x^*)\|^2 \\ &= (1 - \xi_n - \rho_n)^2\|v_n - x^*\|^2 + 2(1 - \xi_n - \rho_n)\xi_n\langle v_n - x^*, z_n - x^*\rangle + \xi_n^2\|z_n - x^*\|^2 \\ &\le (1 - \xi_n - \rho_n)^2\|v_n - x^*\|^2 + 2(1 - \xi_n - \rho_n)\xi_n\|v_n - x^*\|\|z_n - x^*\| + \xi_n^2\|z_n - x^*\|^2 \\ &\le (1 - \xi_n - \rho_n)^2\|v_n - x^*\|^2 + (1 - \xi_n - \rho_n)\xi_n\|v_n - x^*\|^2 \\ &\quad + (1 - \xi_n - \rho_n)\xi_n\|z_n - x^*\|^2 + \xi_n^2\|z_n - x^*\|^2 \\ &= (1 - \xi_n - \rho_n)(1 - \rho_n)\|v_n - x^*\|^2 + (1 - \rho_n)\xi_n\|z_n - x^*\|^2. \end{aligned} \tag{44}$$

Substituting (42) into (44), we obtain

$$\begin{aligned}&\|(1 - \xi_n - \rho_n)(v_n - x^*) + \xi_n(z_n - x^*)\|^2 \\ &\le (1 - \xi_n - \rho_n)(1 - \rho_n)\|v_n - x^*\|^2 + (1 - \rho_n)\xi_n(\|v_n - x^*\| + \rho_n M_1)^2 \\ &\le (1 - \xi_n - \rho_n)(1 - \rho_n)\|v_n - x^*\|^2 + (1 - \rho_n)\xi_n\|v_n - x^*\|^2 \\ &\quad + 2(1 - \rho_n)\xi_n\rho_n\|v_n - x^*\|M_1 + \rho_n^2 M_1^2 \\ &\le (1 - \rho_n)^2\|v_n - x^*\|^2 + 2(1 - \rho_n)\rho_n\|v_n - x^*\|M_1 + \rho_n^2 M_1^2 \\ &= [(1 - \rho_n)\|v_n - x^*\| + \rho_n M_1]^2. \end{aligned}$$

Therefore, we have

$$\|(1 - \xi_n - \rho_n)(v_n - x^*) + \xi_n(z_n - x^*)\| \le (1 - \rho_n)\|v_n - x^*\| + \rho_n M_1. \tag{45}$$

Combining (43) and (45), we get

$$\begin{aligned}
\|v_{n+1} - x^*\| &\leq (1-\rho_n)\|v_n - x^*\| + \rho_n M_1 + \rho_n\|x^*\| \\
&= (1-\rho_n)\|v_n - x^*\| + \rho_n(M_1 + \|x^*\|) \\
&\leq \max\{\|v_n - x^*\|, M_1 + \|x^*\|\} \\
&\leq \cdots \\
&\leq \max\{\|v_0 - x^*\|, M_1 + \|x^*\|\}.
\end{aligned}$$

Thus $\{v_n\}$ is bounded and $\{z_n\}$, $\{q_n\}$ are also bounded.

Claim 2. We show that

$$\begin{aligned}
(1-\rho_n)\xi_n(1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} &+ (1-\rho_n)\xi_n\lambda(1-\beta-\lambda)\|Su_n - u_n\|^2 \\
&\leq \|v_n - x^*\|^2 - \|v_{n+1} - x^*\|^2 + \rho_n M_4,
\end{aligned}$$

for some $M_4 > 0$. Indeed, we have

$$\begin{aligned}
\|v_{n+1} - x^*\|^2 &= \|(1-\xi_n-\rho_n)v_n + \xi_n z_n - x^*\|^2 \\
&= \|(1-\xi_n-\rho_n)(v_n - x^*) + \xi_n(z_n - x^*) - \rho_n x^*\|^2 \\
&= \|(1-\xi_n-\rho_n)(v_n - x^*) + \xi_n(z_n - x^*)\|^2 - 2\rho_n\langle(1-\xi_n-\rho_n)(v_n - x^*) \\
&\quad + \xi_n(z_n - x^*), x^*\rangle + \rho_n^2\|x^*\|^2 \\
&\leq \|(1-\xi_n-\rho_n)(v_n - x^*) + \xi_n(z_n - x^*)\|^2 + \rho_n M_2, \qquad (46)
\end{aligned}$$

