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Characterizations of relativistic magneto-fluid spacetimes admitting Einstein soliton

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Abstract. In this paper, we study some geometric properties of relativistic magneto-fluid spacetime admitting Einstein soliton. Here, we expose the nature of soliton when a relativistic magneto-fluid spacetime and $\mathcal{W}_2$-flat relativistic magneto-fluid spacetime and $\mathcal{Q}$-flat relativistic magneto-fluid spacetime satisfying Einstein field equation. Also we explore magneto-fluid warped product spacetimes that admits magneto-fluid generalized Robertson-Walker spacetimes and magneto-fluid standard static spacetimes in terms of Einstein soliton. Finally, we discuss some results on different types of magneto-fluid spacetimes admitting Einstein soliton.

1. Introduction

In recent years, the Ricci flow have been an engrossing research topic in the area of differential geometry. The concept of Ricci flow was first introduced by Hamilton and then developed especially to answer Thurston's geometric conjecture. A Ricci soliton can be considered as a fixed point of Hamilton's Ricci flow (see [17] for further details) and a natural generalization of the Einstein metric (i.e., the Ricci tensor denoted by Ric is a constant multiple of the pseudo-Riemannian metric g), defined on a pseudo-Riemannian manifold (M, g) by

$$\frac{1}{2}\mathfrak{L}_V g + \mathrm{Ric} = \Omega g,$$

where $\mathfrak{L}_V$ denotes the Lie-derivative in the direction of $V \in \mathfrak{X}(M)$, Ric is the Ricci tensor of g and Ω is a constant. The Ricci soliton is said to be shrinking, steady, and expanding according to Ω is negative, zero, and positive, respectively. Otherwise, it will be called as indefinite. A Ricci soliton is trivial if V is either zero or Killing on M. In [33] Pigoli et al. considered the soliton constant Ω to be a smooth function on M and Ricci soliton becomes Ricci almost soliton. After that, Barros et al. studied Ricci almost soliton in

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[9, 10]. In a recent paper [15], Cho-Kimura generalized the notion of Ricci soliton to η-Ricci soliton and Calin-Crasmareanu studied this concept in Hopf hypersurfaces of complex space forms in [14].

In 2016, Einstein soliton was introduced to the literature by G. Catino and L. Mazzieri (see [12]) which generates self-similar solutions to the Einstein flow, i.e,

$$\frac{\partial g}{\partial t} = -2\Big(\mathcal{S} - \frac{\mathcal{R}}{2} g\Big), \tag{1}$$

where $\mathcal{S}$ is Ricci tensor, g is Riemannian metric and $\mathcal{R}$ is the scalar curvature.
The equation of the Einstein soliton [7, 36] is given by,

$$\mathfrak{L}_W g + 2\mathcal{S} + (2\Omega - \mathcal{R})g = 0, \tag{2}$$

where $\mathfrak{L}_W$ is the Lie derivative along the vector field W, $\mathcal{S}$ is the Ricci tensor, $\mathcal{R}$ is the scalar curvature of the Riemannian metric g, and Ω is a real constant.

Furthermore, it is said to be expanding, steady or shrinking according as $\Omega \gtreqless 0$ respectively.

In [1], authors demonstrated that a spacetime can be characterized in terms of its Ricci soliton. Venkatesha et al. [42] also examined Ricci solitons on perfect fluid space time. Several authors (see [2, 8, 13]) investigated spacetime with some Ricci soliton extensively in different ways. Very recently, some authors in (see [19–26, 34, 35]), have studied Ricci soliton and Einstein soliton and their generalizations on contact and complex manifolds. Also many authors have been investigated the spacetime with the use of different techniques (see more details [3, 38]). The aim of the current research article is to investigate the solitonic attributes of relativistic magneto-fluid spacetime (MFS) if its metric is Einstein soliton. Here, we explore the nature of the soliton if the relativistic magneto-fluid spacetime (MFS) satisfying Einstein field equation (EFE). We also demonstrate magneto fluid generalized Robertson-Walker spacetimes and magneto-fluid standard static spacetimes admitting Einstein soliton. With the help of this geometric analysis, we can find out some physical accomplishments like general relativity, relativistic fluids models and try to scrutinize how the spacetime manifold becomes magnetized when certain magnetic properties are present in connection with solitonic nature.

2. Geometrical behavior of magneto-fluid spacetime(MFS)

In a magneto-fluid spacetime, the magnetic energy momentum tensor (EMT in short) $\mathcal{T}$ is of the following form [30, 31, 41],

$$\mathcal{T}(U,V) = pg(U,V) + (\rho + p)\eta(U)\eta(V) + \mu\Big\{H\Big[\eta(U)\eta(V) + \frac{1}{2}g(U,V)\Big] - \gamma(U)\gamma(V)\Big\}, \tag{3}$$

