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New symmetric midpoint type inequalities for convex functions

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Abstract. In this study, we first develop symmetric quantum integral identity utilizing the derivatives and integrals of symmetric quantum types. Then, by using this identity, we establish modified versions of midpoint-type inequalities for differentiable convex functions. To obtain recent results, a few basic inequalities such as power mean and Hölder's, have been utilized. We create links between our results and previous findings in the literature taking $q \to 1$. For a better understanding and validation of the results, we present numerical results and some graphs. Finally, we provide some examples to illustrate the validity of newly obtained symmetric quantum inequalities. The concepts and methods presented in this work can inspire more investigation.

1. Introduction

Calculus is one of the main area of mathematics, which focuses on the study of functions and their ongoing modifications. Gottfried Wilhelm Leibniz and Isaac Newton made contributions to its present evolution in the 17th century. The concept of quantum calculus (a calculus without limits) was initially introduced after the 17th century by Euler (1707–1783), who established a link between mathematics and physics. In the early years of 20th century F. H. Jackson and others advanced quantum calculus. Quantum calculus deals with the analysis of difference equations and offers numerical results to numerous dynamical system issues. There are two subtypes of quantum calculus: h-calculus and q-calculus. In addition, it might be stated that quantum calculus summarizes the classical results by observing the derivative and integration of calculus when q aproaches to 1. Renowned mathematician Euler (1707–1783), who also created q-calculus, originally proposed the quantum parameter in Newton's infinite series. Jackson [1] expanded on Euler's 1910 concept to develop the quantum integral and quantum derivative of continuous functions on the interval $(0,\infty)$, also referred to as calculus without boundaries. In 1966 Al-Salam [2] conducted research on the ideas of quantum fractional and quantum Riemann-Liouville fractional integral inequalities. Kac and Cheung outlined the essential ideas of q-calculus in their book [3]; additionally, refer to [4]-[6]. Afterwards, the qunantum integral and quantum derivative on finite intervals were introduced in [7] by Tariboon and Ntouyas in particular. Furthermore, a few writers have examined the existence theory for q-boundary value

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issues in the past few years [8]–[10].
After that, it was expanded to more useful areas like particle physics, theory of relativity, discrete mathematics, hypergeometric series, quantum mechanics, and combinatorics. The sources [11] can be cited in order to completely comprehend and retain the q-calculus notions. Various quantum boundary value problems (BVPs) and initial value problems (IVPs) have recently been presented and addressed using approaches such as iteration, lower-upper solutions, and fixed-point theorems. Among the examples of such applications are oscillation on q-difference inclusions [12], multi-order q-BVPs [13], p Laplacian q-difference equations [14], q-symmetric problems [15], singular q-problems [16], q-integro-equations [17], q-delay equations [18], q-intego-equations on time scales [19], and so forth [20, 21].
It is well known that convexity in terms of integral inequalities is utilized, either directly or indirectly, in contemporary research. Due to this and its wide range of applications, the concept of convex sets has spread widely across several domains. Convex functions are useful instruments for proving a variety of inequalities. A broader range of functions, including quasi-convex functions [22–24], log convex functions [25], coordinated convex functions [26, 27], harmonically convex functions [28], GA-convex functions [29, 30], and (α, m)-convex functions [31], have been studied convex functions in the modern era.
Many well-known inequalities, including the Hermite-Hadamard, trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss, and Grüss-Cebyvsev for q-calculus, were introduced by Tariboon and Ntouyas in [32]. Alp et al. (2018), however, presented the initial updated form of the q-Hermite-Hadamard inequality in [33] by taking support lines and the notion of convex functions into account.
In the area of quantum calculus, Bermudo et al. presented another helpful strategy in 2020 in [34].
There exist substantial literature, introduced by evolution of useful inequalities in quantum calculus. Some trapezoid-type inequalities for ${}_{b_0}q-$integrals were established by Noor et al. in [35]. However, for ${}^{b_1}q$-integrals, Budak et al. provided a number of midpoint and trapezoid type inequalities in [36, 37]. A few quantum inequalities of the Simpson and Newton types are shown in [38]–[41]; for the coordinate case, see [42, 43]. Numerous mathematicians have studied the interesting field of quantum calculus. Readers with an interest can check [44]–[47].
In this paper, we will derive inequalities of midpoint type in the symmetrical sense. Da Cruz et al. were the ones who initially proposed the concept of symmetric quantum calculus [48]. In quantum mechanics, the q-symmetric calculus is crucial. It is essential to the formulation of the generalized linear Schrödinger equation, the quantum dynamical equation, and basic hypergeometric functions in quantum mechanics [49]. We can write the symmetrical sense of q and h-differentials, as stated in [3]. For $q \neq 1$ and h $\neq 0$,

$$\tilde{d}_q f(\lambda) = f(q\lambda) - f(q^{-1}\lambda), \qquad \lambda \neq a_0.$$
$$\tilde{d}_h g(\lambda) = g(\lambda + h) - g(\lambda - h), \qquad \lambda \neq a_0.$$

This work is organized as follows: Section 2 provides a brief overview of the fundamentals $\tilde{q}-$calculus. In Section 3, we develop key identity that are essential to the development of the paper's main findings with he help of $\tilde{q}$-integrals. Additionally, Symmetric q-integrals are used in this Section to derive the midpoint-type inequality for $\tilde{q}-$differentiable functions. It is also taken into consideration how the results presented in this research connect to related findings in the literature. Section 4 provide some examples, figures and numerical results. In Section 5, a summary of the results is addressed.

2. Prelimaries and defintions of $\tilde{q}$-calculus

For a real parameter $0 < q < 1$, in symmetric quantum we have $[n]_q$ as:

$$[n]_q = \frac{1 - q^{2n}}{1 - q^2}, \quad n \in \mathbb{R}.$$

Definition 2.1. *[3] Let h be a real function on J. Then*

$$\tilde{D}_q h(t) = \frac{h(qt) - h(q^{-1}t)}{(q - q^{-1})t}, \quad if \ t \in J^q \setminus \{0\}.$$

Definition 2.2. *[3] Let $a_0, a_1, \in J$ and $a_0 < a_1$. For $h : J \to \mathbb{R}$ and for $0 < q < 1$ we have,*

$$\int_{a_0}^{a_1} h(t)\, \tilde{d}_q t = \int_0^{a_1} h(t)\, \tilde{d}_q t - \int_0^{a_0} h(t)\, \tilde{d}_q t$$

where

$$\int_0^u h(t)\, \tilde{d}_q t = (q^{-1} - q)u \sum_{n=0}^{\infty} q^{2n+1} h(q^{2n+1}u)$$
$$= (1 - q^2)u \sum_{n=0}^{\infty} q^{2n} h(q^{2n+1}u), \qquad u \in J.$$

