Filomat 39:4 (2025), 1373–1388
https://doi.org/10.2298/FIL2504373S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fractional integral inequalities for m-polynomial exponential type s-convex functions

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Abstract. In this article, we derive Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities by utilizing Riemann-Liouville fractional integrals with inclusion of m-polynomial exponential type s-convex functions. The Hölders and power mean inequalities are used to establish the results that have strong applicability across a wide range of disciplines, including stochastic processes, computer science and engineering. We employ particular functions to investigate these inequalities and displaying their 2D and 3D graphs along with relevant table values. This presentation serves as evidence supporting the validity of the results obtained. We introduce trapezoid bounds as applications serving as error estimates for the developed result.

1. Introduction

Fractional calculus offers a flexible mathematical framework for comprehending and analyzing phenomena that traditional integer-order calculus cannot adequately describe. A subfield of mathematics known as fractional calculus expands the classical operations of differentiation and integration to non-integer orders. In fractional calculus, these operations are extended to encompass fractional or non-integer orders, whereas ordinary calculus primarily concentrates on derivatives and integrals of integer orders. The importance of fractional calculus lies in its wide-ranging applicability and relevance across diverse scientific domains and practical situations. Fractional calculus is utilized in modeling physiological processes, biomedical systems, signal processing, control theory, financial mathematics and various branches of physics and engineering. The core concept of fractional calculus is to broaden the concepts of differentiation and integration beyond whole numbers.

Moreover, correlation between convexity and fractional calculus presents an appealing research field, linking classical calculus and mathematical analysis with the advancements made in fractional calculus. Convex functions exhibit characteristics such as non-decreasing slopes and always lying above their chords. Fractional calculus provides techniques to understand and represent these characteristics, expanding beyond the limitations of conventional calculus methods. The theory of convex functions falls within the

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2020 *Mathematics Subject Classification*. Primary 26D15; Secondary 26A33.

Keywords. Hermite-Hadamard inequality; Hermite-Hadamard-Fejér inequality; Hölder's inequality; Power-mean integral inequality; m-polynomial exponential type s-convexity; Riemann-Liouville fractional integrals; Trapezoid type error bounds

Received: 23 June 2024; Revised: 27 September 2024; Accepted: 31 October 2024

Communicated by Miodrag Spalević

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broader scope of convexity. A convex function is defined by having a convex epigraph. However, it stands as a crucial theory with implications across various branches of mathematics. Graphical analysis often introduces the concept of convexity in mathematics. Additionally, calculus provides a valuable tool for identifying convexity through the second-derivative test. Convexity theory provides a structured framework for constructing highly efficient, compelling and robust numerical methods to address and solve a wide array of problems across different mathematical domains. Convexity is applied across a wide range of fields including optimization [15, 38], economics [12], geometry [13, 21], finance [36], statistics [6, 9, 35], control theory [46], signal processing and machine learning [16, 39]. The growth of inequality theory was also influenced by convexity theory. The theory of convexity provides a basis for broadening and generalizing classical inequalities. It has been shown that fractional integral inequalities are among the most effective tools for advancing various fields in both pure and applied mathematics. It is useful in many different areas of mathematical research. Inequalities are used in calculus [31], trigonometry [5], physics [37], engineering [10], economics [32], finance [19], computer science [14], information theory [40] and differential equations [1]. Some significant inequalities that have strong applicability include Jensen's inequality [24], Hölder's inequality [48], mean inequality [23], Hardy-type inequality [28], Gagliardo-Nirenberg inequality [18], Ostrowski-type inequality [26, 29], Olsen inequality [41], Steklov inequality [33] and Grüss type inequalities [11] have strong applications in diverse fields of science.

The Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities [7, 27, 30, 34] are widely known inequalities relevant to a convex function [4]. The Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities are useful tools in mathematical analysis [43]. These inequalities are widely recognized within the domain of convex functions and have been thoroughly explored and broadened across different convexities, encompassing diverse scenarios and parameters. These inequalities have been utilized to tackle diverse problems within the domain of fractional calculus [20]. Mathematician have investigated inequalities involving different convexities, we refer the reader to [3, 25, 49].

In the present work, we utilize the definition of m-polynomial exponential type s-convex functions via fractional calculus to establish Hermite-Hadamard-type and Hermite-Hadamard-Fejér type inequalities. First, we need to recall the following definitions to understand the main finding of the present work.

The concept of s-convexity can be categorized into two ways with the condition $0 < s \leq 1$, which are stated as follows.

Definition 1.1. *[2] A function $\Upsilon : [0, \infty) \to \Re$ is known as s-convex in the first sense if*

$$\Upsilon(\varsigma\zeta_1 + (1-\varsigma)\zeta_2) \leq \varsigma^s \Upsilon(\zeta_1) + (1-\varsigma)^s \Upsilon(\zeta_2),$$

where $\zeta_1, \zeta_2 \in [0, \infty)$ and $\varsigma^s + (1-\varsigma)^s = 1$ for all $\varsigma \geq 0$.

Definition 1.2. *[45] A function $\Upsilon : [0, \infty) \to \Re$ is known as s-convex in the second sense if*

$$\Upsilon(\varsigma\zeta_1 + (1-\varsigma)\zeta_2) \leq \varsigma^s \Upsilon(\zeta_1) + (1-\varsigma)^s \Upsilon(\zeta_2),$$

where $\zeta_1, \zeta_2 \in [0, \infty)$ and $\varsigma + (1-\varsigma) = 1$ for all $\varsigma \geq 0$.

Definition 1.3. *[45] If $\varsigma \in [0, 1]$, ζ_1, $\zeta_2 \in I$, then the function $\Upsilon : I \subset \Re \to \Re$ is said to be exponential type convex if*

$$\Upsilon(\varsigma\zeta_1 + (1-\varsigma)\zeta_2) \leq (e^{\varsigma} - 1)\Upsilon(\zeta_1) + (e^{1-\varsigma} - 1)\Upsilon(\zeta_2), \tag{1}$$

holds.

Definition 1.4. *[8] A non-negative function $\Upsilon : I \subset \Re \to \Re$ is called as m-polynomial convex if*

$$\Upsilon(\varsigma\zeta_1 + (1-\varsigma)\zeta_2) \leq \frac{1}{m}\sum_{i=1}^{m}[1-(1-\varsigma)]^i \Upsilon(\zeta_1) + \frac{1}{m}\sum_{i=1}^{m}[1-\varsigma^i]\Upsilon(\zeta_2), \tag{2}$$

is true for all ζ_1, $\zeta_2 \in I$ and $\varsigma \in [0, 1]$ where $m \in N$.

Definition 1.5. *[42] The real valued function $\Upsilon : I \subset \mathfrak{R} \to \mathfrak{R}$ is called as m-polynomial exponential type s-convex function if the inequality*

$$\Upsilon(\varsigma\zeta_1 + (1-\varsigma)\zeta_2) \leq \frac{1}{m}\sum_{i=1}^{m}(e^{s\varsigma}-1)^i\Upsilon(\zeta_1) + \frac{1}{m}\sum_{i=1}^{m}(e^{s(1-\varsigma)}-1)^i\Upsilon(\zeta_2), \tag{3}$$

is true for all $\zeta_1, \zeta_2 \in I$ and $\varsigma \in [0,1]$ where $m \in N$ and $s \in [\ln 2.4, 1]$.

