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A study on lacunary strong $\hat{E}(r,t)$-convergence according to F^k

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Abstract. In this manuscript, by using a double band matrix such as Lucas band matrix, a lacunary sequence and a sequence of modulus functions a new generalization is studied in which the concept of lacunary strong $\hat{E}(r,t)-$convergence according to F^k is introduced. Then this concept is to be base to create the sequence space $N_\theta(\hat{E}(r,t), F^k)$, where for instance

$$N_\theta(\hat{E}(r,t), F^k) = \left\{ x = (x_i) : \lim_{r\mapsto\infty} \frac{1}{h_r} \sum_{i\in I_r} f_i^k \left(\left| \hat{E}_i(r,t)(x) - s \right| \right) = 0 \ \text{ for some } s \in \mathbb{C} \right\}.$$

Furthermore, some inclusion relationships are given on this generalization. Finally, the connections between the sequence spaces $N_\theta(\hat{E}(r,t), F^k)$ and $w(\hat{E}(r,t), F^k)$ are investigated.

1. Introduction

Assume that ω is the space containing all sequences of real and complex numbers. A linear subspace of ω is referred to as a sequence space. The symbols c_0, c and ℓ_∞ represent the sequence spaces of all null, convergent and bounded sequences respectively, normed by $\|x\|_\infty = \sup_i |x_i|$. For the first time, the difference sequence spaces were presented by Kizmaz [1] in the form of $X(\Delta) = \{x \in \omega : x_i - x_{i-1} \in X\}$, $X = \ell_\infty$, c, c_0. After that, Et and Colak [2] has created a generalization on these sequence spaces such as $X(\Delta^r) = \{x \in \omega : \Delta^r x \in X\}$. Kirisci and Basar [3] have recently defined and investigated the difference sequence spaces $\hat{X} = \{x \in \omega : B(r,t)x \in X\}$, $X = \ell_\infty$, c, c_0, where $B(r,t) = (rx_i + tx_{i-1})$; $r, t \neq 0$. At the same time, some other mathematicians have investigated the difference sequence spaces in their studies.

Consider $\Psi = (\psi_{mi})$ as an infinite matrix and assume that X and Y are two sequence spaces. Then, Ψ provides a matrix mapping from X into Y if, for every $x = (x_i)$ in X, there exists the sequence $\Psi x = (\psi_m(x)) \in Y$, where

$$\psi_m(x) = \sum_{i=0}^{\infty} \psi_{mi} x_i \quad (m \in \mathbb{N}) \tag{1.1}$$

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The set of all matrices Ψ that have the property $\Psi : X \to Y$ is represented by (X, Y). For that, $\Psi \in (X : Y)$ if and only if the right hand of the equality (1.1) mentioned above is convergent for each $m \in \mathbb{N}$ and $x \in X$. The notion of matrix domain X_Ψ of Ψ in X is expressed by

$$X_\Psi = \{x \in \omega : x_\Psi \in X\} \tag{1.2}$$

that represents a sequence space. In recent years, a few mathematicians have developed certain sequence spaces by use of the matrix domain for an infinite matrix, follow in ([4, 5]).

In 1876, the sequence $\{L_i\}_{i=1}^{\infty}$ of Lucas numbers $1, 3, 4, 7, 11, 18, ...$ was introduced by Edouard Lucas which is given by the Fibonacci recurrence relation in the form $L_i = L_{i-1} + L_{i-2}; i \geq 2$ with different initial conditions $L_0 = 2$ and $L_1 = 1$ where L_i is the *ith* term of the sequence $\{L_i\}_{i=1}^{\infty}$. Lucas numbers have various formulas and several properties, follow in ([6]).

By using Lucas numbers with two real numbers r and t such that $r, t \neq 0$, the Lucas band matrix $\hat{E}(r,t) = (\hat{E}_{im}(r,t))$ has been established as follows:

$$\hat{E}_{im}(r,t) = \begin{cases} t\frac{L_i}{L_{i-1}} & (m = i-1) \\ r\frac{L_{i-1}}{L_i} & (m = i) \\ 0 & m > i \text{ or } 0 \leq m < i-1. \end{cases} \tag{1.3}$$

With the help of (1.3) the $\hat{E}-$transform for a sequence $x = (x_i)$ is formed by

$$\hat{E}_i(r,t)(x) = r\frac{L_{i-1}}{L_i}x_i + t\frac{L_i}{L_{i-1}}x_{i-1}, \quad i \geq 1. \tag{1.4}$$

Recently, several mathematicians have used Lucas numbers and Lucas band matrix in constructing some sequence spaces in their studies, follow in ([7, 8, 9]).

