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The spectral rank and Drazin inverse in J-semisimple and torsion-free rings

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Abstract. In a recent paper [1], Brits, Schulz and the second-named author introduced and studied the spectral rank in the setting of J-semisimple and torsion-free rings. In the present paper, we investigate the relationship between this spectral rank and Drazin invertibility in J-semisimple and torsion-free rings and prove generalizations of results established in [5]. Some open problems are also mentioned.

1. Introduction and Preliminaries

Throughout this paper, R will denote an associative ring with additive identity $\mathbf{0}$ and multiplicative identity $\mathbf{1}$, whose group of units and Jacobson radical will be indicated by $\mathcal{U}(R)$ and $\operatorname{rad}(R)$, respectively. By ([8], Lemma 4.3),

$$\operatorname{rad}(R) = \{x \in R \,|\, \mathbf{1} + yx \in \mathcal{U}(R) \text{ for all } y \in R\}.$$

In [3], Aupetit and Mouton introduced a generalization of the notion of rank in the context of complex unital Banach algebras. We recall that the *(spectral) rank* of an element a of a complex unital Banach algebra A is defined as

$$\operatorname{rank}(a) := \sup_{x \in A} \#\sigma'(xa),$$

where $\sigma'(a)$ represents the nonzero spectrum of a, and the notation $\#K$ denotes the (possibly infinite) number of distinct elements in the set K. Recently (see [1]), the concept of the spectral rank was further extended to the setting of rings. In particular, it was shown in [1, Theorem 4.14] that the algebraic rank (as defined by Stopar in [11]) and the spectral rank coincide (on the socle elements) in J-semisimple and torsion-free rings, where a ring R is said to be *J-semisimple* if $\operatorname{rad}(R) = \{\mathbf{0}\}$ and *torsion-free* if for all $t \in \mathbb{Z}$ and $a \in R$, the

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condition $ta = \mathbf{0}$ implies that either $t = 0$ or $a = \mathbf{0}$. Alternatively, torsion-free rings can be characterized as rings that do not contain any cyclic additive subgroups of finite order. The *spectral rank* of an element $a \in R$, as introduced in [1], is defined as

$$\operatorname{rank}_R^\sigma(a) := \sup_{x\in R} \#\{t \in \mathbb{Z} \,|\, \mathbf{1} + txa \notin \mathcal{U}(R)\}.$$

Whenever the ring R is clear from the context we will simply write $\operatorname{rank}^\sigma(a)$. It is easy to show that the inequality

$$\operatorname{rank}^\sigma(ab) \leq \min\{\operatorname{rank}^\sigma(a), \operatorname{rank}^\sigma(b)\}$$

holds for all $a, b \in R$. This implies that $\operatorname{rank}^\sigma(a) = \operatorname{rank}^\sigma(\mathbf{1})$ for all $a \in \mathcal{U}(R)$; i.e. the spectral ranks of units agree with the spectral rank of the multiplicative identity. We refer the reader to [1] for a more comprehensive study of the spectral rank in rings.

Moving on to the concept of the *left* (respectively, *right*) *socle* of a ring R. As defined in [8], it is the sum of all minimal left (respectively, right) ideals of R, which in fact form a two-sided ideal of R. In the case where R lacks minimal left (respectively, right) ideals, its left (respectively, right) socle is $\{\mathbf{0}\}$. Note that, in general, the left and right socles of a ring may not coincide. By [8, Lemma 11.9], they will coincide whenever R is *semiprime*, i.e. the condition $aRa = \{\mathbf{0}\}$ implies that $a = \mathbf{0}$ for all $a \in R$. In this case, we simply refer to it as the *socle* of R and denote it by $\operatorname{soc}(R)$. It is a well known fact that every J-semisimple ring is semiprime.