Davis stated the definition of the gamma function in [17] which is as follows.

Definition 1.6. *The gamma function is stated as*

$$\Gamma(\zeta) = \int_0^\infty \varsigma^{\zeta-1}e^{-\varsigma}d\varsigma, \tag{4}$$

for $Re(\zeta) > 0$.

Definition 1.7. *[44] If $\Upsilon \in L[\aleph_1, \aleph_2]$, then the Riemann-Liouville fractional integrals with order $\sigma > 0$ are expressed as*

$$(\daleth^{\sigma}_{\aleph_1^+}\Upsilon)(\zeta) = \frac{1}{\Gamma(\sigma)}\int_{\aleph_1}^{\zeta}(\zeta-\varsigma)^{\sigma-1}\Upsilon(\varsigma)d\varsigma, \quad (0 \leq \aleph_1 < \zeta),$$

and

$$(\daleth^{\sigma}_{\aleph_2^-}\Upsilon)(\zeta) = \frac{1}{\Gamma(\sigma)}\int_{\zeta}^{\aleph_2}(\varsigma-\zeta)^{\sigma-1}\Upsilon(\varsigma)d\varsigma, \quad (0 \leq \zeta < \aleph_2).$$

These expressions are recognized as the left- and right-sided Riemann-Liouville fractional integrals respectively.

Theorem 1.8. *(Hölder's inequality [22]) Let Ω, $\Upsilon : [\aleph_1, \aleph_2] \to \mathfrak{R}$ be functions such that $|\Omega|^p$, $|\Upsilon|^q \in L[\aleph_1, \aleph_2]$. Then the following inequality*

$$\int_{\varrho_1}^{\varrho_2}|\Omega(\varsigma)\Upsilon(\varsigma)|d\varsigma \leq \left(\int_{\varrho_1}^{\varrho_2}|\Omega(\varsigma)|^p d\varsigma\right)^{\frac{1}{p}}\left(\int_{\varrho_1}^{\varrho_2}|\Upsilon(\varsigma)|^q d\varsigma\right)^{\frac{1}{q}},$$

holds where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.9. *(Power mean inequality [22]) Let Ω, $\Upsilon : [\aleph_1, \aleph_2] \to \mathfrak{R}$ be functions such that $|\Omega|$, $|\Upsilon|^q \in L[\aleph_1, \aleph_2]$. Then the following relation*

$$\int_{\varrho_1}^{\varrho_2}|\Omega(\varsigma)\Upsilon(\varsigma)|d\varsigma \leq \left(\int_{\varrho_1}^{\varrho_2}|\Omega(\varsigma)|d\varsigma\right)^{1-\frac{1}{q}}\left(\int_{\varrho_1}^{\varrho_2}|\Omega(\varsigma)||\Upsilon(\varsigma)|^q d\varsigma\right)^{\frac{1}{q}},$$

holds where $q \geq 1$.

Lemma 1.10. *[47] Let $\Upsilon : [\aleph_1, \aleph_2] \to \mathfrak{R}$ be a function such that Υ'' exists on $(\aleph_1, \aleph_2)$. If $\Upsilon'' \in L[\aleph_1, \aleph_2]$, then the following identity*

$$\begin{aligned}
&\frac{\Upsilon(\aleph_1) + \Upsilon(\aleph_2)}{2} - \frac{\Gamma(\sigma+1)}{2(\aleph_2-\aleph_1)^{\sigma}}\left[\daleth^{\sigma}_{\aleph_1^+}\Upsilon(\aleph_2) + \daleth^{\sigma}_{\aleph_2^-}\Upsilon(\aleph_1)\right] \\
&= \frac{(\aleph_2-\aleph_1)^2}{2}\int_0^1 \frac{1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}}{\sigma+1}\Upsilon''(\varsigma\aleph_1 + (1-\varsigma)\aleph_2)d\varsigma
\end{aligned}$$

with fractional integrals exists.

Studying generalized versions of inequalities make possible to develop greater flexibility and applications across a range of mathematical and scientific fields. It often results in novel findings and deeper understanding. The purpose of this study is to investigate new refinements of Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities in terms of m-polynomial exponential type s-convex functions via Riemann-Liouville fractional integrals. Some useful applications to trapezoid bounds are provided that are needful in developing effective method for approximating areas under curves, making them valuable across various fields where integration is involved.

2. Main Result

This section focuses on employing m-polynomial exponential type s-convexity to investigate novel mean inequalities. For this purpose, we utilize Hölder's inequality and power mean inequality to obtain the following results.

Theorem 2.1. *Suppose that $\Upsilon : [0, \aleph_2] \to \Re$ is a twice differentiable function. If the function $|\Upsilon''|^q$ is integrable and m-polynomial exponential type s-convex on $[\aleph_1, \aleph_2]$ for some fixed $q \geq 1$, $0 \leq \aleph_1 < \aleph_2$ and $m \in N$ with $s \in [\ln 2.4, 1]$, then the following inequality*

$$\left|\frac{\Upsilon(\aleph_1) + \Upsilon(\aleph_2)}{2} - \frac{\Gamma(\sigma+1)}{2(\aleph_2 - \aleph_1)^\sigma}\left[\daleth^{\sigma}_{\aleph_1^+}\Upsilon(\aleph_2) + \daleth^{\sigma}_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|$$
$$\leq \frac{\sigma(\aleph_2 - \aleph_1)^2}{2(\sigma+1)(\sigma+2)}\left(\frac{1}{m}\sum_{i=1}^{m}(e^s - 1)^i\left(\left|\Upsilon''(\aleph_1)\right|^q + \left|\Upsilon''(\aleph_2)\right|^q\right)\right)^{\frac{1}{q}} \tag{5}$$

is true.

Proof. First, we consider the case $q = 1$. Utilizing Lemma 1.10, we acquire

$$\left|\frac{\Upsilon(\aleph_1) + \Upsilon(\aleph_2)}{2} - \frac{\Gamma(\sigma+1)}{2(\aleph_2 - \aleph_1)^\sigma}\left[\daleth^{\sigma}_{\aleph_1^+}\Upsilon(\aleph_2) + \daleth^{\sigma}_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|$$
$$\leq \frac{(\aleph_2 - \aleph_1)^2}{2}\int_0^1\left|\frac{1 - (1-\varsigma)^{\sigma+1} - \varsigma^{\sigma+1}}{\sigma+1}\right|\left|\Upsilon''(\varsigma\aleph_1 + (1-\varsigma)\aleph_2)\right| d\varsigma. \tag{6}$$

