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On contractions involving an auxiliary mapping and fixed-point results

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Abstract. For a given metric space (M, d), we introduce two classes of mappings $F : M \to M$ satisfying contractions involving an auxiliary mapping $S : M \times M \to M$. For each class, we study the existence and uniqueness of fixed points. Iterative algorithms converging to the fixed points, as well as the size of the convergence errors, are also provided. For particular choices of the auxiliary mapping S, we recover the Banach and Kannan fixed-point theorems. Some examples illustrating the obtained results and an application to cyclic contractions are given.

1. Introduction

Let (M, d) be a metric space. A mapping $F : M \to M$ is called a contraction on (M, d), if there exists $\kappa \in [0, 1)$ such that $d(Fu, Fv) \leq \kappa d(u, v)$ for every $u, v \in M$. From Banach's fixed-point theorem [3], if (M, d) is complete, then any contraction on (M, d) possesses a unique fixed point, and for all $u \in M$, the Picard sequence $\{F^n u\}$ converges to this unique fixed point. This theorem is a fundamental result in analysis, and has several applications in pure and applied mathematics. In particular, it provides a very powerful and useful tool to the study of existence and uniqueness of solutions for various kinds of equations such as integral equations, differential equations, partial differential equations, and evolution equations. For some works related to the applications of Banach's fixed-point theorem, we refer to [1, 5, 9, 14, 28]. Clearly, being continuous, is a necessary condition for a mapping F to be a contraction.

In 1968, Kannan [11] introduced the class of mappings $F : (M, d) \to (M, d)$ satisfying

$$d(Fu, Fv) \leq \kappa \left[d(u, Fu) + d(v, Fv)\right] \tag{1}$$

for all $u, v \in M$, where $\kappa \in [0, 1/2)$ is a constant. Kannan proved that, if (M, d) is a complete metric space and F satisfies (1), then F possesses a unique fixed point, and for all $u \in M$, the Picard sequence $\{F^n u\}$ converges to this unique fixed point. Unlike the class of contractions, a mapping F satisfying (1) is not necessarily continuous. Throughout this paper, any mapping $F : (M, d) \to (M, d)$ satisfying (1) is called a Kannan-contraction. The class of Kannan-contractions plays an important role in metrical fixed point theory. Indeed, apart from the fact that a Kannan-contraction is not necessarily continuous, the Kannan contraction principle provides a characterization of the metric completeness (see Subrahmanyam [27]), while the Banach

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contraction principle does not (see Conell [6]). Due to these facts, the establishment of generalizations and extensions of Kannan's fixed-point theorem attracted a lot of interest. Some contributions related to Kannan's fixed-point theorem can be found in [2, 4, 7, 13, 15–17, 19, 23, 24].

In 2003, Kirk et al. [12] introduced the class of cyclic contractions. Namely, let $\{C_i\}_{i=1}^p$ be a family of nonempty and closed subsets of a metric space (M, d). A cyclic contraction is a mapping $F : \bigcup_{i=1}^p C_i \to \bigcup_{i=1}^p C_i$ satisfying the following conditions:

(i) $F(C_i) \subset C_{i+1}$ for all $1 \leq i \leq p$ (with $C_{p+1} = C_1$).

(ii) There exists $\kappa \in [0, 1)$ such that

$$d(Fu, Fv) \leq \kappa d(u, v)$$

for all $(u, v) \in C_i \times C_{i+1}$ and $1 \leq i \leq p$.

It was shown in [12]that, if (M, d) is a complete metric space and $F : \bigcup_{i=1}^p C_i \to \bigcup_{i=1}^p C_i$ is a cyclic contraction, then F possesses a unique fixed point in $\bigcap_{i=1}^p C_i$. The literature includes numerous generalizations and extensions of this result. For instance, Păcurar and Rus [20] studied the class of cyclic φ-contractions. In [18], Păcurar introduced the class of r-cyclic operators with respect to a covering of a metric space and investigated their behavior under a Banach-type generalized contraction. Other related contributions can be found in [1, 8, 10, 21, 22, 25, 26].

In this paper, we study the existence and uniqueness of fixed points for two new classes of mappings $F : (M, d) \to (M, d)$ satisfying contractions involving an auxiliary mapping $S : M \times M \to M$. The first family of mappings includes the class of contractions, while the second one includes the class of Kannan-contractions. Namely, for particular choices of S, we recover the Banach and Kannan fixed-point theorems. Our obtained results are supported by examples. Next, an application to cyclic contractions involving an auxiliary mapping is provided.

The rest of the paper includes three sections. In Section 2, we introduce the class of S^B-contractions, which includes for a special choice of S, the class of contractions. A fixed-point theorem is established for the introduced mappings. We also provide an example where our obtained result can be used, while Banach's fixed-point theorem is inapplicable. In Section 3, we introduce the class of S^K-contractions, which includes for a special choice of S, the class of Kannan-contractions. A fixed-point result for S^K-contractions is established. Next, an example supporting our obtained result is given. We also show that in this example, Kannan's fixed point theorem is inapplicable. Finally, in Section 4, we apply our results to the study of fixed points for cyclic contractions involving an auxiliary mapping.