Following [1], an element $a \in R$ is said to be *left* (respectively, *right*) *semipotent* if every nonzero left (respectively, right) ideal of R contained in Ra (respectively, aR) contains a nonzero *idempotent*, i.e. an element e satisfying $e^2 = e$. For our purposes, as was done in [1], we shall focus on left semipotent elements and refer to them simply as *semipotent*. Moreover, [1, Theorem 4.14] establishes that the socle of a J-semisimple and torsion-free ring R consists of all semipotent elements a satisfying $\operatorname{rank}^\sigma(a) < \infty$, i.e.

$$\operatorname{soc}(R) = \{a \in R \,|\, \operatorname{rank}^\sigma(a) < \infty \text{ and } a \text{ is semipotent}\}.$$

In 1958, Drazin introduced pseudo-inverses in associative rings and semigroups, as documented in [7]. This pioneering work paved the way for the study of Drazin invertible elements, which are defined as follows:

Definition 1.1. *[7, p.507] An element $a \in R$ is said to be Drazin invertible if there exists some $b \in R$ and a positive integer k such that the following conditions hold:*

(i) $ab = ba$,
(ii) $bab = b$,
(iii) $a^k ba = a^k$.

The unique element b [10, p.55], satisfying conditions (i)-(iii), is referred to as the *Drazin inverse* of a. As a customary notation, we denote the Drazin inverse of a Drazin invertible element a by a^D, and call the smallest positive integer k for which condition (iii) holds the *Drazin index* of a. In the case where $k = 1$, a is called *group invertible*, and its group inverse will be denoted by a^g. Examples of group invertible elements include the units (with group inverses given by their multiplicative inverses) and the idempotents (with group inverses given by themselves). The sets of all group invertible and Drazin invertible elements of R will be denoted by $\mathcal{G}(R)$ and $\mathcal{D}(R)$, respectively. In general, the strict containments $U(R) \subsetneq \mathcal{G}(R) \subsetneq \mathcal{D}(R)$ hold. We further point out that, for any $a \in \mathcal{D}(R)$, $a^D \in \mathcal{G}(R)$, with its group inverse given by a^2a^D. This fact will be used throughout the paper without specific reference. For recent and interesting developments regarding the Drazin inverse, see [9], [13], [6] and [12].

In [5], Brits, Lindeboom, and Raubenheimer investigated the relationship between the spectral rank (introduced by Aupetit and Mouton) and the Drazin index of elements belonging to the socle of a complex unital semisimple Banach algebra A. Their study yielded intriguing results, two of them are stated next, which we will generalize to the setting of J-semisimple and torsion-free rings in the present paper.

(i) If $a \in \mathrm{soc}(A)$ with a Drazin inverse a^D, then $a \in \mathcal{G}(A)$ if and only if $\mathrm{rank}(a) = \mathrm{rank}(a^D)$. [5, Theorem 2.3]
(ii) If $a \in \mathrm{soc}(A)$, then $a \in \mathcal{G}(A)$ if and only if $\mathrm{rank}(a) = \mathrm{rank}(a^k)$ for each $k \in \mathbb{N}$. [5, Theorem 2.4]

Note the application of the inclusion $\mathrm{soc}(A) \subseteq \mathcal{D}(A)$, established in [4, Theorem 9] for a complex unital semisimple Banach algebra A, in (i) and (ii) above. The authors do not know whether such containment holds true in general J-semisimple (and torsion-free) rings. However, we point out next that in the case of commutative J-semisimple rings, every element of the socle is Drazin invertible (in fact group invertible). This is not a new fact in the setting of Banach algebras, since it is an immediate consequence of [4, Theorem 11], whose proof is not applicable to the setting of rings as it relies on spectral-theoretic arguments.

We recall that an element $a \in R$ is said to be *quasinilpotent* if $\mathbf{1} - ax \in U(R)$ for all $x \in R$ commuting with a. It is well known that the set of all quasinilpotent elements (which generally contains the Jacobson radical) coincides with $\mathrm{rad}(R)$ whenever R is a commutative ring.

Proposition 1.2. *Let R be a commutative J-semisimple ring. Then* $\mathrm{soc}(A) \subseteq \mathcal{G}(A) = \mathcal{D}(A)$.