Here we have $(1-\varsigma)^{\sigma+1} - \varsigma^{\sigma+1} \leq 1$ for each $\varsigma \in [\aleph_1, \aleph_2]$. Since the function $\left|\Upsilon''\right|$ is m-polynomial exponential type s-convex defined on $[\aleph_1, \aleph_2]$ and by utilizing the facts $e^{s\sigma} \leq e^s$ and $e^{s(1-\sigma)} \leq e^s$ that holds for any $0 \leq \sigma \leq 1$, we arrive at

$$\left|\frac{\Upsilon(\aleph_1) + \Upsilon(\aleph_2)}{2} - \frac{\Gamma(\sigma+1)}{2(\aleph_2 - \aleph_1)^\sigma}\left[\daleth^{\sigma}_{\aleph_1^+}\Upsilon(\aleph_2) + \daleth^{\sigma}_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|$$
$$\leq \frac{(\aleph_2 - \aleph_1)^2}{2(\sigma+1)}\int_0^1\left|1 - (1-\varsigma)^{\sigma+1} - \varsigma^{\sigma+1}\right|\left(\frac{1}{m}\sum_{i=1}^{m}(e^{s\varsigma} - 1)^i\left|\Upsilon''(\aleph_1)\right| + \frac{1}{m}\sum_{i=1}^{m}(e^{s(1-\varsigma)} - 1)^i\left|\Upsilon''(\aleph_2)\right|\right)d\varsigma$$
$$\leq \frac{(\aleph_2 - \aleph_1)^2}{2n(\sigma+1)}\int_0^1\left|1 - (1-\varsigma)^{\sigma+1} - \varsigma^{\sigma+1}\right|\left(\sum_{i=1}^{m}(e^{s} - 1)^i\left|\Upsilon''(\aleph_1)\right| + \sum_{i=1}^{m}(e^{s} - 1)^i\left|\Upsilon''(\aleph_2)\right|\right)d\varsigma$$
$$= \frac{\sigma(\aleph_2 - \aleph_1)^2}{2n(\sigma+1)(\sigma+2)}\sum_{i=1}^{m}(e^s - 1)^i\left(\left|\Upsilon''(\aleph_1)\right| + \left|\Upsilon''(\aleph_2)\right|\right).$$

This confirms the result for $q = 1$.
Now, we consider the case $q > 1$. By employing Lemma 1.10 and then power mean inequality, we attain

$$\left|\frac{\Upsilon(\aleph_1) + \Upsilon(\aleph_2)}{2} - \frac{\Gamma(\sigma+1)}{2(\aleph_2 - \aleph_1)^\sigma}\left[\daleth^{\sigma}_{\aleph_1^+}\Upsilon(\aleph_2) + \daleth^{\sigma}_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|$$

$$\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\int_0^1 \left|(1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1})\right|\left|\Upsilon''(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\right|d\varsigma$$

$$\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(\int_0^1 |\left(1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}\right)|\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left|\left(1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}\right)\right|\left|\Upsilon''(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\right|^q d\varsigma\right)^{\frac{1}{q}}. \tag{7}$$

Since $\left|\Upsilon''\right|$ is m-polynomial exponential type s-convex, therefore we are able to have

$$\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2}-\frac{\Gamma(\sigma+1)}{2(\aleph_2-\aleph_1)^\sigma}\left[\daleth^\sigma_{\aleph_1^+}\Upsilon(\aleph_2)+\daleth^\sigma_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|$$

$$\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(\int_0^1 \left|\left(1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma}+1\right)\right|\right)^{1-\frac{1}{q}}\left(\int_0^1\left(1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}\right)\right.$$

$$\left.\times\left[\frac{1}{m}\sum_{i=1}^m (e^{s\varsigma}-1)^i\left|\Upsilon''(\aleph_1)\right|^q+\frac{1}{m}\sum_{i=1}^m (e^{s(1-\varsigma)}-1)^i\left|\Upsilon''(\aleph_2)\right|^q\right]d\varsigma\right)^{\frac{1}{q}}$$

$$\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(\frac{\sigma}{\sigma+2}\right)^{1-\frac{1}{q}}\left(\int_0^1\left(1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}\right)\right.$$

$$\left.\times\left[\frac{1}{m}\sum_{i=1}^m (e^{s}-1)^i\left|\Upsilon''(\aleph_1)\right|^q+\frac{1}{m}\sum_{i=1}^m (e^{s}-1)^i\left|\Upsilon''(\aleph_2)\right|^q\right]d\varsigma\right)^{\frac{1}{q}}$$

$$=\frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(\frac{\sigma}{\sigma+2}\right)^{1-\frac{1}{q}}\left(\frac{1}{m}\sum_{i=1}^m (e^{s}-1)^i\left(\left|\Upsilon''(\aleph_1)\right|^q+\left|\Upsilon''(\aleph_2)\right|^q\right)\left(\int_0^1\left(1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}\right)d\varsigma\right)\right)^{\frac{1}{q}}$$

$$=\frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(\frac{\sigma}{\sigma+2}\right)^{1-\frac{1}{q}}\left(\frac{\sigma}{\sigma+2}\right)^{\frac{1}{q}}\left(\frac{1}{m}\sum_{i=1}^m (e^{s}-1)^i\left(\left|\Upsilon''(\aleph_1)\right|^q+\left|\Upsilon''(\aleph_2)\right|^q\right)\right)^{\frac{1}{q}}$$

$$=\frac{\sigma(\aleph_2-\aleph_1)^2}{2(\sigma+1)(\sigma+2)}\left(\frac{1}{m}\sum_{i=1}^m (e^{s}-1)^i\left(\left|\Upsilon''(\aleph_1)\right|^q+\left|\Upsilon''(\aleph_2)\right|^q\right)\right)^{\frac{1}{q}}$$

The proof is done. □

Remark 2.2. *If we substitute $m = 1$ and $s = 1$ in Theorem 2.1, we get the following inequality for exponential convex functions*

$$\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2}-\frac{\Gamma(\sigma+1)}{2(\aleph_2-\aleph_1)^\sigma}\left[\daleth^\sigma_{\aleph_1^+}\Upsilon(\aleph_2)+\daleth^\sigma_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|$$

$$\leq \frac{\sigma(\aleph_2-\aleph_1)^2}{2(\sigma+1)(\sigma+2)}\left((e-1)\left(\left|\Upsilon''(\aleph_1)\right|^q+\left|\Upsilon''(\aleph_2)\right|^q\right)\right)^{\frac{1}{q}} \tag{8}$$

Example 2.3. *Here, we verify the validity of Theorem 2.1 through graphical representations. To do this, we substitute $\Upsilon(\varsigma) = e^\varsigma$ to get the following integral values*

$$\daleth^\sigma_{\aleph_1^+}e^{\aleph_2}=\frac{1}{\Gamma(\sigma)}\int_{\aleph_1}^{\aleph_2}(\aleph_2-\varsigma)^{\sigma-1}e^\varsigma d\varsigma, \tag{9}$$

and

$$\daleth^\sigma_{\aleph_2^-}e^{\aleph_1}=\frac{1}{\Gamma(\sigma)}\int_{\aleph_1}^{\aleph_2}(\varsigma-\aleph_1)^{\sigma-1}e^\varsigma d\varsigma. \tag{10}$$

By utilizing the expressions (9) *and* (10) *in Theorem 2.1, we get*

$$\begin{aligned}
&\frac{e^{\aleph_1}+e^{\aleph_2}}{2}-\frac{\sigma(\aleph_2-\aleph_1)^2}{2(\sigma+1)(\sigma+2)}\left[\frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i\left(\left|e^{\aleph_1}\right|^q+\left|e^{\aleph_2}\right|^q\right)\right]^{\frac{1}{q}}\\
&\leq\frac{\sigma}{2(\aleph_2-\aleph_1)^{\sigma}}\int_{\aleph_1}^{\aleph_2}\left[(\aleph_2-\varsigma)^{\sigma-1}+(\varsigma-\aleph_1)^{\sigma-1}\right]e^{\varsigma}d\varsigma\\
&\leq\frac{e^{\aleph_1}+e^{\aleph_2}}{2}+\frac{\sigma(\aleph_2-\aleph_1)^2}{2(\sigma+1)(\sigma+2)}\left[\frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i\left(\left|e^{\aleph_1}\right|^q+\left|e^{\aleph_2}\right|^q\right)\right]^{\frac{1}{q}}. \qquad (11)
\end{aligned}$$