A subset E of $\mathbb{N}$ has a natural density $\delta(E)$ that is given by

$$\delta(E) = \lim_{k\to\infty} \frac{1}{k} |E_k|$$

where $|E_k| = \{i \leq k : i \in E\}$ specifies the number of elements of E not greater than k. It's evident that $\delta(\mathbb{N}) = 1$ and $\delta(E) = 0$ if E is a subset of $\mathbb{N}$ that is finite and $\delta(\mathbb{N} \setminus E) = 1 - \delta(E)$.

From Freedman [10], the sequence $\theta = (k_r)$ of non-negative integers is referred to be a lacunary sequence such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. And the periods given by θ are identified by $I_r = (k_{r-1}, k_r]$ and $\frac{k_r}{k_{r-1}}$ can be recognized by q_r. These notations are going to be used during the article.

Lacunary sequences have been investigated by several authors in ([11, 12-16]).

The meaning that Orhan and Fridy [17] intended from lacunary statistical convergence is stated in the following expression.

Suppose that $\theta = (k_r)$ is a given lacunary sequence. A sequence (x_i) of numbers is referred to be lacunary statistically convergent (or S_θ-convergent) to ℓ, if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : |x_i - \ell| \geq \varepsilon\}| = 0$$

for each given $\varepsilon > 0$. In this context, we shall build $x_i \to \ell\,(S_\theta)$ or $S_\theta - \lim x_i = \ell$. From the paper entirely, the class of S_θ-convergent sequences is identified by S_θ, that is

$$S_\theta = \left\{(x_i) : \lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : |x_i - \ell| \geq \varepsilon\}| \text{ for some } \ell \in \mathbb{C}\right\}.$$

Lacunary statistically convergence has also been talked in ([12, 13, 17-20]) and studied by many other authors.

According to Freedman [10], the space N_θ of lacunary strongly convergent sequences is presented as follows:

$$N_\theta = \left\{ (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} |x_i - \ell| = 0 \text{ for some } \ell \right\}.$$

This space becomes a BK-space with the norm presented below.

$$\|x\|_\theta = \sup_r \frac{1}{h_r} \sum_{i\in I_r} |x_i| .$$

The space N_θ^0 defines the collection of all sequences containing $\ell = 0$ in the definition of N_θ. It becomes also a BK-space with the presented norm $\|\cdot\|_\theta$. After that, Pehlivan and Fisher [14] developed the idea of lacunary strong convergence by the use of a modulus function.

In 1953, Nakano [21] established the concept of the modulus function. We remember that $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ is a modulus function such that $f(x) = 0$ iff $x = 0$, f is increasing and continuous from 0^+, and $f(x+y) \le f(x) + f(y)$ for every x, y in $\mathbb{R}^+ \cup \{0\}$. A modulus can either be bound or unbound. For illustration, $f(x) = \frac{1}{1+x}$ is a bounded modulus, but $f(x) = x^p$,$(0 < p \le 1)$ is being unbounded. The notion of the modulus function has also been discussed in ([5, 12, 19]).

Since $\left| f(c_1) - f(c_2) \right| \le f(|c_1 - c_2|)$, then from the continuity of f on 0^+ it shall be clear that f is continuous on $\mathbb{R}^+ \cup \{0\}$. Additionally, since $f(x+y) \le f(x) + f(y)$ for every x, y in $\mathbb{R}^+ \cup \{0\}$, then we can have $f(mc) \le mf(c)$ for every $m \in \mathbb{N}$ and for all $c \in \mathbb{R}^+ \cup \{0\}$.

Lemma 1.1 For any $k \in \mathbb{N}$, the function $f^k = f \circ f ... \circ f$ (k times) is a modulus whenever f is a modulus function. The notion f^k has been used in some monographs to create new sequence spaces and established new concepts, follow in ([11, 20]).

