Filomat 39:4 (2025), 1355–1372
https://doi.org/10.2298/FIL2504355M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Complete convergence for the weighted sums of random variables

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Abstract. In the present paper, we study the complete convergence of the weighted sums of independent and identically distributed random variables, which include and improve some known results.

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [9] as follows. A sequence $\{U_n, n \geq 1\}$ of random variables converges completely to the constant μ if

$$\sum_{n=1}^{\infty} \mathbb{P}(|U_n - \mu| > \varepsilon) < \infty \quad \text{for all} \quad \varepsilon > 0,$$

which, from the Borel-Cantelli lemma, implies that $U_n \xrightarrow{a.s.} \mu$. The converse is true if $\{U_n, n \geq 1\}$ are independent random variables.

Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables and $S_n = X_1 + X_2 + \cdots + X_n$. Hsu and Robbins [9] proved that if $\mathbb{E}X = 0$ and $\mathbb{E}X^2 < \infty$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|S_n| > \varepsilon n\right) < \infty. \tag{1.1}$$

The converse was proved by Erdös [7]. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory and has been generalized and extended in several directions. Spitzer [16] proved that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\left(|S_n| > \varepsilon n\right) < \infty \tag{1.2}$$

if and only if $\mathbb{E}X = 0$ and $\mathbb{E}|X| < \infty$. Katz [13] and Baum and Katz [2] generalized the work of Spitzer [16] and obtained that for $0 < p < 2$ and $r \geq p$,

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2} \mathbb{P}\left(|S_n| > \varepsilon n^{1/p}\right) < \infty, \quad \text{for} \quad \varepsilon > 0 \tag{1.3}$$

2020 *Mathematics Subject Classification*. Primary 60F15.
Keywords. Complete convergence, weighted sums, independent and identically distributed random variables.
Received: 13 June 2024; Revised: 06 December 2024; Accepted: 16 December 2024
Communicated by Biljana Popović
Research supported by National Natural Science Foundation of China (NSFC-11971154).
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if and only if $\mathbb{E}|X|^r < \infty$ and when $r \geq 1$, $\mathbb{E}X = 0$.

Let $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$ be an array of constants. The strong convergence results for weighted sums $\sum_{k=1}^{n} a_{n,k} X_k$ have been studied by many authors (see, for example, Bai and Cheng [1], Chen and Gan [4], Cai [3]). Many useful linear statistics, for example, least-squares (LS) estimators, non parametric regression function estimators and jackknife estimates are of the form of the weighted sums. Chow [5] has established the following complete convergence result for the weighted sums.

Theorem 1.1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with $\mathbb{E}X = 0$ and $\mathbb{E}|X|^p < \infty$ for some $p \geq 2$. Let $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of constants. If for each $n \geq 1$ and for some finite constant K not depending on n,*

$$\sum_{k=1}^{n} a_{n,k}^2 \leq K \quad \text{and} \quad n^{1/p} \max_{1 \leq k \leq n} |a_{n,k}| \leq K,$$

then for all $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left| \frac{1}{n^{1/p}} \sum_{k=1}^{n} a_{n,k} X_k \right| > \varepsilon \right) < \infty.$$

Li et al. [10] improved Theorem 1.1, and obtained

Theorem 1.2. *Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with $\mathbb{E}X = 0$, and let $\{a_{n,k}, k \in \mathbb{Z}, n \geq 1\}$ be a sequence of real numbers.*

(1) If

$$\sum_{k \in \mathbb{Z}} a_{n,k}^2 = O(1) \text{ and } \mathbb{E}X^2 \log(1 + |X|) < \infty,$$

then for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left| \sum_{k \in \mathbb{Z}} a_{n,k} X_k \right| > \varepsilon n^{1/2} \right) < \infty.$$

(2) Let $p > 2$. If $\mathbb{E}|X|^p < \infty$ and for some $0 < \delta < 2/p$, $2 \leq q < p$, let

$$\sum_{k \in \mathbb{Z}} a_{n,k}^2 = O(n^{\delta}) \quad \text{and} \quad \sum_{k \in \mathbb{Z}} |a_{n,k}|^q = O(1),$$

then for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left| \sum_{k \in \mathbb{Z}} a_{n,k} X_k \right| > \varepsilon n^{1/p} \right) < \infty.$$

Theorem 1.2 has been extended and improved by many authors. Wang et al. [21] and Sung [19] further generalized Theorem 1.2, and in particular, Wang et al. [21] discussed its necessity. Liang and Su [12] and Liang [11] discussed the complete convergence for weighted sums of negatively associated random variables, which generalized and extended the results on independent and identically distributed case of Li et al. [10]. Wang et al. [20] generalized and improved the corresponding ones of Li et al. [10] for independent sequences to the case of negatively superadditive dependent sequences. Miao et al. [15] considered a sequence of independent identically distributed random variables with mild moment condition and general dependence conditions, and established strong law of large numbers and complete convergence for the partial sums. Du and Miao [6] and Miao and Shao [14] separately studied the strong laws for weighted sums of some dependent random variables and the complete convergence of the weighted sums for martingale differences. The aim of this paper is to continue to study the complete convergence for the weighted sums of independent random variables, which include and improve some known results. Throughout this paper, let C be a constant not depending on n, which may be different in different places.

2. A general result

We begin with a general result on the complete convergence.

Theorem 2.1. *For each $n \geq 1$, let $\{X_{n,k}, 1 \leq k \leq n, n \geq 1\}$ be a sequence of independent real random variables. Assume that the following condition holds:*

$$\sum_{k=1}^{n} X_{n,k} \xrightarrow{\mathbb{P}} 0. \tag{2.1}$$

Let $\{b_n, n \geq 1\}$ be a sequence of positive constants. For every $\varepsilon > 0$, assume that there exists a sequence of positive constants $\{c_n, n \geq 1\}$, which may depend on ε, and define $Y_{n,k} = X_{n,k} I_{\{|X_{n,k}| \leq c_n\}}$ for every $1 \leq k \leq n$ and $n \geq 1$, such that the following conditions are satisfied:

$$\sum_{n=1}^{\infty} b_n \inf_{t>0} \exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^{n} \mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\sum_{k=1}^{n} \mathbb{E}Y_{n,k}^4 + \frac{t^5 e^{tc_n}}{5!}\sum_{k=1}^{n} \mathbb{E}|Y_{n,k}|^5\right) < \infty \tag{2.2}$$

and

$$\sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} \mathbb{P}\left(|X_{n,k}| > c_n\right) < \infty. \tag{2.3}$$

Then for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} b_n \mathbb{P}\left(\left|\sum_{k=1}^{n} X_{n,k}\right| > \varepsilon\right) < \infty. \tag{2.4}$$

Remark 2.1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with $\mathbb{E}X = 0$ and $\mathbb{E}X^2 < \infty$. For any $\varepsilon > 0$, each $n \geq 1$ and $1 \leq k \leq n$, let $X_{n,k} = n^{-1}X_k$, $b_n \equiv 1$ and $c_n \equiv \varepsilon/4$, then $Y_{n,k} = n^{-1}X_k I_{\{|X_k| \leq n\varepsilon/4\}}$. Hence it is easy to check that*

$$\sum_{k=1}^{n} X_{n,k} \xrightarrow{\mathbb{P}} 0$$

and for any $\varepsilon > 0$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \inf_{t>0} \exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^{n} \mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\sum_{k=1}^{n} \mathbb{E}Y_{n,k}^4 + \frac{t^5 e^{tc_n}}{5!}\sum_{k=1}^{n} \mathbb{E}|Y_{n,k}|^5\right) \\
&\quad \leq \sum_{n=1}^{\infty} \inf_{t>0} \exp\left(-t\varepsilon + \frac{t^2}{2n}\mathbb{E}X^2 + \frac{t^4\varepsilon^2}{4!4^2 n}\mathbb{E}X^2 + \frac{t^5\varepsilon^3 e^{t\varepsilon/4}}{5!4^3 n}\mathbb{E}X^2\right) \\
&\quad \leq \sum_{n=1}^{\infty} \exp\left(-2\log n + \left(\frac{2\log^2 n}{\varepsilon^2 n} + \frac{\log^4 n}{4!\varepsilon^2 n} + \frac{\log^5 n}{5!4\varepsilon^2 n^{1/2}}\right)\mathbb{E}X^2\right) \\
&\quad < \infty,
\end{aligned}$$

where we take $t = (2/\varepsilon)\log n$. Furthermore, we have

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \mathbb{P}\left(|X_{n,k}| > c_n\right) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \mathbb{P}\left(|X_k| > \frac{\varepsilon n}{4}\right)$$

$$\le \sum_{n=1}^{\infty} n\mathbb{P}\left(|X| > \frac{\varepsilon n}{4}\right) \le C\mathbb{E}X^2 < \infty.$$