By specifying the parameters $m=1$, $\aleph_1=0$, $\aleph_2=1$, $q=2$, $s=1$, *in the double inequality* (11), *we derive the following functions*

$$\begin{aligned}
p_o(\sigma)&=1.8591-\frac{3.7967\sigma}{2(\sigma+1)(\sigma+2)},\\
p_1(\sigma)&=\frac{\sigma}{2}\int_0^1\left[(1-\varsigma)^{\sigma-1}+\varsigma^{\sigma-1}\right]e^{\varsigma}d\varsigma,\\
p_2(\sigma)&=1.8591+\frac{3.7967\sigma}{2(\sigma+1)(\sigma+2)}.
\end{aligned}$$

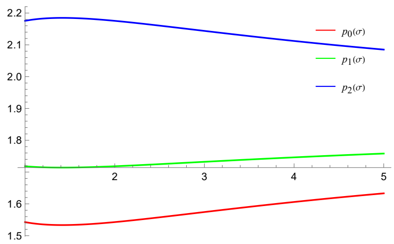

Figure 1: The 2D graph exhibiting the inequality (11) for $1 \leq \sigma \leq 3$ is shown in Figure 1.

Table 1: The Table 1 illustrates the comparative results between the double inequality in Example 2.3.

Functions	1	2	3	4	5
$p_0(\sigma)$	1.54271	1.54271	1.57435	1.60599	1.63311
$p_1(\sigma)$	1.71828	1.71828	1.73226	1.74625	1.75825
$p_2(\sigma)$	2.17549	2.17549	2.14385	2.11221	2.08509

By specifying the parameters $m=1$, $s=1$, $\sigma=1$, $q=2$, *we get the following functions*

$$\begin{aligned}
&\frac{e^{\aleph_1}+e^{\aleph_2}}{2}-\frac{(\aleph_2-\aleph_1)^2}{12}\left((e-1)|e^{\aleph_1}|^2+|e^{\aleph_2}|^2\right)^{\frac{1}{2}}\\
&\leq\frac{1}{(\aleph_2-\aleph_1)}\int_{\aleph_1}^{\aleph_2}e^{\varsigma}d\varsigma\\
&\leq\frac{e^{\aleph_1}+e^{\aleph_2}}{2}+\frac{(\aleph_2-\aleph_1)^2}{12}\left((e-1)|e^{\aleph_1}|^2+|e^{\aleph_2}|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

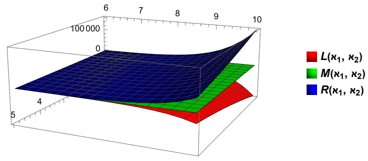

Figure 2: The 3D graph exhibiting the inequality (11) for $6 \leq \aleph_1 \leq 10$, $11 \leq \aleph_2 \leq 15$ is shown in Figure 2.

Theorem 2.4. *Suppose that $\Upsilon : [0, \aleph_2] \to \Re$ is a twice differentiable function. If the function $|\Upsilon''|^q$ is integrable and m-polynomial exponential type s-convex on $[\aleph_1, \aleph_2]$ for some fixed $q \geq 1$, $0 \leq \aleph_1 < \aleph_2$ and $m \in N$ with $s \in [\ln 2.4, 1]$, then the following inequality*

$$\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2} - \frac{\Gamma(\sigma+k)}{2(\aleph_2-\aleph_1)^{\frac{\sigma}{k}}}\left[\daleth^{\sigma}_{\aleph_1^+,k}\Upsilon(\aleph_2) + \daleth^{\sigma}_{\aleph_2^-,k}\Upsilon(\aleph_1)\right]\right|$$
$$\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(1-\frac{2}{p(\sigma+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i\left(\left|\Upsilon''(\aleph_1)\right|^q + \left|\Upsilon''(\aleph_2\right|^q)\right)^{\frac{1}{q}}. \tag{12}$$

is true with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying Lemma 1.10 and Hölder's inequality, we get

$$\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2} - \frac{\Gamma(\sigma+1)}{2(\aleph_2-\aleph_1)^{\sigma}}\left[\daleth^{\sigma}_{\aleph_1^+}\Upsilon(\aleph_2) + \daleth^{\sigma}_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|$$
$$\leq \frac{(\aleph_2-\aleph_1)^2}{2}\int_0^1\left|\frac{1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}}{\sigma+1}\right|\left|\Upsilon''(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\right|d\varsigma.$$
$$\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(\int_0^1\left(1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}\right)^p d\varsigma\right)^{\frac{1}{p}}\left(\int_0^1\left|\Upsilon''(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\right|^q d\varsigma\right)^{\frac{1}{q}}.$$

By using the relation

$$\left(1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}\right)^p \leq 1-(1-\varsigma)^{p(\sigma+1)}-\varsigma^{p(\sigma+1)},$$

for any $\varsigma \in (0,1)$ and $p \geq 1$. Since $\left|\Upsilon''\right|$ is m-polynomial exponential type s-convex on $[\aleph_1, \aleph_2]$ and the facts $e^{s\sigma} \leq e^s$ and $e^{s(1-\sigma)} \leq e^s$ are true for any $0 \leq \sigma \leq 1$, therefore for any $0 \leq \sigma \leq 1$, we get

$$\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2} - \frac{\Gamma(\sigma+1)}{2(\aleph_2-\aleph_1)^{\sigma}}\left[\daleth^{\sigma}_{\aleph_1^+}\Upsilon(\aleph_2) + \daleth^{\sigma}_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|$$
$$\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(\int_0^1\left(1-(1-\varsigma)^{p(\sigma+1)}-\varsigma^{p(\sigma+1)}\right)d\varsigma\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left(\frac{1}{m}\sum_{i=1}^{m}(e^{s\varsigma}-1)^i\left|\Upsilon''(\aleph_1)\right|^q + \frac{1}{m}\sum_{i=1}^{m}(e^{s(1-\varsigma)}-1)^i\left|\Upsilon''(\aleph_2)\right|^q\right)d\varsigma\right)^{\frac{1}{q}}$$
$$\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(1-\frac{2}{p(\sigma+1)+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i\left|\Upsilon''(\aleph_1)\right|^q + \frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i\left|\Upsilon''(\aleph_2)\right|^q\right)d\varsigma\right)^{\frac{1}{q}}$$

$$= \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(1-\frac{2}{p(\sigma+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i\left(\left|\Upsilon''(\aleph_1)\right|^q\int_0^1 d\varsigma+\left|\Upsilon''(\aleph_2\right|^q\int_0^1 d\varsigma\right)\right)^{\frac{1}{q}}.$$

$$= \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(1-\frac{2}{p(\sigma+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i\left(\left|\Upsilon''(\aleph_1)\right|^q+\left|\Upsilon''(\aleph_2\right|^q\right)\right)^{\frac{1}{q}}.$$