Proof. Let $a \in \mathrm{soc}(A)$. By [11, Theorem 4.10], a is unit regular, i.e. there exists $b \in U(R)$ such that $a = aba$. Utilizing the fact that A is commutative, one can show that $a \in \mathcal{G}(A)$ with $a^g = bab$, establishing the inclusion. To prove the identity, assume that $a \in \mathcal{D}(A)$. Then there exists a positive integer k such that $a^k a^D a = a^k$, which is equivalent to $a - aa^D a$ being nilpotent. Since every nilpotent element is quasinilpotent, and hence in $\mathrm{rad}(R) = \{\mathbf{0}\}$, we have that $a = aa^D a$, from which we conclude that $a \in \mathcal{G}(A)$. $\square$

In the case of an arbitrary J-semisimple ring R, we introduce the set $\mathcal{S}^D(R)$ as the intersection of the socle and the set of Drazin invertible elements of R, i.e. $\mathcal{S}^D(R) := \mathrm{soc}(R) \cap \mathcal{D}(R)$, which is generally non-empty as (for instance) $\mathbf{0} \in \mathrm{soc}(R)$ is Drazin invertible.

We now recall the following results from [1] which will prove useful to us.

Lemma 1.3 ([1], Lemma 3.7). *Let e be an idempotent in R. Then eRe is a ring with multiplicative identity e. Moreover, for any $a \in R$, we have that*

$$\mathbf{1} + ae \in \mathcal{U}(R) \iff e + eae \in \mathcal{U}(eRe).$$

Furthermore, if R is J-semisimple, then eRe is J-semisimlpe.

Lemma 1.4 ([1], Lemma 3.10). *Let e be a nonzero idempotent in R. Then* $\mathrm{rank}^\sigma_{eRe}(eae) = \mathrm{rank}^\sigma_R(eae)$ *for each $a \in R$.*

We introduce some further terminology and notation, so let R be a J-semisimple and torsion-free ring. For $a \in \mathrm{soc}(R)$, we consider the set

$$E(a) := \{x \in R \,|\, \#\{t \in \mathbb{Z} \,|\, \mathbf{1} + txa \notin \mathcal{U}(R)\} = \mathrm{rank}^\sigma(a)\}.$$

Note that, by [1, Theorem 4.14], $E(a)$ is nonempty set for each $a \in \mathrm{soc}(R)$. Moreover, it is easily seen that $E(\mathbf{0}) = R$. We shall call an element $a \in \mathrm{soc}(R)$ *maximal finite-rank* if $\mathbf{1} \in E(a)$, i.e. $\mathrm{rank}^\sigma(a) = \#\{t \in \mathbb{Z} \,|\, \mathbf{1} + ta \notin \mathcal{U}(R)\}$.

2. Main results

This section is devoted to generalizations of results in [5] by Brits, Lindeboom and Raubenheimer in the setting of Banach algebras to the context of J-semisimple and torsion-free rings.

We start by pointing out the well known core-nilpotent decomposition of Drazin invertible elements in rings, which will be useful in establishing the two results that follows.

Lemma 2.1 ([10], p.56). *(core-nilpotent decomposition) Let R be a ring. If $a \in \mathcal{D}(R)$, then a can be uniquely decomposed as a sum of a group invertible element and a nilpotent element which commute and whose product is zero.*

The group invertible element in Lemma 2.1 is given by aa^Da (which has group inverse a^D) and is called the *core of a*, while the nilpotent element $a - aa^Da$ (which has nilpotency index the Drazin index of a) is referred to as the *nilpotent part of a*.

As an immediate consequence of the core-nilpotent decomposition, we have that the core of a maximal finite-rank element is also maximal finite-rank.

Proposition 2.2. *Let R be a J-semisimple and torsion-free ring and $a \in \mathcal{S}^D(R)$. If a is maximal finite-rank, then so is aa^Da and*

$$\mathrm{rank}^\sigma(a) = \mathrm{rank}^\sigma(aa^Da).$$

Proof. Suppose that $a \in \mathcal{S}^D(R)$ is maximal finite-rank and let $e := aa^D$. For each $t \in \mathbb{Z}$, consider $\alpha := \mathbf{1} + tea$ and $\beta := \mathbf{1} + t(\mathbf{1} - e)a$. Then

$$\alpha\beta = \mathbf{1} + tea + t(\mathbf{1} - e)a,$$

and hence $\mathbf{1} + tea + t(\mathbf{1} - e)a \notin \mathcal{U}(R)$ if and only if either $\alpha \notin \mathcal{U}(R)$ or $\beta \notin \mathcal{U}(R)$. In view of Lemma 2.1 (and the remark thereafter) and the fact that $(\mathbf{1} - e)a$ is nilpotent (and hence quasinilpotent), it now follows that