Throughout this paper, the following notations will be used:

- M denotes an arbitrary nonempty set.
- For a given mapping $F : M \to M$ and $z \in M$, we set

$$F^0 z = z, \quad F^{n+1} z = F(F^n z), \ n \geq 0.$$

We denote by $\mathrm{Fix}(F)$, the set of fixed points of F, that is,

$$\mathrm{Fix}(F) = \{z \in M : Fz = z\}.$$

- For a given mapping $S : M \times M \to M$, we denote by $\mathrm{Fix}(S)$, the set of fixed points of S, that is,

$$\mathrm{Fix}(S) = \{z \in M : S(z, z) = z\}.$$

2. The class of S^B-contractions

In this section, we are concerned with the study of fixed points for the following class of mappings.

Definition 2.1. *Let (M,d) be a metric space and $S : M \times M \to M$ be a given mapping. A mapping $F : M \to M$ is called a S^B-contraction, if there exists $\kappa \in [0,1)$ such that*

$$d(Fx, S(Fx,Fy)) + d(Fy, S(Fx,Fy)) \leq \kappa \left[d(x, S(x,y)) + d(y, S(x,y))\right] \tag{2}$$

for every $x, y \in M$.

Remark 2.2. *Notice that, if $S(u,v) = v$ for all $u, v \in M$, then* (2) *reduces to*

$$d(Fx,Fy) \leq \kappa d(x,y)$$

for every $x, y \in M$. Then, a contraction is a S^B-contraction with $S(u,v) = v$.

We have the following fixed-point result.

Theorem 2.3. *Let (M,d) be a complete metric space and $S : M \times M \to M$ be a given mapping. Assume that the following conditions hold:*

(i) $F : M \to M$ is a S^B-contraction for some $\kappa \in [0,1)$.

(ii) For all $u, v \in M$, we have

$$\lim_{n\to\infty} d(F^n u, v) = 0 \implies \lim_{n\to\infty} d(F(F^n u), Fv) = 0.$$

Then,

(I) For all $x_0 \in M$, the sequence $\{F^n x_0\}$ converges to a fixed point of F.

(II) F admits a unique fixed point $x^* \in M$.

(III) $x^* \in \mathrm{Fix}(S)$.

(IV) For all $x_0 \in M$ and $n \geq 0$, we have

$$d(F^n x_0, x^*) \leq \frac{\kappa^n}{1-\kappa}\left[d(x_0, S(x_0, Fx_0)) + d(Fx_0, S(x_0, Fx_0))\right].$$

Proof. (I). Let $x_0 \in M$ and $\{x_n\}$ be the sequence defined by

$$x_n = F^n x_0, \quad n \geq 0.$$

Using (2) with $(x,y) = (x_0, x_1)$, we obtain

$$d(Fx_0, S(Fx_0, Fx_1)) + d(Fx_1, S(Fx_0, Fx_1)) \leq \kappa\left[d(x_0, S(x_0,x_1)) + d(x_1, S(x_0,x_1))\right],$$

that is,

$$d(x_1, S(x_1,x_2)) + d(x_2, S(x_1,x_2)) \leq \kappa\left[d(x_0, S(x_0,x_1)) + d(x_1, S(x_0,x_1))\right]. \tag{3}$$

Similarly, using (2) with $(x,y) = (x_1, x_2)$, we obtain

$$d(Fx_1, S(Fx_1, Fx_2)) + d(Fx_2, S(Fx_1, Fx_2)) \leq \kappa\left[d(x_1, S(x_1,x_2)) + d(x_2, S(x_1,x_2))\right],$$

that is,

$$d(x_2, S(x_2,x_3)) + d(x_3, S(x_2,x_3)) \leq \kappa\left[d(x_1, S(x_1,x_2)) + d(x_2, S(x_1,x_2))\right]. \tag{4}$$

Then, combining both inequalities (3) and (4), we obtain

$$d(x_2, S(x_2,x_3)) + d(x_3, S(x_2,x_3)) \leq \kappa^2\left[d(x_0, S(x_0,x_1)) + d(x_1, S(x_0,x_1))\right].$$

Continuing in the same way, by induction, we obtain

$$d(x_n, S(x_n, x_{n+1})) + d(x_{n+1}, S(x_n, x_{n+1})) \leq \delta_0 \kappa^n, \quad n \geq 0, \tag{5}$$

where

$$\delta_0 = d(x_0, S(x_0, x_1)) + d(x_1, S(x_0, x_1)). \tag{6}$$

On the other hand, by the triangle inequality, for all $n \geq 0$, we have

$$d(x_n, x_{n+1}) \leq d(x_n, S(x_n, x_{n+1})) + d(x_{n+1}, S(x_n, x_{n+1})). \tag{7}$$

Then, it follows from (5) and (7) that

$$d(x_n, x_{n+1}) \leq \kappa^n \delta_0, \quad n \geq 0,$$

which implies by the triangle inequality that

$$d(x_n, x_{n+m}) \leq \frac{\kappa^n}{1-\kappa} \delta_0, \quad n \geq 0,\ m \geq 1. \tag{8}$$

Since $\kappa \in [0, 1)$, we deduce that $\{x_n\}$ is a Cauchy sequence. Then, due to the completeness of (M, d), there exists $x^* \in M$ such that

$$\lim_{n\to\infty} d(F^n x_0, x^*) = 0, \tag{9}$$

which implies by (ii) that

$$\lim_{n\to\infty} d(F^{n+1} x_0, Fx^*) = \lim_{n\to\infty} d(F(F^n x_0), Fx^*) = 0.$$

Hence, by the uniqueness of the limit, we obtain that $x^* \in \mathrm{Fix}(F)$. This proves part (I).