Hence the result is proved. □

Remark 2.5. *If we substitute $m = 1$ and $s = 1$ in Theorem 2.4 we get following relation for exponential convex functions*

$$\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2}-\frac{\Gamma(\sigma+1)}{2(\aleph_2-\aleph_1)^\sigma}\left[\daleth^{\sigma}_{\aleph_1^+}\Upsilon(\aleph_2)+\daleth^{\sigma}_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|$$
$$\le \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(1-\frac{2}{p(\sigma+1)+1}\right)^{\frac{1}{p}}\left((e-1)\left(\left|\Upsilon''(\aleph_1)\right|^q+\left|\Upsilon''(\aleph_2\right|^q\right)\right)^{\frac{1}{q}}. \tag{13}$$

Example 2.6. *Here, we verify the validity of the Theorem 2.4 through graphical representations. To do this, we substitute $\Upsilon(\varsigma)=\varsigma^2$ to get the following integral values*

$$\daleth^{\sigma}_{\aleph_1^+}\varsigma^{\aleph_2}=\frac{1}{\Gamma(\sigma)}\int_{\aleph_1}^{\aleph_2}(\aleph_2-\varsigma)^{\sigma-1}\varsigma^2 d\varsigma, \tag{14}$$

and

$$\daleth^{\sigma}_{\aleph_2^-}\varsigma^{\aleph_1}=\frac{1}{\Gamma(\sigma)}\int_{\aleph_1}^{\aleph_2}(\varsigma-\aleph_1)^{\sigma-1}\varsigma^2 d\varsigma. \tag{15}$$

By utilizing the expressions (14) *and* (15) *in Theorem 2.4, we achieve*

$$\frac{\aleph_1^2+\aleph_2^2}{2}-\frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left[1-\frac{2}{p(\sigma+1)+1}\right]^{\frac{1}{p}}\left[\frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i((2)^q+(2)^q)\right]^{\frac{1}{q}}$$
$$\le \frac{\sigma}{2(\aleph_2-\aleph_1)^\sigma}\int_{\aleph_1}^{\aleph_2}\left[(\aleph_2-\varsigma)^{\sigma-1}+(\varsigma-\aleph_1)^{\sigma-1}\right]\varsigma^2 d\varsigma$$
$$\le \frac{\aleph_1^2+\aleph_2^2}{2}+\frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left[1-\frac{2}{p(\sigma+1)+1}\right]^{\frac{1}{p}}\left[\frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i((2)^q+(2)^q)\right]^{\frac{1}{q}}. \tag{16}$$

By specifying the parameters $m = 1$, $s = 1$, $\aleph_1 = 0$, $\aleph_2 = 1$, $p = 2$, $q = 2$, in the double inequality (16), *we derive the following functions*

$$p_o(\sigma)=\frac{1}{2}-\frac{1.8538}{\sigma+1}\left[\frac{(2\sigma+1)}{2\sigma+3}\right]^{\frac{1}{2}},$$
$$p_1(\sigma)=\frac{\sigma}{2}\int_0^1\left[(1-\varsigma)^{\sigma-1}+\varsigma^{\sigma-1}\right]\varsigma^2 d\varsigma,$$
$$p_2(\sigma)=\frac{1}{2}+\frac{1.8538}{\sigma+1}\left[\frac{(2\sigma+1)}{2\sigma+3}\right]^{\frac{1}{2}}.$$

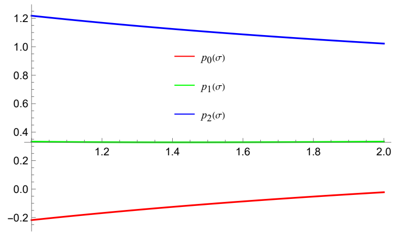

Figure 3: The 2D graph exhibiting the inequality (16) for $1 \leq \sigma \leq 2$ is shown in Figure 3.

Table 2: The Table 2 illustrates the comparative results between the double inequality in Example 2.6.

Functions	1	1.2	1.4	1.6	1.8	2
$p_0(\sigma)$	−0.21797	−0.16863	−0.12522	−0.08684	−0.05273	−0.02225
$p_1(\sigma)$	0.33333	0.32955	0.32843	0.32906	0.33083	0.33333
$p_2(\sigma)$	1.21797	1.16863	1.12522	1.08684	1.05273	1.02225

By specifying the parameters $m = 1$, $s = 1$, $\sigma = 1$, $q = 2$, $q = 2$, we get the following functions

$$\begin{aligned}&\frac{\aleph_1^2+\aleph_2^2}{2} - 0.9269(\aleph_2-\aleph_1)^2\left(\frac{3}{5}\right)^{\frac{1}{2}}\\ &\leq \frac{1}{(\aleph_2-\aleph_1)}\int_{\aleph_1}^{\aleph_2}\varsigma^2 d\varsigma\\ &\leq \frac{\aleph_1^2+\aleph_2^2}{2} + 0.9269(\aleph_2-\aleph_1)^2\left(\frac{3}{5}\right)^{\frac{1}{2}}.\end{aligned}$$

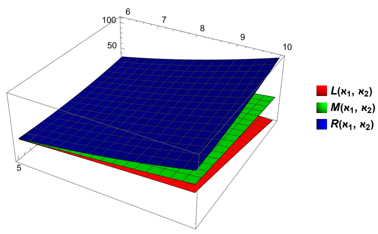

Figure 4: The 3D graph exhibiting the inequality (16) for $1 \leq \aleph_1 \leq 5$, $6 \leq \aleph_2 \leq 10$ is shown in Figure 4.

Theorem 2.7. *Suppose that $\Upsilon : [0, \aleph_2] \to \Re$ is a twice differentiable mapping. If the function $|\Upsilon''|^q$ is integrable and m-polynomial exponential type s-convex on $[\aleph_1, \aleph_2]$ for some fixed $q \geq 1$, $0 \leq \aleph_1 < \aleph_2$ and $m \in N$ with $s \in [\ln 2.4, 1]$, then the following inequality*

$$\begin{aligned}&\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2} - \frac{\Gamma(\sigma+1)}{2(\aleph_2-\aleph_1)^\sigma}\left[\daleth^\sigma_{\aleph_1^+}\Upsilon(\aleph_2) + \daleth^\sigma_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|\\ &\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left[\left(1-\frac{2}{q(\sigma+1)+1}\right)\left(\frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i\left(\left|\Upsilon''(\aleph_1)\right|^q + \left|\Upsilon''(\aleph_2)\right|^q\right)\right)\right]^{\frac{1}{q}}.\end{aligned}$$

holds.