where ρ defines magneto-fluid density, p denotes the pressure, μ defines magnetic permeability, γ is the magnetic flux and H connotes the strength of magnetic field such that $\eta(U) = g(U,\xi)$ and $g(V,\zeta) = \gamma(V)$ are two non-zero 1-form. Also, ξ is a unit time-like vector field $g(\xi,\xi) = -1$ and ζ is a unit space-like vector field $g(\zeta,\zeta) = 1$. Therefore, ξ and ζ are orthogonal vector fields carrying out the underlying magneto-fluid spacetime.
Moreover, EMT of electromagnetism in addition to the scalar field theory is considered as an example of EMT. Einstein's gravitational equation with cosmological constant is given by [31, 41],

$$\mathcal{S}(U,V) + \Big[\lambda - \frac{\mathcal{R}}{2}\Big]g(U,V) = \kappa\mathcal{T}(U,V), \tag{4}$$

for any vector fields U, V, where λ refers to cosmological constant, κ remits to gravitational constant (that is sorted out as $8\pi G$, such that G is a universal gravitational constant), $\mathcal{S}, \mathcal{R}$ denote for the Ricci tensor and

scalar curvature of the underlying spacetime, respectively.
Einstein's equation with cosmological constant is used to yield both $\mathcal{S}$ and $\mathcal{R}$ for the purpose of having static universe, using Einstein's idea. In conformity with the current cosmology theories, it is considered as a candidate of dark energy which escalate the macrocosm diversification.
Now, in view of the identities (3) and (4), we obtain the Einstein's field equation (EFE) for a MFS as,

$$\mathcal{S}(U,V) = \Big[-\lambda + \frac{\mathcal{R}}{2} + \kappa\Big(p + \frac{\mu H}{2}\Big)\Big]g(U,V) + \kappa(\mu H + p + \rho)\eta(U)\eta(V) - \mu\kappa\gamma(U)\gamma(V). \tag{5}$$

Now, we contract (5) to obtain

$$\mathcal{R} = 4\lambda - \kappa[(H-1)\mu + 3p - \rho]. \tag{6}$$

3. Some results on the MFS admitting the Einstein soliton

We contract the identity (2) and using (6), it yields:

$$\Omega = \lambda - \frac{\kappa}{4}[(H-1)\mu + 3p - \rho] - \frac{\mathrm{div}W}{4}, \tag{7}$$

where $\mathrm{div}W$ is the divergence of the vector field W.

This leads to the following:

Theorem 3.1. *If a relativistic MFS obeying EFE, admits an Einstein soliton (g, W, Ω), then the soliton is expanding, steady or shrinking according to $\lambda \gtreqless \frac{\kappa}{4}[(H-1)\mu + 3p - \rho] + \frac{\mathrm{div}W}{4}$.*

From this theorem, we can also state,

Corollary 3.2. *If a relativistic MFS obeying EFE, admits an Einstein soliton (g, W, Ω), then the vector field W is solenoidal if and only if $\Omega = \lambda - \frac{\kappa}{4}[(H-1)\mu + 3p - \rho]$. Also the soliton is expanding, steady or shrinking according to $\lambda \gtreqless \frac{\kappa}{4}[(H-1)\mu + 3p - \rho]$*

Using the property of Lie derivative one can write,

$$(\mathfrak{L}_W g)(X,Y) = g(\nabla_X W, Y) + g(\nabla_Y W, X). \tag{8}$$

Then making the use of (5), (6) and (2), the previous identity reads

$$\begin{aligned} g(\nabla_X W, Y) + g(\nabla_Y W, X) &= -2\Big[\Omega - \lambda + \kappa\Big(p + \frac{\mu H}{2}\Big)\Big]g(X,Y) \\ &- 2\kappa(\mu H + p + \rho)\eta(X)\eta(Y) + 2\mu\kappa\gamma(X)\gamma(Y). \end{aligned} \tag{9}$$

for any vector fields X, Y.

Suppose θ is a 1-form, which is metrically equivalent to W and is given by $\theta(X) = g(X, W)$ for an arbitrary vector field X. Then the exterior derivative $d\theta$ of θ can be written as:

$$2(d\theta)(X,Y) = g(\nabla_X W, Y) - g(\nabla_Y W, X). \tag{10}$$

As $d\theta$ is skew-symmetric, so if we define a tensor field F of type (1,1) by,

$$(d\theta)(X,Y) = g(X, FY), \tag{11}$$

then F is skew self-adjoint i.e. $g(X, FY) = -g(FX, Y)$. So (11) can be written as:

$$(d\theta)(X,Y) = -g(FX,Y) \tag{12}$$

In view of (12), (10) provides

$$g(\nabla_X W, Y) - g(\nabla_Y W, X) = -2g(FX, Y). \tag{13}$$

Now we have two identities (13) and (9) side by side and factoring out Y so we find that,

$$\nabla_X W = -FX - \Big[\Omega - \lambda + \kappa\Big(p + \frac{\mu H}{2}\Big)\Big]X - \kappa(\mu H + p + \rho)\eta(X)\xi + \mu\kappa\gamma(X)\zeta. \tag{14}$$