Definition 2.3. *[50] Let $h : [a_0, a_1] \to \mathbb{R}$ be a continuous function. Then, ${}_{a_0}q$-symmetric derivative at $\lambda \in [a_0, a_1]$ is given as*

$${}_{a_0}\tilde{D}_q h(\lambda) = \frac{\tilde{d}_q h(\lambda)}{\tilde{d}_q \lambda} = \frac{h(q\lambda + (1-q)a_0) - h(q^{-1}\lambda + (1 - q^{-1})a_0)}{(q - q^{-1})(\lambda - a_0)}, \qquad \lambda \neq a_0.$$

which implies that

$$\tilde{D}_q h(\lambda) = \frac{\tilde{d}_q h(\lambda)}{\tilde{d}_q \lambda} = \frac{h(q\lambda) - h(q^{-1}\lambda)}{(q - q^{-1})\lambda}, \qquad \lambda \neq 0.$$

Definition 2.4. *[50] Let $h : [a_0, a_1] \to \mathbb{R}$ be a continuous function. Then,*

$$\int_{a_0}^{\lambda} h(t)\, {}_{a_0}\tilde{d}_q t = (q^{-1} - q)(\lambda - a_0) \sum_{n=0}^{\infty} q^{2n+1} h(q^{2n+1}\lambda + (1 - q^{2n+1})a_0).$$

Here, $\lambda \in [a_0, a_1]$, or

$$\int_{a_0}^{\lambda} h(t)\, {}_{a_0}\tilde{d}_q t = (1 - q^2)(\lambda - a_0) \sum_{n=0}^{\infty} q^{2n} h(q^{2n+1}\lambda + (1 - q^{2n+1})a_0). \tag{1}$$

Consider $h(t) = 1$, then (1) becomes

$$\int_{a_0}^{\lambda} h(t)\, {}_{a_0}\tilde{d}_q t = \lambda - a_0.$$

and if $a_0 = 0$ in (1), then

$$\int_0^{\lambda} h(t)\, {}_0\tilde{d}_q t = \int_0^{\lambda} h(t) \tilde{d}_q t.$$

or

$$\int_0^{\lambda} h(t)\, {}_0\tilde{d}_q t = \int_0^{\lambda} h(t)\, \tilde{d}_q t = (1 - q^2)\lambda \sum_{n=0}^{\infty} q^{2n} h(q^{2n+1}\lambda).$$

and is called $\tilde{q}-$ *integral.*
If $p \in (a_0, \lambda)$, *then the* ${}_{a_0}\tilde{q}$*-definite integral on* $[p, \lambda]$ *is expressed as*

$$\int_{p}^{\lambda} h(t)\, {}_{a_0}\tilde{d}_q t = \int_{a_0}^{\lambda} h(t)\, {}_{a_0}\tilde{d}_q t - \int_{a_0}^{p} h(t)\, {}_{a_0}\tilde{d}_q t.$$

Definition 2.5. *Let* $h : [a_0, a_1] \to \mathbb{R}$ *be a continuous function. Then,* ${}^{a_1}q$*-symmetric derivative at* $\lambda \in [a_0, a_1]$ *is given as*

$$ {}^{a_1}\tilde{D}_q h(\lambda) = \frac{\tilde{d}_q h(\lambda)}{\tilde{d}_q \lambda} = \frac{h(q\lambda + (1-q)a_1) - h(q^{-1}\lambda + (1-q^{-1})a_1)}{(q^{-1}-q)(a_1-\lambda)}, \qquad \lambda \neq a_1. \tag{2}$$

Definition 2.6. *Let* $h : [a_0, a_1] \to \mathbb{R}$ *be a continuous function. Then, the* ${}^{a_1}\tilde{q}$*-definite integral on* $[a_0, a_1]$ *is given as*

$$\int_{\lambda}^{a_1} h(t)\, {}^{a_1}\tilde{d}_q t = (q^{-1}-q)(a_1-\lambda) \sum_{n=0}^{\infty} q^{2n+1} h(q^{2n+1}\lambda + (1-q^{2n+1})a_1)$$

for $x \in [a_0, a_1]$, *or*

$$\int_{x}^{a_1} f(t)\, {}^{a_1}\tilde{d}_q t = (1-q^2)(a_1-x) \sum_{n=0}^{\infty} q^{2n} f(q^{2n+1}x + (1-q^{2n+1})a_1). \tag{3}$$

Remark 2.7. *It is important to note this here that the monotonicity property*

$$f(\lambda) \leq g(\lambda) \implies \int_{a_0}^{a_1} f(\lambda)\, d_q\lambda \leq \int_{a_0}^{a_1} g(\lambda)\, d_q\lambda,$$

is not always accurate in q calculus $\forall\ \lambda \in [a_0, a_1]$. *A counter-example in the context of Hahn calculus may be found in [51] so that, if* $f \leq g$ *on* $[a_0, a_1]$, *then*

$$\int_{a_0}^{a_1} f(\lambda)\, d_{(q,w)}\lambda > \int_{a_0}^{a_1} g(\lambda)\, d_{(q,w)}\lambda.$$

Cardoso et al. conducted some new research in this area in [52], where they provide more generalizations of the Hahn difference operator utilizing β*-integrals. Consequently, to avoid such uncertainty in this study, we shall take into account the monotonicity property for every function inside the framework of symmetric q-calculus in every one of our theorems.*

$$\int_{a_0}^{a_1} f(\lambda)\, \tilde{d}_q\lambda \leq \int_{a_0}^{a_1} g(\lambda)\, \tilde{d}_q\lambda,$$

$\forall\ \lambda \in [a_0, a_1]$.

3. Midpoint-type inequalities for symmetric quantum calculus

In this section, we will establish a lemma and utilizing this lemma, we will develop the inequalities for midpoint-type in symmetric q-calculus for convex functions at a_1. Note that, in all theorems the monotonicity property, given in Remark (2.7), is satisfied for the function $f : [a_0, a_1] \to \mathbb{R}$.

Lemma 3.1. *For any convex* ${}^{a_1}q-$*symmetric differentiable function* $f : [a_0, a_1] \to \mathbb{R}$ *on* (a_0, a_1), *if its first* ${}^{a_1}q-$*symmetric derivative is continuous and integrable on* $[a_0, a_1]$, *then*

$$\begin{aligned} \frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x)\, {}^{a_1}\tilde{d}_q x - f\left(\frac{qa_0 + (1-q+q^2)a_1}{1+q^2}\right) &= q^2(a_1-a_0)\left[\int_0^{\frac{1}{1+q^2}} t\, {}^{a_1}\tilde{D}_q f\Big(qta_0 + (1-qt)a_1\Big)\, \tilde{d}_q t \right. \\ &\left. + \int_{\frac{1}{1+q^2}}^{1} \left(t - \frac{1}{q^2}\right) {}^{a_1}\tilde{D}_q f\Big(qta_0 + (1-qt)a_1\Big)\, \tilde{d}_q t\right] \end{aligned} \tag{4}$$

holds for $q \in (0,1)$. ***Proof:*** *Applying the definition of $\tilde{q}-$derivative from def.(2.5), we have*

$$
\begin{aligned}
{}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big) &= \frac{f\Big(q(qta_0+(1-qt)a_1+(1-q)a_1\Big)-f\Big(q^{-1}(qta_0+(1-qt)a_1)+(1-q^{-1})a_1\Big)}{(q^{-1}-q)(a_1-qta_0-(1-qt)a_1)}\\
&= \frac{f\Big(q^2ta_0+a_1(1-q^2t)\Big)-f\Big(ta_0+a_1(1-t)\Big)}{(1-q^2)(a_1-a_0)t}.
\end{aligned}
$$