(II). From (I), we know that $\mathrm{Fix}(F) \neq \emptyset$. We now show that F has a unique fixed point. Indeed, suppose that $x, y \in \mathrm{Fix}(F)$. Then, by (2), we have

$$d(x, S(x, y)) + d(y, S(x, y)) \leq \kappa \left[d(x, S(x, y)) + d(y, S(x, y)) \right],$$

which implies (since $\kappa \in [0, 1)$) that

$$d(x, S(x, y)) + d(y, S(x, y)) = 0.$$

Consequently, we obtain

$$x = S(x, y) = y, \tag{10}$$

which proves that F admits a unique fixed point. This proves part (II).

(III). From (I) and (II), we deduce that $\mathrm{Fix}(F) = \{x^*\}$, where x^* is given by (9). Notice that due to the uniqueness of the fixed point, x^* is independent of the choice of x_0. Taking $x = y = x^*$ in (10), we obtain

$$x^* = S(x^*, x^*),$$

which shows that $x^* \in \mathrm{Fix}(S)$. This proves part (III).

(IV). Passing to the limit as $m \to \infty$ in (8), using (6) and (9), we obtain (IV).
The proof of Theorem 2.3 is then completed. □

Remark 2.4. *Notice that, if $F : (M, d) \to (M, d)$ is continuous, then F satisfies condition (ii) of Theorem 2.3.*

Remark 2.5. *From Remark 2.2, taking $S(u,v)=v$ in Theorem 2.3, we obtain the Banach fixed-point theorem.*

An example illustrating Theorem 2.3 is given below.

Example 2.6. *Let $M=\{x_1,x_2,x_3\}$ and $S: M\times M\to M$ be the mapping defined by*

$$S(x_i,x_i)=x_i,\ \ S(x_i,x_j)=S(x_j,x_i),\quad i,j\in\{1,2,3\}$$

and

$$S(x_1,x_2)=x_1,\ \ S(x_1,x_3)=x_2,\ \ S(x_2,x_3)=x_1.$$

Let d be the metric on M defined by

$$d(x_i,x_j)=\begin{cases} 1 & if \quad i\neq j,\\ 0 & if \quad i=j.\end{cases} \tag{11}$$

We consider the mapping $F: M\to M$ defined by

$$Fx_1=x_1,\ \ Fx_2=x_1,\ \ Fx_3=x_2.$$

We claim that for all $i,j\in\{1,2,3\}$,

$$d(Fx_i,S(Fx_i,Fx_j))+d(Fx_j,S(Fx_i,Fx_j))\leq \frac{1}{2}\left[d(x_i,S(x_i,x_j))+d(x_j,S(x_i,x_j))\right]. \tag{12}$$

Notice that for all $i\in\{1,2,3\}$, we have

$$\begin{aligned} d(Fx_i,S(Fx_i,Fx_i))+d(Fx_i,S(Fx_i,Fx_i)) &= d(Fx_i,Fx_i)+d(Fx_i,Fx_i)\\ &=0, \end{aligned}$$

which shows that (12) *holds for all $i=j\in\{1,2,3\}$. On the other hand, due to the symmetry of S, we have just to show that* (12) *holds for $(i,j)\in\{(1,2),(1,3),(2,3)\}$.*
Case 1: $(i,j)=(1,2)$. In this case, we have

$$\begin{aligned} d(Fx_i,S(Fx_i,Fx_j))+d(Fx_j,S(Fx_i,Fx_j)) &= d(Fx_1,S(Fx_1,Fx_2))+d(Fx_2,S(Fx_1,Fx_2))\\ &= d(x_1,S(x_1,x_1))+d(x_1,S(x_1,x_1))\\ &= d(x_1,x_1)+d(x_1,x_1)\\ &=0, \end{aligned}$$

which shows that (12) *holds.*
Case 2: $(i,j)=(1,3)$. In this case, we have

$$\begin{aligned} \frac{d(Fx_i,S(Fx_i,Fx_j))+d(Fx_j,S(Fx_i,Fx_j))}{d(x_i,S(x_i,x_j))+d(x_j,S(x_i,x_j))} &= \frac{d(Fx_1,S(Fx_1,Fx_3))+d(Fx_3,S(Fx_1,Fx_3))}{d(x_1,S(x_1,x_3))+d(x_3,S(x_1,x_3))}\\ &= \frac{d(x_1,S(x_1,x_2))+d(x_2,S(x_1,x_2))}{d(x_1,S(x_1,x_3))+d(x_3,S(x_1,x_3))}\\ &= \frac{d(x_1,x_1)+d(x_2,x_1)}{d(x_1,x_2)+d(x_3,x_2)}\\ &= \frac{1}{2}, \end{aligned}$$