Proof. Utilizing Lemma 1.10 and Hölder's inequality, we attain

$$\begin{aligned}
&\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2}-\frac{\Gamma(\sigma+1)}{2(\aleph_2-\aleph_1)^{\sigma}}\left[\daleth^{\sigma}_{\aleph_1^+}\Upsilon(\aleph_2)+\daleth^{\sigma}_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|\\
&\leq \frac{(\aleph_2-\aleph_1)^2}{2}\int_0^1\left|\frac{1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}}{\sigma+1}\right|\left|\Upsilon''(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\right|d\varsigma.\\
&\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left(\int_0^1 1d\varsigma\right)^{\frac{1}{p}}\left(\int_0^1\left(1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}\right)^q\left|\Upsilon''(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\right|^q d\varsigma\right)^{\frac{1}{q}}.
\end{aligned}$$

By using the relation

$$\left(1-(1-\varsigma)^{\sigma+1}-\varsigma^{\sigma+1}\right)^q\leq 1-(1-\varsigma)^{q(\sigma+1)}-\varsigma^{q(\sigma+1)}$$

for any $\varsigma\in(0,1)$ and $q\geq 1$. Since $\left|\Upsilon''\right|$ is m-polynomial exponential type s-convex on $[\aleph_1,\aleph_2]$ and the facts $e^{s\sigma}\leq e^s$ and $e^{s(1-\sigma)}\leq e^s$ are true for any $0\leq\sigma\leq 1$, therefore for any $0\leq\sigma\leq 1$, we get

$$\begin{aligned}
&\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left[\int_0^1\left(1-(1-\varsigma)^{q(\sigma+1)}-\varsigma^{q(\sigma+1)}\right)\right.\\
&\left.\times\left(\frac{1}{m}\sum_{i=1}^{m}(e^{s\varsigma}-1)^i\left|\Upsilon''(\aleph_1)\right|^q+\frac{1}{m}\sum_{i=1}^{m}(e^{s(1-\varsigma)}-1)^i\left|\Upsilon''(\aleph_2)\right|^q\right)\right]^{\frac{1}{q}}\\
&\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left[\left(1-\frac{2}{q(\sigma+1)+1}\right)\right.\\
&\left.\times\left(\frac{1}{m}\sum_{i=1}^{m}(e^{s}-1)^i\left|\Upsilon''(\aleph_1)\right|^q+\frac{1}{m}\sum_{i=1}^{m}(e^{s}-1)^i\left|\Upsilon''(\aleph_2)\right|^q\right)\right]^{\frac{1}{q}}\\
&= \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left[\left(1-\frac{2}{q(\sigma+1)+1}\right)\left(\frac{1}{m}\sum_{i=1}^{m}(e^{s}-1)^i\left(\left|\Upsilon''(\aleph_1)\right|^q+\left|\Upsilon''(\aleph_2)\right|^q\right)\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

Thus, we obtain the desired outcomes. □

Remark 2.8. *The following result is a special case for exponential convex functions, corresponding to the choice of parameters $m = 1$ and $s = 1$ in Theorem 2.7.*

$$\begin{aligned}
&\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2}-\frac{\Gamma(\sigma+1)}{2(\aleph_2-\aleph_1)^{\sigma}}\left[\daleth^{\sigma}_{\aleph_1^+}\Upsilon(\aleph_2)+\daleth^{\sigma}_{\aleph_2^-}\Upsilon(\aleph_1)\right]\right|\\
&\leq \frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left[\left(1-\frac{2}{q(\sigma+1)+1}\right)\left((e-1)\left(\left|\Upsilon''(\aleph_1)\right|^q+\left|\Upsilon''(\aleph_2)\right|^q\right)\right)\right]^{\frac{1}{q}}. \qquad (17)
\end{aligned}$$

Example 2.9. *To verify the validity of the Theorem 2.7 through graphical representations, we make substitution $\Upsilon(\varsigma)=\varsigma^3$ to get the following integral values*

$$\daleth^{\sigma}_{\aleph_1^+}\varsigma^{\aleph_2}=\frac{1}{\Gamma(\sigma)}\int_{\aleph_1}^{\aleph_2}(\aleph_2-\varsigma)^{\sigma-1}\varsigma^3 d\varsigma, \qquad (18)$$

and

$$\daleth^{\sigma}_{\aleph_2^-}\varsigma^{\aleph_1}=\frac{1}{\Gamma(\sigma)}\int_{\aleph_1}^{\aleph_2}(\varsigma-\aleph_1)^{\sigma-1}\varsigma^3 d\varsigma. \qquad (19)$$

By utilizing the expressions (18) *and* (19) *in Theorem 2.7, we achieve*

$$\begin{aligned}&\frac{\aleph_1^3+\aleph_2^3}{2}-\frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left[\frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i\left(1-\frac{2}{q(\sigma+1)+1}\right)((6\aleph_1)^q+(6\aleph_2)^q)\right]^{\frac{1}{q}}\\&\leq\frac{\sigma}{2(\aleph_2-\aleph_1)^\sigma}\int_{\aleph_1}^{\aleph_2}\left[(\aleph_2-\varsigma)^{\sigma-1}+(\varsigma-\aleph_1)^{\sigma-1}\right]\varsigma^3d\varsigma\\&\leq\frac{\aleph_1^3+\aleph_2^3}{2}+\frac{(\aleph_2-\aleph_1)^2}{2(\sigma+1)}\left[\frac{1}{m}\sum_{i=1}^{m}(e^s-1)^i\left(1-\frac{2}{q(\sigma+1)+1}\right)((6\aleph_1)^q+(6\aleph_2)^q)\right]^{\frac{1}{q}}.\end{aligned}\tag{20}$$

By specifying the parameters $m=1$, $s=1$, $\aleph_1=0$, $\aleph_2=1$, $q=2$, in the graph of the double inequality (20)*, we acquire the following functions*

$$\begin{aligned}p_o(\sigma)&=\frac{1}{2}-\frac{3.9325}{\sigma+1}\left(\frac{2\sigma+1}{2\sigma+3}\right)^{\frac{1}{2}},\\p_1(\sigma)&=\frac{\sigma}{2}\int_0^1\left[(1-\varsigma)^{\sigma-1}+\varsigma^{\sigma-1}\right]\varsigma^3d\varsigma,\\p_2(\sigma)&=\frac{1}{2}+\frac{3.9325}{\sigma+1}\left(\frac{2\sigma+1}{2\sigma+3}\right)^{\frac{1}{2}}.\end{aligned}$$

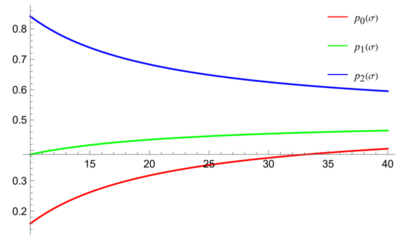

Figure 5: The 2D graph exhibiting the inequality (20) for $10\leq\sigma\leq40$ is shown in Figure 5.

Table 3: The Table 3 illustrates the comparative results between the double inequality in Example 2.9.

Functions	10	20	30	40	50
$p_0(\sigma)$	0.15839	0.31715	0.37518	0.40525	0.42365
$p_1(\sigma)$	0.38636	0.43506	0.45464	0.46516	0.47172
$p_2(\sigma)$	0.84161	0.68286	0.62483	0.59476	0.57636

By specifying the parameters $m=1$, $s=1$, $\sigma=1$, $q=2$, we get the following functions

$$\begin{aligned}&\frac{\aleph_1^3+\aleph_2^3}{2}-\frac{3(\aleph_2-\aleph_1)^2}{2}\left(1.03097(\aleph_1^2+\aleph_2^2)\right)^{\frac{1}{2}}\\&\leq\frac{1}{(\aleph_2-\aleph_1)}\int_{\aleph_1}^{\aleph_2}\varsigma^3d\varsigma\\&\leq\frac{\aleph_1^3+\aleph_2^3}{2}+\frac{3(\aleph_2-\aleph_1)^2}{2}\left(1.03097(\aleph_1^2+\aleph_2^2)\right)^{\frac{1}{2}}.\end{aligned}$$

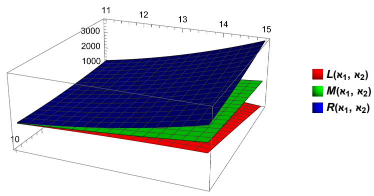

Figure 6: The 3D graph exhibiting the inequality (20) for $6 \leq \aleph_1 \leq 10$, $11 \leq \aleph_2 \leq 15$ is shown in Figure 6.