Definition 1.2 [20] Suppose that f is a modulus function and let $\theta = (k_r)$ be given as a lacunary sequence. Let $\ell \in \mathbb{C}$ be a number, then the sequence (x_i) in $\mathbb{C}$ is referred to be lacunary strongly convergent to ℓ according to f^k (or $N_\theta(f^k)$-strongly convergent), if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} f^k(|x_i - \ell|) = 0.$$

In this manner, it is written as $x_i \to \ell(N_\theta(f^k))$ or $N_\theta(f^k) - \lim x_i = \ell$. The set of all $N_\theta(f^k)$-strongly convergent sequences is identified by $N_\theta(f^k)$. That is

$$N_\theta(f^k) = \left\{ (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} f^k(|x_i - \ell|) = 0 \text{ for some } \ell \in \mathbb{C} \right\}.$$

It is clear to see that if $k = 1$, then the space $N_\theta(f^k)$ will become the same as $N_\theta(f)$ of Pehlivan and Fisher [14]. In the special case of $\theta = (2^r)$ and $f(u) = u$, we see that $N_\theta(f^k) = |\sigma_1|$, where $|\sigma_1| = w$ is the set of strongly Cesaro summable sequences. We use the notation (x_i) as a sequence of complex numbers throughout the next.

2. Main results

In this part of our manuscript, we introduce a concept and several sequence spaces based on the Lucas band matrix $\hat{E}(r,t)$ and a sequence of modulus functions $F = (f_i)$, and study some interesting results through newly sequence spaces. Let the set of all sequences of modulus functions $F = (f_i)$ be identified by $\mathcal{F}$ such that $\lim_{u\to 0^+}\sup_i f_i(u) = 0$. The sequence of modulus functions determined by F is indicated by $F = (f_i) \in \mathcal{F}$. We let $F^k = (f_i^k) = \left\{f_1^k, f_2^k, ...\right\}$ $(k \in \mathbb{N})$ to be a sequence of composite modulus functions. We use these notations throughout this study.

Definition 2.1 Assume that $F = (f_i)$ is a sequence of modulus functions in $\mathcal{F}$ and let $\theta = (k_r)$ be given as a lacunary sequence. Let s be a number in $\mathbb{C}$, then the sequence (x_i) in $\mathbb{C}$ is referred to be lacunary strongly $\hat{E}(r,t)$-convergent to s according to F^k (or $N_\theta(\hat{E}(r,t), F^k)$-strongly convergent), if

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) = 0.$$

In this manner, we refer it to as $x_i \to s(N_\theta(\hat{E}(r,t), F^k))$ or $N_\theta(\hat{E}(r,t), F^k) - \lim x_i = s$. The set of all $N_\theta(\hat{E}(r,t), F^k)$-strongly convergent sequences is identified by $N_\theta(\hat{E}(r,t), F^k)$. That is

$$N_\theta(\hat{E}(r,t), F^k) = \left\{(x_i) : \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) = 0 \text{ for some } s \in \mathbb{C}\right\}.$$

The set of all sequences in which $s = 0$ in the definition of $N_\theta(\hat{E}(r,t), F^k)$ is identified by $N_\theta^0(\hat{E}(r,t), F^k)$. That is

$$N_\theta^0(\hat{E}(r,t), F^k) = \left\{(x_i) : \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x)\right|\right) = 0\right\}.$$

Observe that the functions in F are not necessary to be unbounded modulus in this definition. If we put $f_i(u) = u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then the $N_\theta(\hat{E}(r,t), F^k)$-strong convergence is reduced to the $N_\theta(\hat{E}(r,t))$-strong convergence and it is so for the $N_\theta^0(\hat{E}(r,t), F^k)$-strong convergence. Also, it is clear to see that if $k = 1$, then from the spaces $N_\theta(\hat{E}(r,t), F^k)$ and $N_\theta^0(\hat{E}(r,t), F^k)$ we have $N_\theta(\hat{E}(r,t), F)$ and $N_\theta^0(\hat{E}(r,t), F)$, respectively. As well as if $f_i(u) = f(u)$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then instead of $N_\theta(\hat{E}(r,t), F^k)$ and $N_\theta^0(\hat{E}(r,t), F^k)$ we shall write $N_\theta(\hat{E}(r,t), f^k)$ and $N_\theta^0(\hat{E}(r,t), f^k)$, respectively.

Theorem 2.2 Assume that $F = (f_i)$ is a sequence of modulus functions in $\mathcal{F}$, and let $\theta = (k_r)$ be given as a lacunary sequence. Then the sets $N_\theta(\hat{E}(r,t), F^k)$ and $N_\theta^0(\hat{E}(r,t), F^k)$ are linear spaces.