$$\begin{aligned}
\mathrm{rank}^\sigma(a) &= \#\{t \in \mathbb{Z} \,|\, \mathbf{1} + ta \notin \mathcal{U}(R)\} \\
&= \#\{t \in \mathbb{Z} \,|\, \mathbf{1} + t[ea + (\mathbf{1} - e)a] \notin \mathcal{U}(R)\} \\
&= \#\left[\{t \in \mathbb{Z} \,|\, \alpha \notin \mathcal{U}(R)\} \cup \{t \in \mathbb{Z} \,|\, \beta \notin \mathcal{U}(R)\}\right] \\
&= \#\left[\{t \in \mathbb{Z} \,|\, \alpha \notin \mathcal{U}(R)\} \cup \emptyset\right] \\
&\leq \mathrm{rank}^\sigma(ea) \\
&\leq \mathrm{rank}^\sigma(a),
\end{aligned}$$

which shows that $\mathrm{rank}^\sigma(a) = \mathrm{rank}^\sigma(ea) = \#\{t \in \mathbb{Z} \mid \alpha \notin \mathcal{U}(R)\}$, i.e. $ea \in \mathcal{S}^D(R)$ is maximal finite-rank. □

Next we give a basic fact about Drazin invertible elements of rings. Its proof is a mere copy of the proof of [5, Lemma 2.5] but uses the core-nilpotent decomposition of Drazin invertible elements in rings.

Lemma 2.3. *Let R be a ring. If $a \in \mathcal{D}(R)$, then the Drazin index of a, say k, is the least nonnegative integer such that $a^k \in \mathcal{G}(R)$.*

Proof. Let $a \in \mathcal{D}(R)$ with Drazin index k. Then it is easily seen that $a^k \in \mathcal{G}(R)$ with group inverse $(a^D)^k$. Suppose that j is another integer strictly smaller than k satisfying $a^j \in \mathcal{G}(R)$. Utilizing Lemma 2.1 and the fact that aa^D (and hence also $\mathbf{1} - aa^D$) is an idempotent, we have that

$$a^j = (aa^Da + a - aa^Da)^j = (aa^Da)^j + a^j(\mathbf{1} - aa^D).$$

Hence $a^j(\mathbf{1} - aa^D) = \mathbf{0}$ since $a^j \in \mathcal{G}(R)$. But this contradicts our assumption that k is the least nonnegative integer satisfying $a^k = a^ka^Da$. Consequently, the result follows. □

The following lemma is crucial for proving Theorem 2.5 and is derived from the proof of [1, Propositon 3.4]. It's worth noting that while not explicitly established in that particular proof, the argument originates from it, requiring only a minor adjustment.

Lemma 2.4. *Let R be a J-semisimple and torsion-free ring and $a_1, a_2 \in \mathrm{soc}(R)$. Then there exist $x_1 \in E(a_1)$ and $x_2 \in E(a_2)$ such that*

$$\{t \in \mathbb{Z} \,|\, \mathbf{1} + tx_1a_1 \notin \mathcal{U}(R)\} \cap \{t \in \mathbb{Z} \,|\, \mathbf{1} + tx_2a_2 \notin \mathcal{U}(R)\} = \varnothing.$$

Proof. Let $a_1, a_2 \in \mathrm{soc}(R)$. Then there exist $y_1, y_2 \in R$ such that

$$\mathrm{rank}^\sigma(a_1) = \#\{t \in \mathbb{Z} \,|\, \mathbf{1} + ty_1a_1 \notin \mathcal{U}(R)\}$$

and

$$\operatorname{rank}^{\sigma}(a_2) = \#\{t \in \mathbb{Z} \,|\, \mathbf{1} + ty_2a_2 \notin \mathcal{U}(R)\}.$$

Let m be the largest integer $t \in \mathbb{Z}$ for which $\mathbf{1} + ty_2a_2 \notin \mathcal{U}(R)$. Now, choose an integer s such that $s > m$ and

$$m - s < \min\{t \in \mathbb{Z} \,|\, \mathbf{1} + ty_1a_1 \notin \mathcal{U}(R)\}. \tag{1}$$