for some $M_2 > 0$. Substituting (44) into (46), we get

$$\|v_{n+1} - x^*\|^2 \leq (1-\xi_n-\rho_n)(1-\rho_n)\|v_n - x^*\|^2 + (1-\rho_n)\xi_n\|z_n - x^*\|^2 + \rho_n M_2, \qquad (47)$$

which implies from (38) and (47) that

$$\begin{aligned}
&\|v_{n+1} - x^*\|^2 \\
&\leq (1-\xi_n-\rho_n)(1-\rho_n)\|v_n - x^*\|^2 + (1-\rho_n)\xi_n\|q_n - x^*\|^2 \\
&\quad - (1-\rho_n)\xi_n(1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} - (1-\rho_n)\xi_n\lambda(1-\beta-\lambda)\|Su_n - u_n\|^2 + \rho_n M_2. \\
&\qquad (48)
\end{aligned}$$

Since $\{v_n\}$ is bounded and $\|q_n - x^*\| \leq \|v_n - x^*\| + \rho_n M_1$, we get

$$\|q_n - x^*\|^2 \leq \|v_n - x^*\|^2 + \rho_n M_3, \qquad (49)$$

for some $M_3 > 0$. Substituting (49) into (48), we obtain

$$\begin{aligned}
&\|v_{n+1} - x^*\|^2 \\
&\leq (1-\xi_n-\rho_n)(1-\rho_n)\|v_n - x^*\|^2 + (1-\rho_n)\xi_n\|v_n - x^*\|^2 + (1-\rho_n)\xi_n\rho_n M_3 \\
&\quad - (1-\rho_n)\xi_n(1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} - (1-\rho_n)\xi_n\lambda(1-\beta-\lambda)\|Su_n - u_n\|^2 + \rho_n M_2 \\
&= (1-\rho_n)^2\|v_n - x^*\|^2 - (1-\rho_n)\xi_n(1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} \\
&\quad - (1-\rho_n)\xi_n\lambda(1-\beta-\lambda)\|Su_n - u_n\|^2 + \rho_n[(1-\rho_n)\xi_n M_3 + M_2] \\
&\leq \|v_n - x^*\|^2 - (1-\rho_n)\xi_n(1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} \\
&\quad - (1-\rho_n)\xi_n\lambda(1-\beta-\lambda)\|Su_n - u_n\|^2 + \rho_n M_4,
\end{aligned}$$

for some $M_4 > 0$. Therefore, we have

$$(1-\rho_n)\xi_n(1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_n\|^4}{\|q_n - Sq_n\|^2 + \|A^*(I-T)Aq_n\|^2} + (1-\rho_n)\xi_n\lambda(1-\beta-\lambda)\|Su_n - u_n\|^2 \\ \leq \|v_n - x^*\|^2 - \|v_{n+1} - x^*\|^2 + \rho_n M_4.$$

Claim 3.

$$\|v_{n+1} - x^*\|^2 \leq (1-\rho_n)\|v_n - x^*\|^2 \\ + \rho_n\Big[\frac{\alpha_n}{\rho_n}\|v_n - v_{n-1}\|(1-\rho_n)M_5 + 2\xi_n\|v_n - z_n\|\|v_n - x^*\| + 2\langle x^*, x^* - v_{n+1}\rangle\Big].$$

Indeed, we have

$$v_{n+1} = (1-\xi_n-\rho_n)v_n + \xi_n z_n = (1-\xi_n)v_n + \xi_n z_n - \rho_n v_n.$$