Hence the following Robbins-Hsu theorem holds: for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\sum_{k=1}^{n} X_k\right| > \varepsilon n\right) < \infty.$$

Remark 2.2. *Let* $\{X, X_n, n \ge 1\}$ *be a sequence of independent and identically distributed random variables with* $\mathbb{E}X = 0$ *and* $\mathbb{E}(|X|\log^+|X|) < \infty$. *For any* $\varepsilon > 0$, *each* $n \ge 1$ *and* $1 \le k \le n$, *let* $X_{n,k} = n^{-1}X_k$, $b_n = n^{-1}\log n$ *and* $c_n \equiv \varepsilon/8$, *then* $Y_{n,k} = n^{-1}X_k I_{\{|X_k| \le n\varepsilon/8\}}$. *Hence it is easy to check that*

$$\sum_{k=1}^{n} X_{n,k} \xrightarrow{\mathbb{P}} 0$$

and for any $\varepsilon > 0$, *we have*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\log n}{n} \inf_{t>0} \exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^4 + \frac{t^5 e^{tc_n}}{5!}\sum_{k=1}^{n}\mathbb{E}|Y_{n,k}|^5\right)\\
\le &\sum_{n=1}^{\infty} \frac{\log n}{n} \inf_{t>0} \exp\left(-t\varepsilon + \frac{t^2}{2n}\frac{Cnc_n}{\log(nc_n)}\mathbb{E}(|X|\log^+|X|) + \frac{t^4}{4!n^3}\frac{C(nc_n)^3}{\log(nc_n)}\mathbb{E}(|X|\log^+|X|)\right.\\
&\qquad\left. + \frac{t^5 e^{tc_n}}{5!n^4}\frac{C(nc_n)^4}{\log(nc_n)}\mathbb{E}(|X|\log^+|X|)\right)\\
\le &\sum_{n=1}^{\infty} \frac{\log n}{n} \exp\left(-4\log\log n + \frac{C(4/\varepsilon)^2(\varepsilon/8)}{2\log(n\varepsilon/8)}(\log\log n)^2\mathbb{E}(|X|\log^+|X|)\right.\\
&\qquad + \frac{C(4/\varepsilon)^4(\varepsilon/8)^3}{4!\log(n\varepsilon/8)}(\log\log n)^4\mathbb{E}(|X|\log^+|X|)\\
&\qquad\left. + \frac{C(4/\varepsilon)^5(\varepsilon/8)^4(\log n)^{1/2}}{5!\log(n\varepsilon/8)}(\log\log n)^5\mathbb{E}(|X|\log^+|X|)\right)\\
< &\infty,
\end{aligned}$$

where we take $t = (4/\varepsilon)\log\log n$. *Furthermore, we have*

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\log n}{n}\sum_{k=1}^{n}\mathbb{P}\left(|X_{n,k}| > c_n\right) &= \sum_{n=1}^{\infty} \frac{\log n}{n}\sum_{k=1}^{n}\mathbb{P}\left(|X_k| > \frac{\varepsilon n}{8}\right)\\
&\le \sum_{n=1}^{\infty} \log n\mathbb{P}\left(|X| > \frac{\varepsilon n}{8}\right) \le C\mathbb{E}(|X|\log^+|X|) < \infty.
\end{aligned}$$

Hence the following Baum-Katz theorem holds: for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{\log n}{n}\mathbb{P}\left(\left|\sum_{k=1}^{n} X_k\right| > \varepsilon n\right) < \infty.$$

Remark 2.3. *Let* $r > 1$, $p \in (0, 2)$ *and* $\{X, X_n, n \ge 1\}$ *be a sequence of independent and identically distributed random variables with* $\mathbb{E}|X|^{pr} < \infty$ *(let* $\mathbb{E}X = 0$ *for* $pr \ge 1$*). For each* $n \ge 1$ *and* $1 \le k \le n$, *let* $X_{n,k} = n^{-1/p}X_k$. *From* c_r*-inequality for* $0 < pr \le 1$, *von Bahr-Esseen inequality for* $1 \le pr \le 2$ *and Rosenthal inequality for* $pr \ge 2$, *we get*

$$\mathbb{E}\left|\sum_{k=1}^{n} X_{n,k}\right|^{pr} = \frac{1}{n^r}\mathbb{E}\left|\sum_{k=1}^{n} X_k\right|^{pr}$$

$$\leq \begin{cases} \frac{1}{n^r}\sum_{k=1}^n \mathbb{E}\,|X_k|^{pr} = \frac{1}{n^{r-1}}\mathbb{E}|X|^{pr}, & \text{for } 0 < pr \leq 1 \\ \frac{C}{n^r}\sum_{k=1}^n \mathbb{E}\,|X_k|^{pr} = \frac{C}{n^{r-1}}\mathbb{E}|X|^{pr}, & \text{for } 1 \leq pr \leq 2 \\ \frac{C}{n^r}\left(\left(\sum_{k=1}^n \mathbb{E}X_k^2\right)^{pr/2} + \sum_{k=1}^n \mathbb{E}|X_k|^{pr}\right) \leq \frac{C}{n^{r(1-p/2)}}\mathbb{E}|X|^{pr}, & \text{for } pr \geq 2 \end{cases},$$

which implies that

$$\sum_{k=1}^n X_{n,k} \xrightarrow{\mathbb{P}} 0.$$

Let $b_n = n^{r-2}$. *For the case* $pr \leq 2$ *and for any* $\varepsilon > 0$, *let* $c_n \equiv \varepsilon/2$, $r-1 < \alpha < 2(r-1)$, *and* $Y_{n,k} = n^{-1/p}X_k I_{\{|X_k| \leq n^{1/p}\varepsilon/2\}}$ *for each* $n \geq 1$ *and* $1 \leq k \leq n$. *Then we have*

$$\begin{aligned}
&\sum_{n=1}^\infty n^{r-2}\inf_{t>0}\exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^n \mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\sum_{k=1}^n \mathbb{E}Y_{n,k}^4 + \frac{t^5 e^{tc_n}}{5!}\sum_{k=1}^n \mathbb{E}|Y_{n,k}|^5\right) \\
&\leq \sum_{n=1}^\infty n^{r-2}\inf_{t>0}\exp\left(-t\varepsilon + \frac{t^2(\varepsilon/2)^{2-pr}}{2n^{r-1}}\mathbb{E}|X|^{pr} + \frac{t^4(\varepsilon/2)^{4-pr}}{4!n^{r-1}}\mathbb{E}|X|^{pr} + \frac{t^5(\varepsilon/2)^{5-pr}e^{t\varepsilon/2}}{5!n^{r-1}}\mathbb{E}|X|^{pr}\right) \\
&\leq \sum_{n=1}^\infty n^{r-2}\exp\left(-\alpha\log n + \left(\frac{(\alpha/\varepsilon)^2(\varepsilon/2)^{2-pr}\log^2 n}{2n^{r-1}} + \frac{(\alpha/\varepsilon)^4(\varepsilon/2)^{4-pr}(\log n)^4}{4!n^{r-1}}\right.\right. \\
&\qquad\left.\left. + \frac{(\alpha/\varepsilon)^5(\varepsilon/2)^{5-pr}(\log n)^5 n^{\alpha/2}}{5!n^{r-1}}\right)\mathbb{E}|X|^{pr}\right) \\
&< \infty,
\end{aligned}$$

where we take $t = (\alpha/\varepsilon)\log n$. *For the case* $pr > 2$, *let* $c_n = \dfrac{\varepsilon}{2\alpha}\left(\dfrac{2}{p} - 1\right)$ *for some* $\alpha > r-1$, *and* $Y_{n,k} = n^{-1/p}X_k I_{\{|X_k| \leq n^{1/p}\varepsilon/p\alpha\}}$ *for each* $n \geq 1$ *and* $1 \leq k \leq n$. *Then we have*