Substituting the above equation in $R(X, Y)W = \nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]}W$, we have,

$$\begin{aligned} R(X, Y)W &= (\nabla_Y F)X - (\nabla_X F)Y \\ &+ \kappa(\mu H + p + \rho)[Y\eta(X) - X\eta(Y)] - \mu\kappa[Y\gamma(X) - X\gamma(Y)]. \end{aligned} \tag{15}$$

Noting that $d\theta$ is closed, we obtain,

$$g(X, (\nabla_Z F)Y) + g(Y, (\nabla_X F)Z) + g(Z, (\nabla_Y F)X) = 0. \tag{16}$$

We take inner product of (15) with respect to Z, it is acquired that

$$\begin{aligned} g(R(X, Y)W, Z) &= g((\nabla_Y F)X, Z) - g((\nabla_X F)Y, Z) \\ &+ \kappa(\mu H + p + \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ &- \mu\kappa[g(Y, Z)\gamma(X) - g(X, Z)\gamma(Y)]. \end{aligned} \tag{17}$$

As F is skew self-adjiont, then $\nabla_X F$ is also skew self-adjiont. Then using (16), (17) takes the form,

$$\begin{aligned} g(R(X, Y)W, Z) &= \kappa(\mu H + p + \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ &- \mu\kappa[g(Y, Z)\gamma(X) - g(X, Z)\gamma(Y)] - g(X, (\nabla_Z F)Y). \end{aligned} \tag{18}$$

Now we set $X = Z = e_i$ in the above equation, where $\{e_i\}$ is a local orthonormal frame and summing over $i = 1, 2, 3, 4$ to obtain,

$$S(Y, W) = -3\kappa(\mu H + p + \rho)\eta(Y) + 3\mu\kappa\gamma(Y) - (divF)Y, \tag{19}$$

where div$F$ is the divergence of the tensor field F.
Now, we equate the two identities (5) and (19) to get,

$$\begin{aligned} (\mathrm{div}F)Y &= -\kappa(\mu H + p + \rho)(3 + \eta(W))\eta(Y) + \mu\kappa(3 + \gamma(W))\gamma(Y) \\ &- \Big[\lambda - \frac{\kappa}{2}(p - \rho - \mu)\Big]\theta(Y). \end{aligned} \tag{20}$$

We compute the covariant derivative of the squared g-norm of W by using (14), we have the followings,

$$\begin{aligned} \nabla_X ||W||^2 &= 2g(\nabla_X W, W) \\ &- 2g(FX, W) - 2\Big[\Omega - \lambda + \kappa\Big(p + \frac{\mu H}{2}\Big)\Big]g(X, W) \\ &- 2\kappa(\mu H + p + \rho)\eta(X)\eta(W) + 2\mu\kappa\gamma(X)\gamma(W). \end{aligned} \tag{21}$$

From (5), (6), (2) becomes,

$$\begin{aligned} &(\mathfrak{L}_W g)(X, W) \\ &= -2g(FX, W) - 2\Big[\Omega - \lambda + \kappa\Big(p + \frac{\mu H}{2}\Big)\Big]g(X, W) - 2\kappa(\mu H + p + \rho)\eta(X)\eta(W) + 2\mu\kappa\gamma(X)\gamma(W). \end{aligned} \tag{22}$$

Utilizing the above equation, (21) takes the form,

$$\nabla_X ||W||^2 + 2g(FX, W) - (\mathfrak{L}_W g)(X, W) = 0. \tag{23}$$

So we can state the following,

Theorem 3.3. *If a relativistic MFS obeying EFE, admits an Einstein soliton (g, W, Ω), then the vector W and its metric dual 1-form θ satisfies the relations* (20) *and* (23)*.*

In 1970, G. P. Pokhariyal and R. S. Mishra (see [32]) introduced a new curvature tensor and studied its properties. This (0,4) type tensor in an n-dimensional Riemannian manifold, denoted as $\mathcal{W}_2$, is defined by,

$$\mathcal{W}_2(X, Y, Z, P) = \grave{R}(X, Y, Z, P) + \frac{1}{n-1}[g(X, Z)\mathcal{S}(Y, P) - g(Y, Z)\mathcal{S}(X, P)], \tag{24}$$

for all vector fields X, Y, Z, P and $\grave{R}(X, Y, Z, P) = g(R(X, Y)Z, P)$.

Definition 3.4. *A spacetime is said to be $\mathcal{W}_2$-flat if its $\mathcal{W}_2$-curvature tensor vanishes identically.*

Let us assume that we have a $\mathcal{W}_2$-flat relativistic MFS that obeys EFE.
Then from equation (24) and the definition of $\mathcal{W}_2$-flat, we have,

$$\grave{R}(X, Y, Z, P) = -\frac{1}{3}[g(X, Z)\mathcal{S}(Y, P) - g(Y, Z)\mathcal{S}(X, P)]. \tag{25}$$

We substitute $X = P = e_i$ in the above equation, where e_i's are a local orthonormal frame, summing over $i = 1, 2, 3, 4$ to yield

$$\mathcal{S}(Y, Z) = \frac{\mathcal{R}}{4}g(Y, Z). \tag{26}$$

Using (26), (2) takes the form,

$$(\mathfrak{L}_W g)(Y, Z) + (2\Omega - \frac{\mathcal{R}}{2})g(Y, Z) = 0. \tag{27}$$