Let $t_n = (1-\xi_n)v_n + \xi_n z_n$. Then we have

$$\begin{aligned}
&\|t_n - x^*\|^2 \\
&= \|(1-\xi_n)v_n + \xi_n z_n - x^*\|^2 \\
&= \|(1-\xi_n)(v_n - x^*) + \xi_n(z_n - x^*)\|^2 \\
&= (1-\xi_n)^2\|v_n - x^*\|^2 + \xi_n^2\|z_n - x^*\|^2 + 2(1-\xi_n)\xi_n\langle v_n - x^*, z_n - x^*\rangle \\
&\leq (1-\xi_n)^2\|v_n - x^*\|^2 + \xi_n^2\|z_n - x^*\|^2 + 2(1-\xi_n)\xi_n\|v_n - x^*\|\|z_n - x^*\| \\
&\leq (1-\xi_n)^2\|v_n - x^*\|^2 + \xi_n^2\|z_n - x^*\|^2 + (1-\xi_n)\xi_n\|v_n - x^*\|^2 + (1-\xi_n)\xi_n\|z_n - x^*\|^2 \\
&= (1-\xi_n)\|v_n - x^*\|^2 + \xi_n\|z_n - x^*\|^2 \\
&\leq (1-\xi_n)\|v_n - x^*\|^2 + \xi_n\|q_n - x^*\|^2. \qquad (50)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\|q_n - x^*\|^2 &= \|v_n + \alpha_n(v_n - v_{n-1}) - x^*\|^2 \\
&= \|(v_n - x^*) + \alpha_n(v_n - v_{n-1})\|^2 \\
&= \|v_n - x^*\|^2 + 2\alpha_n\langle v_n - x^*, v_n - v_{n-1}\rangle + \alpha_n^2\|v_n - v_{n-1}\|^2 \\
&\leq \|v_n - x^*\|^2 + 2\alpha_n\|v_n - x^*\|\|v_n - v_{n-1}\| + \alpha_n^2\|v_n - v_{n-1}\|^2 \\
&= \|v_n - x^*\|^2 + \alpha_n\|v_n - v_{n-1}\|\Big[2\|v_n - x^*\| + \alpha_n\|v_n - v_{n-1}\|\Big] \\
&\leq \|v_n - x^*\|^2 + \alpha_n\|v_n - v_{n-1}\|M_5, \qquad (51)
\end{aligned}$$

for some $M_5 > 0$. Combining (50) and (51), we obtain

$$\begin{aligned}
\|t_n - x^*\|^2 &\leq (1-\xi_n)\|v_n - x^*\|^2 + \xi_n\|v_n - x^*\|^2 + \alpha_n\xi_n\|v_n - v_{n-1}\|M_5 \\
&\leq \|v_n - x^*\|^2 + \alpha_n\|v_n - v_{n-1}\|M_5. \qquad (52)
\end{aligned}$$

Since $t_n = (1-\xi_n)v_n + \xi_n z_n$, we have $v_n - t_n = \xi_n(v_n - z_n)$. Therefore, it follows that

$$v_{n+1} = t_n - \rho_n v_n = (1-\rho_n)t_n - \rho_n(v_n - t_n) = (1-\rho_n)t_n - \rho_n\xi_n(v_n - z_n).$$

This implies that

$$\begin{aligned}
&\|v_{n+1} - x^*\|^2 \\
&= \|(1-\rho_n)t_n - \rho_n\xi_n(v_n - z_n) - x^*\|^2 \\
&= \|(1-\rho_n)(t_n - x^*) - (\rho_n\xi_n(v_n - z_n) + \rho_n x^*)\|^2 \\
&\le (1-\rho_n)^2\|t_n - x^*\|^2 - 2\langle \rho_n\xi_n(v_n - z_n) + \rho_n x^*, v_{n+1} - x^*\rangle \\
&\le (1-\rho_n)\|t_n - x^*\|^2 - 2\langle \rho_n\xi_n(v_n - z_n), v_{n+1} - x^*\rangle - 2\rho_n\langle x^*, v_{n+1} - x^*\rangle \\
&\le (1-\rho_n)\|t_n - x^*\|^2 + 2\rho_n\xi_n\|v_n - z_n\|\|v_{n+1} - x^*\| - 2\rho_n\langle x^*, v_{n+1} - x^*\rangle. \qquad (53)
\end{aligned}$$