From R.H.S of (4),

$$
\begin{aligned}
& q^2(a_1-a_0)\Bigg[\int_0^{\frac{1}{1+q^2}} t\ {}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\,\tilde{d}_qt\\
&+\int_{\frac{1}{1+q^2}}^1 t\,{}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\,\tilde{d}_qt-\frac{1}{q^2}\int_{\frac{1}{1+q^2}}^1 {}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\,\tilde{d}_qt\Bigg]\\
&=q^2(a_1-a_0)\Bigg[\int_0^{\frac{1}{1+q^2}} t\,{}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\,\tilde{d}_qt\\
&\quad+\int_{\frac{1}{1+q^2}}^1 t\,{}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\,\tilde{d}_qt-\frac{1}{q^2}\int_{\frac{1}{1+q^2}}^1 {}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\,\tilde{d}_qt\\
&\quad-\frac{1}{q^2}\int_0^{\frac{1}{1+q^2}} {}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\,\tilde{d}_qt+\frac{1}{q^2}\int_0^{\frac{1}{1+q^2}} {}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\,\tilde{d}_qt\Bigg]\\
&=q^2(a_1-a_0)\Bigg[\int_0^1 t\,{}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\,\tilde{d}_qt\\
&\quad-\frac{1}{q^2}\int_0^1 {}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\,\tilde{d}_qt+\frac{1}{q^2}\int_0^{\frac{1}{1+q^2}} {}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\,\tilde{d}_qt\Bigg]\\
&=q^2(a_1-a_0)\Bigg[\int_0^1 \frac{f\Big(q^2ta_0+(1-q^2t)a_1\Big)-f\Big(ta_0+(1-t)a_1\Big)}{(1-q^2)(a_1-a_0)}\tilde{d}_qt\\
&\quad-\frac{1}{q^2}\int_0^1 \frac{f\Big(q^2ta_0+(1-q^2t)a_1\Big)-f\Big(ta_0+(1-t)a_1\Big)}{(1-q^2)(a_1-a_0)t}\tilde{d}_qt\\
&\quad+\frac{1}{q^2}\int_0^{\frac{1}{1+q^2}} \frac{f\Big(q^2ta_0+(1-q^2t)a_1\Big)-f\Big(ta_0+(1-t)a_1\Big)}{(1-q^2)(a_1-a_0)t}\tilde{d}_qt\Bigg]\\
&=q^2(a_1-a_0)\Bigg[\frac{1}{(1-q^2)(a_1-a_0)}\Bigg\{\int_0^1 f\Big(q^2ta_0+(1-q^2t)a_1\Big)\tilde{d}_qt-\int_0^1 f\Big(ta_0+(1-t)a_1\Big)\tilde{d}_qt\Bigg\}\\
&\quad-\frac{1}{q^2(1-q^2)(a_1-a_0)}\Bigg\{\int_0^1 \frac{f\Big(q^2ta_0+(1-q^2t)a_1\Big)}{t}\tilde{d}_qt-\int_0^1 \frac{f\Big(ta_0+(1-t)a_1\Big)}{t}\tilde{d}_qt\Bigg\}\\
&\quad+\frac{1}{q^2(1-q^2)(a_1-a_0)}\Bigg\{\int_0^{\frac{1}{1+q^2}} \frac{f\Big(q^2ta_0+(1-q^2t)a_1\Big)}{t}\tilde{d}_qt-\int_0^{\frac{1}{1+q^2}} \frac{f\Big(ta_0+(1-t)a_1\Big)}{t}\tilde{d}_qt\Bigg\}\Bigg]
\end{aligned}
$$

Applying the definition of q̃-integrals,

$$=\frac{q^2}{1-q^2}\Bigg[(1-q^2)\sum_{n=0}^{\infty}q^{2n}f\Big(q^{2n+3}a_0+(1-q^{2n+3})a_1\Big)-(1-q^2)\sum_{n=0}^{\infty}q^{2n}f\Big(q^{2n+1}a_0+(1-q^{2n+1})a_1\Big)\Bigg]$$
$$-\frac{1}{1-q^2}\Bigg[(1-q^2)\sum_{n=0}^{\infty}f\Big(q^{2n+3}a_0+(1-q^{2n+3})a_1\Big)-(1-q^2)\sum_{n=0}^{\infty}f\Big(q^{2n+1}a_0+(1-q^{2n+1})a_1\Big)\Bigg]$$
$$+\frac{1}{1-q^2}\Bigg[(1-q^2)\sum_{n=0}^{\infty}f\Big(\frac{q^{2n+3}a_0}{1+q^2}+(1-\frac{q^{2n+3}}{1+q^2})a_1\Big)-(1-q^2)\sum_{n=0}^{\infty}f\Big(\frac{q^{2n+1}a_0}{1+q^2}+(1-\frac{q^{2n+1}}{1+q^2})a_1\Big)\Bigg]$$

$$=\frac{q^2}{1-q^2}\Bigg[(1-q^2)\sum_{n=0}^{\infty}q^{2n}f\Big(q^{2n+3}a_0+(1-q^{2n+3})a_1\Big)-(1-q^2)\sum_{n=0}^{\infty}q^{2n}f\Big(q^{2n+1}a_0+(1-q^{2n+1})a_1\Big)\Bigg]$$
$$-\frac{1}{1-q^2}\Bigg[(1-q^2)\sum_{n=0}^{\infty}f\Big(q^{2n+3}a_0+(1-q^{2n+3})a_1\Big)-(1-q^2)\sum_{n=0}^{\infty}f\Big(q^{2n+1}a_0+(1-q^{2n+1})a_1\Big)\Bigg]$$
$$+\frac{1}{1-q^2}\Bigg[(1-q^2)\sum_{n=0}^{\infty}f\Big(\frac{q^{2n+3}a_0}{1+q^2}+(1-\frac{q^{2n+3}}{1+q^2})a_1\Big)-(1-q^2)\sum_{n=0}^{\infty}f\Big(\frac{q^{2n+1}a_0}{1+q^2}+(1-\frac{q^{2n+1}}{1+q^2})a_1\Big)\Bigg]$$

$$=q^2\Bigg[\sum_{n=0}^{\infty}q^{2n}f\Big(q^{2n+3}a_0+(1-q^{2n+3})a_1\Big)-\sum_{n=0}^{\infty}q^{2n}f\Big(q^{2n+1}a_0+(1-q^{2n+1})a_1\Big)\Bigg]$$
$$-\Bigg[\sum_{n=0}^{\infty}f\Big(q^{2n+3}a_0+(1-q^{2n+3})a_1\Big)-\sum_{n=0}^{\infty}f\Big(q^{2n+1}a_0+(1-q^{2n+1})a_1\Big)\Bigg]$$
$$+[\sum_{n=0}^{\infty}f\Big(\frac{q^{2n+3}a_0}{1+q^2}+(1-\frac{q^{2n+3}}{1+q^2})a_1\Big)-\sum_{n=0}^{\infty}f\Big(\frac{q^{2n+1}a_0}{1+q^2}+(1-\frac{q^{2n+1}}{1+q^2})a_1\Big)\Bigg]$$