$$\begin{aligned}
&\sum_{n=1}^\infty n^{r-2}\inf_{t>0}\exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^n \mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\sum_{k=1}^n \mathbb{E}Y_{n,k}^4 + \frac{t^5 e^{tc_n}}{5!}\sum_{k=1}^n \mathbb{E}|Y_{n,k}|^5\right) \\
&\leq \sum_{n=1}^\infty n^{r-2}\inf_{t>0}\exp\left(-t\varepsilon + \frac{t^2}{2n^{2/p-1}}\mathbb{E}|X|^2 + \frac{t^4 c_n^2}{4!n^{2/p-1}}\mathbb{E}|X|^2 + \frac{t^5 c_n^3 e^{tc_n}}{5!n^{2/p-1}}\mathbb{E}|X|^2\right) \\
&\leq \sum_{n=1}^\infty n^{r-2}\exp\left(-\alpha\log n + \left(\frac{(\alpha/\varepsilon)^2\log^2 n}{2n^{2/p-1}} + \frac{(\alpha/\varepsilon)^4 c_n^2(\log n)^4}{4!n^{2/p-1}}\right.\right. \\
&\qquad\left.\left. + \frac{(\alpha/\varepsilon)^5 c_n^3(\log n)^5 n^{2^{-1}(2p^{-1}-1)}}{5!n^{2/p-1}}\right)\mathbb{E}|X|^2\right) \\
&< \infty,
\end{aligned}$$

where we take $t = (\alpha/\varepsilon)\log n$. *Furthermore, we have*

$$\begin{aligned}
\sum_{n=1}^\infty n^{r-2}\sum_{k=1}^n \mathbb{P}\left(|X_{n,k}| > c_n\right) &= \sum_{n=1}^\infty n^{r-2}\sum_{k=1}^n \mathbb{P}\left(|X_k| > \frac{\varepsilon}{2\alpha}\left(\frac{2}{p} - 1\right)n^{1/p}\right) \\
&\leq \sum_{n=1}^\infty n^{r-1}\mathbb{P}\left(|X| > \frac{\varepsilon}{2\alpha}\left(\frac{2}{p} - 1\right)n^{1/p}\right) \leq C\mathbb{E}|X|^{pr} < \infty.
\end{aligned}$$

Hence we can obtain the following Baum-Katz result: for any $\varepsilon > 0$,

$$\sum_{n=1}^\infty n^{r-2}\mathbb{P}\left(\left|\sum_{k=1}^n X_k\right| > \varepsilon n^{1/p}\right) < \infty.$$

In order to prove Theorem 2.1, we need to recall the following lemma.

Lemma 2.1. *[8] If $\{Y_n, n \geq 1\}$ is a sequence of random variables with $Y_n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, then for all $t > 0$ and sufficiently large n,*

$$\mathbb{P}(|Y_n| > t) \leq 2\mathbb{P}(|Y_n^s| > t/2),$$

where $Y_n^s = Y_n - \tilde{Y}_n$ and $\tilde{Y}_n$ is an independent copy of Y_n.

Proof. **[Proof of Theorem 2.1]** Applying Lemma 2.1, we can assume that for each $n \geq 1$, $\{X_{n,k}, 1 \leq k \leq n\}$ is a sequence of symmetric random variables. From the condition (2.3), it is enough to check that

$$\sum_{n=1}^{\infty} b_n \mathbb{P}\left(\left|\sum_{k=1}^{n} Y_{n,k}\right| > \varepsilon\right) < \infty. \tag{2.5}$$

First we note that for $x > 0$,

$$\begin{aligned} e^x =& 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + x^5 \sum_{k=5}^{\infty} \frac{1}{k!}x^{k-5} \\ \leq& 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 e^x. \end{aligned}$$

For every $1 \leq k \leq n$ and $n \geq 1$, since $Y_{n,k}$ is a symmetric random variable, then for any $t > 0$, we have

$$\begin{aligned} \mathbb{E}e^{tY_{n,k}} \leq& 1 + 0 + \frac{t^2}{2}\mathbb{E}Y_{n,k}^2 + 0 + \frac{t^4}{4!}\mathbb{E}Y_{n,k}^4 + \frac{t^5}{5!}\mathbb{E}\left(|Y_{n,k}|^5 e^{t|Y_{n,k}|}\right) \\ \leq& \exp\left(\frac{t^2}{2}\mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\mathbb{E}Y_{n,k}^4 + \frac{t^5}{5!}e^{tc_n}\mathbb{E}|Y_{n,k}|^5\right). \end{aligned}$$

Hence the following upper exponential inequality holds:

$$\mathbb{P}\left(\left|\sum_{k=1}^{n} Y_{n,k}\right| > \varepsilon\right) \leq 2\inf_{t>0}\exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^4 + \frac{t^5 e^{tc_n}}{5!}\sum_{k=1}^{n}\mathbb{E}|Y_{n,k}|^5\right),$$

which, by the condition (2.2), implies the claim (2.5). □

3. Complete convergence of weighted sums

3.1. Weighted sums

In the subsection, we state some results on complete convergence for weighted sums of independent and identically distributed random variables.

Theorem 3.1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with $\mathbb{E}X = 0$ and $\{a_{n,k}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of real numbers. Let $p \geq 2$, $r \geq 1$, $\lambda > 0$ and assume that there exist two finite constants $C > 0$ and $0 < \delta < 2/p$ not depending on n such that for all $n \geq 1$,*

$$\sum_{k=1}^{n} a_{n,k}^2 \leq Cn^{\delta} \tag{3.1}$$

and

$$n^{\lambda} \max_{1\leq k\leq n} |a_{n,k}| \leq C. \tag{3.2}$$

If one of the following conditions holds:

(1) Let $0 < \lambda \le (pr + p - 2)/2p$ *and* $\mathbb{E}|X|^{p(r+1)/(1+p\lambda)} < \infty$*,*

(2) $\mathbb{E}|X|^{p(r+\delta+2\lambda)/(1+p\lambda)} < \infty$*,*

then for any $\varepsilon > 0$*, we have*

$$\sum_{n=1}^{\infty} n^{r-1}\mathbb{P}\left(\left|\sum_{k=1}^{n} a_{n,k}X_k\right| > \varepsilon n^{1/p}\right) < \infty. \tag{3.3}$$

Remark 3.1. *In the proof of Theorem 3.1, we need the condition* $\mathbb{E}|X|^2 < \infty$*. The condition* $0 < \lambda \le (pr + p - 2)/2p$ *in (1) is to guarantee* $p(r + 1)/(1 + p\lambda) \ge 2$*. For the condition (2), by the condition* $r \ge 1$*, it is easy to check* $p(r + \delta + 2\lambda)/(1 + p\lambda) \ge 2$ *for all* $\lambda > 0$*. Furthermore, under the condition* $0 < \lambda \le (pr + p - 2)/2p$*, if* $\mathbb{E}|X|^{\beta} < \infty$*, where*

$$\beta = \max\left\{p(r + 1)/(1 + p\lambda),\ p(r + \delta + 2\lambda)/(1 + p\lambda)\right\},$$

then (3.3) holds.

In fact, the condition $r \ge 1$ *is only used for the case (2). If* $(r + \delta)p \ge 2$*, then* $p(r + \delta + 2\lambda)/(1 + p\lambda) \ge 2$ *for all* $\lambda > 0$*. So we can weaken the restriction on the condition* $r \ge 1$*.*

Remark 3.2. *Theorem 3.1 is an interesting supplement to Theorem 1.2 by comparing their conditions. Let* $r = 1$*. For the case* $p > 2$ *(*$p = 2$*), if the parameters* δ *and* λ *satisfy* $\delta < (p - 2)\lambda$ *(for any* $0 < \delta < 1$*), then from (2) in Theorem 3.1, the moment condition in Theorem 3.1 is weaker than Theorem 1.2.*

Theorem 3.2. *Let* $\{X, X_n, n \ge 1\}$ *be a sequence of independent and identically distributed random variables with* $\mathbb{E}X = 0$ *and* $\{a_{n,k}, 1 \le k \le n, n \ge 1\}$ *be a triangular array of real numbers. Let* $1 \le p < 2$*,* $r \ge 1$*,* $\lambda > 0$ *and assume that there exist two finite constants* $C > 0$ *and* $0 < \delta < 1$ *not depending on n such that for all* $n \ge 1$*,*

$$\sum_{k=1}^{n} |a_{n,k}|^p \le Cn^{\delta} \tag{3.4}$$

and

$$n^{\lambda} \max_{1\le k\le n} |a_{n,k}| \le C. \tag{3.5}$$

If one of the following conditions holds:

(1) Let $0 < \lambda \le r/p$ *and* $\mathbb{E}|X|^{p(r+1)/(1+p\lambda)} < \infty$*,*

(2) $\mathbb{E}|X|^{p(r+\delta+p\lambda)/(1+p\lambda)} < \infty$*,*

then for any $\varepsilon > 0$*, we have*

$$\sum_{n=1}^{\infty} n^{r-1}\mathbb{P}\left(\left|\sum_{k=1}^{n} a_{n,k}X_k\right| > \varepsilon n^{1/p}\right) < \infty. \tag{3.6}$$

Remark 3.3. *In the proof of Theorem 3.2, we need the condition* $\mathbb{E}|X|^p < \infty$*. The condition* $0 < \lambda \le r/p$ *in (1) is to guarantee* $p(r + 1)/(1 + p\lambda) \ge p$*. For the condition (2), since* $r \ge 1$ *and* $0 < \delta < 1$*, it is easy to check* $p(r + \delta + p\lambda)/(1 + p\lambda) \ge p$ *for all* $\lambda > 0$*. Furthermore, under the condition* $0 < \lambda \le r/p$*, if* $\mathbb{E}|X|^{\beta} < \infty$*, where*

$$\beta = \max\left\{p(r + 1)/(1 + p\lambda),\ p(r + \delta + p\lambda)/(1 + p\lambda)\right\},$$

then (3.6) holds.

Theorem 3.3. *Let* $\{X, X_n, n \ge 1\}$ *be a sequence of independent and identically distributed random variables with* $\mathbb{E}X = 0$*,* $\mathbb{E}X^2 = 1$ *and* $\{a_{n,k}, 1 \le k \le n, n \ge 1\}$ *be a triangular array of real numbers. Let* $p \ge 2$*,* $r > 0$*,* $\lambda > 0$ *and assume that there exist two finite constants* $C > 0$ *and* $\rho \ge 0$ *not depending on n such that for all* $n \ge 1$*,*

$$\lim_{n\to\infty} \log n \sum_{k=1}^{n} a_{n,k}^2 = \rho \tag{3.7}$$

and

$$n^{\lambda}\max_{1\le k\le n}|a_{n,k}|\le C. \tag{3.8}$$

If one of the following conditions holds:

(1) Let $0<\lambda\le(r+1)/2$ *and* $\mathbb{E}[|X|\log^{+}|X|]^{(r+1)/\lambda}<\infty$,

(2) $\mathbb{E}\left[|X|^{2+(r/\lambda)}(\log^{+}|X|)^{1+(r/\lambda)}\right]<\infty$,

then for any $\varepsilon>\sqrt{2r(1+12^{-1}+e\cdot 60^{-1})\rho}$, *we have*

$$\sum_{n=1}^{\infty}n^{r-1}\mathbb{P}\left(\left|\sum_{k=1}^{n}a_{n,k}X_k\right|>\varepsilon\right)<\infty.$$

Remark 3.4. *In [18, Theorem 4.1.3] and [17, p. 1556], Stout proved the following result. Let* $\{X,X_n,n\ge1\}$ *be a sequence of independent and identically distributed random variables with* $\mathbb{E}X=0$ *and* $\mathbb{E}|X|^{2/\alpha}<\infty$ *for some* $0<\alpha\le1$. *Suppose that* $\{a_{n,k},1\le k\le n,n\ge1\}$ *is a sequence of constants with*

$$\max_{1\le k\le n}|a_{n,k}|\le Cn^{-\alpha}$$

for some $C>0$ *and*

$$\lim_{n\to\infty}\log n\sum_{k=1}^{n}a_{n,k}^{2}=0.$$

Then for any $\varepsilon>0$, *we have*

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\left|\sum_{k=1}^{n}a_{n,k}X_k\right|>\varepsilon\right)<\infty.$$

For the case $\lambda\in(0,2^{-1})$, *it is easy to check that*

$$2+\frac{r}{\lambda}<\frac{r+1}{\lambda},$$

then, from Theorem 3.3, the moment condition in (2) is weaker than (1). When $r=1$ *and* $\rho=0$, *for the case* $\lambda\in(0,2^{-1})$, *the moment condition in Theorem 3.3 is weaker than the above Stout's result.*

Remark 3.5. *By the Borel-Cantelli lemma, Theorem 3.3 (for the case* $r=1$*) implies*

$$\limsup_{n\to\infty}\left|\sum_{k=1}^{n}a_{n,k}X_k\right|\le\sqrt{2(1+12^{-1}+e\cdot 60^{-1})\rho}\ \text{ a.s.}$$

Theorem 3.4. *Let* $\{X,X_n,n\ge1\}$ *be a sequence of independent and identically distributed random variables with* $\mathbb{E}X=0$ *and* $\{a_{n,k},1\le k\le n,n\ge1\}$ *be a triangular array of real numbers. Let* $1<p<2$, $r\ge1$, $\lambda>0$ *and assume that there exist two finite constants* $C>0$ *and* $\rho\ge0$ *not depending on n such that for all* $n\ge1$,

$$\lim_{n\to\infty}(\log n)^{p-1}\sum_{k=1}^{n}|a_{n,k}|^{p}=\rho \tag{3.9}$$

and

$$n^{\lambda}\max_{1\le k\le n}|a_{n,k}|\le C. \tag{3.10}$$

If one of the following conditions holds:

(1) Let $0<\lambda\le(r+1)/p$ *and* $\mathbb{E}[|X|\log^{+}|X|]^{(r+1)/\lambda}<\infty$,

(2) $\mathbb{E}\left[|X|^{p+(r/\lambda)}(\log^+|X|)^{1+(r/\lambda)}\right] < \infty$,
then for any $\varepsilon > \varepsilon_0$, *we have*

$$\sum_{n=1}^{\infty} n^{r-1}\mathbb{P}\left(\left|\sum_{k=1}^{n} a_{n,k}X_k\right| > \varepsilon\right) < \infty,$$

where ε_0 *is the minimum value satisfying the following inequality*

$$r - \frac{\varepsilon^2}{\Delta} + \frac{\varepsilon^p\Delta^{1-p}}{2} < 0 \tag{3.11}$$

and

$$\Delta = \left(1 + 12^{-1} + e\cdot 60^{-1}\right)\rho\mathbb{E}|X|^p.$$

Remark 3.6. *From the proof of Theorem 3.4, for the case* $\rho = 0$, *we can take* $\varepsilon_0 = 0$.

Next we consider a special weighted sum, which was studied in Li et al. [10]. Let $\beta > -1$ and $\{a_{n,k}, 1 \le k \le n, n \ge 1\}$ be a triangular array of real numbers such that

$$\sum_{k=1}^{n} a_{n,k} = 1 \quad \text{for all} \quad n \ge 1 \tag{3.12}$$

and we can write

$$a_{n,k} = c_{n,k}(k/n)^{\beta}(1/n) \tag{3.13}$$

with $0 < c \le c_{n,k} \le C < \infty$ for every $1 \le k \le n$ for some constants c and C.

Theorem 3.5. *Let* $\{X, X_n, n \ge 1\}$ *be a sequence of independent and identically distributed random variables with* $\mathbb{E}X = 0$ *and* $\{a_{n,k}, 1 \le k \le n, n \ge 1\}$ *be a triangular array of real numbers satisfying (3.12) and (3.13). Let* $\mathbb{E}|X|^p < \infty$ *for some* $1 < p \le 2$ *and let* $0 < r < p - 1$, *then for any* $\varepsilon > 0$, *we have*

$$\sum_{n=1}^{\infty} n^{r-1}\mathbb{P}\left(\left|\sum_{k=1}^{n} a_{n,k}X_k\right| > \varepsilon\right) < \infty. \tag{3.14}$$

Theorem 3.6. *Let* $\{X, X_n, n \ge 1\}$ *be a sequence of independent and identically distributed random variables with* $\mathbb{E}X = 0$ *and* $\{a_{n,k}, 1 \le k \le n, n \ge 1\}$ *be a triangular array of real numbers satisfying (3.12) and (3.13). Let* $r > 0$ *and*

$$\begin{aligned} \mathbb{E}|X|^{\frac{r}{1+\beta}}\log^+|X| < \infty \quad &\textit{for} \quad \frac{r\beta}{1+\beta} = -1, \\ \mathbb{E}|X|^{\frac{r}{1+\beta}} < \infty \quad &\textit{for} \quad \frac{r\beta}{1+\beta} < -1, \\ \mathbb{E}|X|^{r+1} < \infty \quad &\textit{for} \quad \frac{r\beta}{1+\beta} > -1. \end{aligned}$$