From (52) and (53), it follows that

$$\begin{aligned}
\|v_{n+1} - x^*\|^2 &\le (1-\rho_n)\|v_n - x^*\|^2 + (1-\rho_n)\alpha_n\|v_n - v_{n-1}\|M_5 \\
&\quad + 2\rho_n\xi_n\|v_n - z_n\|\|x^* - v_{n+1}\| - 2\rho_n\langle x^*, v_{n+1} - x^*\rangle \\
&\le (1-\rho_n)\|v_n - x^*\|^2 + \rho_n\Big[\frac{\alpha_n}{\rho_n}\|v_n - v_{n-1}\|(1-\rho_n)M_5 \\
&\quad + 2\xi_n\|v_n - z_n\|\|x^* - v_{n+1}\| - 2\langle x^*, v_{n+1-x^*}\rangle\Big].
\end{aligned}$$

Claim 4. $\{\|v_n - x^*\|^2\}$ converges to zero. Indeed, by Lemma 2.4 it suffices to show that

$$\limsup_{k\to\infty}\langle x^*, v_{n_k+1} - x^*\rangle \le 0 \text{ and } \lim_{k\to\infty}\|v_{n_k} - z_{n_k}\| = 0$$

for every subsequence $\{\|v_{n_k} - x^*\|\}$ of $\{\|v_n - x^*\|\}$ satisfying

$$\liminf_{k\to\infty}(\|v_{n_k+1} - x^*\| - \|v_{n_k} - x^*\|) \ge 0.$$

For this, suppose that $\{\|v_{n_k} - x^*\|\}$ is a subsequence of $\{\|v_n - x^*\|\}$ such that $\liminf_{k\to\infty}(\|v_{n_k+1} - x^*\| - \|v_{n_k} - x^*\|) \ge 0$. Then

$$\liminf_{k\to\infty}(\|v_{n_k+1} - x^*\|^2 - \|v_{n_k} - x^*\|^2) = \liminf_{k\to\infty}[(\|v_{n_k+1} - x^*\| - \|v_{n_k} - x^*\|)(\|v_{n_k+1} - x^*\| + \|v_{n_k} - x^*\|)] \ge 0.$$

By Claim 2 we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\Big((1-\rho_{n_k})\xi_{n_k}(1-\mu)^2\gamma(1-\gamma)\frac{\|(I-T)Aq_{n_k}\|^4}{\|q_{n_k} - Sq_{n_k}\|^2 + \|A^*(I-T)Aq_{n_k}\|^2} \quad + (1-\rho_{n_k})\xi_{n_k}\lambda(1-\beta-\lambda)\|Su_{n_k} - u_{n_k}\|^2\Big) \\
&\le \limsup_{k\to\infty}[\|v_{n_k} - x^*\|^2 - \|v_{n_k+1} - x^*\|^2 + \rho_{n_k}M_4] \\
&\le \limsup_{k\to\infty}[\|v_{n_k} - x^*\|^2 - \|v_{n_k+1} - x^*\|^2] + \limsup_{k\to\infty}\rho_{n_k}M_4 \\
&= -\liminf_{k\to\infty}[\|v_{n_k+1} - x^*\|^2 - \|v_{n_k} - x^*\|^2] \\
&\le 0.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\frac{\|(I-T)Aq_{n_k}\|^4}{\|q_{n_k} - Sq_{n_k}\|^2 + \|A^*(I-T)Aq_{n_k}\|^2} = 0 \text{ and } \lim_{k\to\infty}\|Su_{n_k} - u_{n_k}\|^2 = 0. \qquad (54)$$