Proof. Here, we only consider $N_\theta(\hat{E}(r,t), F^k)$. Suppose $x_i \to s_1(N_\theta(\hat{E}(r,t), F^k))$ and $y_i \to s_2(N_\theta(\hat{E}(r,t), F^k))$ as $i \to \infty$, and let $\alpha, \beta \in \mathbb{C}$. Then there are positive integers N_α and M_β such that $|\alpha| \le N_\alpha$ and $|\beta| \le M_\beta$. Since f_i is a modulus for each $i \in \mathbb{N}$, we may have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(\alpha x + \beta y) - (\alpha s_1 + \beta s_2)\right|\right)$$

$$= \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\alpha\left(\hat{E}_i(r,t)(x) - s_1\right) + \beta\left(\hat{E}_i(r,t)(y) - s_2\right)\right|\right)$$

$$\le \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(|\alpha|\left|\hat{E}_i(r,t)(x) - s_1\right| + |\beta|\left|\hat{E}_i(r,t)(y) - s_2\right|\right)$$

$$\le N_\alpha \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s_1\right|\right) + M_\beta \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(y) - s_2\right|\right).$$

Putting the limits into both given sides as r tends to ∞, we then have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(\alpha x+\beta y)-(\alpha s_1+\beta s_2)\right|\right)=0.$$

This implies that $(x_i+y_i)\to(s_1+s_2)\,(N_\theta(\hat{E}(r,t),F^k))$ as $i\to\infty$. Therefore, the space $N_\theta(\hat{E}(r,t),F^k)$ is indeed linear. □

Theorem 2.3 Assume that $F=(f_i)$ is a sequence of modulus functions in $\mathcal{F}$, and let $\theta=(k_r)$ be given as a lacunary sequence. Then every $N_\theta^0(F^k)$-strongly convergent sequence implies $N_\theta^0(\hat{E}(r,t),F^k)$-strongly convergent, i.e, $N_\theta^0(F^k)\subset N_\theta^0(\hat{E}(r,t),F^k)$.

Proof. From the Lucas sequence, we see that $(L_{i-1}/L_i)\le 2$ and $(L_i/L_{i-1})\le 3$ $(\forall i\in\mathbb{N})$. Then by using (1.4), we have

$$\left|\hat{E}_i(r,t)(x)\right|\le 36\max\{|r|,|t|\}(|x_i|+|x_{i-1}|).$$

Let $x=(x_i)$ be an $N_\theta^0(F^k)$-strongly convergent sequence. Since f_i is increasing for each $i\in\mathbb{N}$, then we may write

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x)\right|\right)&\le 36\max\{T_r,T_t\}\frac{1}{h_r}\sum_{i\in I_r} f_i^k(|x_i|+|x_{i-1}|)\\&\le 36\max\{T_r,T_t\}\left(\frac{1}{h_r}\sum_{i\in I_r} f_i^k(|x_i|)+\frac{1}{h_r}\sum_{i\in I_r} f_i^k(|x_{i-1}|)\right)\end{aligned}$$

where T_r and T_t are natural values such that $|r|\le T_r$ and $|t|\le T_t$. Thus, we have

$$\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x)\right|\right)\to 0 \text{ as } r\to\infty.$$

This concludes that $x\in N_\theta^0(\hat{E}(r,t),F^k)$. Hence the proof. □

Corollary 2.4 Assume that $F=(f_i)$ is a sequence of modulus functions in $\mathcal{F}$ and let $\theta=(k_r)$ be given as a lacunary sequence. We have the results below

(i) If $f_i(u)\le u$ for all $i\in\mathbb{N}$ and for every $u\in\mathbb{R}^+\cup\{0\}$, then $N_\theta^0\subset N_\theta^0(\hat{E}(r,t),F^k)$.

(ii) If $f_i(u)\ge u$ for all $i\in\mathbb{N}$ and for every $u\in\mathbb{R}^+\cup\{0\}$, then $N_\theta^0(F^k)\subset N_\theta^0(\hat{E}(r,t))$.

(iii) If $f_i(u)=u$ for all $i\in\mathbb{N}$ and for every $u\in\mathbb{R}^+\cup\{0\}$, then $N_\theta^0\subset N_\theta^0(\hat{E}(r,t))$.