which shows that (12) *holds.*
Case 3: $(i,j)=(2,3)$*. In this case, we have*

$$\begin{aligned}\frac{d(Fx_i,S(Fx_i,Fx_j))+d(Fx_j,S(Fx_i,Fx_j))}{d(x_i,S(x_i,x_j))+d(x_j,S(x_i,x_j))}&=\frac{d(Fx_2,S(Fx_2,Fx_3))+d(Fx_3,S(Fx_2,Fx_3))}{d(x_2,S(x_2,x_3))+d(x_3,S(x_2,x_3))}\\&=\frac{d(x_1,S(x_1,x_2))+d(x_2,S(x_1,x_2))}{d(x_2,S(x_2,x_3))+d(x_3,S(x_2,x_3))}\\&=\frac{d(x_1,x_1)+d(x_2,x_1)}{d(x_2,x_1)+d(x_3,x_1)}\\&=\frac{1}{2},\end{aligned}$$

which shows that (12) *holds.*

Consequently (12) *is satisfied for all* $i,j\in\{1,2,3\}$*, which shows that* F *is a* S^B*-contraction with* $\kappa=1/2$*. On the other hand, we have*

$$\mathrm{Fix}(F)=\{x_1\},\quad S(x_1,x_1)=x_1,$$

which confirms Theorem 2.3.

We point out that Banach's fixed-point theorem is not applicable in this case. Indeed, we have

$$\begin{aligned}\frac{d(Fx_1,Fx_3)}{d(x_1,x_3)}&=\frac{d(x_1,x_2)}{d(x_1,x_3)}\\&=1,\end{aligned}$$

which shows that F *is not a contraction.*

3. The class of S^K-contractions

In this section, we introduce the class of S^K-contractions, which includes Kannan-contractions.

Definition 3.1. *Let* (M,d) *be a metric space and* $S:M\times M\to M$ *be a given mapping. A mapping* $F:M\to M$ *is called a* S^K*-contraction, if there exists* $0\le\kappa<\frac{1}{2}$ *such that*

$$d(Fx,S(Fx,Fy))+d(Fy,S(Fx,Fy))\le\kappa\left[d(x,S(x,Fx))+d(Fx,S(x,Fx))+d(y,S(y,Fy))+d(Fy,S(y,Fy))\right]\quad(13)$$

for every $x,y\in M$*.*

Remark 3.2. *Notice that, if* $S(u,v)=u$ *for all* $u,v\in M$*, then* (13) *reduces to* (1)*. Then, a Kannan-contraction is a* S^K*-contraction with* $S(u,v)=u$*.*

We have the following fixed-point result.

Theorem 3.3. *Let* (M,d) *be a complete metric space and* $S:M\times M\to M$ *be a given mapping. Assume that the following conditions hold:*

(i) $F:M\to M$ *is a* S^K*-contraction for some* $0\le\kappa<\frac{1}{2}$*.*

(ii) $S(x,x)=x$ *for all* $x\in M$*.*

(iii) For all $u,v\in M$*, we have*

$$\lim_{n\to\infty}d(F^nu,v)=0\implies\lim_{n\to\infty}d(F(F^nu),Fv)=0.$$

Then

(I) For all $x_0\in M$*, the sequence* $\{F^nx_0\}$ *converges to a fixed point of* F*.*

(II) F admits a unique fixed point $x^* \in M$.

(III) For all $x_0 \in M$ and $n \geq 0$, we have

$$d(F^n x_0, x^*) \leq \frac{\lambda^n}{1-\lambda}\left[d(x_0, S(x_0, Fx_0)) + d(Fx_0, S(x_0, Fx_0))\right],$$

where $\lambda = \frac{\kappa}{1-\kappa}$.

Proof. (I). For $x_0 \in M$, let $\{x_n\} \subset M$ be the Picard sequence defined by

$$x_n = F^n x_0, \quad n \geq 0.$$

Using (13) with $(x, y) = (x_0, x_1)$, we obtain

$$\begin{aligned} & d(Fx_0, S(Fx_0, Fx_1)) + d(Fx_1, S(Fx_0, Fx_1)) \\ & \leq \kappa\left[d(x_0, S(x_0, Fx_0)) + d(Fx_0, S(x_0, Fx_0)) + d(x_1, S(x_1, Fx_1)) + d(Fx_1, S(x_1, Fx_1))\right], \end{aligned}$$

that is,

$$d(x_1, S(x_1, x_2)) + d(x_2, S(x_1, x_2)) \leq \kappa\left[d(x_0, S(x_0, x_1)) + d(x_1, S(x_0, x_1)) + d(x_1, S(x_1, x_2)) + d(x_2, S(x_1, x_2))\right],$$

which yields

$$d(x_1, S(x_1, x_2)) + d(x_2, S(x_1, x_2)) \leq \lambda\left[d(x_0, S(x_0, x_1)) + d(x_1, S(x_0, x_1))\right]. \tag{14}$$