It follows from (54) that

$$\lim_{k\to\infty}\|(I-T)Aq_{n_k}\| = 0 \text{ and } \lim_{k\to\infty}\|u_{n_k} - Su_{n_k}\| = 0. \qquad (55)$$

Now, we show that

$$\|v_{n_k+1} - v_{n_k}\| \to 0 \text{ as } n \to \infty.$$

Indeed, using the definition of $\{u_{n_k}\}$, see that

$$\begin{aligned}\|u_{n_k} - q_{n_k}\| &= \tau_{n_k}\|A^*(I-T)Aq_{n_k}\| \\ &\leq \frac{\|(I-T)Aq_{n_k}\|^2}{\|q_{n_k} - Sq_{n_k}\|^2 + \|A^*(I-T)Aq_{n_k}\|^2}\|A^*\|\|(I-T)Aq_{n_k}\| \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{56}$$

Using (55) and the definition of $\{z_n\}$, we get

$$\|u_{n_k} - z_{n_k}\| = \lambda\|Su_{n_k} - u_{n_k}\| \to 0 \text{ as } n \to \infty. \tag{57}$$

Combining (56) and (57), we deduce that

$$\lim_{k\to+\infty} \|z_{n_k} - q_{n_k}\| = 0. \tag{58}$$

Moreover, we have

$$\|v_{n_k} - q_{n_k}\| = \alpha_{n_k}\|v_{n_k} - v_{n_k-1}\| = (1-\rho_{n_k})\frac{\alpha_{n_k}}{1-\rho_{n_k}}\|v_{n_k} - v_{n_k-1}\| \to 0. \tag{59}$$

Combining (58) and (59), we deduce that

$$\lim_{k\to+\infty} \|z_{n_k} - v_{n_k}\| = 0.$$

Therefore, we have

$$\begin{aligned}\|v_{n_k+1} - v_{n_k}\| &= \|(1-\xi_{n_k} - \rho_{n_k})v_{n_k} + \xi_{n_k}z_{n_k} - v_{n_k}\| \\ &= \|\xi_{n_k}(z_{n_k} - v_{n_k}) - \rho_{n_k}v_{n_k}\| \\ &\leq \xi_{n_k}\|z_{n_k} - v_{n_k}\| + \rho_{n_k}\|v_{n_k}\| \to 0.\end{aligned} \tag{60}$$

Since the sequence $\{v_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{v_{n_{k_j}}\}$ of $\{v_{n_k}\}$, which converges weakly to some $z \in H$, such that

$$\limsup_{k\to\infty}\langle x^*, v_{n_k} - x^*\rangle = \lim_{j\to\infty}\langle x^*, v_{n_{k_j}} - x^*\rangle = \langle x^*, z - x^*\rangle. \tag{61}$$

From (59) and (56) we get

$$q_{n_k} \rightharpoonup z \text{ and } u_{n_k} \rightharpoonup z,$$

this together with (55) and and the demiclosedness of $I - T$ and $I - S$ at zero, we have $z \in \Omega$ and, from (61) and the definition of $x^* = P_\Omega(0)$, we have

$$\limsup_{k\to\infty}\langle x^*, v_{n_k} - x^*\rangle = \langle x^*, z - x^*\rangle \leq 0. \tag{62}$$

Combining (60) and (62), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle x^*, v_{n_k+1} - x^*\rangle &\leq \limsup_{k\to\infty}\langle x^*, v_{n_k} - x^*\rangle \\ &= \langle x^*, z - x^*\rangle \\ &\leq 0.\end{aligned} \tag{63}$$

Hence, by (63), $\lim_{n\to\infty}\frac{\alpha_n}{\rho_n}\|v_n - v_{n-1}\| = 0$, Claim 3 and Lemma 2.4, we have $\lim_{n\to\infty}\|v_n - x^*\| = 0$. This is the desired result.