Now if the vector field W is concircular vector field [16], then we know for any smooth function ν,

$$\nabla_X W = \nu X, \tag{28}$$

for all vector fields X.
Using the property of Lie derivative and (28), one can have,

$$(\mathfrak{L}_W g)(Y, Z) = 2\nu g(Y, Z). \tag{29}$$

Then with the help of the identities (29) and (27), we achieve,

$$\Omega + \nu = \frac{\mathcal{R}}{4}. \tag{30}$$

In light of (5), the previous equation becomes,

$$\Omega = \lambda - \frac{\kappa}{4}[(H - 1)\mu + 3p - \rho] - \nu. \tag{31}$$

This leads to the following,

Theorem 3.5. *If a $\mathcal{W}_2$-flat relativistic MFS obeying EFE admits an Einstein soliton (g, W, Ω), where W is a concircular vector field, then the soliton is expanding, steady or shrinking according as $\lambda \gtreqless \frac{\kappa}{4}[(H-1)\mu + 3p - \rho] + \nu$.*

In a recent paper [27], Mantica and Suh introduced a new curvature tensor in an n-dimensional Riemannian manifold which is denoted by $\mathcal{Q}$ and defined as

$$\mathcal{Q}(X, Y)Z = R(X, Y)Z - \frac{\psi}{(n-1)}[g(Y, Z)X - g(X, Z)Y], \tag{32}$$

where ψ is an arbitrary scalar function. Such a tensor $\mathcal{Q}$ is known as Q-curvature tensor.

Definition 3.6. *A spacetime is said to be $\mathcal{Q}$-flat if its $\mathcal{Q}$-curvature tensor vanishes identically.*

Let us assume that (M^4, g) be a relativistic MFS, which is $\mathcal{Q}$-flat. Then taking inner product with U in (32), we get,

$$\grave{R}(X, Y, Z, U) = \frac{\psi}{3}[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]. \tag{33}$$

We substitute $X = U = e_i$ in the above equation and taking summation over $1 \leq i \leq 4$ and then it yields:

$$\mathcal{S}(Y, Z) = \psi g(Y, Z). \tag{34}$$

This shows that the spacetime is Einstein.
In view of (2), the above equation takes the form,

$$(\mathfrak{L}_W g)(Y, Z) + (2\psi + 2\Omega - \mathcal{R})g(Y, Z) = 0, \tag{35}$$

Taking $Y = Z = e_i$ in the above equation, summing over $1 \leq i \leq 4$ and using (6), we get,

$$\Omega = 2\lambda - \frac{\kappa}{2}[(H-1)\mu + 3p + \rho] - \psi - \frac{\mathrm{div}W}{4}, \tag{36}$$

where $\mathrm{div}W$ is the divergence of the vector field W.

Hence we have,

Theorem 3.7. *If a $\mathcal{Q}$-flat relativistic MFS obeying EFE admits an Einstein soliton (g, W, Ω), then the soliton is expanding, steady or shrinking according as $\lambda \gtreqless \frac{\kappa}{4}[(H-1)\mu + 3p - \rho] + \frac{\psi}{2} + \frac{divW}{8}$.*

4. Magneto-fluid warped product spacetimes(MFWPS)

4.1. *Magneto-fluid generalized Robertson-Walker spacetimes (MFGRWS)*

We will recall the definition of a generalized Robertson-Walker spacetimes(GRWS). Let (F, g_F) be an s-dimensional Riemannian manifold and $b\colon I \to (0, \infty)$ be a smooth function. Then the $(s+1)$-dimensional product manifold $I \times_b F$ furnished with the metric tensor

$$g = -\mathrm{d}t^2 \oplus b^2 g_F$$

is called a GRWS and is denoted by $M = I \times_b F$ where I is an open and connected interval in $\mathbb{R}$ and $\mathrm{d}t^2$ is the usual Euclidean metric tensor on I. This structure was introduced to extend Robertson-Walker spacetimes [39, 40] and have been studied by many authors, such as [18, 28, 29]. From now on, $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ is denoted by ∂_t. The following result can be obtained by [31].

Theorem 4.1. *Let $M = I \times_b F$ be a GRWS with the metric tensor $g = -\mathrm{d}t^2 \oplus b^2 g_F$. If $\partial_t \in \mathfrak{X}(I)$ and $V, W \in \mathfrak{X}(F)$, then*

1. $\mathrm{Ric}(\partial_t, \partial_t) = -s\dfrac{b''}{b}$
2. $\mathrm{Ric}(\partial_t, V) = 0$
3. $\mathrm{Ric}(V, W) = \mathrm{Ric}_F(V, W) + \left[bb'' + (s-1)(b')^2\right]g_F(V, W)$

where $s = \dim(F)$.

Let $\xi = \partial_t$ be a unit timelike vector field and ζ be a unit spacelike vector field on a GRWS of the form $M = I \times_b F$ with the metric tensor $g = -\mathrm{d}t^2 \oplus b^2 g_F$.

