Filomat 39:4 (2025), 1181–1196
https://doi.org/10.2298/FIL2504181J

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Three-space properties in paratopological gyrogroups

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Abstract. In this paper, we consider the three-space properties in paratopological gyrogroups. The following are the established conclusions: (1) metrizability of compact (resp., sequentially compact, countably compact) subsets is a three-space property in the class of k-gentle paratopological gyrogroups; (2) let G be a strongly paratopological gyrocommutative gyrogroup and let H be a second-countable invariant topological subgyrogroup of G. If the paratopological gyrogroup G/H has a countable network, then so does G; (3) let H be a compact strongly L-subgyrogroup of a paratopological gyrogroup G. If H and G/H have countable tightness, then G has countable tightness.

1. Introduction

A structure similar to a group, referred to as a gyrogroup, is characterized by the absence of the associative law (as defined in Definition 2.1). A *paratopological gyrogroup* G is a gyrogroup G with a topology such that its binary operation is jointly continuous [4]. If G is a paratopological gyrogroup and the inverse operation of G is continuous, then G is a *topological gyrogroup* [3]. Significant recent advancements in the study of topological gyrogroups and paratopological gyrogroups are detailed in the review article [3, 4, 10, 15].

A topological-algebraic property $\mathcal{P}$ is a *three-space property* in the class of topological (paratopological) groups provided that for every topological (paratopological) group G and a closed invariant subgroup N of G, the fact that both N and G/N have $\mathcal{P}$ implies that G also has $\mathcal{P}$. Now there are many conclusions about the three-space properties of topological groups and partopological groups [2]. In 2006, M. Bruguera and M. Tkachenko [9] studied some properties of compact, countably compact, pseudocompact, and functionally bounded sets which are preserved or destroyed when taking extensions of topological groups. In 2010, O. Ravsky [21] proved that being a topological group is a three-space property in the class of paratopological groups. In 2015, S. Lin, F. Lin, and L.H. Xie conducted a study on the convergence phenomena observed

2020 *Mathematics Subject Classification*. Primary 22A22; Secondary 20N05, 22A30.
Keywords. Three-space property, paratopological group, countable networks, metrizability, second-countable.
Received: 03 June 2024; Revised: 29 October 2024; Accepted: 06 November 2024
Communicated by Ljubiša D. R. Kočinac
Research supported by the Natural Science Foundation of Guangdong Province under Grant (Nos. 2021A1515010381; 2020A1515110458). The Innovation Project of Department of Education of Guangdong Province (No. 2022KTSCX145), and Scientific research project of Guangzhou Panyu Polytechnic (No. 2022KJ02).
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within the extensions of topological groups, as referenced in [18]. In 2015, the research by L.H. Xie and S. Lin, as cited in [27], demonstrated that characteristics like local compactness, compactness, and connectedness can be considered as of three-space properties in regular paratopological groups. Additionally, in 2017, the work of M. Fernández and I. Sánchezin, as referenced in [12], showed that if H is an invariant topological subgroup of a paratopological group G such that H is second-countable and G/H has countable network, then G has countable network as well.

Given that gyrogroups are an extension of groups, it prompts us to investigate the potential extension of three-space properties in paratopological groups to paratopological gyrogroups in terms of their topological-algebraic characteristics. As indicated in [17], the understanding of three-space properties within paratopological gyrogroups is not as comprehensive when contrasted with what is known about paratopological groups. In [17], they have proved the following conclusion: being a strongly topological gyrogroup is a three-space property in the class of strongly paratopological gyrogroups.

The focus of our paper is to explore how certain characteristics are altered or maintained by extensions of paratopological gyrogroups. The structure of the paper is outlined as follows. The main aim of Section 2 is to present the pertinent concepts and conclusions that are necessary for understanding the content of this article. In Section 3, we investigate the three-space properties for compact type sets. We show that the properties of compactness, connection, etc, are three-space properties in paratopological gyrogroups (see Theorem 3.11), and that metrizability of compact (resp., sequentially compact, countably compact) subsets is a three-space property in the class of k-gentle paratopological gyrogroups (see Theorem 3.14). In Section 4, we study the three-space property for paratopological gyrocommutative gyrogroups. Let G be a strongly paratopological gyrocommutative gyrogroup and let H be a second-countable invariant topological subgyrogroup of G. If the paratopological gyrogroup G/H has a countable network, then so does G (see Theorem 4.6).