□

4. Numerical results

We consider an application of our algorithms to inverse problems occurring from signal processing. For example, we consider the following equation:

$$y = Ax + e, \tag{64}$$

where $x \in \mathbb{R}^N$ is recovered, $y \in \mathbb{R}^K$ is noisy observations, $A : \mathbb{R}^N \to \mathbb{R}^K$ is a bounded linear observation operator. It determines a process with loss of information. For finding solutions of the linear inverse problems (64), a successful one of some models is the convex unconstrained minimization problem:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|^2 + \lambda\|x\|_1. \tag{65}$$

It is well known that the problem 65 is equivalent to the convex constrained optimization problem

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2, \quad \text{s.t. } \|x\|_1 \le t, \tag{66}$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ and $t > 0$ is a given constant.
When $C := \{x \in \mathbb{R}^n : \|x\|_1 \le t\}$, $Q = \{b\}$ and $U = P_C, T = P_Q$, problem (66) is a particular case of the problem 2. Therefore, we can solve the problem with the proposed algorithms.

For numerical experiments, we will solve (66) by our algorithms and Algorithm 3.6 of Yao et al. [27], Algorithm 4.1 of Wang et al. [21]. In what follows, we will perform a sparse signal recovery experiment to demonstrate the efficiency of our algorithm. In our experiment, the original signal $x \in \mathbb{R}^n$ contains K randomly placed ±1 spikes. The matrix $A \in \mathbb{R}^{n\times m}$ is generated from a normal distribution with mean zero and one variance and then orthonormalizing the columns. The restoration accuracy is measured using the Mean Squared Error MSE $=\frac{1}{N}\|x - x^\star\|^2$, where $x^\star$ is an estimated signal of x.

The process is started with the initial $v_0 = v_1 = (0, ..., 0)^T \in \mathbb{R}^m, t = K$ for all agorithms. We choose

$$\alpha_n = \begin{cases} \min\{\alpha, \dfrac{1}{n^{1.1}\|v_n - v_{n-1}\|}\} & \text{if } v_n \ne v_{n-1}, \\ \alpha & \text{otherwise.} \end{cases}$$

with $\alpha = 0.6$ and $\gamma = 0.6, \lambda = 0.5$ for Algorithm 3.1, Algorithm 3.2, $\gamma = 0.6$ for Algorithm 3.6.
We take $\rho_n = 1 - \frac{1}{20*(n+1)}, \xi_n = \frac{\rho_n}{2}$ for Algorithm 3.1, $\rho_n = \frac{1}{20*(n+1)}, \xi_n = 0.98 - \rho_n$ for Algorithm 3.2 , $\alpha_n = \beta_n = \frac{1}{20*(n+1)}, u = v_0$ for Algorithm 4.1 and Algorithm 3.6. To terminate algorithms, we use the condition $MSE \le \epsilon$ with $\epsilon = 10^{-6}$ or the number of iterations ≥ 500 for all algorithms.

All the numerical experiments are performed on an HP laptop with Intel(R) Core(TM)i5-6200U CPU 2.3GHz with 4 GB RAM. All the programs are written in Matlab2015a. The numerical results are reported in Figs 1-4.