Similarly, taking $(x, y) = (x_1, x_2)$ in (13), we obtain

$$\begin{aligned} & d(Fx_1, S(Fx_1, Fx_2)) + d(Fx_2, S(Fx_1, Fx_2)) \\ & \leq \kappa\left[d(x_1, S(x_1, Fx_1)) + d(Fx_1, S(x_1, Fx_1)) + d(x_2, S(x_2, Fx_2)) + d(Fx_2, S(x_2, Fx_2))\right], \end{aligned}$$

that is,

$$d(x_2, S(x_2, x_3)) + d(x_3, S(x_2, x_3)) \leq \kappa\left[d(x_1, S(x_1, x_2)) + d(x_2, S(x_1, x_2)) + d(x_2, S(x_2, x_3)) + d(x_3, S(x_2, x_3))\right],$$

which implies that

$$d(x_2, S(x_2, x_3)) + d(x_3, S(x_2, x_3)) \leq \lambda\left[d(x_1, S(x_1, x_2)) + d(x_2, S(x_1, x_2))\right]. \tag{15}$$

Then, it follows from (14) and (15) that

$$d(x_2, S(x_2, x_3)) + d(x_3, S(x_2, x_3)) \leq \lambda^2\left[d(x_0, S(x_0, x_1)) + d(x_1, S(x_0, x_1))\right].$$

Continuing in the same way, by induction, we obtain

$$d(x_n, S(x_n, x_{n+1})) + d(x_{n+1}, S(x_n, x_{n+1})) \leq \lambda^n \delta_0, \quad n \geq 0, \tag{16}$$

where δ_0 is given by (6). On the other hand, by the triangle inequality, we have

$$d(x_n, x_{n+1}) \leq d(x_n, S(x_n, x_{n+1})) + d(x_{n+1}, S(x_n, x_{n+1})), \quad n \geq 0. \tag{17}$$

Hence, in view of (16) and (17), it holds that

$$d(x_n, x_{n+1}) \leq \lambda^n \delta_0, \quad n \geq 0.$$

Notice that, since $0 \leq \kappa < \frac{1}{2}$, then $0 \leq \lambda < 1$. Making use of the triangle inequality, we obtain

$$d(x_n, x_{n+m}) \leq \frac{\lambda^n}{1-\lambda}\delta_0, \quad n \geq 0,\ m \geq 1, \tag{18}$$

which implies that $\{x_n\}$ is a Cauchy sequence in the complete metric space (M, d). Consequently, there exists $x^* \in M$ such that

$$\lim_{n\to\infty} d(F^n x_0, x^*) = 0, \tag{19}$$

which implies by (iii) that $x^* \in \mathrm{Fix}(F)$. This proves part (I).

(II). From (I), we know that $\mathrm{Fix}(F) \neq \emptyset$. Let us suppose that $x, y \in \mathrm{Fix}(F)$. Then, making use of (13) and (ii), we obtain

$$\begin{aligned} d(x, S(x, y)) + d(y, S(x, y)) &\leq 2\kappa\,[d(x, S(x, x)) + d(y, S(y, y))] \\ &= 2\kappa\,[d(x, x) + d(y, y)] \\ &= 0, \end{aligned}$$

which implies that

$$d(x, S(x, y)) = d(y, S(x, y)) = 0.$$

Consequently, we have $x = y$, which proves that F admits a unique fixed point. This proves part (II).

(III). From (I) and (II), we deduce that $\mathrm{Fix}(F) = \{x^*\}$, where x^* is given by (19). Passing to the limit as $m \to \infty$ in (18) and using (19), we obtain (III).

The proof of Theorem 3.3 is then completed. □

Remark 3.4. *Taking $S(u, v) = u$ for all $u, v \in M$ (see Remark 3.2),* (13) *reduces to*

$$d(Fx, Fy) \leq \kappa\,[d(x, Fx) + d(y, Fy)]$$

for every $x, y \in M$. Clearly, the mapping S satisfies condition (ii) of Theorem 3.3. Furthermore, for all $u, v \in M$, we have

$$d(F^{n+1}u, Fv) = d(F(F^n u), Fv) \leq \kappa\left[d(F^n u, F^{n+1}u) + d(v, Fv)\right], \quad n \geq 0. \tag{20}$$

So, if

$$\lim_{n\to\infty} d(F^n u, v) = 0,$$

then passing to the limit as $n \to \infty$ in (20)*, we obtain*

$$d(v, Fv) \leq \kappa d(v, Fv) \leq \frac{d(v, Fv)}{2},$$

which yields

$$d(v, Fv) = 0.$$

Consequently, we obtain

$$\lim_{n\to\infty} d(F(F^n u), Fv) = \lim_{n\to\infty} d(F^{n+1}u, Fv) = \lim_{n\to\infty} d(F^{n+1}u, v) = 0,$$

which shows that condition (iii) of Theorem 3.3 is satisfied. Then, Theorem 3.3 includes Kannan's fixed-point theorem [11].