Then for any $\varepsilon > 0$, *we have*

$$\sum_{n=1}^{\infty} n^{r-1}\mathbb{P}\left(\left|\sum_{k=1}^{n} a_{n,k}X_k\right| > \varepsilon\right) < \infty. \tag{3.15}$$

Remark 3.7. *Li et al. [10] proved the following result. Let* $\{X, X_n, n \ge 1\}$ *be a sequence of independent and identically distributed random variables with* $\mathbb{E}X = 0$ *and* $\{a_{n,k}, 1 \le k \le n, n \ge 1\}$ *be a triangular array of real numbers satisfying (3.12) and (3.13). If*

$$\begin{cases} \mathbb{E}|X|^{1/(1+\beta)} < \infty & \textit{for } \beta \in (-1, -1/2) \\ \mathbb{E}X^2\log(1+|X|) < \infty & \textit{for } \beta = -1/2 \\ \mathbb{E}|X|^2 < \infty & \textit{for } \beta > -1/2 \end{cases},$$

then for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\sum_{k=1}^{n} a_{n,k}X_k\right| > \varepsilon\right) < \infty. \tag{3.16}$$

Theorem 3.5 gives the convergent rate of the complete convergence, which shows that the moment conditions are independent on the parameter β in the weight $\{a_{n,k}, 1 \le k \le n, n \ge 1\}$. Theorem 3.6 is a meaningful supplement of the work in Li et al. [10]. When $r = 1$, Theorem 3.6 deduces (3.16).

3.2. Proofs

In this subsection, we shall give the proofs of those theorems in the subsection 3.1.

Proof. **[Proof of Theorem 3.1]** For any $\varepsilon > 0$, each $n \ge 1$ and $1 \le k \le n$, let

$$X_{n,k} = \frac{a_{n,k}}{n^{1/p}}X_k,\ \ b_n = n^{r-1},\ \ c_n \equiv \frac{\varepsilon}{4r}\left(\frac{2}{p} - \delta\right) \text{ and } Y_{n,k} = X_{n,k}I_{\{|X_{n,k}|\le c_n\}}.$$

From the condition (3.1), we get

$$\mathbb{E}\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n} a_{n,k}X_k\right|^2 \le \frac{C}{n^{2/p}}\sum_{k=1}^{n} a_{n,k}^2\mathbb{E}X_k^2 = o(1)$$

which implies

$$\sum_{k=1}^{n} X_{n,k} \xrightarrow{\mathbb{P}} 0.$$

Moreover, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1}\inf_{t>0}\exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^4 + \frac{t^5e^{tc_n}}{5!}\sum_{k=1}^{n}\mathbb{E}|Y_{n,k}|^5\right)\\
\le&\sum_{n=1}^{\infty} n^{r-1}\inf_{t>0}\exp\left(-t\varepsilon + \frac{t^2\mathbb{E}X^2}{2n^{2/p}}\sum_{k=1}^{n}a_{n,k}^2 + \frac{t^4c_n^2\mathbb{E}X^2}{4!n^{2/p}}\sum_{k=1}^{n}a_{n,k}^2 + \frac{t^5c_n^3\mathbb{E}X^2e^{tc_n}}{5!n^{2/p}}\sum_{k=1}^{n}a_{n,k}^2\right)\\
\le&\sum_{n=1}^{\infty} n^{-r-1}\exp\left(\left(\frac{(2r/\varepsilon)^2\log^2 n}{2n^{2/p}} + \frac{(2r/\varepsilon)^4c_n^2\log^4 n}{4!n^{2/p}} + \frac{(2r/\varepsilon)^5c_n^3(\log n)^5n^{\frac{1}{2}\left(\frac{2}{p}-\delta\right)}}{5!n^{2/p}}\right)\mathbb{E}X^2\sum_{k=1}^{n}a_{n,k}^2\right)\\
<&\infty,
\end{aligned}$$

where we take $t = (2r/\varepsilon)\log n$. Furthermore, from the condition (3.2), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n}\mathbb{P}\left(|X_{n,k}| > c_n\right)\\
=&\sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n}\mathbb{P}\left(\frac{1}{n^{1/p}}|a_{n,k}X_k| > \frac{\varepsilon}{4r}\left(\frac{2}{p} - \delta\right)\right)\\
\le&\sum_{n=1}^{\infty} n^{r}\mathbb{P}\left(C|X| > \frac{\varepsilon}{4r}\left(\frac{2}{p} - \delta\right)n^{(1/p)+\lambda}\right)\\
\le& C\mathbb{E}|X|^{p(r+1)/(1+p\lambda)} < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathbb{P}\left(|X_{n,k}| > c_n\right) \\
=&\sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathbb{P}\left(\frac{1}{n^{1/p}}|a_{n,k}X_k| > \frac{\varepsilon}{4r}\left(\frac{2}{p}-\delta\right)\right) \\
\leq& C\sum_{n=1}^{\infty} n^{r-1-2/p} \sum_{k=1}^{n} a_{n,k}^2 \mathbb{E}X^2 I_{\{|X|>Cn^{(1/p)+\lambda}\}} \\
\leq& C\sum_{n=1}^{\infty} n^{r-1+\delta-2/p} \sum_{k=n}^{\infty} \mathbb{E}X^2 I_{\{Ck^{(1/p)+\lambda}<|X|\leq C(k+1)^{(1/p)+\lambda}\}} \\
\leq& C\sum_{k=1}^{\infty} k^{r+\delta-2/p} \mathbb{E}X^2 I_{\{Ck^{(1/p)+\lambda}<|X|\leq C(k+1)^{(1/p)+\lambda}\}} \\
\leq& C\mathbb{E}|X|^{p(r+\delta+2\lambda)/(1+p\lambda)} < \infty.
\end{aligned}$$

Hence from Theorem 2.1, the desired results can be obtained. □

Proof. **[Proof of Theorem 3.2]** For any $\varepsilon > 0$, each $n \geq 1$ and $1 \leq k \leq n$, let

$$X_{n,k} = \frac{a_{n,k}}{n^{1/p}} X_k, \quad b_n = n^{r-1}, \quad c_n \equiv \frac{\varepsilon}{4r}(1-\delta) \quad \text{and} \quad Y_{n,k} = X_{n,k} I_{\{|X_{n,k}|\leq c_n\}}.$$

From the von Bahr-Esseen inequality and the condition (3.4), we get

$$\mathbb{E}\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n} a_{n,k}X_k\right|^p \leq \frac{C}{n}\sum_{k=1}^{n}|a_{n,k}|^p = o(1),$$

which implies

$$\sum_{k=1}^{n} X_{n,k} \xrightarrow{\mathbb{P}} 0.$$

Moreover, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1} \inf_{t>0} \exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^4 + \frac{t^5 e^{tc_n}}{5!}\sum_{k=1}^{n}\mathbb{E}|Y_{n,k}|^5\right) \\
\leq& \sum_{n=1}^{\infty} n^{r-1} \inf_{t>0} \exp\left(-t\varepsilon + \frac{t^2 c_n^{2-p}\mathbb{E}|X|^p}{2n}\sum_{k=1}^{n}|a_{n,k}|^p \right.\\
&\qquad\left. + \frac{t^4 c_n^{4-p}\mathbb{E}|X|^p}{4!n}\sum_{k=1}^{n}|a_{n,k}|^p + \frac{t^5 c_n^{5-p}\mathbb{E}|X|^p e^{tc_n}}{5!n}\sum_{k=1}^{n}|a_{n,k}|^p\right) \\
\leq& \sum_{n=1}^{\infty} n^{-r-1} \exp\left(\left(\frac{(2r/\varepsilon)^2 c_n^{2-p}\log^2 n}{2n} + \frac{(2r/\varepsilon)^4 c_n^{4-p}\log^4 n}{4!n}\right.\right.\\
&\qquad\left.\left. + \frac{(2r/\varepsilon)^5 c_n^{5-p}(\log n)^5 n^{\frac{1}{2}(1-\delta)}}{5!n}\right)\mathbb{E}|X|^p\sum_{k=1}^{n}|a_{n,k}|^p\right) \\
<&\infty,
\end{aligned}$$