2. Definitions and preliminaries

In this section, we introduce necessary notation, terminology and facts about topological gyrogroups and paratopological gyrogroups. The binary operation in a given set is known as the set operation. The set of all automorphisms of a groupoid $(S, \oplus)$, denoted $Aut(S, \oplus)$, forms a group with group operation given by bijection composition. Throughout this paper, all topological spaces are assumed to be Hausdorff, unless otherwise is explicitly stated. The family of open neighborhoods of the neutral element 0 in a (para)topological gyrogroup G will be denoted by $\mathcal{U}$.

Definition 2.1. ([26, Definition 2.7]) Let $(G, \oplus)$ be a nonempty groupoid. We say that $(G, \oplus)$ or just G (when it is clear from the context) is a gyrogroup if the followings hold:

(1) There is an identity element $0 \in G$ such that $0 \oplus x = x = x \oplus 0$ for all $x \in G$.
(2) For each $x \in G$, there exists an *inverse element* $\ominus x \in G$ such that $\ominus x \oplus x = 0 = x \oplus (\ominus x)$.
(3) For any $x, y \in G$, there exists an *gyroautomorphism* $\mathrm{gyr}[x, y] \in Aut(G, \oplus)$ such that $x \oplus (y \oplus z) = (x \oplus y) \oplus \mathrm{gyr}[x, y](z)$ for all $z \in G$;
(4) For any $x, y \in G$, $\mathrm{gyr}[x \oplus y, y] = \mathrm{gyr}[x, y]$.

Definition 2.2. ([26, Definition 2.9]) Let $(G, \oplus)$ be a gyrogroup with gyrogroup operation (or, addition) $\oplus$. The gyrogroup cooperation (or, coaddition) $\boxplus$ is a second binary operation in G given by the equation $a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b$ for all $a, b \in G$. The groupoid $(G, \boxplus)$ is called a cogyrogroup, and is said to be the cogyrogroup associated with the gyrogroup $(G, \oplus)$.

Replacing b by $\ominus b$ in $a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b$, we have the identity $a \boxminus b = a \ominus \mathrm{gyr}[a, b]b$ for all $a, b \in G$, where we use the obvious notation, $a \boxminus b = a \boxplus (\ominus b)$.

Theorem 2.3. ([26, Table 2.2]) *Let $(G, \oplus)$ be a gyrogroup. Then, for any $a, b, c \in G$ we have*

(1) $(a \oplus b) \oplus c = a \oplus (b \oplus gyr[b, a]c)$; *Right Gyroassociative Law*
(2) $gyr[a, b] = gyr[a, b \oplus a]$; *Right Loop Property*

(3) $(\ominus a) \oplus (a \oplus b) = b$;
(4) $(a \ominus b) \boxplus b = a$;
(5) $(a \boxminus b) \oplus b = a$; *Right cancellation*
(6) $gyr[a, b](c) = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c))$;
(7) $\ominus(a \oplus b) = gyr[a, b](\ominus b \ominus a)$; *Gyrosum Inversion*
(8) $gyr[a, b](\ominus x) = \ominus gyr[a, b]x$;
(9) $gyr^{-1}[a, b] = gyr[b, a]$; *Inversive symmetry*
(10) $\ominus(a \boxplus b) = (\ominus b) \boxplus (\ominus a)$. *The Cogyroautomorphic Inverse Theorem*

Definition 2.4. ([26, Definition 2.8]) A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation obeys the gyrocommutative law $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$ for all $a, b \in G$.

Theorem 2.5. ([26, Table 3.1]) *Let $(G, \oplus)$ be a gyrocommutative gyrogroup. Then, for any $a, b \in G$, we have*

(1) $\ominus(a \oplus b) = \ominus a \ominus b$; *Gyroautomorphic Inverse Property*
(2) $a \boxplus b = b \boxplus a$;
(3) $a \boxplus b = a \oplus ((\ominus a \oplus b) \oplus a)$.

Definition 2.6. ([24, Definition 4]) Let G be a gyrogroup. A nonempty subset H of G is a subgyrogroup, written $H \leq G$, if H is a gyrogroup under the operation inherited from G and the restriction of $\mathrm{gyr}[a, b]$ to H becomes an automorphism of H for all $a, b \in H$.

Furthermore, a subgyrogroup H of G is said to be an *L-subgyrogroup* [24, Definition 8], denoted by $H \leq_L G$, if $\mathrm{gyr}[a, h](H) = H$ for all $a \in G$ and $h \in H$.

Definition 2.7. ([24, Definition 9]) A subgyrogroup H of a gyrogroup G is normal in G, written $H \trianglelefteq G$, if it is the kernel of a gyrogroup homomorphism of G.