An example illustrating Theorem 3.3 is provided below.

Example 3.5. *Let $M = [0, 1]$ and d be the standard metric on M, i.e.,*

$$d(x, y) = |x - y|, \quad x, y \in M.$$

We consider the mapping $F : M \to M$ defined by

$$Fx = \begin{cases} 0 & \text{if} \quad 0 \leq x < 1, \\ \frac{1}{2} & \text{if} \quad x = 1. \end{cases}$$

Notice that F is not a Kannan-contraction. Indeed, for $(x, y) = \left(\frac{1}{2}, 1\right)$, we have

$$\frac{d(Fx, Fy)}{d(x, Fx) + d(y, Fy)} = \frac{\left|0 - \frac{1}{2}\right|}{\left|\frac{1}{2} - 0\right| + \left|1 - \frac{1}{2}\right|} = \frac{1}{2}.$$

So, there is no $\kappa \in \left[0, \frac{1}{2}\right)$ such that

$$d(Fx, Fy) \leq \kappa \left[d(x, Fx) + d(y, Fy)\right]$$

for every $x, y \in M$.

We now introduce the mapping $S : M \times M \to M$ defined by

$$S(x, y) = \begin{cases} x & \text{if} \quad x = y, \\ 1 - \sin\left[\frac{\pi(x+y)}{3}\right] & \text{if} \quad x \neq y. \end{cases}$$

Clearly, the mapping S satisfies condition (ii) of Theorem 3.3. We claim that

$$d(Fx, S(Fx, Fy)) + d(Fy, S(Fx, Fy)) \leq \frac{1}{3}\left[d(x, S(x, Fx)) + d(Fx, S(x, Fx)) + d(y, S(y, Fy)) + d(Fy, S(y, Fy))\right] \quad (21)$$

for every $x, y \in M$. Notice that, if $x = y$, then

$$\begin{aligned} d(Fx, S(Fx, Fy)) + d(Fy, S(Fx, Fy)) &= d(Fx, S(Fx, Fx)) + d(Fx, S(Fx, Fx)) \\ &= 2d(Fx, Fx) \\ &= 0, \end{aligned}$$

which shows that (21) *holds. So, taking into consideration the symmetry of S, we just have to check that* (21) *is satisfied for $0 \leq x < y < 1$ and $0 \leq x < 1$, $y = 1$.*

Case 1: $0 \leq x < y < 1$. In this case, we have

$$\begin{aligned} d(Fx, S(Fx, Fy)) + d(Fy, S(Fx, Fy)) &= d(0, S(0, 0)) + d(0, S(0, 0)) \\ &= 2d(0, 0) \\ &= 0, \end{aligned}$$

which shows that (21) *holds.*

Case 2: $0 \leq x < 1$, $y = 1$. In this case, we have

$$\begin{aligned} d(Fx, S(Fx, Fy)) + d(Fy, S(Fx, Fy)) &= d\left(0, S\left(0, \frac{1}{2}\right)\right) + d\left(\frac{1}{2}, S\left(0, \frac{1}{2}\right)\right) \\ &= S\left(0, \frac{1}{2}\right) + \left|\frac{1}{2} - S\left(0, \frac{1}{2}\right)\right| \\ &= 1 - \sin\left(\frac{\pi}{6}\right) + \left|\frac{1}{2} - 1 + \sin\left(\frac{\pi}{6}\right)\right| \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned}
d(y, S(y, Fy)) + d(Fy, S(y, Fy)) &= d\left(1, S\left(1, \frac{1}{2}\right)\right) + d\left(\frac{1}{2}, S\left(1, \frac{1}{2}\right)\right) \\
&= \left|1 - S\left(1, \frac{1}{2}\right)\right| + \left|\frac{1}{2} - S\left(1, \frac{1}{2}\right)\right| \\
&= \left|1 - 1 + \sin\left(\frac{\pi}{2}\right)\right| + \left|\frac{1}{2} - 1 + \sin\left(\frac{\pi}{2}\right)\right| \\
&= \frac{3}{2}.
\end{aligned}$$

Consequently, it holds that

$$\begin{aligned}
& d(Fx, S(Fx, Fy)) + d(Fy, S(Fx, Fy)) \\
&= \frac{1}{3}\left[d(y, S(y, Fy)) + d(Fy, S(y, Fy))\right] \\
&\leq \frac{1}{3}\left[d(x, S(x, Fx)) + d(Fx, S(x, Fx)) + d(y, S(y, Fy)) + d(Fy, S(y, Fy))\right],
\end{aligned}$$

which shows that (21) *holds.*
Then, condition (i) of Theorem 3.3 is satisfied with $\kappa = \frac{1}{3}$.

On the other hand, by the definition of F*, for all* $u \in M$*, we have*

$$F^n u = 0, \quad n \geq 2.$$

So, if for some $v \in M$*, we have*

$$\lim_{n\to\infty} |F^n u - v| = 0,$$

then $v = 0$ *and*

$$\lim_{n\to\infty} |F(F^n u) - Fv| = |F0 - F0| = 0.$$

This shows that condition (iii) of Theorem 3.3 is also satisfied.