where we take $t = (2r/\varepsilon)\log n$. Furthermore, from the condition (3.5), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathbb{P}\left(|X_{n,k}| > c_n\right) \\
=&\sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathbb{P}\left(\frac{1}{n^{1/p}}|a_{n,k}X_k| > \frac{\varepsilon}{4r}(1-\delta)\right) \\
\leq &\sum_{n=1}^{\infty} n^{r} \mathbb{P}\left(C|X| > \frac{\varepsilon}{4}(1-\delta)\, n^{(1/p)+\lambda}\right) \\
\leq &C\mathbb{E}|X|^{p(r+1)/(1+p\lambda)} < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathbb{P}\left(|X_{n,k}| > c_n\right) \\
=&\sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathbb{P}\left(\frac{1}{n^{1/p}}|a_{n,k}X_k| > \frac{\varepsilon}{4r}(1-\delta)\right) \\
\leq &\sum_{n=1}^{\infty} n^{r-2} \sum_{k=1}^{n} |a_{n,k}|^p \mathbb{E}|X|^p I_{\{|X| > Cn^{(1/p)+\lambda}\}} \\
\leq &\sum_{n=1}^{\infty} n^{r+\delta-2} \sum_{k=n}^{\infty} \mathbb{E}|X|^p I_{\{k^{(1/p)+\lambda} < |X| \leq C(k+1)^{(1/p)+\lambda}\}} \\
\leq &\sum_{k=1}^{\infty} k^{r+\delta-1} \mathbb{E}|X|^p I_{\{k^{(1/p)+\lambda} < |X| \leq C(k+1)^{(1/p)+\lambda}\}} \\
\leq &C\mathbb{E}|X|^{p(r+\delta+p\lambda)/(1+p\lambda)} < \infty.
\end{aligned}$$

Hence from Theorem 2.1, the desired results can be obtained. □

Proof. **[Proof of Theorem 3.3]** For any $\varepsilon > \sqrt{2r(1 + 12^{-1} + e \cdot 60^{-1})\rho}$, each $n \geq 1$ and $1 \leq k \leq n$, let

$$X_{n,k} = a_{n,k}X_k,\ \ b_n = n^{r-1},\ \ c_n = \frac{\varepsilon}{b\log n} \text{ and } Y_{n,k} = X_{n,k} I_{\{|X_{n,k}| \leq c_n\}},$$

where

$$b = \frac{\varepsilon^2}{(1 + 12^{-1} + e \cdot 60^{-1})\rho}.$$

From the condition (3.7), we get

$$\mathbb{E}\left|\sum_{k=1}^{n} a_{n,k}X_k\right|^2 = \sum_{k=1}^{n} a_{n,k}^2 = o(1)$$

which implies

$$\sum_{k=1}^{n} X_{n,k} \xrightarrow{\mathbb{P}} 0.$$

Since $\varepsilon > \sqrt{2r(1 + 12^{-1} + e \cdot 60^{-1})\rho}$, there exists a positive constant $0 < \varepsilon_1 < 1$ such that

$$\varepsilon > \sqrt{\frac{2r(1 + 12^{-1} + e \cdot 60^{-1})\rho}{1 - \varepsilon_1}},$$

which yields

$$r - 1 - b + \frac{(b/\varepsilon)^2}{2}\left(1 + \frac{1}{12} + \frac{e}{60}\right)\rho(1 + \varepsilon_1) < -1.$$

Hence for this ε_1 and all n large enough, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1} \inf_{t>0} \exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^{n} \mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\sum_{k=1}^{n} \mathbb{E}Y_{n,k}^4 + \frac{t^5 e^{tc_n}}{5!}\sum_{k=1}^{n} \mathbb{E}|Y_{n,k}|^5\right)\\
\le &\sum_{n=1}^{\infty} n^{r-1} \inf_{t>0} \exp\left(-t\varepsilon + \frac{t^2 \mathbb{E}X^2}{2}\sum_{k=1}^{n} a_{n,k}^2 + \frac{t^4 c_n^2 \mathbb{E}X^2}{4!}\sum_{k=1}^{n} a_{n,k}^2 + \frac{t^5 c_n^3 \mathbb{E}X^2 e^{tc_n}}{5!}\sum_{k=1}^{n} a_{n,k}^2\right)\\
\le &C\sum_{n=1}^{\infty} n^{r-1} \exp\left(-b\log n + \frac{(b/\varepsilon)^2}{2}\left(1 + \frac{1}{12} + \frac{e}{60}\right)\rho(1 + \varepsilon_1)\log n\right)\\
<&\infty,
\end{aligned}$$

where we take $t = (b/\varepsilon)\log n$. Furthermore, from the condition (3.8), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathbb{P}\left(|X_{n,k}| > c_n\right)\\
= &\sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathbb{P}\left(|a_{n,k}X_k| > \frac{\varepsilon}{b\log n}\right)\\
\le &\sum_{n=1}^{\infty} n^{r} \mathbb{P}\left(|X| > \frac{\varepsilon n^{\lambda}}{Cb\log n}\right)\\
\le &C\sum_{k=1}^{\infty} k^{r+1} \mathbb{P}\left(\frac{\varepsilon k^{\lambda}}{Cb\log k} < |X| \le \frac{\varepsilon (k+1)^{\lambda}}{Cb\log(k+1)}\right)\\
\le &C\mathbb{E}[|X|\log^+|X|]^{(r+1)/\lambda} < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathbb{P}\left(|X_{n,k}| > c_n\right)\\
= &\sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} \mathbb{P}\left(|a_{n,k}X_k| > \frac{\varepsilon}{b\log n}\right)\\
\le &C\sum_{n=1}^{\infty} n^{r-1} \log^2 n \sum_{k=1}^{n} a_{n,k}^2 \mathbb{E}X^2 I_{\{|X| > \frac{\varepsilon n^{\lambda}}{Cb\log n}\}}\\
\le &C\sum_{n=1}^{\infty} n^{r-1} \log n \sum_{k=n}^{\infty} \mathbb{E}X^2 I_{\{\frac{\varepsilon k^{\lambda}}{Cb\log k} < |X| \le \frac{\varepsilon (k+1)^{\lambda}}{Cb\log(k+1)}\}}\\
\le &C\sum_{k=1}^{\infty} k^{r} \log k \mathbb{E}X^2 I_{\{\frac{\varepsilon k^{\lambda}}{Cb\log k} < |X| \le \frac{\varepsilon (k+1)^{\lambda}}{Cb\log(k+1)}\}}\\
\le &C\sum_{k=1}^{\infty} \frac{k^{r+2\lambda}}{\log k} \mathbb{P}\left(\frac{\varepsilon k^{\lambda}}{Cb\log k} < |X| \le \frac{\varepsilon (k+1)^{\lambda}}{Cb\log(k+1)}\right)\\
\le &C\mathbb{E}\left[|X|^{2+(r/\lambda)}(\log^+|X|)^{1+(r/\lambda)}\right] < \infty.
\end{aligned}$$

Hence from Theorem 2.1, the desired results can be obtained. □

Proof. **[Proof of Theorem 3.4]** For any $\varepsilon > 0$, each $n \geq 1$ and $1 \leq k \leq n$, let

$$X_{n,k} = a_{n,k}X_k, \ \ b_n = n^{r-1}, \ \ c_n = \frac{\varepsilon}{b\log n} \ \text{ and } \ Y_{n,k} = X_{n,k}I_{\{|X_{n,k}|\leq c_n\}},$$

where

$$b = \frac{\varepsilon^2}{(1 + 12^{-1} + e \cdot 60^{-1})\,\rho\mathbb{E}|X|^p}.$$

From the von Bahr-Esseen inequality and the condition (3.9), we get

$$\mathbb{E}\left|\sum_{k=1}^{n} a_{n,k}X_k\right|^p \leq C\sum_{k=1}^{n}|a_{n,k}|^p = o(1),$$

which implies

$$\sum_{k=1}^{n} X_{n,k} \xrightarrow{\mathbb{P}} 0.$$

From the inequality (3.11), if $\varepsilon > \varepsilon_0$, then there exists a positive constant $\varepsilon_1 > 0$ such that

$$r - \frac{\varepsilon^2}{\Delta} + \frac{\varepsilon^p\Delta^{1-p}}{2}(1+\varepsilon_1) < 0,$$

which yields

$$r - 1 - b + \frac{(b/\varepsilon)^p}{2}\left(1 + \frac{1}{12} + \frac{e}{60}\right)\rho(1+\varepsilon_1)\mathbb{E}|X|^p < -1.$$