$$=q^2\Bigg[\frac{1}{q^2}\sum_{n=0}^{\infty}q^{2n+2}f\Big(q^{2n+2+1}a_0+(1-q^{2n+2+1})a_1\Big)-\sum_{n=0}^{\infty}q^{2n}f\Big(q^{2n+1}a_0+(1-q^{2n+1})a_1\Big)\Bigg]$$
$$+f\Big(qa_0+(1-q)a_1\Big)-f\Big(\frac{qa_0}{1+q^2}+(1-\frac{q}{1+q^2})a_1\Big)\Bigg]$$

$$=q^2\Bigg[\frac{1}{q^2}\sum_{n=0}^{\infty}q^{2m}f\Big(q^{2m+1}a_0+(1-q^{2m+1})a_1\Big)-\sum_{n=0}^{\infty}q^{2n}f\Big(q^{2n+1}a_0+(1-q^{2n+1})a_1\Big)\Bigg]$$
$$+f\Big(qa_0+(1-q)a_1\Big)-f\Big(\frac{qa_0}{1+q^2}+(1-\frac{q}{1+q^2})a_1\Big)$$
$$=q^2\Bigg[\frac{1}{q^2}\sum_{n=0}^{\infty}q^{2n}f\Big(q^{2n+1}a_0+(1-q^{2n+1})a_1\Big)-\frac{1}{q^2}f\Big(qa_0+(1-q)a_1\Big)$$
$$-\sum_{n=0}^{\infty}q^{2n}f\Big(q^{2n+1}a_0+(1-q^{2n+1})a_1\Big)\Bigg]+f\Big(qa_0+(1-q)a_1\Big)$$
$$-f\Big(\frac{qa_0+(1-q+q^2)a_1}{1+q^2}\Big)$$

$$=\sum_{n=0}^{\infty} q^{2n} f\Big(q^{2n+1}a_0 + (1-q^{2n+1})a_1\Big) - f\Big(qa_0 + (1-q)a_1\Big) - q^2 \sum_{n=0}^{\infty} q^{2n} f\Big(q^{2n+1}a_0 + (1-q^{2n+1})a_1\Big)$$
$$+ f\Big(qa_0 + (1-q)a_1\Big) - f\Big(\frac{qa_0 + (1-q+q^2)a_1}{1+q^2}\Big)$$
$$=(1-q^2)\sum_{n=0}^{\infty} q^{2n} f\Big(q^{2n+1}a_0 + (1-q^{2n+1})a_1\Big) - f\Big(\frac{qa_0 + (1-q+q^2)a_1}{1+q^2}\Big)$$
$$=\frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x)\tilde{d}_q x - f\Big(\frac{qa_0 + (1-q+q^2)a_1}{1+q^2}\Big).$$

which completes the proof.

Remark 3.2. *If q approaches 1 in Lemma (3.1), it will become Lemma (2.1) in [53].*

Theorem 3.3. *For any convex ${}^{a_1}q$−symmetric differentiable function $f : [a_0, a_1] \to \mathbb{R}$ on (a_0, a_1), if its first ${}^{a_1}q$−symmetric derivative is continuous and integrable on $[a_0, a_1]$, then*

$$\left|\frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x)\, {}^{a_1}\tilde{d}_q x - f\Big(\frac{qa_0 + (1-q+q^2)a_1}{1+q^2}\Big)\right| \leq q^2(a_1-a_0)\Bigg[\left|{}^{a_1}\tilde{D}_q f(a_0)\right|\frac{\gamma_1(q)}{(1+q^2)^3(1+q^2+q^4)} + \left|{}^{a_1}\tilde{D}_q f(a_1)\right|\frac{\gamma_2(q)}{q^2(1+q^2)^3(1+q^2+q^4)}\Bigg] \tag{5}$$

where

$$\gamma_1(q) = 2q^2 + q^3 + 3q^4 - 3q^5 + 3q^6 - 3q^7 + q^8 - q^9.$$
$$\gamma_2(q) = q^2 + q^3 + q^4 - 2q^5 + q^6 + q^7.$$

Proof: *Applying the modulus in Lemma (3.1), we have*

$$\left|\frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x) {}^{a_1}\tilde{d}_q x - f\Big(\frac{qa_0 + (1-q+q^2)a_1}{1+q^2}\Big)\right|$$
$$\leq q^2(a_1-a_0)\Bigg[\int_0^{\frac{1}{1+q^2}} t\left|{}^{a_1}\tilde{D}_q f\Big(qta_0 + (1-qt)a_1\Big)\right| \tilde{d}_q t + \int_{\frac{1}{1+q^2}}^1 \Big(\frac{1}{q^2} - t\Big)\left|{}^{a_1}\tilde{D}_q f\Big(qta_0 + (1-qt)a_1\Big)\right| \tilde{d}_q t\Bigg]$$

$$\leq q^2(a_1-a_0)\Bigg[\int_0^{\frac{1}{1+q^2}} t\Big\{(1-qt)\left|{}^{a_1}\tilde{D}_q f(a_1)\right| + qt\left|{}^{a_1}\tilde{D}_q f(a_0)\right|\Big\} \tilde{d}_q t + \int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2} - t)\Big\{(1-qt)\left|{}^{a_1}\tilde{D}_q f(a_1)\right| + qt\left|{}^{a_1}\tilde{D}_q f(a_0)\right|\Big\} \tilde{d}_q t\Bigg]$$

$$\leq q^2(a_1-a_0)\Bigg[\left|{}^{a_1}\tilde{D}_q f(a_1)\right|\int_0^{\frac{1}{1+q^2}} t(1-qt)\, \tilde{d}_q t + \left|{}^{a_1}\tilde{D}_q f(a_0)\right|\int_0^{\frac{1}{1+q^2}} qt^2\, \tilde{d}_q t$$
$$+\left|{}^{a_1}\tilde{D}_q f(a_1)\right|\int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2} - t)(1-qt)\, \tilde{d}_q t + \left|{}^{a_1}\tilde{D}_q f(a_0)\right|\int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2} - t)qt\, \tilde{d}_q t\Bigg]. \tag{6}$$