Consequently, Theorem 3.3 applies. Moreover, 0 *is the unique fixed point of* F*, which confirms Theorem 3.3.*

4. Cyclic contractions with an auxiliary mapping

In this section, making use of Theorem 2.3, we establish the existence and uniqueness of fixed points for the following class of mappings.

Definition 4.1. *Let* $\{C_i\}_{i=1}^p$ *be a family of nonempty and closed subsets of a metric space* (M, d)*. Let*

$$S : \bigcup_{i=1}^p C_i \times \bigcup_{i=1}^p C_i \to \bigcup_{i=1}^p C_i$$

be a given mapping. A mapping

$$F : \bigcup_{i=1}^p C_i \to \bigcup_{i=1}^p C_i$$

is called a cyclic S^B*-contraction, if the following conditions hold:*

(a) $F(C_i) \subset C_{i+1}$ *for all* $1 \leq i \leq p$ *(with* $C_{p+1} = C_1$*).*

(b) There exists $\kappa \in [0, 1)$ *such that for all* $(x, y) \in C_i \times C_{i+1}$ *and* $1 \leq i \leq p$*, we have*

$$d(Fx, S(Fx, Fy)) + d(Fy, S(Fx, Fy)) \leq \kappa\left[d(x, S(x, y)) + d(y, S(x, y))\right].$$

We have the following result.

Theorem 4.2. *Let $\{C_i\}_{i=1}^p$ be a family of nonempty and closed subsets of a complete metric space (M,d). Let $S : \bigcup_{i=1}^p C_i \times \bigcup_{i=1}^p C_i \to \bigcup_{i=1}^p C_i$ be a given mapping. Assume that*

(i) $F : \bigcup_{i=1}^p C_i \to \bigcup_{i=1}^p C_i$ is a cyclic S^B-contraction for some $\kappa \in [0,1)$.

Then, $\bigcap_{i=1}^p C_i \neq \emptyset$. Moreover, if

(ii) for all $u,v \in \bigcap_{i=1}^p C_i$, we have

$$\lim_{n\to\infty} d(F^n u, v) = 0 \implies \lim_{n\to\infty} d(F(F^n u), Fv) = 0,$$

then

(I) F admits a unique fixed point $x^* \in \bigcap_{i=1}^p C_i$.

(II) $x^* \in \mathrm{Fix}(S)$.

(III) For all $x_0 \in \bigcup_{i=1}^p C_i$ and $n \geq 0$, we have

$$d(F^n x_0, x^*) \leq \frac{\kappa^n}{1-\kappa}\left[d(x_0, S(x_0, Fx_0)) + d(Fx_0, S(x_0, Fx_0))\right] \tag{22}$$

and

$$d(F^n x_0, x^*) \leq \kappa^n \left[d(x_0, S(x_0, x^*)) + d(x^*, S(x_0, x^*))\right]. \tag{23}$$

Proof. Let $u_0 \in \bigcup_{i=1}^p C_i$ and

$$u_n = F^n u_0, \quad n \geq 0.$$

By (a), for every $n \geq 0$, there exists $1 \leq i_n \leq p$ such that $(u_n, u_{n+1}) \in C_{i_n} \times C_{i_n+1}$. Then, making use of (b) with $(x,y) = (u_0, u_1)$, we obtain

$$d(Fu_0, S(Fu_0, Fu_1)) + d(Fu_1, S(Fu_0, Fu_1)) \leq \kappa\left[d(u_0, S(u_0, u_1)) + d(u_1, S(u_0, u_1))\right],$$

that is,

$$d(u_1, S(u_1, u_2) + d(u_2, S(u_1, u_2)) \leq \kappa\gamma_0, \tag{24}$$

where

$$\gamma_0 = d(u_0, S(u_0, u_1)) + d(u_1, S(u_0, u_1). \tag{25}$$

Similarly, making use of (b) with $(x,y) = (u_1, u_2)$, we obtain

$$d(u_2, S(u_2, u_3)) + d(u_3, S(u_2, u_3)) \leq \kappa\left[d(u_1, S(u_1, u_2)) + d(u_2, S(u_1, u_2))\right],$$

which implies by (24) that

$$d(u_2, S(u_2, u_3)) + d(u_3, S(u_2, u_3)) \leq \kappa^2\gamma_0.$$

Continuing in the same way, we obtain

$$d(u_n, S(u_n, u_{n+1})) + d(u_{n+1}, S(u_n, u_{n+1})) \leq \kappa^n\gamma_0, \quad n \geq 0,$$

which implies by the triangle inequality that

$$d(u_n, u_{n+1}) \leq \kappa^n\gamma_0, \quad n \geq 0. \tag{26}$$

Consequently, $\{u_n\}$ is a Cauchy sequence in $\left(\bigcup_{i=1}^p C_i, d\right)$. On the other hand, since $\bigcup_{i=1}^p C_i$ is a closed subset of the complete metric space (M, d), then $\left(\bigcup_{i=1}^p C_i, d\right)$ is a complete metric space. Therefore, there exists $u^* \in \bigcup_{i=1}^p C_i$ such that