Theorem 2.5 Assume that $F=(f_i)$ is a sequence of modulus functions in $\mathcal{F}$, and let the lacunary sequences $\theta=(k_r)$ and $\vartheta=(s_r)$ be given such that $I_r\subseteq J_r$ for all $r\in\mathbb{N}$. If $\lim_{r\to\infty}\frac{\ell_r}{h_r}=1$ and $(x_i)\in\ell_\infty(\hat{E}(r,t))$ then $N_\theta(\hat{E}(r,t),F^k)\subset N_\vartheta(\hat{E}(r,t),F^k)$, where

$$\ell_\infty(\hat{E}(r,t))=\left\{(x_i):\sup_i\left|\hat{E}_i(r,t)(x)\right|<\infty\right\}.$$

Proof. Let $(x_i)\in\ell_\infty(\hat{E}(r,t))\cap N_\theta(\hat{E}(r,t),F^k)$. Then $(\hat{E}_i(r,t)(x))$ is a bounded sequence, so there has some

$H > 0$ such that $\left|\hat{E}_i(r,t)(x) - s\right| \leq H$ for all $i \in \mathbb{N}$. Since $I_r \subseteq J_r$ and $h_r \leq \ell_r$ for all $r \in \mathbb{N}$, we are going to write

$$\begin{aligned}\frac{1}{\ell_r}\sum_{i\in J_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) &= \frac{1}{\ell_r}\sum_{i\in J_r - I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) + \frac{1}{\ell_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right)\\ &\leq \left(\frac{\ell_r - h_r}{\ell_r}\right)\sup_i f_i^k(H) + \frac{1}{\ell_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right)\\ &\leq \left(\frac{\ell_r}{h_r} - 1\right)\sup_i f_i^k(H) + \frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right).\end{aligned}$$

Putting the limits into both given sides as r tends to ∞, we then have

$$\frac{1}{\ell_r}\sum_{i\in J_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) = 0$$

since $\lim_{r\to\infty}\frac{\ell_r}{h_r} = 1$. Therefore, we obtain that $(x_i) \in N_\vartheta(\hat{E}(r,t), F^k)$. Hence the proof. □

Corollary 2.6 Assume that $F = (f_i)$ is given as a sequence of modulus functions in $\mathcal{F}$ and $k < n$. If $f_i(u) \leq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then Theorem 2.5, provides the following results:
(i) $N_\theta(\hat{E}(r,t)) \subset N_\vartheta(\hat{E}(r,t), F^k)$.
(ii) $N_\theta(\hat{E}(r,t), F^k) \subset N_\vartheta(\hat{E}(r,t), F^n)$.

Corollary 2.7 Assume that $F = (f_i)$ is given as a sequence of modulus functions in $\mathcal{F}$ and $k < n$. If $f_i(u) \geq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then Theorem 2.5, provides the following results:
(i) $N_\theta(\hat{E}(r,t), F^k) \subset N_\vartheta(\hat{E}(r,t))$.
(ii) $N_\theta(\hat{E}(r,t), F^n) \subset N_\vartheta(\hat{E}(r,t), F^k)$.

Theorem 2.8 Assume that $F = (f_i)$ is a sequence of modulus functions in $\mathcal{F}$, and let the lacunary sequences $\theta = (k_r)$ and $\vartheta = (s_r)$ be given such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$. If $\lim_{r\to\infty}\sup\frac{\ell_r}{h_r} < \infty$, then $N_\vartheta(\hat{E}(r,t), F^k) \subset N_\theta(\hat{E}(r,t), F^k)$.

Proof. Let $(x_i) \in N_\vartheta(\hat{E}(r,t), F^k)$. Since $I_r \subseteq J_r$ we may write

$$\frac{1}{\ell_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) \leq \frac{1}{\ell_r}\sum_{i\in J_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right)$$

and then

$$\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) \leq \frac{\ell_r}{h_r}\frac{1}{\ell_r}\sum_{i\in J_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right).$$

Putting the limits into both given sides as r tends to ∞, we then have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) = 0.$$

Therefore, we obtain that $(x_i) \in N_\theta(\hat{E}(r,t), F^k)$. Hence the proof. □

Corollary 2.9 Assume that $F = (f_i)$ is given as a sequence of modulus functions in $\mathcal{F}$ and $k < n$. If $f_i(u) \leq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then Theorem 2.8, provides the following results.
(i) $N_\vartheta(\hat{E}(r,t)) \subset N_\theta(\hat{E}(r,t), F^k)$.
(ii) $N_\vartheta(\hat{E}(r,t), F^k) \subset N_\theta(\hat{E}(r,t), F^n)$.