Consider,

$$\begin{aligned}\int_0^{\frac{1}{1+q^2}} t(1-qt)\,\tilde{d}_q t &= \int_0^{\frac{1}{1+q^2}} t\,\tilde{d}_q t - \int_0^{\frac{1}{1+q^2}} qt^2\,\tilde{d}_q t\\ &= (1-q^2)\frac{1}{1+q^2}\sum_{n=0}^{\infty} q^{2n}\frac{q^{2n+1}}{1+q^2} - \frac{q^3}{(1+q^2)^3(1+q^2+q^4)}\\ &= (1-q^2)\frac{q}{(1+q^2)^2}\sum_{n=0}^{\infty} q^{4n} - \frac{q^3}{(1+q^2)^3(1+q^2+q^4)}\\ &= \frac{q}{(1+q^2)^3} - \frac{q^3}{(1+q^2)^3(1+q^2+q^4)}\\ &= \frac{q+q^5}{(1+q^2)^3(1+q^2+q^4)}.\end{aligned} \tag{7}$$

$$\begin{aligned}\int_0^{\frac{1}{1+q^2}} qt^2\,\tilde{d}_q t &= q(1-q^2)\frac{1}{1+q^2}\sum_{n=0}^{\infty} q^{2n}\Big(\frac{q^{2n+1}}{1+q^2}\Big)^2\\ &= q^3(1-q^2)\frac{1}{(1+q^2)^3}\sum_{n=0}^{\infty} q^{6n}\\ &= \frac{q^3}{(1+q^2)^3(1+q^2+q^4)}.\end{aligned} \tag{8}$$

$$\begin{aligned}\int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2}-t)(1-qt)\,\tilde{d}_q t &= \int_0^1 (\frac{1}{q^2}-t)(1-qt)\,\tilde{d}_q t - \int_0^{\frac{1}{1+q^2}} (\frac{1}{q^2}-t)(1-qt)\,\tilde{d}_q t\\ &= \frac{1}{q^2}\int_0^1 1\,\tilde{d}_q t - \frac{1}{q}\int_0^1 t\,\tilde{d}_q t - \int_0^1 t\,\tilde{d}_q t + \int_0^1 qt^2\,\tilde{d}_q t - \frac{1}{q^2}\int_0^{\frac{1}{1+q^2}} 1\,\tilde{d}_q t\\ &\quad + \frac{1}{q}\int_0^{\frac{1}{1+q^2}} t\,\tilde{d}_q t + \int_0^{\frac{1}{1+q^2}} t\,\tilde{d}_q t - \int_0^{\frac{1}{1+q^2}} qt^2\,\tilde{d}_q t\\ &= \frac{1}{q^2} - \frac{1-q^2}{q}\sum_{n=0}^{\infty} q^{2n}q^{2n+1} - (1-q^2)\sum_{n=0}^{\infty} q^{2n}q^{2n+1}\\ &\quad + \frac{q^3}{1+q^2+q^4} - \frac{1}{q^2(1+q^2)} + \frac{1}{(1+q^2)^3} + \frac{q}{(1+q^2)^3}\\ &\quad - \frac{q^3}{(1+q^2)^3(1+q^2+q^4)}\\ &= \frac{q^2+q^4-2q^5+q^6}{q^2(1+q^2)^3(1+q^2+q^4)}.\end{aligned} \tag{9}$$

$$\begin{aligned}\int_{\frac{1}{1+q^2}}^{1}(\frac{1}{q^2}-t)qt\,\tilde{d}_qt &= \int_0^1(\frac{1}{q^2}-t)qt\,\tilde{d}_qt-\int_0^{\frac{1}{1+q^2}}(\frac{1}{q^2}-t)qt\,\tilde{d}_qt\\ &= \frac{1}{q}\int_0^1 t\,\tilde{d}_qt-\int_0^1 qt^2\,\tilde{d}_qt-\frac{1}{q}\int_0^{\frac{1}{1+q^2}} t\,\tilde{d}_qt-\int_0^{\frac{1}{1+q^2}} qt^2\,\tilde{d}_qt\\ &= \frac{1}{q}(1-q^2)\sum_{n=0}^{\infty}q^{2n}q^{2n+1}-q(1-q^2)\sum_{n=0}^{\infty}q^{2n}q^{(2n+1)2}-\frac{1}{(1+q^2)^3}\\ &+\frac{q^3}{(1+q^2)^3(1+q^2+q^4)}\\ &= (1-q^2)\sum_{n=0}^{\infty}q^{4n}-q^3(1-q^2)\sum_{n=0}^{\infty}q^{6n}-\frac{1}{(1+q^2)^3}+\frac{q^3}{(1+q^2)^3(1+q^2+q^4)}\\ &= \frac{2q^2+3q^4-3q^5+3q^6-3q^7+q^8-q^9}{(1+q^2)^3(1+q^2+q^4)}.\end{aligned} \tag{10}$$

Adding (7),(8),(9) and (10) and then putting in (6),we get the required result.

Remark 3.4. *If q approaches 1 in Theorem (3.3), it will become Theorem (2.2) in [53].*

Theorem 3.5. *For any convex ${}^{a_1}q$−symmetric differentiable function $f : [a_0, a_1] \to \mathbb{R}$ on (a_0, a_1), if its first ${}^{a_1}q$−symmetric derivative is continuous and integrable on $[a_0, a_1]$ and if $|{}^{a_1}\tilde{D}_q f|$ is convex on $[a_0, a_1]$.*

$$\begin{aligned}&\left|\frac{1}{a_1-a_0}\int_{a_0}^{a_1}f(x)\,{}^{a_1}\tilde{d}_qx-f\Big(\frac{qa_0+(1-q+q^2)a_1}{1+q^2}\Big)\right|\\ &\le \frac{q^2(a_1-a_0)}{(1+q^2)^{3-\frac{3}{r}}}\Bigg[\alpha_1(q)\Big\{\Big|{}^{a_1}\tilde{D}_qf(a_0)\Big|^r\theta_1(q)+\Big|{}^{a_1}\tilde{D}_qf(a_1)\Big|^r\theta_2(q)\Big\}^{\frac{1}{r}}\\ &+\alpha_2(q)\Big\{\Big|{}^{a_1}\tilde{D}_qf(a_0)\Big|^r\theta_3(q)+\Big|{}^{a_1}\tilde{D}_qf(a_1)\Big|^r\theta_4(q)\Big\}^{\frac{1}{r}}\Bigg]\end{aligned} \tag{11}$$

where

$$\alpha_1(q) = q^{1-\frac{1}{r}}.$$
$$\alpha_2(q) = (1+2q^2-2q^3+q^4-q^5)^{1-\frac{1}{r}}.$$
$$\theta_1(q) = \int_0^{\frac{1}{1+q^2}} qt^2\,\tilde{d}_qt = \frac{q^3}{(1+q^2)^3(1+q^2+q^4)}.$$
$$\theta_2(q) = \int_0^{\frac{1}{1+q^2}} t(1-qt)\,\tilde{d}_qt = \frac{q+q^5}{(1+q^2)^3(1+q^2+q^4)}.$$
$$\theta_3(q) = \int_{\frac{1}{1+q^2}}^1(\frac{1}{q^2}-t)qt\,\tilde{d}_qt = \frac{2q^2+3q^4-3q^5+3q^6-3q^7+q^8-q^9}{(1+q^2)^3(1+q^2+q^4)}.$$

and

$$\theta_4(q) = \int_{\frac{1}{1+q^2}}^1(\frac{1}{q^2}-t)(1-qt)\,\tilde{d}_qt = \frac{q^2+q^4-2q^5+q^6}{q^2(1+q^2)^3(1+q^2+q^4)}.$$

Proof: *Taking the modulus in Lemma (3.1), we have*

$$\left|\frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x)\,{}^{a_1}\tilde{d}_q x - f\Big(\frac{qa_0+(1-q+q^2)a_1}{1+q^2}\Big)\right|$$
$$\leq q^2(a_1-a_0)\Big[\int_0^{\frac{1}{1+q^2}} t\left|{}^{a_1}\tilde{D}_q f(qta_0+(1-qt)a_1)\right|\tilde{d}_q t + \int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2}-t)\left|{}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\right|\tilde{d}_q t\Big]$$