Denote $\eta(U) = g(U, \xi)$ and $\gamma(V) = g(V, \zeta)$ where U and V are vector on M.

By using Equation 5 and Theorem 4.1, we obtain the followings:

$$-s\frac{b''}{b} = [-\lambda + \frac{\tau}{2} + \kappa(p + \frac{\mu H}{2})] + \kappa(\mu H + p + \rho), \tag{37}$$

since $\eta(\partial_t) = -1$ and $\gamma(\partial_t) = 0$.

$$0 = [-\lambda + \frac{\tau}{2} + \kappa(p + \frac{\mu H}{2})] \tag{38}$$

since $\eta(V) = 0$ and $\gamma(V) = g(V, \zeta)$.

By combining equations 37 and 38, we have:

$$-s\frac{b''}{b} = \kappa(\mu H + p + \rho). \tag{39}$$

Moreover,

$$\mathrm{Ric}_F(V, W) + \left[bb'' + (s-1)(b')^2\right] g_F(V, W) = -s\frac{b''}{b} - \mu\kappa b^4 g_F(U, \zeta) g_F(V, \zeta), \tag{40}$$

since $\eta(U) = 0$ and $\eta(V) = 0$.

Using Theorem 9 of [37] and Theorem 3.5, we can state:

Theorem 4.2. *Let $M = I \times_b F$ be a GRWS with the metric tensor $g = -\mathrm{d}t^2 \oplus b^2 g_F$. Suppose that (M, g) has constant sectional curvature $K = -(b')^2$. If (M, g) is a relativistic MFS obeying EFE admits an Einstein soliton (g, W, Ω), where W is a concircular vector field, then the soliton is expanding, steady or shrinking according as $\lambda \gtreqless \frac{\kappa}{4}[(H-1)\mu + 3p - \rho] + \nu$.*

Note that under the hypothesis of the last theorem, Equation 39 takes the following form:

$$\kappa(\mu H + p + \rho) = 0 \tag{41}$$

and Equation 40 becomes:

$$\mathrm{Ric}_F(V, W) - 2K g_F(V, W) = -s\frac{b''}{b} - \mu\kappa b^4 g_F(U, \zeta) g_F(V, \zeta). \tag{42}$$

4.2. Magneto-fluid standard static spacetimes (MFSSS)

Let (F, g_F) be an s-dimensional Riemannian manifold and $f\colon F \to (0, \infty)$ be a smooth function. Then recall that a standard static spacetimes (SSS) can be defined as the $(s+1)$-dimensional product manifold ${}_fI \times F$ furnished with the metric tensor

$$g = -f^2 \mathrm{d}t^2 \oplus g_F$$

is called a standard static spacetime and is denoted by $M =_f I \times F$ where I is an open and connected subinterval in $\mathbb{R}$ and dt^2 is the usual Euclidean metric tensor on I.

Remark that SSS can be considered as a generalization of the Einstein static universe (see [4–6, 11]) and many spacetime models that characterize the universe and the solutions of EFE are known to have this structure. The next result can be obtained by [31].

Theorem 4.3. *Let $M =_f I \times F$ be a SSS with the metric tensor $g = -f^2 \mathrm{d}t^2 \oplus g_F$. If $\partial_t \in \mathfrak{X}(I)$ and $V, W \in \mathfrak{X}(F)$, then*

1. $\mathrm{Ric}(\partial_t, \partial_t) = f\nabla^F(f)$
2. $\mathrm{Ric}(\partial_t, V) = 0$
3. $\mathrm{Ric}(V, W) = \mathrm{Ric}^F(V, W) - \frac{1}{f}\mathrm{H}^F_f(V, W)$

where $s = \dim(F)$.

By using Equation 5 and Theorem 4.3, we obtain the followings:

$$f\nabla^F(f) = [-\lambda + \frac{\tau}{2} + \kappa(p + \frac{\mu H}{2})] + \kappa(\mu H + p + \rho)f^2, \tag{43}$$

since $\eta(\partial_t) = g(\partial_t, (1/f)\partial_t) = -f$ and $\gamma(\partial_t) = 0$.

$$0 = [-\lambda + \frac{\tau}{2} + \kappa(p + \frac{\mu H}{2})] \tag{44}$$

since $\eta(V) = 0$ and $\gamma(\partial_t) = 0$.
By combining equations 43 and 44, we have:

$$f\nabla^F(f) = \kappa(\mu H + p + \rho)f^2 \tag{45}$$

Moreover,

$$\mathrm{Ric}_F(V, W) - \frac{1}{f}\mathrm{H}^f_F(V, W) = -g_F(U, \zeta)g_F(V, \zeta) \tag{46}$$

since $\eta(U) = 0$, $\eta(V) = 0$ and $\gamma(U) = g_F(U, \zeta)$.
Using Theorem 11 of [37] and Theorem 3.5, we can state:

Theorem 4.4. *Let $M =_f I \times F$ be a SSS with the metric tensor $g = -f^2\mathrm{d}t^2 \oplus g_F$. Suppose that (F, g_F) is of flat curvature satisfying $\mathrm{H}^f_F = -(1/3)\Delta_F(f)g_F$. If (M, g) is a relativistic MFS obeying EFE admits an Einstein soliton (g, W, Ω), where W is a concircular vector field, then the soliton is expanding, steady or shrinking according as $\lambda \gtreqless \frac{\kappa}{4}[(H-1)\mu + 3p - \rho] + \nu$.*

Note that under the hypothesis of the last theorem, Equation 45 takes the following form:

$$\kappa(\mu H + p + \rho) = 0 \tag{47}$$

and Equation 46 becomes:

$$\mathrm{Ric}_F(V, W) + \frac{\Delta_F(f)}{3f}g_F(V, W) = -g_F(U, \zeta)g_F(V, \zeta). \tag{48}$$

5. Some different types of magneto-fluid spacetime (MFS)

In this section we discuss some different types of MFS by changing their EMT and find the results when the MFS obeying the new formed EFE, admits the Einstein soliton.
Now in the equation of EMT i.e. (3), when we put,
• **Case-I**: $p = 0$, we have,

$$\mathcal{T}(U, V) = \rho\eta(U)\eta(V) + \mu\Big\{H\Big[\eta(U)\eta(V) + \frac{1}{2}g(U, V)\Big] - \gamma(U)\gamma(V)\Big\}.$$

Then using the above equation, (4) gives a new EFE, which is,

$$\mathcal{S}(U, V) = \Big[-\lambda + \frac{\mathcal{R}}{2} + \frac{\kappa\mu H}{2}\Big]g(U, V) + \kappa(\mu H + \rho)\eta(U)\eta(V) - \mu\kappa\gamma(U)\gamma(V).$$

Hence by tracing the above equation, we get,

$$\mathcal{R} = 4\lambda - \kappa[(H-1)\mu - \rho]. \tag{49}$$

Now contracting (2) and using (49), we obtain,

$$\Omega = \lambda - \frac{\kappa}{4}[(H-1)\mu - \rho] - \frac{\mathrm{div}W}{4},$$

where $\mathrm{div}W$ is the divergence of the vector field W.
Hence we can state:

Theorem 5.1. *If a relativistic pressureless MFS obeying EFE, admits an Einstein soliton (g, W, Ω), then the soliton is expanding, steady or shrinking according to $\lambda \gtreqless \frac{\kappa}{4}[(H-1)\mu - \rho] + \frac{\mathrm{div}W}{4}$.*

- **Case-II**: $p + \rho = 0$, we have,

$$\mathcal{T}(U, V) = pg(U, V) + \mu\Big\{H\Big[\eta(U)\eta(V) + \frac{1}{2}g(U, V)\Big] - \gamma(U)\gamma(V)\Big\}.$$

Then using the above equation, (4) gives a new EFE, which is,

$$\mathcal{S}(U, V) = \Big[-\lambda + \frac{\mathcal{R}}{2} + \kappa\Big(p + \frac{\mu H}{2}\Big)\Big]g(U, V) + \kappa\mu H\eta(U)\eta(V) - \mu\kappa\gamma(U)\gamma(V).$$

Hence by tracing the above equation, we get,

$$\mathcal{R} = 4\lambda - \kappa[(H-1)\mu + 4p]. \tag{50}$$

Now contracting (2) and using (50), we obtain,

$$\Omega = \lambda - \frac{\kappa}{4}[(H-1)\mu + 4p] - \frac{\mathrm{div}W}{4},$$

where $\mathrm{div}W$ is the divergence of the vector field W.

Theorem 5.2. *Let a relativistic MFS obeying EFE, admit an Einstein soliton (g, W, Ω). If the sum of the pressure and density of the fluid is zero, then the soliton is expanding, steady or shrinking according to $\lambda \gtreqless \frac{\kappa}{4}[(H-1)\mu + 4p] + \frac{\mathrm{div}W}{4}$.*

- **Case-III**: $\rho = 3p$, we have,

$$\begin{aligned} \mathcal{T}(U, V) &= pg(U, V) + 4p\eta(U)\eta(V) \\ &+ \mu\Big\{H\Big[\eta(U)\eta(V) + \frac{1}{2}g(U, V)\Big] - \gamma(U)\gamma(V)\Big\}, \end{aligned} \tag{51}$$

Then using the above equation, (4) gives a new EFE, which is,

$$\mathcal{S}(U, V) = \Big[-\lambda + \frac{\mathcal{R}}{2} + \kappa\Big(p + \frac{\mu H}{2}\Big)\Big]g(U, V) + \kappa(\mu H + 4p)\eta(U)\eta(V) - \mu\kappa\gamma(U)\gamma(V).$$

Hence by tracing the above equation, we get,

$$\mathcal{R} = 4\lambda - (H-1)\kappa\mu. \tag{52}$$

Now contracting (2) and using (52), we obtain,

$$\Omega = \lambda - \frac{(H-1)\kappa\mu}{4} - \frac{\mathrm{div}W}{4},$$

where $\mathrm{div}W$ is the divergence of the vector field W.