With this choice of s, we have that $\mathbf{1} + sy_2a_2 \in \mathcal{U}(R)$. Notice also that, for $t \in \mathbb{Z}$, the condition

$$\mathbf{1} + t(\mathbf{1} + sy_2a_2)^{-1}y_2a_2 \notin \mathcal{U}(R)$$

is equivalent to

$$\mathbf{1} + (s+t)y_2a_2 \notin \mathcal{U}(R)$$

since $(\mathbf{1} + sy_2a_2)(\mathbf{1} + t(\mathbf{1} + sy_2a_2)^{-1}y_2a_2) = \mathbf{1} + (s+t)y_2a_2$.

From (1) and this equivalence, it then follows that the sets

$$\{t \in \mathbb{Z} \,|\, \mathbf{1} + ty_1a_1 \notin \mathcal{U}(R)\}$$

and

$$\{t \in \mathbb{Z} \,|\, \mathbf{1} + t(\mathbf{1} + sy_2a_2)^{-1}y_2a_2 \notin \mathcal{U}(R)\}$$

are disjoint, and that

$$\operatorname{rank}^{\sigma}(a_2) = \#\{t \in \mathbb{Z} \,|\, \mathbf{1} + t(\mathbf{1} + sy_2a_2)^{-1}y_2a_2 \notin \mathcal{U}(R)\}.$$

By choosing $x_1 = y_1 \in E(a_1)$ and $x_2 = (\mathbf{1} + sy_2a_2)^{-1}y_2 \in E(a_2)$, we have established our result. □

By refining the techniques in ([5], Theorem 2.3), where ([2], Exercise 9, p.66) - not applicable to our setting - were utilized, we give next a necessary and sufficient condition for elements of $\mathcal{S}^D(R)$ to be group invertible, which is that the spectral ranks of the elements and their Drazin inverses must coincide. This result generalizes Theorem 2.3 in [5] to the setting of J-semisimple and torsion-free rings.

Theorem 2.5. *Let R be a J-semisimple and torsion-free ring and $a \in \mathcal{S}^D(R)$. Then $a \in \mathcal{G}(R)$ if and only if $\operatorname{rank}^{\sigma}(a) = \operatorname{rank}^{\sigma}(a^D)$.*

Proof. We begin with the forward implication, so suppose that $a \in \mathcal{G}(R)$ with group inverse a^D. Then

$$\operatorname{rank}^{\sigma}(a) = \operatorname{rank}^{\sigma}(a^2a^D) \leq \operatorname{rank}^{\sigma}(a^D) = \operatorname{rank}^{\sigma}(a(a^D)^2) \leq \operatorname{rank}^{\sigma}(a),$$

establishing the identity $\operatorname{rank}^{\sigma}(a) = \operatorname{rank}^{\sigma}(a^D)$.

For the reverse implication, suppose that $\operatorname{rank}^{\sigma}(a) = \operatorname{rank}^{\sigma}(a^D)$. We proceed to prove the identity $a = a^2a^D$, from which it will follow that $a \in \mathcal{G}(R)$. Now observe that a^D and aa^Da, which belong to $\mathcal{G}(R)$, are both invertible in the J-semisimple ring eRe (with multiplicative identity the idempotent $e := aa^D$). Hence

$$\operatorname{rank}^{\sigma}(a) = \operatorname{rank}^{\sigma}(a^D) = \operatorname{rank}^{\sigma}(aa^Da) = \operatorname{rank}^{\sigma}(aa^D).$$

By Lemma 2.4, for $ea, (\mathbf{1} - e)a \in \operatorname{soc}(R)$, there exist $x \in E(ea)$ and $y \in E((\mathbf{1} - e)a)$ such that the sets

$$\{t \in \mathbb{Z} \,|\, \mathbf{1} + txea \notin \mathcal{U}(R)\} \;\; (= \{t \in \mathbb{Z} \,|\, \mathbf{1} + teaxe \notin \mathcal{U}(R)\})$$

and

$$\{t \in \mathbb{Z} \,|\, \mathbf{1} + ty(\mathbf{1} - e)a \notin \mathcal{U}(R)\} \;\; (= \{t \in \mathbb{Z} \,|\, \mathbf{1} + t(\mathbf{1} - e)ay(\mathbf{1} - e) \notin \mathcal{U}(R)\})$$

are disjoint (where the parts between brackets follow from the well known Jacobson's Lemma).