By the definition of L-subgyrogroup in [24], it is easy to see that a normal subgyrogroup of a gyrogroup G is an L-subgyrogroup.

Theorem 2.8. ([24]) *Let H be a subgyrogroup of a gyrogroup G. Then $H \trianglelefteq G$ if and only if the operation on the coset space G/H given by $(a \oplus H) \oplus (b \oplus H) = (a \oplus b) \oplus H$ for any $a, b \in G$ is well defined.*

In fact, this operation is independent of the choice of representatives for the left cosets, that is, it is a well-defined operation, and the coset space G/H forms a gyrogroup, called a *quotient gyrogroup* [24]. We represent the mapping from G to G/H as π with the form $a \mapsto a \oplus H$. It is evident that for any $a, b \in G$, we obtain $\pi(a \oplus b) = \pi(a) \oplus \pi(b)$, and for every element $a \in G$, the equation $\pi^{-1}(\pi(a)) = a \oplus H$ holds.

Theorem 2.9. ([24]) *Let H be a subgyrogroup of a gyrogroup G. Then H is a normal subgyrogroup in G if and only if $(a \oplus b) \oplus H = a \oplus (H \oplus b) = (a \oplus H) \oplus b$ for all $a, b \in G$.*

Proposition 2.10. ([24]) *Let G be a gyrogroup. If $H \trianglelefteq G$, then $gyr[a, b](H) = H$ for all $a, b \in G$.*

Since in Topology 'normal' refers to a separation property of spaces, we will use the term 'invariant' to denote this property of subgyrogroups.

Definition 2.11. ([16]) A subgyrogroup H of a gyrogroup G is said to be a strongly L-subgyrogroup [1)], denoted by $H \leqslant_{SL} G$, if $\mathrm{gyr}[a, b](H) \subset H$ for all $a, b \in G$.

Proposition 2.12. ([16]) *Let G be a gyrogroup. If $H \leqslant_{SL} G$, then $a \oplus (b \oplus H) = (a \oplus b) \oplus H$ for all $a, b \in G$.*

Definition 2.13. ([3]) A triple $(G, \tau, \oplus)$ is called a topological gyrogroup if and only if

[1)]In [7, Definition 3.9] it is called a strongly subgyrogroup.

(1) (G, τ) is a topological space;
(2) $(G, \oplus)$ is a gyrogroup;
(3) The binary operation $\oplus : G \times G \to G$ is continuous where $G \times G$ is endowed with the product topology and the operation of taking the inverse $\ominus(\cdot) : G \to G$, i.e. $x \to \ominus x$, is continuous.

If a triple $(G, \tau, \oplus)$ satisfies the first two conditions and its binary operation is continuous, we call such triple a *paratopological gyrogroup* [4]. Sometimes we will just say that G is a topological gyrogroup (paratopological gyrogroup) if the binary operation and the topology are clear from the context.

Definition 2.14. A triple $(G, \tau, \oplus)$ is called a right topological gyrogroup if and only if

(1) (G, τ) is a topological space;
(2) $(G, \oplus)$ is a gyrogroup;
(3) For all $a \in G$, the right action $R_a : G \to G$, where $R_a(x) = x \oplus a$ for each $a \in G$, is a continuous mapping.

If a triple $(G, \tau, \oplus)$ satisfies the first two conditions and for all $a \in G$, the left action $L_a : G \to G$, where $L_a(x) = a \oplus x$ for each $a \in G$, is a continuous mapping, we call such triple a left topological gyrogroup. A semitopological gyrogroup is a left topological gyrogroup which is also a right topological gyrogroup.

The relationships between the aforementioned definitions can be represented as follows.
topological gyrogroup $\Rightarrow$ paratopological gyrogroup $\Rightarrow$ semitopological gyrogroup $\Rightarrow$ left (right) topological gyrogroup.

Definition 2.15. ([5]) Let $(G, \tau, \oplus)$ be a topological gyrogroup. We say that G is a strongly topological gyrogroup if there exists a neighborhood base $\mathcal{U}$ of the identity 0 in G such that, for every $U \in \mathcal{U}$, $\mathrm{gyr}[x, y](U) = U$ holds for any $x, y \in G$. For convenience, we say that G is a strongly topological gyrogroup with a neighborhood base $\mathcal{U}$ of 0. Clearly, we may assume that U is symmetric for each $U \in \mathcal{U}$.

For a paratopological gyrogroup $(G, \tau, \oplus)$, we called $(G, \tau, \oplus)$ a *strongly paratopological gyrogroup* if there exists a neighborhood base $\mathcal{U}$ of the identity 0 in G such that, for every $U \in \mathcal{U}$, $\mathrm{gyr}[x, y](U) = U$ holds for any $x, y \in G$.