Using power mean inequality,

$$\leq q^2(a_1-a_0)\Big[\Big(\int_0^{\frac{1}{1+q^2}} t\,\tilde{d}_q t\Big)^{1-\frac{1}{r}}\Big(\int_0^{\frac{1}{1+q^2}} t\left|{}^{a_1}\tilde{D}_q f(qta_0+(1-qt)a_1)\right|^r\tilde{d}_q t\Big)^{\frac{1}{r}}$$
$$+\Big(\int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2}-t)\,\tilde{d}_q t\Big)^{1-\frac{1}{r}}\Big(\int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2}-t)\left|{}^{a_1}\tilde{D}_q f(qta_0+(1-qt)a_1)\right|^r\tilde{d}_q t\Big)^{\frac{1}{r}}\Big]$$

$$\leq q^2(a_1-a_0)\Big[\frac{q^{1-\frac{1}{r}}}{(1+q^2)^{3-\frac{3}{r}}}\Big(\int_0^{\frac{1}{1+q^2}} t\Big\{qt\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r+(1-qt)\left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r\Big\}\tilde{d}_q t\Big)^{\frac{1}{r}}$$
$$+\frac{(1+2q^2-2q^3+q^4-q^5)^{1-\frac{1}{r}}}{(1+q^2)^{3-\frac{3}{r}}}\Big(\int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2}-t)\Big\{qt\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r+(1-qt)\left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r\Big\}\tilde{d}_q t\Big)^{\frac{1}{r}}\Big]$$
$$\leq \frac{q^2(a_1-a_0)}{(1+q^2)^{3-\frac{3}{r}}}\Big[q^{1-\frac{1}{r}}\Big(\Big\{\int_0^{\frac{1}{1+q^2}} qt^2\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r+t(1-qt)\left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r\Big\}\tilde{d}_q t\Big)^{\frac{1}{r}}$$
$$+(1+2q^2-2q^3+q^4-q^5)^{1-\frac{1}{r}}\Big(\Big\{\int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2}-t)qt\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r+(\frac{1}{q^2}-t)(1-qt)\left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r\Big\}\tilde{d}_q t\Big)^{\frac{1}{r}}\Big]$$

$$\leq \frac{q^2(a_1-a_0)}{(1+q^2)^{3-\frac{3}{r}}}\Big[\alpha_1(q)\Big\{\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r\int_0^{\frac{1}{1+q^2}} qt^2\,\tilde{d}_q t+\left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r\int_0^{\frac{1}{1+q^2}} t(1-qt)\,\tilde{d}_q t\Big\}^{\frac{1}{r}}$$
$$+\alpha_2(q)\Big\{\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r\int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2}-t)qt\,\tilde{d}_q t+\left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r\int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2}-t)(1-qt)\,\tilde{d}_q t\Big\}^{\frac{1}{r}}\Big]. \quad (12)$$

Using $(7-10)$ *from Theorem (3.3) in (12), we get the required result.*

Theorem 3.6. *For any convex* ${}^{a_1}q$*−symmetric differentiable function* $f:[a_0,a_1]\to\mathbb{R}$ *on* (a_0,a_1)*, if its first* ${}^{a_1}q$*−symmetric derivative is continuous and integrable on* $[a_0,a_1]$ *and if* $|{}^{a_1}\tilde{D}_q f|$ *is convex on* $[a_0,a_1]$*.Then*

$$\left|\frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x)\,{}^{a_1}\tilde{d}_q x - f\Big(\frac{qa_0+(1-q+q^2)a_1}{1+q^2}\Big)\right|$$
$$\leq q^2(a_1-a_0)\Big[\zeta_1(q)\Big\{\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r\upsilon_1(q)+\left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r\upsilon_2(q)\Big\}^{\frac{1}{r}} \quad (13)$$
$$+\zeta_2(q)\Big\{\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r\upsilon_3(q)+\left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r\upsilon_4(q)\Big\}^{\frac{1}{r}}\Big]$$

where

$$\zeta_1(q) = \int_0^{\frac{1}{1+q^2}} t^s \, \tilde{d}_q t = \frac{q^s(1-q^2)}{(1+q^2)^{s+1}(1-q^{2s+2})}.$$
$$\zeta_2(q) = \int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2} - t)^s \, \tilde{d}_q t)^{\frac{1}{s}}.$$
$$\upsilon_1(q) = \int_0^{\frac{1}{1+q^2}} qt \, \tilde{d}_q t = \frac{q^2}{(1+q^2)^3}.$$
$$\upsilon_2(q) = \int_0^{\frac{1}{1+q^2}} (1-qt) \, \tilde{d}_q t = \frac{1+q^2+q^4}{(1+q^2)^3}.$$
$$\upsilon_3(q) = \int_{\frac{1}{1+q^2}}^1 qt \, \tilde{d}_q t = \frac{2q^4+q^6}{(1+q^2)^3}.$$
$$\upsilon_4(q) = \int_{\frac{1}{1+q^2}}^1 (1-qt) \, \tilde{d}_q t = \frac{q2}{(1+q^2)^3}. \qquad (s^{-1}+r^{-1}=1).$$

Proof *Using the absolute value of Lemma* (3.1), *we have*

$$\left|\frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x)\, {}^{a_1}\tilde{d}_q x - f\Big(\frac{qa_0+(1-q+q^2)a_1}{1+q^2}\Big)\right|$$
$$\leq q^2(a_1-a_0)\Bigg[\int_0^{\frac{1}{1+q^2}} t \left|{}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\right| \tilde{d}_q t + \int_{\frac{1}{1+q^2}}^1 (\frac{1}{q^2}-t)\left|{}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\right| \tilde{d}_q t\Bigg]$$