In the upcoming theorem, we establish a new general form of Fejér-type inequality in the context of m-polynomial exponential type s-convex function.

Theorem 2.10. *Let the function $\Upsilon : I = [\aleph_1, \aleph_2] \subset \Re \to \Re$ be m-polynomial exponential type s-convex with $m \in N$, $s \in [\ln 2.4, 1]$. Then the inequality*

$$\frac{(2-e^{\frac{s}{2}})m}{(e^{\frac{s}{2}}-1)(1-(e^{\frac{s}{2}}-1)^m)}\Upsilon\left(\frac{\aleph_1+\aleph_2}{2}\right)\int_{\aleph_1}^{\aleph_2} w(\zeta_1)d\zeta_1 - \Upsilon(\zeta_2)\int_{\aleph_1}^{\aleph_2} w(\zeta_1)d\zeta_1$$
$$\leq \int_{\aleph_1}^{\aleph_2} \Upsilon(\zeta_1)w(\zeta_1)d\zeta_1$$
$$\leq \frac{1}{m}\sum_{i=1}^{m}\left[\Upsilon(\aleph_1)\int_{\aleph_1}^{\aleph_2}(e^{\frac{s(\aleph_2-\zeta_1)}{\aleph_2-\aleph_1}}-1)^i w(\zeta_1)d\zeta_1 + \Upsilon(\aleph_2)\int_{\aleph_1}^{\aleph_2}(e^{\frac{s(\zeta_1-\aleph_1)}{\aleph_2-\aleph_1}}-1)^i w(\zeta_1)d\zeta_1\right]. \tag{21}$$

is satisfied, where $\zeta_1, \zeta_2 \in I$, $\varsigma \in [0, 1]$ and w is a non-negative, symmetric and integrable function.

Proof. Substituting $\varsigma = \frac{1}{2}$ into the Definition 1.5, we obtain

$$\Upsilon\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \frac{1}{m}\sum_{i=1}^{m}(e^{\frac{s}{2}}-1)^i\Upsilon(\zeta_1) + \frac{1}{m}\sum_{i=1}^{m}(e^{\frac{s}{2}}-1)^i\Upsilon(\zeta_2)$$
$$= \frac{1}{m}\sum_{i=1}^{m}(e^{\frac{s}{2}}-1)^i\left(\Upsilon(\zeta_1)+\Upsilon(\zeta_2)\right).$$

Now we substitute $\zeta_1 = \varsigma\aleph_1 + (1-\varsigma)\aleph_2$, $\zeta_2 = \varsigma\aleph_2 + (1-\varsigma)\aleph_1$, therefore

$$\Upsilon\left(\frac{\aleph_1+\aleph_2}{2}\right) \leq \frac{1}{m}\sum_{i=1}^{m}(e^{\frac{s}{2}}-1)^i\left[\Upsilon(\varsigma\aleph_1+(1-\varsigma)\aleph_2) + \Upsilon(\varsigma\aleph_2+(1-\varsigma)\aleph_1)\right].$$

Since w is non-negative, symmetric and integrable function, we have

$$\Upsilon\left(\frac{\aleph_1+\aleph_2}{2}\right)w(\varsigma\aleph_1+(1-\varsigma)\aleph_2)$$
$$\leq \frac{(e^{\frac{s}{2}}-1)(1-(e^{\frac{s}{2}}-1)^m)}{(2-e^{\frac{s}{2}})m}\left[\Upsilon(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\,w(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\right.$$
$$\left.+\Upsilon(\varsigma\aleph_2+(1-\varsigma)\aleph_1)\,w(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\right]. \tag{22}$$

Integrating the inequality (22) with respect to ς over $[0, 1]$, we acquire

$$\int_0^1 \Upsilon\left(\frac{\aleph_1+\aleph_2}{2}\right)w(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\,d\varsigma$$

$$\leq \frac{(e^{\frac{s}{2}}-1)(1-(e^{\frac{s}{2}}-1)^m)}{(2-e^{\frac{s}{2}})m}\left[\int_0^1 \Upsilon(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\, w(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\, d\varsigma \right.$$
$$\left. + \int_0^1 \Upsilon(\varsigma\aleph_2+(1-\varsigma)\aleph_1)\, w(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\, d\varsigma\right].$$

It can also be write as

$$\Upsilon\left(\frac{\aleph_1+\aleph_2}{2}\right)\int_{\aleph_2}^{\aleph_1}\frac{w(\zeta_1)}{\aleph_1-\aleph_2}d\zeta_1$$
$$\leq \frac{(e^{\frac{s}{2}}-1)(1-(e^{\frac{s}{2}}-1)^m)}{(2-e^{\frac{s}{2}})m}\left[\int_{\aleph_2}^{\aleph_1}\frac{\Upsilon(\zeta_1)w(\zeta_1)}{\aleph_1-\aleph_2}d\zeta_1+\int_{\aleph_2}^{\aleph_1}\frac{\Upsilon(\zeta_2)w(\zeta_1)}{\aleph_1-\aleph_2}d\zeta_1\right]$$

$$\frac{(2-e^{\frac{s}{2}})m}{(e^{\frac{s}{2}}-1)(1-(e^{\frac{s}{2}}-1)^m)}\Upsilon\left(\frac{\aleph_1+\aleph_2}{2}\right)\int_{\aleph_1}^{\aleph_2}w(\zeta_1)d\zeta_1-\Upsilon(\zeta_2)\int_{\aleph_1}^{\aleph_2}w(\zeta_1)d\zeta_1$$
$$\leq \int_{\aleph_1}^{\aleph_2}\Upsilon(\zeta_1)w(\zeta_1)d\zeta_1. \tag{23}$$

which is the left side of the inequality (21). Now for the right side of the inequality (21), substituting $\zeta_1=\aleph_1$, $\zeta_2=\aleph_2$ in Definition 1.5, we can write

$$\Upsilon(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\leq \frac{1}{m}\sum_{i=1}^{m}(e^{s\varsigma}-1)^i\Upsilon(\aleph_1)+\frac{1}{m}\sum_{i=1}^{m}(e^{s(1-\varsigma)}-1)^i\Upsilon(\aleph_2).$$

Since w is symmetric and integrable function, we have

$$\Upsilon(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\, w(\varsigma\aleph_1+(1-\varsigma)\aleph_2)$$
$$\leq \frac{1}{m}\sum_{i=1}^{m}(e^{s\varsigma}-1)^i\Upsilon(\aleph_1)w(\varsigma\aleph_1+(1-\varsigma)\aleph_2)+\frac{1}{m}\sum_{i=1}^{m}(e^{s(1-\varsigma)}-1)^i\Upsilon(\aleph_2)w(\varsigma\aleph_1+(1-\varsigma)\aleph_2). \tag{24}$$