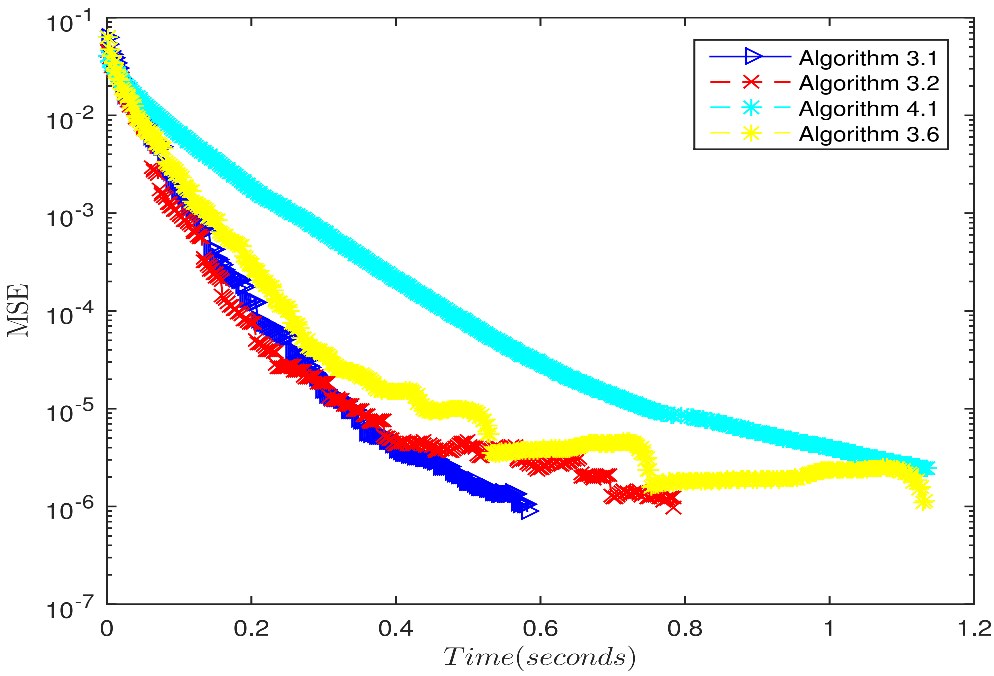

Figure 1: MSE of times generated by all agorithms with m=1024, n=2048 and K=128,

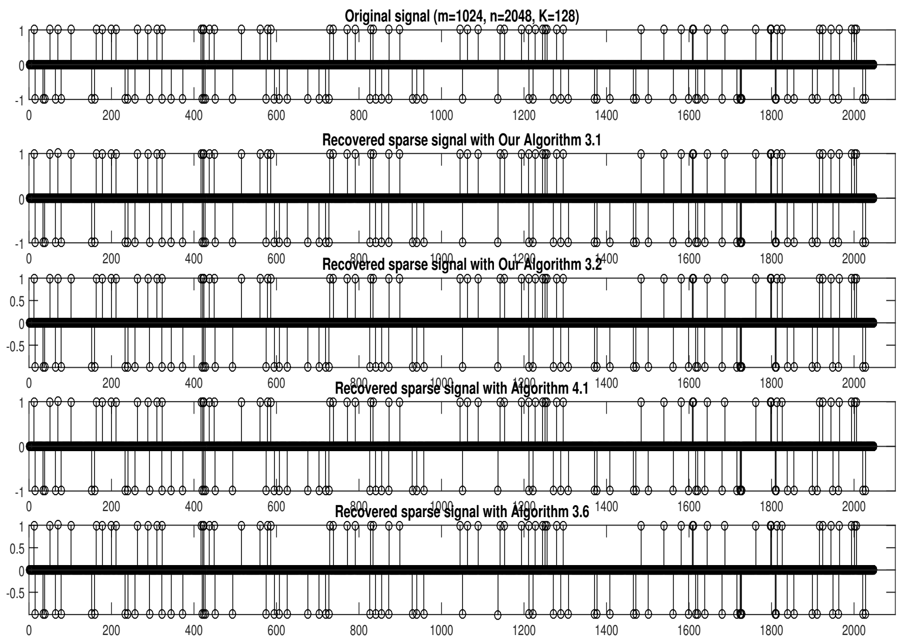

Figure 2: Original signal and recovered signal by all algorithms

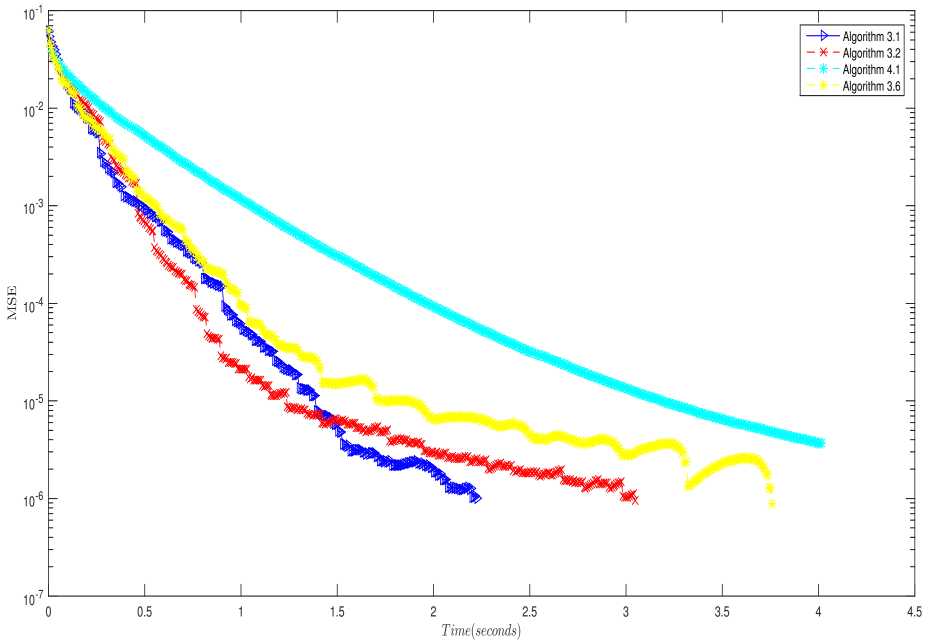

Figure 3: MSE of times generated by all agorithms with m=2048, n=4096 and K=256

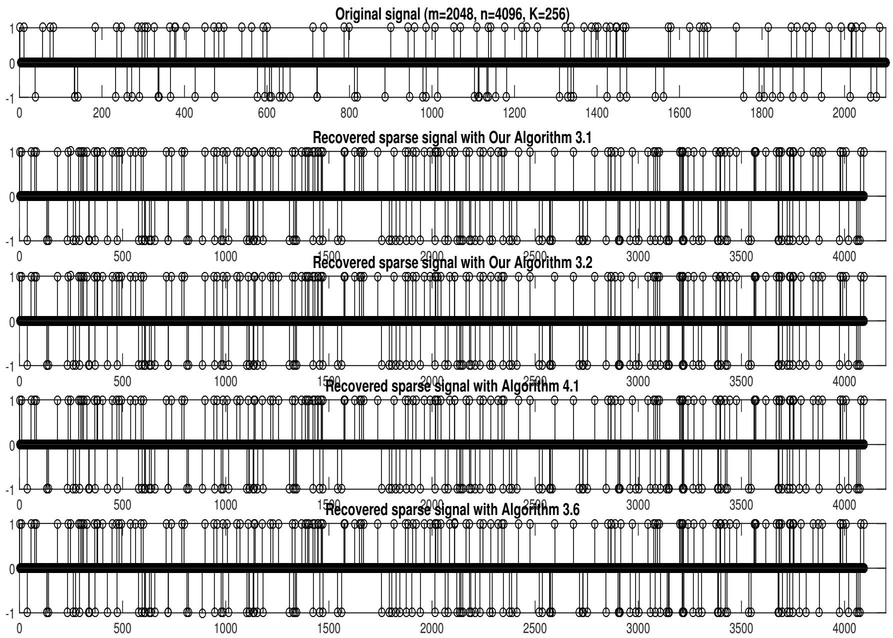

Figure 4: Original signal and recovered signal by all algorithms

Figures 1, 2, 3, 4 give the mean squared error of the Algorithm 3.6 of Yao et al. [27], Algorithm 4.1 of Wang et al. [21] and our Algorithms as well as their execution times. We see that our Algorithm 3.1, our Algorithm 3.2 are less time consuming and more accurate than those of Yao et al. [27] and et al. [21].

5. Conclusions

In this paper, we propose two new iteration algorithms for split common fixed point problems. Our main results are an extension of the related results announced in this direction. This paper's research highlights include novel algorithms and their analysis techniques, which use inertia to improve algorithm convergence rates.

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