By Hölder's inequality,

$$\leq q^2(a_1-a_0)\Bigg[\Big\{\int_0^{\frac{1}{1+q^2}} t^s \, \tilde{d}_q t\Big\}^{\frac{1}{s}}\Big\{\int_0^{\frac{1}{1+q^2}} \left|{}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\right|^r \tilde{d}_q t\Big\}^{\frac{1}{r}}$$
$$+\Big\{\int_{\frac{1}{1+q^2}}^1 \Big(\frac{1}{q^2}-t\Big)^s \tilde{d}_q t\Big\}^{\frac{1}{s}}\Big\{\int_{\frac{1}{1+q^2}}^1 \left|{}^{a_1}\tilde{D}_q f\Big(qta_0+(1-qt)a_1\Big)\right|^r \tilde{d}_q t\Big\}^{\frac{1}{r}}\Bigg]$$
$$\leq q^2(a_1-a_0)\Bigg[\Big\{\int_0^{\frac{1}{1+q^2}} t^s \, \tilde{d}_q t\Big\}^{\frac{1}{s}}\Big\{\int_0^{\frac{1}{1+q^2}} \Big(qt\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r + (1-qt)\left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r\Big)\tilde{d}_q t\Big\}^{\frac{1}{r}}$$
$$+\Big\{\int_{\frac{1}{1+q^2}}^1 \Big(\frac{1}{q^2}-t\Big)^s \tilde{d}_q t\Big\}^{\frac{1}{s}}\Big\{\int_{\frac{1}{1+q^2}}^1 (qt\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r + (1-qt)\left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r\Big)\tilde{d}_q t\Big\}^{\frac{1}{r}}\Bigg]$$
$$\leq q^2(a_1-a_0)\Bigg[\Big\{\int_0^{\frac{1}{1+q^2}} t^s \, \tilde{d}_q t\Big\}^{\frac{1}{s}}\Big\{\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r \int_0^{\frac{1}{1+q^2}} qt \, \tilde{d}_q t + \left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r \int_0^{\frac{1}{1+q^2}} (1-qt) \, \tilde{d}_q t\Big\}^{\frac{1}{r}}$$
$$+\Big\{\int_{\frac{1}{1+q^2}}^1 \Big(\frac{1}{q^2}-t\Big)^s \tilde{d}_q t\Big\}^{\frac{1}{s}}\Big\{\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r \int_{\frac{1}{1+q^2}}^1 qt \, \tilde{d}_q t + \left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r \int_{\frac{1}{1+q^2}}^1 (1-qt) \, \tilde{d}_q t\Big\}^{\frac{1}{r}}\Bigg]. \tag{14}$$

Now consider,

$$\int_0^{\frac{1}{1+q^2}} t^s \, \tilde{d}_q t = (1-q^2)\frac{1}{1+q^2}\sum_{n=0}^{\infty} q^{2n}\Big(\frac{q^{2n+1}}{1+q^2}\Big)^s$$
$$= \frac{q^s(1-q^2)}{(1+q^2)^{s+1}(1-q^{2s+2})}. \tag{15}$$

$$\int_0^{\frac{1}{1+q^2}} qt\,\tilde{d}_q t = q\frac{(1-q^2)}{(1+q^2)}\sum_{n=0}^{\infty} q^{2n}\Big(\frac{q^{2n+1}}{1+q^2}\Big)$$
$$= \frac{q^2}{(1+q^2)^3}. \tag{16}$$

$$\int_0^{\frac{1}{1+q^2}} (1-qt)\,\tilde{d}_q t = \frac{1}{1+q^2} - \frac{q^2}{(1+q^2)^3}$$
$$= \frac{1+q^2+q^4}{(1+q^2)^3}. \tag{17}$$

$$\int_{\frac{1}{1+q^2}}^{1} qt\,\tilde{d}_q t = q(1-q^2)\sum_{n=0}^{\infty} q^{2n}(q^{2n+1}) - \frac{q^2}{(1+q^2)^3}$$
$$= \frac{2q^4+q^6}{(1+q^2)^3}. \tag{18}$$

$$\int_{\frac{1}{1+q^2}}^{1} (1-qt)\,\tilde{d}_q t = 1 - \frac{q^2}{1+q^2} - \frac{1}{1+q^2} + \frac{q^2}{(1+q^2)^3}$$
$$= \frac{q^2}{(1+q^2)^3}. \tag{19}$$

Putting values from (15-19) in (14) we get desired result.

Remark 3.7. *If q approaches 1 in Theorem (3.6), it will become Theorem (2.3) in [53].*

4. Applications

Now we provide some examples which verify our main results.

Example 4.1. *Consider a convex function* $f : [a_0, a_1] \to \mathbb{R}$ *given by* $f(z) = z^2$.
Using Theorem (3.3), left hand side of (5) becomes

$$\left|\frac{1}{1-0}\int_0^1 (z^2)\ {}^1\tilde{d}_q z - f\Big(\frac{q(0)+(1-q+q^2)(1)}{1+q^2}\Big)\right|$$
$$= \left|\{(1-q^2)\sum_{n=0}^{\infty} q^{2n}(1-q^{2n+1})^2\} - \Big(\frac{1-q+q^2}{1+q^2}\Big)^2\right|$$
$$= \left|1 - \frac{2q}{1+q^2} + \frac{q^2}{1+q^2+q^4} - (\frac{1-q+q^2}{1+q^2})^2\right|.$$

Moreover,

Table 1: Numerical results of γ_1, γ_2 for different values of q.

	$q = 0.1$	$q = 0.2$	$q = 0.3$	$q = 0.4$	$q = 0.5$
γ_1	0.021	0.091	0.225	0.437	0.744
γ_2	0.011	0.049	0.121	0.234	0.398
$\frac{\gamma_1}{(1+q^2)^3(1+q^2+q^4)}$	0.020	0.077	0.158	0.236	0.290
$\frac{\gamma_2}{q^2(1+q^2)^3(1+q^2+q^4)}$	1.1	1.065	0.952	0.790	0.640

$$\left|{}^{1}\tilde{D}_q f(z)\right| = \left|\frac{5z-1}{2}\right|.$$

$$\left|{}^{1}\tilde{D}_q f(0)\right| = 0.5.$$

$$\left|{}^{1}\tilde{D}_q f(1)\right| = 2.$$

Now, it's basic to check that

$$\left|\frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x)\,{}^{a_1}\tilde{d}_q x - f\Big(\frac{qa_0+(1-q+q^2)a_1}{1+q^2}\Big)\right| \leq q^2(a_1-a_0)\Big[\left|{}^{a_1}\tilde{D}_q f(a_0)\right|\frac{\gamma_1(q)}{(1+q^2)^3(1+q^2+q^4)} + \left|{}^{a_1}\tilde{D}_q f(a_1)\right|\frac{\gamma_2(q)}{q^2(1+q^2)^3(1+q^2+q^4)}\Big].$$

Also, the graph is presented in figure 1.

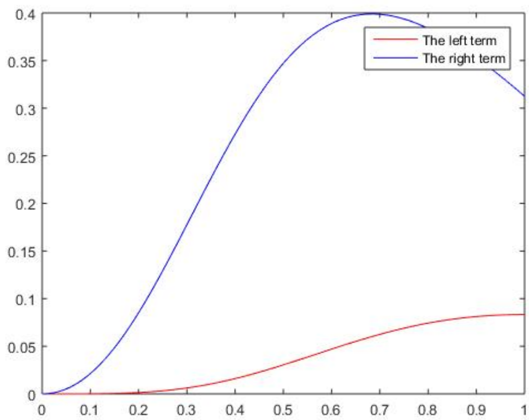

Figure 1: An example to inequality (5).

Example 4.2. *Consider a convex function $f : [a_0, a_1] \to \mathbb{R}$ given by $f(z) = z^2$ and taking r=2. Using Theorem (3.5), left hand side of (11) becomes*

$$\begin{aligned}
&\left|\frac{1}{1-0}\int_0^1 (z^2)\ {}^{1}\tilde{d}_q z - f\Big(\frac{q(0)+(1-q+q^2)(1)}{1+q^2}\Big)\right| \\
&= \left|\Big\{(1-q^2)\sum_{n=0}^{\infty} q^{2n}(1-q^{2n+1})^2\Big\} - \Big(\frac{1-q+q^2}{1+q^2}\Big)^2\right| \\
&= \left|1 - \frac{2q}{1+q^2} + \frac{q^2}{1+q^2+q^4} - (\frac{1-q+q^2}{1+q^2})^2\right|.
\end{aligned}$$

Moreover,

$$\left|{}^{1}\tilde{D}_q f(z)\right| = \left|\frac{5z-1}{2}\right|.$$

Table 2: Numerical results of $\alpha_1, \alpha_2, \theta_1, \theta_2, \theta_3, \theta_4$ for different values of q.