Let $\alpha := eaxe$ and $\beta := (\mathbf{1}-e)ay(\mathbf{1}-e)$. Observe that

$$(\mathbf{1}+t\alpha)(\mathbf{1}+t\beta) = \mathbf{1}+t\alpha+t\beta$$

for all $t \in \mathbb{Z}$, since $e(\mathbf{1}-e) = \mathbf{0}$. Hence $\mathbf{1}+t\alpha+t\beta \notin \mathcal{U}(R)$ if and only if either $\mathbf{1}+t\alpha \notin \mathcal{U}(R)$ or $\mathbf{1}+t\beta \notin \mathcal{U}(R)$. It now follows that

$$\begin{aligned}
& \operatorname{rank}^\sigma(a) + \operatorname{rank}^\sigma((\mathbf{1}-aa^D)a) \\
= \; & \operatorname{rank}^\sigma(aa^Da) + \operatorname{rank}^\sigma((\mathbf{1}-aa^D)a) \\
= \; & \operatorname{rank}^\sigma(ea) + \operatorname{rank}^\sigma((\mathbf{1}-e)a) \\
= \; & \#\{t \in \mathbb{Z} \,|\, \mathbf{1}+txea \notin \mathcal{U}(R)\} + \#\{t \in \mathbb{Z} \,|\, \mathbf{1}+ty(\mathbf{1}-e)a \notin \mathcal{U}(R)\} \\
= \; & \#\{t \in \mathbb{Z} \,|\, \mathbf{1}+t\alpha \notin \mathcal{U}(R)\} + \#\{t \in \mathbb{Z} \,|\, \mathbf{1}+t\beta \notin \mathcal{U}(R)\} \\
= \; & \#\left[\{t \in \mathbb{Z} \,|\, \mathbf{1}+t\alpha \notin \mathcal{U}(R)\} \cup \{t \in \mathbb{Z} \,|\, \mathbf{1}+t\beta \notin \mathcal{U}(R)\}\right] \\
= \; & \#\{t \in \mathbb{Z} \,|\, \mathbf{1}+t(\alpha+\beta) \notin \mathcal{U}(R)\} \\
\leq \; & \operatorname{rank}^\sigma(\alpha+\beta) \\
= \; & \operatorname{rank}^\sigma(a[exe+(\mathbf{1}-e)y(\mathbf{1}-e)]) \\
\leq \; & \operatorname{rank}^\sigma(a).
\end{aligned}$$

Since the rank is non-negative, we have that $\operatorname{rank}^\sigma((\mathbf{1}-aa^D)a) = 0$, and hence $(\mathbf{1}-aa^D)a = \mathbf{0}$. Consequently, $a \in \mathcal{G}(R)$, which completes the proof. □

Using Theorem 2.5, we now characterize group invertibility of an element in $\mathcal{S}^D(R)$ in terms of the spectral ranks of the elements and their powers. This result generalizes Theorem 2.4 in [5] to the setting of J-semisimple and torsion-free rings.

Theorem 2.6. *Let R be a J-semisimple and torsion-free ring and $a \in \mathcal{S}^D(R)$. Then $a \in \mathcal{G}(R)$ if and only if* $\operatorname{rank}^\sigma(a) = \operatorname{rank}^\sigma(a^k)$ *for each $k \in \mathbb{N}$.*

Proof. To prove the forward implication, suppose that $a \in \mathcal{G}(R)$. Then, for each $k \in \mathbb{N}$, $a^k \in \mathcal{G}(R)$ with group inverse $(a^g)^k$. Let $e := aa^g$. Observe that a, a^g, a^k and $(a^g)^k$ are all invertible in the J-semisimple ring eRe, and hence all of them have rank equal to $\operatorname{rank}^\sigma(e)$. From this and Lemma 1.4 we conclude that $\operatorname{rank}^\sigma(a) = \operatorname{rank}^\sigma(a^k)$ for each $k \in \mathbb{N}$. For the reverse implication, suppose that $\operatorname{rank}^\sigma(a) = \operatorname{rank}^\sigma(a^k)$ for each $k \in \mathbb{N}$ and that a has Drazin index $k_1 \in \mathbb{N}$. Using the facts that $a^D \in \mathcal{G}(R)$ and $a^{k_1} \in \mathcal{G}(R)$ with $(a^{k_1})^g = (a^D)^{k_1}$, we have from the first part of the proof that $\operatorname{rank}^\sigma(a^D) = \operatorname{rank}^\sigma((a^D)^{k_1}) = \operatorname{rank}^\sigma((a^{k_1})^g)$. In view of the hypothesis and Theorem 2.5, it now follows that