Theorem 5.3. *Let a relativistic MFS obeying EFE, admit an Einstein soliton (g, W, Ω). If the density of the fluid is thrice of the pressure, then the soliton is expanding, steady or shrinking according to $\lambda \gtreqless \frac{(H-1)\kappa\mu}{4} + \frac{\mathrm{div}W}{4}$.*

- **Case-IV**: $\mu = 0$, we have,

$$\mathcal{T}(U, V) = pg(U, V) + (\rho + p)\eta(U)\eta(V),$$

which is the EMT of perfect fluid [43].
Then using the above equation, (4) gives a new EFE, which is,

$$\mathcal{S}(U, V) = \Big[-\lambda + \frac{\mathcal{R}}{2} + \kappa p\Big]g(U, V) + \kappa(p + \rho)\eta(U)\eta(V).$$

Hence by tracing the above equation, we get,

$$\mathcal{R} = 4\lambda - \kappa\Big[3p - \rho\Big]. \tag{53}$$

Now contracting (2) and using (53), we obtain,

$$\Omega = \lambda - \frac{\kappa}{4}\Big[3p - \rho\Big] - \frac{\mathrm{div}W}{4},$$

where $\mathrm{div}W$ is the divergence of the vector field W.

Theorem 5.4. *Let a relativistic MFS obeying EFE, admit an Einstein soliton (g, W, Ω). If in the fluid, the magnetic permeability vanishes, then the spacetime reduces to perfect fluid spacetime and the soliton is expanding, steady or shrinking according to $\lambda \gtreqless \frac{\kappa}{4}\Big[3p - \rho\Big] + \frac{\mathrm{div}W}{4}$.*

Data Availability Statement

The manuscript has no associated data.

Conflict of Interest Statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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References

[1] M. Ali and Z. Ahsan: *Ricci solitons and symmetries of space time manifold of general relativity.* J. Adv. Res. Class. Modern Geom. **1(2)**, 75-84 (2014).
[2] Z. Ahsan and S. A. Siddiqui: *Concircular curvature tensor and fluid spacetimes.* Int. J. Theor. Phys. **48**, 3202-3212 (2009).
[3] L. Alias, A. Romero, M. Sanchez: *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetime*, Gen Relativ. Gravit **30** (1995), 915-932.
[4] D.E. Allison: *Energy conditions in standard static spacetimes*, General Relativity and Gravitation, **20** (1988) 115-122.
[5] D.E. Allison: *Geodesic Completeness in Static spacetimes*, Geometriae Dedicata **26** (1988) 85-97.
[6] D. E. Allison and B. Ünal: *Geodesic Structure of Standard Static spacetimes*, Journal of Geometry and Physics **46** (2003) 193-200.

[7] A. M. Blaga: *Solitons and geometrical structures in a perfect fluid spacetime*, Rocky Mountain Journal of Mathematics, **50(1)**, 41-53 (2020). https://doi.org/10.1216/rmj.2020.50.41

[8] A. M. Blaga and B. Y. Chen: *Harmonic forms and generalized solitons.* Results in Mathematics, **15(1)**, 16 (2024). https://doi.org/10.1007/s00025-023-02041-y

[9] A. Barros and Jr. E. Ribeiro: *Some characterizations for compact almost Ricci solitons*, Proc. Amer. Math. Soc., **140(3)**, 1033-1040 (2012).

[10] A. Barros, R. Batista and Jr, E. Ribeiro: *Compact almost Ricci solitons with constant scalar curvature are gradient*, Monatsh Math., **174(1)**, 29-39 (2014).

[11] A. L. Besse: *Einstein Manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, (2008).

[12] G. Catino and L. Mazzieri: *Gradient Einstein solitons*, Nonlinear Anal., **132** (2016), 66-94.

[13] M. Chaki and C. Ray: *Spacetimes with covariant constant energy momentum tensor.* Int. J. Theor. Phys. **35(5)**, 1027-1032 (1996).

[14] C. Calin and M. Crasmareanu: *η-Ricci solitons on hopf hypersurfaces in complex space forms*, Rev. Roumaine Math. Pures Appl., **57(1)**, 55-63 (2012).

[15] J. T. Cho and M. Kimura: *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Mathematical Journal, Second Series, **61(2)**, 205-212 (2009).

[16] B. Y. Chen: *Some results on concircular vector fields and their applications to Ricci solitons*, Bulletin of the Korean Mathematical Society, **52(5)**, 1535-1547.

[17] R. S. Hamilton: The Ricci flow on surfaces, volume **71**. Amer. Math. Soc., 1988.

[18] B.-Y. Chen: *A simple characterization of generalized Robertson-Walker spacetimes*, Gen. Relativ. Gravit. **46** (2014) 1833.

[19] S. Dey and S. Uddin: *Conformal η-Ricci almost solitons on Kenmotsu manifolds*, International Journal of Geometric Methods in Modern Physics, **19(08)**(2022), 2250121.