	$q = 0.1$	$q = 0.2$	$q = 0.3$	$q = 0.4$	$q = 0.5$	$q = 0.6$
α_1	0.316	0.447	0.547	0.632	0.707	0.774
α_2	1.009	1.032	1.063	1.098	1.131	1.157
θ_1	0.0009	0.068	0.018	0.035	0.048	0.0631
θ_2	0.096	0.170	0.212	0.227	0.207	0.197
θ_3	0.019	0.070	0.139	0.207	0.241	0.274
θ_4	0.969	0.875	0.734	0.587	0.414	0.310

$$\left| {}^{1}\tilde{D}_q f(0) \right| = 0.5.$$

$$\left| {}^{1}\tilde{D}_q f(1) \right| = 2.$$

Now, it's basic to check that

$$\begin{aligned}
&\left| \frac{1}{a_1 - a_0} \int_{a_0}^{a_1} f(x)\, {}^{a_1}\tilde{d}_q x - f\left(\frac{q a_0 + (1 - q + q^2) a_1}{1 + q^2} \right) \right| \\
&\leq \frac{q^2 (a_1 - a_0)}{(1 + q^2)^{3 - \frac{3}{r}}} \left[\alpha_1(q) \left\{ \left| {}^{a_1}\tilde{D}_q f(a_0) \right|^r \theta_1(q) + \left| {}^{a_1}\tilde{D}_q f(a_1) \right|^r \theta_2(q) \right\}^{\frac{1}{r}} \right. \\
&\left. + \alpha_2(q) \left\{ \left| {}^{a_1}\tilde{D}_q f(a_0) \right|^r \theta_3(q) + \left| {}^{a_1}\tilde{D}_q f(a_1) \right|^r \theta_4(q) \right\}^{\frac{1}{r}} \right].
\end{aligned}$$

Also, the graph is presented in figure 2.

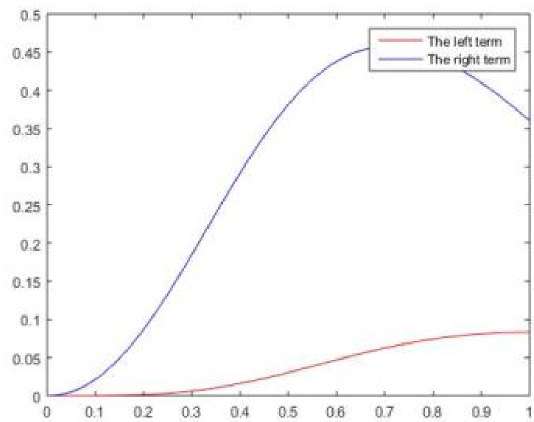

Figure 2: An example to inequality (11).

Example 4.3. *Consider a convex function $f : [a_0, a_1] \to \mathbb{R}$ given by $f(z) = z^2$ and taking s=r=2.*

Using Theorem (3.6), left hand side of (13) becomes

$$\left|\frac{1}{1-0}\int_0^1 (z^2)\ {}^1\tilde{d}_q z - f\Big(\frac{q(0)+(1-q+q^2)(1)}{1+q^2}\Big)\right|$$
$$= \left|\Big\{(1-q^2)\sum_{n=0}^{\infty} q^{2n}(1-q^{2n+1})^2\Big\} - \Big(\frac{1-q+q^2}{1+q^2}\Big)^2\right|$$
$$= \left|1 - \frac{2q}{1+q^2} + \frac{q^2}{1+q^2+q^4} - \Big(\frac{1-q+q^2}{1+q^2}\Big)^2\right|.$$

Moreover,

Table 3: Numerical results of $\zeta_1, \zeta_2, \upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4$ for different values of q.

	$q = 0.1$	$q = 0.2$	$q = 0.3$	$q = 0.4$	$q = 0.5$	$q = 0.6$
ζ_1	0.009	0.034	0.063	0.086	0.097	0.105
ζ_2	0.986	0.932	0.831	0.692	0.534	0.481
υ_1	0.009	0.035	0.069	0.102	0.128	0.143
υ_2	0.980	0.926	0.847	0.759	0.6717	0.592
υ_3	0.0001	0.002	0.012	0.035	0.071	0.121
υ_4	0.009	0.035	0.069	0.102	0.128	0.143

$$\left|{}^1\tilde{D}_q f(z)\right| = \left|\frac{5z-1}{2}\right|.$$

$$\left|{}^1\tilde{D}_q f(0)\right| = 0.5.$$

$$\left|{}^1\tilde{D}_q f(1)\right| = 2.$$

Now, it's basic to check that

$$\left|\frac{1}{a_1-a_0}\int_{a_0}^{a_1} f(x)\,{}^{a_1}\tilde{d}_q x - f\Big(\frac{qa_0+(1-q+q^2)a_1}{1+q^2}\Big)\right|$$
$$\le q^2(a_1-a_0)\Bigg[\zeta_1(q)\Big\{\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r \upsilon_1(q) + \left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r \upsilon_2(q)\Big\}^{\frac{1}{r}}$$
$$+ \zeta_2(q)\Big\{\left|{}^{a_1}\tilde{D}_q f(a_0)\right|^r \upsilon_3(q) + \left|{}^{a_1}\tilde{D}_q f(a_1)\right|^r \upsilon_4(q)\Big\}^{\frac{1}{r}}\Bigg].$$

Also, the graph is presented in figure 3.

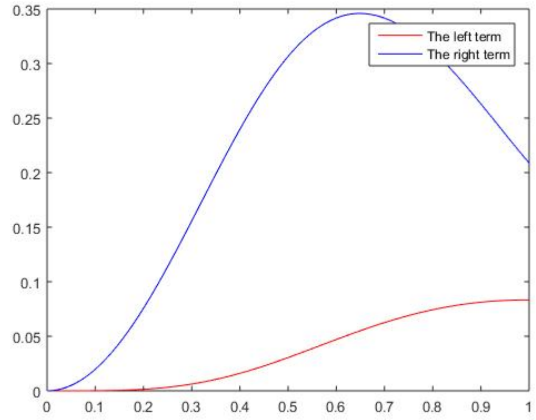

Figure 3: An example to inequality (13).

5. Conclusion

In this article, new variants of symmetric midpoint-type inequalities for differentiable convex functions in the frame work $\tilde{q}$-calculus are developed. We also used power mean and Hölder's inequalities to find symmetric q-type midpoint-type inequalities in consideration of $\tilde{q}$-differentiable convex mappings. It is an interesting concept that the other mathematician in this area can derive new inequalities for symmetric quantum coordinated convex mappings. Examples and numerical results are also provided to validate our main outcomes.

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