Integrating the inequality (24) with respect to ς over $[0,1]$, we acquire

$$\int_0^1\Upsilon(\varsigma\aleph_1+(1-\varsigma)\aleph_2)\, w(\varsigma\aleph_1+(1-\varsigma)\aleph_2)d\varsigma$$
$$\leq \int_0^1\frac{1}{m}\sum_{i=1}^{m}(e^{s\varsigma}-1)^i\Upsilon(\aleph_1)w(\varsigma\aleph_1+(1-\varsigma)\aleph_2)d\varsigma+\int_0^1\frac{1}{m}\sum_{i=1}^{m}(e^{s(1-\varsigma)}-1)^i\Upsilon(\aleph_2)w(\varsigma\aleph_1+(1-\varsigma)\aleph_2)d\varsigma.$$

Now we substitute $\zeta_1=\varsigma\aleph_1+(1-\varsigma)\aleph_2$ to obtain

$$\int_{\aleph_2}^{\aleph_1}\frac{\Upsilon(\zeta_1)w(\zeta_1)}{\aleph_1-\aleph_2}d\zeta_1$$
$$\leq \frac{1}{m(\aleph_1-\aleph_2)}\sum_{i=1}^{m}\left[\Upsilon(\aleph_1)\int_{\aleph_2}^{\aleph_1}(e^{\frac{s(\aleph_2-\zeta_1)}{\aleph_2-\aleph_1}}-1)^i w(\zeta_1)d\zeta_1+\Upsilon(\aleph_2)\int_{\aleph_2}^{\aleph_1}(e^{\frac{s(\zeta_1-\aleph_1)}{\aleph_2-\aleph_1}}-1)^i w(\zeta_1)d\zeta_1\right]$$

$$\int_{\aleph_1}^{\aleph_2}\Upsilon(\zeta_1)w(\zeta_1)d\zeta_1$$
$$\leq \frac{1}{m}\sum_{i=1}^{m}\left[\Upsilon(\aleph_1)\int_{\aleph_1}^{\aleph_2}(e^{\frac{s(\aleph_2-\zeta_1)}{\aleph_2-\aleph_1}}-1)^i w(\zeta_1)d\zeta_1+\Upsilon(\aleph_2)\int_{\aleph_1}^{\aleph_2}(e^{\frac{s(\zeta_1-\aleph_1)}{\aleph_2-\aleph_1}}-1)^i w(\zeta_1)d\zeta_1\right]. \tag{25}$$

Which is the right side of the inequality (21). Ultimately, by combining the inequalities (23) and (25), we get

$$\begin{aligned}
&\frac{(2-e^{\frac{s}{2}})m}{(e^{\frac{s}{2}}-1)(1-(e^{\frac{s}{2}}-1)^m)}\Upsilon\left(\frac{\aleph_1+\aleph_2}{2}\right)\int_{\aleph_1}^{\aleph_2} w(\zeta_1)d\zeta_1 - \Upsilon(\zeta_2)\int_{\aleph_1}^{\aleph_2} w(\zeta_1)d\zeta_1\\
&\leq \int_{\aleph_1}^{\aleph_2} \Upsilon(\zeta_1)w(\zeta_1)d\zeta_1\\
&\leq \frac{1}{m}\sum_{i=1}^{m}\left[\Upsilon(\aleph_1)\int_{\aleph_1}^{\aleph_2}(e^{\frac{s(\aleph_2-\zeta_1)}{\aleph_2-\aleph_1}}-1)^i w(\zeta_1)d\zeta_1 + \Upsilon(\aleph_2)\int_{\aleph_1}^{\aleph_2}(e^{\frac{s(\zeta_1-\aleph_1)}{\aleph_2-\aleph_1}}-1)^i w(\zeta_1)d\zeta_1\right].
\end{aligned}$$

Hence the proof is done. □

3. Applications

In this section, we establish a connection between the main results and bounds of the trapezoidal type. In numerical computation, trapezoidal inequalities for various functions are helpful. By using these inequalities bounds and constraints on the parameters of trapezoids are obtained. Trapezoid type inequalities play a vital role for deeper understanding of trapezoids. They are extensively applicable in the fields of optimization, physics and analytical geometry. They are helpful tools for solving a lot of mathematical problems and related practical areas.

Proposition 3.1. *From Theorem 2.1, we choose $m = 1$, $s = 1$, and $\vartheta = 1$, then the following trapezoid type inequality*

$$\begin{aligned}
&\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2} - \frac{1}{(\aleph_2-\aleph_1)}\int_{\aleph_1}^{\aleph_2}\Upsilon(v)dv\right|\\
&\leq \frac{(\aleph_2-\aleph_1)^2}{12}\left[(e-1)\left(|\Upsilon''(\aleph_1)|^q + |\Upsilon''(\aleph_2)|^q\right)\right]^{\frac{1}{q}}
\end{aligned}$$

holds.

Proposition 3.2. *From Theorem 2.4, we choose $m = 1$, $s = 1$ and $\vartheta = 1$, then the following trapezoid type inequality*

$$\begin{aligned}
&\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2} - \frac{1}{(\aleph_2-\aleph_1)}\int_{\aleph_1}^{\aleph_2}\Upsilon(v)dv\right|\\
&\leq \frac{(\aleph_2-\aleph_1)^2}{4}\left(\frac{2p-1}{2p+1}\right)^{\frac{1}{p}}\left[(e-1)\left(|\Upsilon''(\aleph_1)|^q + |\Upsilon''(\aleph_2)|^q\right)\right]^{\frac{1}{q}}
\end{aligned}$$

holds.

Proposition 3.3. *From Theorem 2.7 with $m = 1$, $s = 1$, and $\vartheta = 1$, then the following trapezoid type inequality*

$$\begin{aligned}
&\left|\frac{\Upsilon(\aleph_1)+\Upsilon(\aleph_2)}{2} - \frac{1}{(\aleph_2-\aleph_1)}\int_{\aleph_1}^{\aleph_2}\Upsilon(v)dv\right|\\
&\leq \frac{(\aleph_2-\aleph_1)^2}{4}\left[(e-1)\left(\frac{2q-1}{2q+1}\right)\left(|\Upsilon''(\aleph_1)|^q + |\Upsilon''(\aleph_2)|^q\right)\right]^{\frac{1}{q}}
\end{aligned}$$

holds.

4. Conclusions

In this presented work, we have discovered novel fractional integral inequalities via m-polynomial exponential type s-convex functions. We employ the Hölder's inequality and power mean inequality to arrive at our main goals. Both of these inequalities have remarkable usefulness for studying and analyzing a variety of inequalities. These are major tools across the various fields of practical problems including mathematical economics and data analysis. Moreover, m-polynomial exponential type s-convex functions are used to provide new refinements of Hermite-Hadamard and Fejé Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals which makes it more flexible for different studies. It has wide applications in probability theory, functional analysis and finding of numerical bounds. The newly established findings are presented by 2D and 3D graphs and corresponding tables by utilizing specific functions which confirm our results. Hopefully, our novel results may encourage to study more advanced concepts and development of further generalized inequalities.

5. Acknowledgment

We are grateful to the editors and the anonymous reviewers for their valuable comments and remarks which led to significant improvements on the paper.

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