[20] S. Dey: *Conformal Ricci soliton and almost conformal Ricci soliton in paracontact geometry*, International Journal of Geometric Methods in Modern Physics, **20**(03) (2023), 2350041.

[21] S. Dey: *Certain results of κ-almost gradient Ricci-Bourguignon soliton on pseudo-Riemannian manifolds*, Journal of Geometry and Physics, **184**(2022), 104725.

[22] S. Dey and S. Roy: *Characterization of general relativistic spacetime equipped with η-Ricci-Bourguignon soliton*, Journal of Geometry and Physics, **178**(2022), 104578.

[23] S. Dey: *Certain results on gradient almost η-Ricci-Bourguignon soliton*, Quaestiones Mathematicae, **47(2)** (2024), 263-284.

[24] S .Dey and Y. J. Suh: *Geometry of almost contact metrics as an almost $*$-η-Ricci-Bourguignon solitons*, Reviews in Mathematical Physics, Vol. **35**, No. 7, 2350012(2023).

[25] S. Dey and S. Uddin: *Applications of some types of solitons within the framework of Kählerian spacetime manifolds*, International Journal of Geometric Methods in Modern Physics, **20**(09) (2023), 2350143.

[26] D. Ganguly, S. Dey, A. Ali and A. Bhattacharyya: *Conformal Ricci soliton and Quasi-Yamabe soliton on generalized Sasakian space form*, Journal of Geometry and Physics, Vol. **169** (2021) 104339, https://doi.org/10.1016/j.geomphys.2021.104339.

[27] C. A. Mantica, Y.J. Suh: *Pseudo Q-symmetric Riemannian manifolds*, Int. J. Geom. Methods Mod. Phys **10**, 1350013 (2013).

[28] C. A. Mantica, L. G. Molinari, U. C. De: *A condition for a perfect fluid spacetime to be a generalized Robertson-Walker spacetime*, J. Math. Phys. **57** (2) (2016) 022508.

[29] C. A. Mantica , Y. J. Suh, U. C. De: *A note on generalized Robertson-Walker spacetimes*, Int. J. Geom. Meth. Mod. Phys. **13** (2016) 1650079.

[30] M. Novello and M. J. Reboucas: *The stability of a rotating universe*, Astrophys. J., **225**(1978), 719-724.

[31] B. O'Neill: *Semi-Riemannian Geometry with Application to Relativity*, Academic Press Limited, Londan, 1983.

[32] G. P. Pokhariyal and R. S. Mishra: *The curvature tensor and their relativistic significance*, Yokohoma Math. J. **18**, 105-108 (1970).

[33] S. Pigola, M. Rigoli, M. Rimoldi and A. Setti:*Ricci almost solitons*, Ann. Scuola. Norm. Sup. Pisa. CL Sc., (5) **X**, 757-799 (2011).

[34] S. Roy, S. Dey, A. Bhattacharyya and S. K. Hui: *$*$-Conformal η-Ricci Soliton on Sasakian manifold*, Asian-European Journal of Mathematics, **15**(02) (2022), 2250035.

[35] S. Roy, S. Dey and A. Bhattacharyya: *Conformal Einstein soliton within the framework of para-Kähler manifold*, Differential Geometry-Dynamical Systems, **23** (2021), 235-243.

[36] S. Roy, S. Dey and A. Bhattacharyya: *A Kenmotsu metric as a conformal η-Einstein soliton*, Carpathian Mathematical Publications, **13(1)** (2021), 110-118, https://doi.org/10.15330/cmp.13.1.110-118.

[37] S. Shenawy and B. Ünal: *W_2-Curvature Tensor on Warped Product Manifolds and Applications*, International Journal of Geometric Methods in Modern Physics, **13(7)**, 1650099-1–1650099-14, (2016).

[38] M. Sanchez: *On the geometry of generalized Robertson-Walker sapcetimes: curvature and Killing fields*, Gen. Relativ. Grav., **31** (2016), 1-15.

[39] M. Sánchez: *On the Geometry of Generalized Robertson-Walker Spacetimes: geodesics*, Gen. Relativ. Gravitation **30** (1998) 915-932.

[40] M. Sánchez: *On the Geometry of Generalized Robertson-Walker Spacetimes: Curvature and Killing fields*, J. Geom. Phys. **31** (1999) 1-15.

[41] Mohd. D. Siddiqi and U. C. De: *Relativistic magneto-fluid spacetimes*, Journal of Geometry and Physics, 104370, https://doi.org/10.1016/j.geomphys.2021.104370.

[42] Venkatesha and H. Aruna Kumara: *Ricci solitons and geometrical structure in a perfect fluid spacetime with torse-forming vector field.* Afrika Mathematika **30**, 725-736(2019).

[43] P. Zhang, Y. Li, S. Roy, S. Dey and A. Bhattacharyya: *Geometrical Structure in a Perfect Fluid Spacetime with Conformal Ricci Yamabe Soliton*, Symmetry **14** (2022), 594. https://doi.org/10.3390/sym14030594.