Corollary 2.10 Assume that $F = (f_i)$ is given as a sequence of modulus functions in $\mathcal{F}$ and $k < n$. If $f_i(u) \geq u$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$, then Theorem 2.8, provides the following results.

(i) $N_{\vartheta}(\hat{E}(r,t), F^k) \subset N_{\theta}(\hat{E}(r,t))$.
(ii) $N_{\vartheta}(\hat{E}(r,t), F^n) \subset N_{\theta}(\hat{E}(r,t), F^k)$.

Theorem 2.11 Assume that $F = (f_i)$ and $G = (g_i)$ are two sequences of modulus functions in $\mathcal{F}$, and let $\theta = (k_r)$ be given as a lacunary sequence. If $\sup_{u,i} \frac{f_i(u)}{g_i(u)} < \infty$, then $N_{\theta}(\hat{E}(r,t), F^k \circ G) \subset N_{\theta}(\hat{E}(r,t), F^{k+1})$.

Proof. Let $a \in \mathbb{C}$. Then there is a natural number T_a such that $|a| \le T_a$. Now suppose $0 < \alpha = \sup_{u,i} \frac{f_i(u)}{g_i(u)} < \infty$, then $\alpha \ge \frac{f_i(u)}{g_i(u)}$ and so that $f_i(u) \le \alpha g_i(u)$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$. Since (f_i) is increasing for each $i \in \mathbb{N}$, then we get

$$f_i^{k+1}(u) \le f_i^k(\alpha g_i(u)) \le T_a f_i^k(g_i(u)). \tag{2.1}$$

Now if $(x_i) \in N_{\theta}(\hat{E}(r,t), F^k \circ G)$, then according to (2.1), we shall possess

$$\frac{1}{h_r}\sum_{i\in I_r} f_i^{k+1}\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) \le T_a \frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(g_i\left(\left|\hat{E}_i(r,t)(x) - s\right|\right)\right) \to 0 \text{ as } r \to \infty.$$

This concludes that $(x_i) \in N_{\theta}(\hat{E}(r,t), F^k \circ G)$ implies $(x_i) \in N_{\theta}(\hat{E}(r,t), F^{k+1})$. Hence the proof. □

Theorem 2.12 Assume that $F = (f_i)$ and $G = (g_i)$ are two sequences of modulus functions in $\mathcal{F}$, and let $\theta = (k_r)$ be given as a lacunary sequence. If $\inf_{u,i} \frac{f_i(u)}{g_i(u)} > 0$, then $N_{\theta}(\hat{E}(r,t), F^{k+1}) \subset N_{\theta}(\hat{E}(r,t), F^k \circ G)$.

Proof. Let $b \in \mathbb{C}$ with $b \ne 0$. Then there has a natural value $T_{b^{-1}}$ such that $\left|b^{-1}\right| \le T_{b^{-1}}$. Now suppose $\beta = \inf_{u,i} \frac{f_i(u)}{g_i(u)} > 0$ then $\beta \le \frac{f_i(u)}{g_i(u)}$ and so that $g_i(u) \le \frac{1}{\beta} f_i(u)$ for all $i \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$. Since (f_i) is increasing for each $i \in \mathbb{N}$ and $\beta > 0$, then we get

$$f_i^k(g_i(u)) \le f_i^k\left(\beta^{-1} f_i(u)\right) \le T_{\beta^{-1}} f_i^{k+1}(u). \tag{2.2}$$

Now let $(x_i) \in N_{\theta}(\hat{E}(r,t), F^{k+1})$, then according to (2.2), we shall possess

$$\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(g_i\left(\left|\hat{E}_i(r,t)(x) - s\right|\right)\right) \le T_{\beta^{-1}} \frac{1}{h_r}\sum_{i\in I_r} f_i^{k+1}\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) \to 0 \text{ as } r \to \infty.$$

This concludes that $(x_i) \in N_{\theta}(\hat{E}(r,t), F^{k+1})$ implies $(x_i) \in N_{\theta}(\hat{E}(r,t), F^k \circ G)$. Hence the proof. □

Corollary 2.13 Assume that $F = (f_i)$ and $G = (g_i)$ are two sequences of modulus functions in $\mathcal{F}$, and let $\theta = (k_r)$ be given as a lacunary sequence.