Moreover, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1}\inf_{t>0}\exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^4 + \frac{t^5e^{tc_n}}{5!}\sum_{k=1}^{n}\mathbb{E}|Y_{n,k}|^5\right)\\
\leq&\sum_{n=1}^{\infty} n^{r-1}\inf_{t>0}\exp\left(-t\varepsilon + \frac{t^2c_n^{2-p}\mathbb{E}|X|^p}{2}\sum_{k=1}^{n}|a_{n,k}|^p\right.\\
&\qquad\qquad\qquad\qquad\left.+\frac{t^4c_n^{4-p}\mathbb{E}|X|^p}{4!}\sum_{k=1}^{n}|a_{n,k}|^p + \frac{t^5c_n^{5-p}\mathbb{E}|X|^pe^{tc_n}}{5!}\sum_{k=1}^{n}|a_{n,k}|^p\right)\\
\leq&\sum_{n=1}^{\infty} n^{r-1}\exp\left(-b\log n + \frac{(b/\varepsilon)^p}{2}\left(1 + \frac{1}{12} + \frac{e}{60}\right)\rho(1+\varepsilon_1)\mathbb{E}|X|^p\log n\right)\\
<&\infty,
\end{aligned}$$

where we take $t = (b/\varepsilon)\log n$. Furthermore, from the condition (3.10), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n}\mathbb{P}\left(|X_{n,k}| > c_n\right)\\
=&\sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n}\mathbb{P}\left(|a_{n,k}X_k| > \frac{\varepsilon}{b\log n}\right)\\
\leq&\sum_{n=1}^{\infty} n^{r}\mathbb{P}\left(|X| > \frac{\varepsilon n^{\lambda}}{Cb\log n}\right)
\end{aligned}$$

$$\leq C\sum_{k=1}^{\infty} k^{r+1}\mathbb{P}\left(\frac{\varepsilon k^{\lambda}}{Cb\log k} < |X| \leq \frac{\varepsilon (k+1)^{\lambda}}{Cb\log (k+1)}\right)$$
$$\leq C\mathbb{E}[|X|\log^{+}|X|]^{(r+1)/\lambda} < \infty$$

and

$$\sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n}\mathbb{P}\left(|X_{n,k}| > c_n\right)$$
$$= \sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n}\mathbb{P}\left(|a_{n,k}X_k| > \frac{\varepsilon}{b\log n}\right)$$
$$\leq C\sum_{n=1}^{\infty} n^{r-1}\log^{p} n\sum_{k=1}^{n}|a_{n,k}|^{p}\mathbb{E}|X|^{p}I_{\{|X|>\frac{\varepsilon n^{\lambda}}{Cb\log n}\}}$$
$$\leq C\sum_{n=1}^{\infty} n^{r-1}\log n\sum_{k=n}^{\infty}\mathbb{E}|X|^{p}I_{\{\frac{\varepsilon k^{\lambda}}{Cb\log k}<|X|\leq\frac{\varepsilon (k+1)^{\lambda}}{Cb\log (k+1)}\}}$$
$$\leq C\sum_{k=1}^{\infty} k^{r}\log k\mathbb{E}|X|^{p}I_{\{\frac{\varepsilon k^{\lambda}}{Cb\log k}<|X|\leq\frac{\varepsilon (k+1)^{\lambda}}{Cb\log (k+1)}\}}$$
$$\leq C\sum_{k=1}^{\infty} k^{r+p\lambda}(\log k)^{1-p}\mathbb{P}\left(\frac{\varepsilon k^{\lambda}}{Cb\log k} < |X| \leq \frac{\varepsilon (k+1)^{\lambda}}{Cb\log (k+1)}\right)$$
$$\leq C\mathbb{E}\left[|X|^{p+(r/\lambda)}(\log^{+}|X|)^{1+(r/\lambda)}\right] < \infty.$$

Hence from Theorem 2.1, the desired results can be obtained. □

Proof. **[Proof of Theorem 3.5]** For any $\varepsilon > 0$, each $n \geq 1$ and $1 \leq k \leq n$, let

$$X_{n,k} = a_{n,k}X_k,\;\; b_n = n^{r-1},\;\; c_n \equiv \frac{\varepsilon}{4r}(p-1) \;\text{ and }\; Y_{n,k} = X_{n,k}I_{\{|X_{n,k}|\leq c_n\}}.$$

From the von Bahr-Esseen inequality and the conditions (3.12) and (3.13), we get

$$\mathbb{E}\left|\sum_{k=1}^{n} a_{n,k}X_k\right|^{p} \leq C\sum_{k=1}^{n} a_{n,k}^{p}\mathbb{E}|X_k|^{p} \leq \frac{C}{n^{p-1}}$$

which implies

$$\sum_{k=1}^{n} X_{n,k} \xrightarrow{\mathbb{P}} 0.$$

Moreover, we have

$$\sum_{n=1}^{\infty} n^{r-1}\inf_{t>0}\exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^{2} + \frac{t^4}{4!}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^{4} + \frac{t^5 e^{tc_n}}{5!}\sum_{k=1}^{n}\mathbb{E}|Y_{n,k}|^{5}\right)$$
$$\leq \sum_{n=1}^{\infty} n^{r-1}\inf_{t>0}\exp\left(-t\varepsilon + \frac{t^2 c_n^{2-p}\mathbb{E}|X|^{p}}{2}\sum_{k=1}^{n}|a_{n,k}|^{p}\right.$$
$$\left. + \frac{t^4 c_n^{4-p}\mathbb{E}|X|^{p}}{4!}\sum_{k=1}^{n}|a_{n,k}|^{p} + \frac{t^5 c_n^{5-p}\mathbb{E}|X|^{p}e^{tc_n}}{5!}\sum_{k=1}^{n}|a_{n,k}|^{p}\right)$$

$$
\begin{aligned}
&\le \sum_{n=1}^{\infty} n^{r-1} \inf_{t>0} \exp\left(-t\varepsilon + \left(\frac{(2r/\varepsilon)^2 c_n^{2-p}\log^2 n}{2} + \frac{(2r/\varepsilon)^4 c_n^{4-p}\log^4 n}{4!}\right.\right.\\
&\qquad\qquad \left.\left. + \frac{(2r/\varepsilon)^5 c_n^{5-p}(\log n)^5 e^{tc_n}}{5!}\right)\frac{\mathbb{E}|X|^p}{n^{p-1}}\right)\\
&\le \sum_{n=1}^{\infty} n^{r-1} \exp\left(-2r\log n + \left(\frac{(2r/\varepsilon)^2 c_n^{2-p}\log^2 n}{2} + \frac{(2r/\varepsilon)^4 c_n^{4-p}\log^4 n}{4!}\right.\right.\\
&\qquad\qquad \left.\left. + \frac{(2r/\varepsilon)^5 c_n^{5-p}(\log n)^5 n^{\frac{1}{2}(p-1)}}{5!}\right)\frac{\mathbb{E}|X|^p}{n^{p-1}}\right)\\
&<\infty,
\end{aligned}
$$

where we take $t = (2r/\varepsilon)\log n$. Furthermore, from the conditions (3.12) and (3.13), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n}\mathbb{P}\left(|X_{n,k}| > c_n\right)\\
&= \sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n}\mathbb{P}\left(|a_{n,k}X_k| > \frac{\varepsilon}{4r}(p-1)\right)\\
&\le \begin{cases} C\sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n} a_{n,k}^p \mathbb{E}|X|^p I_{\{|X|>Cn^{1+\beta}\}} & \text{for } -1<\beta\le 0\\ C\sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n} a_{n,k}^p \mathbb{E}|X|^p I_{\{|X|>Cn\}} & \text{for } \beta>0 \end{cases}\\
&\le \begin{cases} C\sum_{n=1}^{\infty} n^{r-p}\sum_{k=n}^{\infty} \mathbb{E}|X|^p I_{\{Ck^{1+\beta}<|X|\le C(k+1)^{1+\beta}\}} & \text{for } -1<\beta\le 0\\ C\sum_{n=1}^{\infty} n^{r-p}\sum_{k=n}^{\infty} \mathbb{E}|X|^p I_{\{Ck<|X|\le C(k+1)\}} & \text{for } \beta>0 \end{cases}\\
&\le \begin{cases} C\sum_{k=1}^{\infty} \mathbb{E}|X|^p I_{\{Ck^{1+\beta}<|X|\le C(k+1)^{1+\beta}\}} & \text{for } -1<\beta\le 0\\ C\sum_{k=1}^{\infty} \mathbb{E}|X|^p I_{\{Ck<|X|\le C(k+1)\}} & \text{for } \beta>0 \end{cases}\\
&\le C\mathbb{E}|X|^p < \infty.
\end{aligned}
$$