$$\operatorname{rank}^\sigma(a) = \operatorname{rank}^\sigma(a^{k_1}) = \operatorname{rank}^\sigma((a^{k_1})^g) = \operatorname{rank}^\sigma(a^D).$$

By utilizing Theorem 2.5 again, we conclude that $a \in \mathcal{G}(R)$, which completes the proof. □

We give two consequences of Theorem 2.6. The first is the left-to-right implication of Theorem 2.7 in [5] in the setting of J-semisimple and torsion-free rings.

Corollary 2.7. *Let R be a J-semisimple and torsion-free ring and $a \in \mathcal{S}^D(R)$. If a has Drazin index k, then k is the least nonnegative integer satisfying* $\operatorname{rank}^\sigma(a^k) = \operatorname{rank}^\sigma(a^{k+1})$.

Proof. Let $a \in \mathcal{S}^D(R)$ with Drazin index k. From Lemma 2.3 it follows that k is the least nonnegative integer such that $a^k \in \mathcal{G}(R)$. In view of Theorem 2.6, we have that $\operatorname{rank}^\sigma(a^k) = \operatorname{rank}^\sigma(a^{2k})$, and hence

$$\operatorname{rank}^\sigma(a^{k+1}) \leq \operatorname{rank}^\sigma(a^k) = \operatorname{rank}^\sigma(a^{2k}) \leq \operatorname{rank}^\sigma(a^{k+1}),$$

which establishes the identity $\operatorname{rank}^\sigma(a^k) = \operatorname{rank}^\sigma(a^{k+1})$. □

An immediate consequence of Corollary 2.7 provides additional insights into an example given by Askes, Brits, and Schulz (cf. [1, Example 3.13]):

Example 2.8. *Consider the (non-commutative) J-semisimple ring R, which is torsion-free and consists of all functions $f : [0,1] \to M_n(\mathbb{C})$ (with pointwise addition and multiplication), where $M_n(\mathbb{C})$ denotes the set of all $n \times n$ matrices with complex entries, and let $g \in \operatorname{soc}(R)$. Then g has Drazin index k if and only if k is the least nonnegative integer satisfying* $\operatorname{rank}^\sigma(g^k) = \operatorname{rank}^\sigma(g^{k+1})$.

In particular, $g \in G(R)$ if and only if $\operatorname{rank}^\sigma(g) = \operatorname{rank}^\sigma(g^2)$

Proof. First note that

$$D(R) = \{f \in R : f(x) \in D(M_n(\mathbb{C})) \text{ for all } x \in [0,1]\} = R$$

since every square matrix is Drazin invertible. Now let $g \in \operatorname{soc}(A)$. We prove the reverse implication, as the forward implication is obvious from Corollary 2.7; hence suppose that k is the least nonnegative integer satisfying $\operatorname{rank}^\sigma(g^k) = \operatorname{rank}^\sigma(g^{k+1})$, i.e.

$$\sum_{t \in \operatorname{supp}(g)} \operatorname{rank}^\sigma(g(t)^k) = \sum_{t \in \operatorname{supp}(g)} \operatorname{rank}^\sigma(g(t)^{k+1}),$$

which gives $\operatorname{rank}^\sigma(g(t))^k = \operatorname{rank}^\sigma(g(t)^{k+1})$ for all $t \in [0,1]$. Then the matrix $g(t)$ (for each $t \in [0,1]$) is Drazin invertible with Drazin index k, from which we obtain that g has Drazin index k. □

Remark 2.9. *Though an if and only if statement holds in Example 2.8, the authors do not know whether the converse statement of Corollary 2.7 is generally true. In fact, we point out that (in [5]) the authors utilized the converse statement of Theorem 2.7 (whose validity in the context of J-semisimple and torsion-free rings is still unknown) to establish Corollary 2.7 in the context of unital semisimple Banach algebras.*

As a second remark, we point out that Example 2.8 could not have been obtained from ([5], Theorem 2.7) as the given R is not a Banach algebra (see Example 3.13 in [1]).