(i) If $\sup_{u,i} \frac{f_i(u)}{g_i(u)} < \infty$, then $N_{\theta}(\hat{E}(r,t), G^{k+1}) \subset N_{\theta}(\hat{E}(r,t), G^k \circ F)$.
(ii) If $\inf_{u,i} \frac{f_i(u)}{g_i(u)} > 0$, then $N_{\theta}(\hat{E}(r,t), G^k \circ F) \subset N_{\theta}(\hat{E}(r,t), G^{k+1})$.

Theorem 2.14 Assume that $F = (f_i)$ is a sequence of modulus functions in $\mathcal{F}$ and let $\theta = (k_r)$ be given as a lacunary sequence. If $\lim_{u\to\infty} \frac{f_i^k(u)}{u} > 0$ for all $i \in \mathbb{N}$ and there has a natural number d such that $f_i(uv) \ge d f_i(u) f_i(v)$ for all $i \in \mathbb{N}$ and for every $u, v \in \mathbb{R}^+ \cup \{0\}$. Then $(x_i) \to s(N_{\theta}(\hat{E}(r,t), F^k))$ implies $(x_i) \to s(S_{\theta}(\hat{E}(r,t), F^k))$, where

$$S_{\theta}(\hat{E}(r,t), F^k) = \left\{(x_i) : \lim_{r\to\infty} \frac{1}{f_i^k(h_r)} f_i^k\left(\left|\left\{i \in I_r : \left|\hat{E}_i(r,t)(x) - s\right| \ge \varepsilon\right\}\right|\right) = 0 \text{ for some } s \in \mathbb{C}\right\}.$$

Proof. Suppose that $(x_i) \to s(N_{\theta}(\hat{E}(r,t), F^k))$, and $\varepsilon > 0$ is given. Let $\sum_A$ and $\sum_B$ be defined on $\left|\hat{E}_i(r,t)(x) - s\right| \ge \varepsilon$ and $\left|\hat{E}_i(r,t)(x) - s\right| < \varepsilon$, respectively. Then from the definition of modulus functions, we may write

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x)-s\right|\right) &= \frac{1}{h_r}\sum_{i\in I_r\setminus A} f_i^k\left(\left|\hat{E}_i(r,t)(x)-s\right|\right)+\frac{1}{h_r}\sum_{i\in I_r\setminus B} f_i^k\left(\left|\hat{E}_i(r,t)(x)-s\right|\right)\\ &\geq \frac{1}{h_r}\sum_{i\in I_r\setminus A} f_i^k\left(\left|\hat{E}_i(r,t)(x)-s\right|\right)\\ &\geq \frac{1}{h_r} f_r^k\left(\sum_{i\in I_r\setminus A}\left|\hat{E}_i(r,t)(x)-s\right|\right)\\ &\geq \frac{1}{h_r} f_r^k\left(\left|\left\{i\in I_r:\left|\hat{E}_i(r,t)(x)-s\right|\geq\varepsilon\right\}\right|\varepsilon\right)\\ &\geq \frac{D}{h_r} f_r^k\left(\left|\left\{i\in I_r:\left|\hat{E}_i(r,t)(x)-s\right|\geq\varepsilon\right\}\right|\right) f_r^k(\varepsilon),\end{aligned}$$

where D is a positive constant. Since $\left|\left\{i\in I_r:\left|\hat{E}_i(r,t)(x)-s\right|\geq\varepsilon\right\}\right|$ is a natural value, then

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x)-s\right|\right) &\geq \frac{D}{h_r} f_r^k\left(\left|\left\{i\in I_r:\left|\hat{E}_i(r,t)(x)-s\right|\geq\varepsilon\right\}\right|\right)\frac{\inf_r f_r^k(\varepsilon)}{\inf_r f_r^k(1)}\\ &= \frac{f_r^k\left(\left|\left\{i\in I_r:\left|\hat{E}_i(r,t)(x)-s\right|\geq\varepsilon\right\}\right|\right)}{f_r^k(h_r)}\frac{f_r^k(h_r)}{h_r}\frac{\inf_r f_r^k(\varepsilon)}{\inf_r f_r^k(1)}D.\end{aligned}$$

Putting the limits on both sides as $r\to\infty$, it concludes that $x_i\to s(S_\theta(\hat{E}(r,t),F^k))$. Hence the proof. □

Theorem 2.15 Assume that $F=(f_i)$ is a sequence of modulus functions in $\mathcal{F}$ and let $\theta=(k_r)$ be given as a lacunary sequence.