$$\lim_{n\to\infty} d(u_n, u^*) = 0. \tag{27}$$

Furthermore, by (a), for all $1 \leq i \leq p$, there exists a subsequence $\{u_{\varphi_i(n)}\}$ of $\{u_n\}$ such that $\{u_{\varphi_i(n)}\} \subset C_i$. Then, in view of (27), we obtain

$$\lim_{n\to\infty} d(u_{\varphi_i(n)}, u^*) = 0$$

for all $1 \leq i \leq p$. Since C_i is closed for all $1 \leq i \leq p$, we deduce that

$$u^* \in \bigcap_{i=1}^p C_i,$$

which shows that $\bigcap_{i=1}^p C_i \neq \emptyset$. Moreover, by (a), we have

$$F\left(\bigcap_{i=1}^p C_i\right) \subset \bigcap_{i=1}^p C_i.$$

(I)-(II). Assume now that (ii) holds. Let us consider the mapping

$$F|_{\bigcap_{i=1}^p C_i} : \bigcap_{i=1}^p C_i \to \bigcap_{i=1}^p C_i.$$

Clearly, the mapping $F|_{\bigcap_{i=1}^p C_i}$ satisfies the assumptions of Theorem 2.3. Consequently, $F|_{\bigcap_{i=1}^p C_i}$ admits a unique fixed point $x^* \in \bigcap_{i=1}^p C_i$ and $x^* \in \mathrm{Fix}(S)$, which proves parts (I) and (II).

(III). We now take an arbitrary element $x_0 \in \bigcup_{i=1}^p C_i$. Since $x^* \in \bigcap_{i=1}^p C_i$, there exists some $1 \leq j \leq p$ such that $(x_0, x^*) \in C_j \times C_{j+1}$. Then, using (b) with $(x, y) = (x_0, x^*)$, we obtain

$$d(Fx_0, S(Fx_0, Fx^*)) + d(Fx^*, S(Fx_0, Fx^*)) \leq \kappa \left[d(x_0, S(x_0, x^*)) + d(x^*, S(x_0, x^*))\right],$$

that is (since $x^* \in \mathrm{Fix}(F)$),

$$d(Fx_0, S(Fx_0, x^*)) + d(x^*, S(Fx_0, x^*)) \leq \kappa \left[d(x_0, S(x_0, x^*)) + d(x^*, S(x_0, x^*))\right]. \tag{28}$$

Similarly, there exists some $1 \leq k \leq p$ such that $(Fx_0, x^*) \in C_k \times C_{k+1}$. Then, using (b) with $(x, y) = (Fx_0, x^*)$, we obtain

$$d(F(Fx_0), S(F(Fx_0), Fx^*)) + d(Fx^*, S(F(Fx_0), Fx^*)) \leq \kappa \left[d(Fx_0, S(Fx_0, x^*)) + d(x^*, S(Fx_0, x^*))\right],$$

that is,

$$d(F^2x_0, S(F^2x_0, x^*)) + d(x^*, S(F^2x_0, x^*)) \leq \kappa \left[d(Fx_0, S(Fx_0, x^*)) + d(x^*, S(Fx_0, x^*))\right],$$

which implies by (28) that

$$d(F^2x_0, S(F^2x_0, x^*)) + d(x^*, S(F^2x_0, x^*)) \leq \kappa^2 \left[d(x_0, S(x_0, x^*)) + d(x^*, S(x_0, x^*))\right].$$

Continuing this process, we obtain by induction that

$$d(F^nx_0, S(F^nx_0, x^*)) + d(x^*, S(F^nx_0, x^*)) \leq \kappa^n \left[d(x_0, S(x_0, x^*)) + d(x^*, S(x_0, x^*))\right], \quad n \geq 0,$$

which implies by the triangle inequality that

$$d(F^n x_0, x^*) \leq \kappa^n \left[d(x_0, S(x_0, x^*)) + d(x^*, S(x_0, x^*))\right], \quad n \geq 0,$$

which proves (23) and that

$$\lim_{n\to\infty} d(F^n x_0, x^*) = 0. \tag{29}$$

On the other hand, using (26) with $u_0 = x_0$, we get thanks to the triangle inequality that

$$d(F^n x_0, F^{n+m} x_0) \leq \frac{\kappa^n}{1-\kappa}\left[d(x_0, S(x_0, Fx_0)) + d(Fx_0, S(x_0, Fx_0))\right], \quad n \geq 0,\ m \geq 1,$$

which implies by passing to the limit as $m \to \infty$ and using (29) that

$$d(F^n x_0, x^*) \leq \frac{\kappa^n}{1-\kappa}\left[d(x_0, S(x_0, Fx_0)) + d(Fx_0, S(x_0, Fx_0))\right], \quad n \geq 0.$$

This proves (22). Thus, part (III) is proved.

The proof of Theorem 4.2 is then completed. □

Remark 4.3. *Taking $S(u, v) = v$ in Theorem 4.2, we obtain the fixed-point result due to Kirk et al. [12].*

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