Our second corollary of Theorem 2.6, which we give next, is a generalization of ([5], Corollary 2.8) to the setting of J-semisimple and torsion-free rings.

Corollary 2.10. *Let R be a J-semisimple and torsion-free ring and $ab, ba \in \mathcal{S}^D(R)$. If $ab \in \mathcal{G}(R)$, then $ba \in \mathcal{G}(R)$ if and only* $\operatorname{rank}^\sigma(ab) = \operatorname{rank}^\sigma(ba)$.

Proof. Let $ab \in \mathcal{G}(R)$.

To prove the forward implication, suppose also that $ba \in \mathcal{G}(R)$. Then by Theorem 2.6 (for $k = 2$), we have that

$$\operatorname{rank}^\sigma(ab) = \operatorname{rank}^\sigma((ab)^2) \leq \operatorname{rank}^\sigma(ba) = \operatorname{rank}^\sigma((ba)^2) \leq \operatorname{rank}^\sigma(ab),$$

and hence $\operatorname{rank}^\sigma(ab) = \operatorname{rank}^\sigma(ba)$.

To prove the reverse implication, suppose that $\operatorname{rank}^\sigma(ab) = \operatorname{rank}^\sigma(ba)$ and let $k \in \mathbb{N}$. In view of Theorem 2.6, we have that

$$\operatorname{rank}^\sigma(ba) = \operatorname{rank}^\sigma(ab) = \operatorname{rank}^\sigma((ab)^k) \leq \operatorname{rank}^\sigma((ba)^{k-1}) \leq \operatorname{rank}^\sigma(ba).$$

Since $\operatorname{rank}^\sigma(ba) = \operatorname{rank}^\sigma((ba)^{k-1})$ and $k > 1$ was arbitrary, using Theorem 2.6 again, it follows that $ba \in \mathcal{G}(R)$. This completes the proof. □

We conclude with a few open questions.

3. Open questions

This paper, which is a first attempt at studying the connections between the spectral rank and generalized inverses in the setting of rings, contains some important open questions which we specify next.

Question 3.1. *Do we have identity* $\mathcal{S}^D(R) = \mathrm{soc}(R)$ *for an arbitrary J-semisimple and torsion-free ring?*

Question 3.2. *Is it the case that, in a J-semisimple and torsion-free ring* R, $a \in \mathcal{S}^D(R)$ *has Drazin index* k *if and only if* k *is the least nonnegative integer such that*

$$\mathrm{rank}^\sigma(a^k) = \mathrm{rank}^\sigma(a^{k+1})?$$

Recall that the above result holds in unital semisimple Banach algebras. Its proof relies on the rank structure decomposition of elements belonging to the socle, a tool we do not have in our setting.
Given that this paper aims to build upon and generalize the concepts and results displayed in [5], it is important to highlight one of its key findings. Specifically, it was demonstrated that if a_n and a are group invertible elements of the socle of a unital semisimple Banach algebra A with $a_n \to a$, then $a_n^D \to a^D$ if and only if there exists some $n_0 \in \mathbb{N}$ such that $\mathrm{rank}(a_n) = \mathrm{rank}(a)$ for all $n \geq n_0$. However, in the context of J-semisimple and torsion-free rings, the concept of convergence of sequences is not well-defined. To address the ensuing question, it is essential to introduce additional structure:

Question 3.3. *Let* R *be a J-semisimple and torsion-free topological ring and suppose that* a_n *(for each* $n \in \mathbb{N}$*) and* a *are group invertible elements in* $\mathcal{S}^D(R)$ *such that* a_n *converges to* a.
Are the following two statements equivalent?
(i) a_n^g *converges to* a^g.
(ii) There exists $n_0 \in \mathbb{N}$ *such that* $\mathrm{rank}^\sigma(a_n) = \mathrm{rank}^\sigma(a)$ *for all* $n \geq n_0$.

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