(i) If $\liminf q_r>1$ then $x_i\to s(w(\hat{E}(r,t),F^k))$ implies $x_i\to s(N_\theta(\hat{E}(r,t),F^k))$.

(ii) If $\limsup q_r<\infty$, then $x_i\to s(N_\theta(\hat{E}(r,t),F^k))$ implies $x_i\to s(w(\hat{E}(r,t),F^k))$.

Proof. (i) Since $\liminf q_r>1$ then there is $\delta>0$ such that

$$q_r=\frac{k_r}{k_{r-1}}\geq 1+\delta$$

for sufficiently large r, then we have that

$$\frac{h_r}{k_r}\geq\frac{\delta}{\delta+1}\text{ and }\frac{k_r}{h_r}\leq\frac{\delta+1}{\delta}$$

Now assume that $x_i\to s(w(\hat{E}(r,t),F^k))$, then we may write

$$\begin{aligned}\frac{1}{k_r}\sum_{i=1}^{k_r} f_i^k\left(\left|\hat{E}_i(r,t)(x)-s\right|\right) &\geq \frac{1}{k_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x)-s\right|\right)\\ &= \frac{h_r}{k_r}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x)-s\right|\right)\\ &\geq \frac{\delta+1}{\delta}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x)-s\right|\right).\end{aligned}$$

Thus

$$\frac{\delta+1}{\delta}\frac{1}{h_r}\sum_{i\in I_r} f_i^k\left(\left|\hat{E}_i(r,t)(x)-s\right|\right)\to 0\text{ as }r\to\infty.$$

This concludes that $x_i \to s(N_\theta(\hat{E}(r,t), F^k))$.

(ii) Since $\limsup q_r < \infty$, then there has a positive number K, say $K = \sup q_r$ so that $q_r < K$ for every $r \in \mathbb{N}$. Assume that $x_i \to s(N_\theta(\hat{E}(r,t), F^k))$ and given $\varepsilon > 0$. There has n_0 such that for every $n > n_0$, we shall have

$$T_n = \frac{1}{h_n}\sum_{i\in I_n} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) < \varepsilon.$$

Also, a positive number S can be found such that $T_n \le S$ for all n. Let m be an arbitrary integer such that $m \in (k_{r-1}, k_r]$. Now we may write

$$\begin{aligned}
\frac{1}{m}\sum_{i=1}^{m} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) &\le \frac{1}{k_r}\sum_{i=1}^{k_r} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right)\\
&\le \frac{1}{k_{r-1}}\left(\sum_{n=1}^{n_0} + \sum_{n=n_0+1}^{k_r}\right)\sum_{i\in I_n} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right)\\
&= \frac{1}{k_{r-1}}\sum_{n=1}^{n_0}\sum_{i\in I_n} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) + \frac{1}{k_{r-1}}\sum_{n=n_0+1}^{k_r}\sum_{i\in I_n} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right)\\
&\le \frac{1}{k_{r-1}}\sum_{n=1}^{n_0}\sum_{i\in I_n} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) + \frac{1}{k_{r-1}}(k_r - k_{n_0})\,\varepsilon\\
&= \frac{1}{k_{r-1}}(h_1T_1 + h_2T_2 + \cdots + h_{n_0}T_{n_0}) + \frac{1}{k_{r-1}}(k_r - k_{n_0})\,\varepsilon\\
&\le \frac{1}{k_{r-1}}\left(\sup_{i\in[1,n_0]} T_i k_{n_0}\right) + \varepsilon K < \frac{1}{k_{r-1}}K_{n_0}S + \varepsilon K.
\end{aligned}$$

Thus

$$\frac{1}{m}\sum_{i=1}^{m} f_i^k\left(\left|\hat{E}_i(r,t)(x) - s\right|\right) \to 0 \text{ as } r \to \infty.$$

This implies that $x_i \to s(w(\hat{E}(r,t), F^k))$. Hence the proof. □

Compliance with ethical standards

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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