A subgyrogroup H of a topological gyrogroup G is called *admissible* [6] if there exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the identity 0 in G such that $U_{n+1} \oplus (U_{n+1} \oplus U_{n+1}) \subset U_n$ for each $n \in \omega$ and $H = \bigcap_{n \in \omega} U_n$. If G is a strongly topological gyrogroup with a symmetric neighborhood base $\mathcal{U}$ at 0 and each $U_n \in \mathcal{U}$, we say that the admissible topological subgyrogroup is generated from $\mathcal{U}$.

Proposition 2.16. ([3]) *Let G be a topological gyrogroup, $x, y \in G$.*

(1) The left translation $L_x : G \to G$, where $L_x(y) = x \oplus y$ for every $y \in G$, is homeomorphism;
(2) The right translation $R_x : G \to G$, where $R_x(y) = y \oplus x$ for every $y \in G$, is homeomorphism.

In section 5 of [10], the authors proved that the right translation of a paratopological loop is homeomorphisms. Since every gyrogroup is a left Bol loop [22], we can get a paratopological gyrogroup is a paratopological loop, it follows that the right translation of a paratopological gyrogroup is homeomorphisms. Next, we will provide a detailed proof of this conclusion.

Proposition 2.17. *Let $(G, \oplus)$ be a gyrogroup. Then $a \boxplus b = b \oplus ((\ominus b \oplus a) \oplus b)$ for all $a, b \in G$.*

Proof. For all $a, b \in G$ we have

$$\begin{aligned} b \oplus ((\ominus b \oplus a) \oplus b) &= (b \oplus (\ominus b \oplus a)) \oplus \mathrm{gyr}[b, \ominus b \oplus a]b \quad \text{by the left gyroassociative law} \\ &= a \oplus \mathrm{gyr}[a, \ominus b \oplus a]b \quad \text{by a left cancellation and a left loop property} \\ &= a \oplus \mathrm{gyr}[a, \ominus b]b \quad \text{by a right loop property} \\ &= a \boxplus b. \qquad \text{by Definition 2.2} \end{aligned}$$

□

Proposition 2.18. ([26, Theorem 2.22]) *Let $(G, \oplus)$ be a gyrogroup, and let $a \in G$. Then L_a and R_a are bijective.*

Proposition 2.19. *Let G be a paratopological gyrogroup, $a \in G$.*

(1) The left translation $L_a : G \to G$, where $L_a(x) = a \oplus x$ for every $a \in G$, is homeomorphism [4];
(2) The right translation $R_a : G \to G$, where $R_a(x) = x \oplus a$ for every $a \in G$, is homeomorphism.

Proof. By Proposition 2.18 we have that R_a is bijective. To prove (2), let $x \in G$ and let U be a neighborhood of $R_a(x) = x \oplus a$. By the joint continuity of G, there exist open subsets U_a, U_x of G such that $a \in U_a$, $x \in U_x$ and $U_x \oplus U_a \subseteq U$. Hence $R_a(U_x) = U_x \oplus a \subseteq U_x \oplus U_a \subseteq U$, which shows that $R_a : G \to G$ is continuous.

To prove that $(R_a)^{-1}$ is continuous, we put $y = x \oplus a$, then $x = y \boxminus a = y \boxplus (\ominus a) = (\ominus a) \oplus ((a \oplus y) \oplus (\ominus a))$ by Theorem 2.17. That is $R_a^{-1}(x) = (\ominus a) \oplus ((a \oplus x) \oplus (\ominus a))$ for any $x \in G$. So, $R_a^{-1} = L_{\ominus a} \circ R_{\ominus a} \circ L_a$ which is continuous by (1). Thus we get R_a is homeomorphism. □

The following theorem shows that the admissible topological subgyrogroup of a topological gyrogroup is an invariant subgyrogroup.

Theorem 2.20. *Suppose that $(G, \tau, \oplus)$ is a topological gyrogroup and $\{U_n : n \in \omega\}$ is a sequence of open symmetric neighborhoods of the identity 0 in G such that $U_{n+1} \oplus (U_{n+1} \oplus U_{n+1}) \subset U_n$ for each $n \in \omega$. Then the admissible topological subgyrogroup $H = \bigcap_{n\in\omega} U_n$ is an invariant subgyrogroup of G.*

Proof. Put $\mathcal{U} = \{U_n : n \in \omega\}$.