Hence from Theorem 2.1, the desired results can be obtained. □

Proof. **[Proof of Theorem 3.6]** The proof is as similar as Theorem 3.5. Firstly, it is easy to check that for the case $\frac{r\beta}{1+\beta} \le -1$, we have $\frac{r}{1+\beta} > 1$. Let $p = \frac{r}{1+\beta}$ for the case $\frac{r\beta}{1+\beta} \le -1$ and $p = r+1$ for the case $\frac{r\beta}{1+\beta} > -1$. For any $\varepsilon > 0$, each $n \ge 1$ and $1 \le k \le n$,

$$
X_{n,k} = a_{n,k}X_k,\ \ b_n = n^{r-1},\ \ c_n \equiv \frac{\varepsilon}{4r}(p-1) \ \text{ and } \ Y_{n,k} = X_{n,k}I_{\{|X_{n,k}|\le c_n\}}.
$$

It is easy to check that

$$
\sum_{k=1}^{n} X_{n,k} \xrightarrow{\mathbb{P}} 0.
$$

and

$$\sum_{n=1}^{\infty} n^{r-1} \inf_{t>0} \exp\left(-t\varepsilon + \frac{t^2}{2}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^2 + \frac{t^4}{4!}\sum_{k=1}^{n}\mathbb{E}Y_{n,k}^4 + \frac{t^5 e^{tc_n}}{5!}\sum_{k=1}^{n}\mathbb{E}|Y_{n,k}|^5\right) < \infty.$$

Furthermore, from the conditions (3.12) and (3.13), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n}\mathbb{P}\left(|X_{n,k}| > c_n\right)\\
\le& C\sum_{n=1}^{\infty} n^{r-1}\sum_{k=1}^{n}\mathbb{P}\left(|X_k| > Cn^{1+\beta}k^{-\beta}\right)\\
\le& C\int_1^{\infty}\int_1^{x} x^{r-1}\mathbb{P}\left(|X| > Cx^{1+\beta}y^{-\beta}\right)dydx.
\end{aligned}$$

Let $u = x^{1+\beta}y^{-\beta}$ and $v = y$, we have

$$\begin{aligned}
&\int_1^{\infty}\int_1^{x} x^{r-1}\mathbb{P}\left(|X| > Cx^{1+\beta}y^{-\beta}\right)dydx\\
=&\frac{1}{1+\beta}\int_1^{\infty}\int_1^{u}(uv^{\beta})^{\frac{r-1}{1+\beta}}\mathbb{P}\left(|X| > Cu\right)(vu^{-1})^{\frac{\beta}{1+\beta}}dvdu\\
=&\frac{1}{1+\beta}\int_1^{\infty}u^{\frac{r-1-\beta}{1+\beta}}\mathbb{P}\left(|X| > Cu\right)\int_1^{u}v^{\frac{r\beta}{1+\beta}}dvdu.
\end{aligned}$$

For the case $\frac{r\beta}{1+\beta} = -1$, then we have

$$\begin{aligned}
&\int_1^{\infty}\int_1^{x} x^{r-1}\mathbb{P}\left(|X| > Cx^{1+\beta}y^{-\beta}\right)dydx\\
=&\frac{1}{1+\beta}\int_1^{\infty}u^{\frac{r-1-\beta}{1+\beta}}\log u\mathbb{P}\left(|X| > Cu\right)du\\
\le& C\mathbb{E}|X|^{\frac{r}{1+\beta}}\log^{+}|X|.
\end{aligned}$$

For the case $\frac{r\beta}{1+\beta} < -1$, then we have

$$\begin{aligned}
&\int_1^{\infty}\int_1^{x} x^{r-1}\mathbb{P}\left(|X| > Cx^{1+\beta}y^{-\beta}\right)dydx\\
=&-\frac{1}{r\beta+1+\beta}\int_1^{\infty}u^{\frac{r-1-\beta}{1+\beta}}\left[1-u^{\frac{r\beta}{1+\beta}+1}\right]\mathbb{P}\left(|X| > Cu\right)du\\
\le& C\mathbb{E}|X|^{\frac{r}{1+\beta}}.
\end{aligned}$$

For the case $\frac{r\beta}{1+\beta} > -1$, then we have

$$\begin{aligned}
&\int_1^{\infty}\int_1^{x} x^{r-1}\mathbb{P}\left(|X| > Cx^{1+\beta}y^{-\beta}\right)dydx\\
=&\frac{1}{r\beta+1+\beta}\int_1^{\infty}u^{\frac{r-1-\beta}{1+\beta}}\left[u^{\frac{r\beta}{1+\beta}+1}-1\right]\mathbb{P}\left(|X| > Cu\right)du\\
\le& C\mathbb{E}|X|^{r+1}.
\end{aligned}$$

Hence from Theorem 2.1, the desired results can be obtained. □

References

[1] Z. D. Bai and P. E. Cheng, *Marcinkiewicz strong laws for linear statistics*, Statist. Probab. Lett. **46** (2000), no. 2, 105-112.

[2] L. E. Baum and M. Katz, *Convergence rates in the law of large numbers*, Trans. Amer. Math. Soc. **120** (1965), 108-123.

[3] G. H. Cai, *Strong laws for weighted sums of NA random variables*, Metrika **68** (2008), no. 3, 323-331.

[4] P. Y. Chen and S. X. Gan, *Limiting behavior of weighted sums of i.i.d. random variables*, Statist. Probab. Lett. **77** (2007), no. 16, 1589-1599.

[5] Y. S. Chow, *Some convergence theorems for independent random variables*, Ann. Math. Statist. **37** (1966), 1482-1493.

[6] M. H. Du and Y. Miao, *Strong laws for weighted sums of some dependent random variables and applications*, Filomat **37** (2023), no. 18, 6161-6176.

[7] P. Erdös, *On a theorem of Hsu and Robbins*, Ann. Math. Statistics **20** (1949), 286-291.

[8] T. C. Hu, D. Szynal and A. I. Volodin, *A note on complete convergence for arrays*, Statist. Probab. Lett. **38** (1998), no. 1, 27-31.

[9] P. L. Hsu and H. Robbins, *Complete convergence and the law of large numbers*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 25-31.

[10] D. L. Li, M. B. Rao, T. F. Jiang and X. C. Wang, *Complete convergence and almost sure convergence of weighted sums of random variables*, J. Theoret. Probab. **8** (1995), no. 1, 49-76.

[11] H. Y. Liang, *Complete convergence for weighted sums of negatively associated random variables*, Statist. Probab. Lett. **48** (2000), no. 4, 317-325.

[12] H. Y. Liang and C. Su, *Complete convergence for weighted sums of NA sequences*, Statist. Probab. Lett. **45** (1999), no. 1, 85-95.

[13] M. L. Katz, *The probability in the tail of a distribution*, Ann. Math. Statist. **34** (1963), 312-318.

[14] Y. Miao and M. Y. Shao, *Complete convergence of weighted sums of martingale differences and statistical applications*, Bull. Malays. Math. Sci. Soc. **46** (2023), no. 4, Paper No. 116.

[15] Y. Miao, J. N. Shi and Z. H. Yu, *On the complete convergence and strong law for dependent random variables with general moment conditions,* Acta Math. Hungar. **168** (2022), no. 2, 425-442.

[16] F. Spitzer, *A combinatorial lemma and its application to probability theory*, Trans. Amer. Math. Soc. **82** (1956), 323-339.

[17] W. F. Stout, *Some results on the complete and almost sure convergence of linear combinations of independent random variables and martingale differences*, Ann. Math. Statist. **39** (1968), 1549-1562.

[18] W. F. Stout, *Almost sure convergence*, Academic Press, New York-London, 1974.

[19] S .H. Sung, *Complete convergence for weighted sums of random variables*, Statist. Probab. Lett. **77** (2007), no. 3, 303-311.

[20] X. J. Wang, A. T. Shen, Z. Y. Chen and S. H. Hu, *Complete convergence for weighted sums of NSD random variables and its application in the EV regression model*, TEST **24** (2015), no. 1, 166-184.

[21] Y. B. Wang, X. G. Liu and C. Su, *Equivalent conditions of complete convergence for independent weighted sums*, Sci. China Ser. A **41** (1998), no. 9, 939-949.