Claim 1. $a \oplus H = H \oplus a$ for each $a \in G$.

For $a \in G$, suppose $f(x) = \ominus a \oplus (x \oplus a)$ for any $x \in G$. So, $f = L_{\ominus a} \circ R_a$ which is homeomorphism by Proposition 2.16. Since $f(0) = \ominus a \oplus (0 \oplus a) = 0$, for $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $f(V) = \ominus a \oplus (V \oplus a) \subset U$. It follows that $\ominus a \oplus (H \oplus a) \subset H$, for each $a \in G$, that is $H \oplus a \subset a \oplus H$. And for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $f^{-1}(V) \subset U$. It follows that $a \oplus H \subset H \oplus a$, for each $a \in G$. So we can get $H \oplus a = a \oplus H$.

Claim 2. $(a \oplus H) \oplus b = (a \oplus b) \oplus H$ for each $a, b \in G$.

For $a, b \in G$, suppose $f(x) = \ominus(a \oplus b) \oplus ((a \oplus x) \oplus b)$ for any $x \in G$. So, $f = L_{\ominus(a\oplus b)} \circ R_b \circ L_a$ which is homeomorphism by Proposition 2.16. Since $f(0) = \ominus(a \oplus b) \oplus ((a \oplus 0) \oplus b) = 0$, for $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $f(V) = \ominus(a \oplus b) \oplus ((a \oplus V) \oplus b) \subset U$. It follows that $\ominus(a \oplus b) \oplus ((a \oplus H) \oplus b) \subset H$, for each $a, b \in G$, that is $(a \oplus H) \oplus b \subset (a \oplus b) \oplus H$. And for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $f^{-1}(V) \subset U$. It follows that $(a \oplus b) \oplus H \subset (a \oplus H) \oplus b$, for each $a, b \in G$. So we can get $(a \oplus H) \oplus b = (a \oplus b) \oplus H$.

Claim 3. $a \oplus (H \oplus b) = (a \oplus b) \oplus H$ for each $a, b \in G$.

For $a, b \in G$, suppose $f(x) = \ominus(a \oplus b) \oplus (a \oplus (b \oplus x)) = \text{gyr}[a, b](x)$ for any $x \in G$. So f is a homeomorphism by Proposition 2.16. Since $f(0) = \text{gyr}[a, b](0) = 0$, for $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $f(V) = \text{gyr}[a, b](V) \subset U$. It follows that $\text{gyr}[a, b](H) \subset H$, for each $a, b \in G$.

For $a, b \in G$, $f^{-1}(x) = \text{gyr}^{-1}[a, b](x) = \text{gyr}[b, a](x)$ for any $x \in G$. According to the proof process above, it can be concluded that $H \subset \text{gyr}[a, b](H)$. Then $\text{gyr}[a, b](H) = H$.

From Claims 1, 2 and 3 it follows that H is an invariant subgyrogroup of G by Theorem 2.9. □

For a gyrogroup G, by $\mathcal{P}(G)$ we denote the set of all paratopologies on the gyrogroup G. For paratopologies $\tau_1, \tau_2 \in \mathcal{P}(G)$ put $\tau_1 \wedge \tau_2 = \sup\{\tau \in \mathcal{P}(G) : \tau \subset \tau_1 \cap \tau_2\}$, $\tau_1 \vee \tau_2 = \inf\{\tau \in \mathcal{P}(G) : \tau \supset \tau_1 \cup \tau_2\}$. $\tau_1 \vee \tau_2$ is a topology generated by taking the union of τ_1 and τ_2 as the subbase.

Proposition 2.21. *Let τ_1, τ_2 be paratopologies on a gyrogroup G with bases at the unit $\mathcal{B}_1, \mathcal{B}_2$ respectively. Then the upper bound $\tau_1 \vee \tau_2$ is a paratopological gyrogroup topology on G with a neighborhood base $\mathcal{B}_1 \vee \mathcal{B}_2 = \{U_1 \cap U_2 : U_i \in \mathcal{B}_i\}$ at the unit.*

Proof. Firstly, we shall prove that $(G, \oplus, \tau_1 \vee \tau_2)$ is a paratopological gyrogroup. It is enough to show that the binary operation $\oplus : (G, \tau_1 \vee \tau_2) \times (G, \tau_1 \vee \tau_2) \to (G, \tau_1 \vee \tau_2)$ is continuous where $(G, \tau_1 \vee \tau_2) \times (G, \tau_1 \vee \tau_2)$ is endowed with the product topology. Take any $x, y \in G$ and any neighborhood U of $x \oplus y$. We can assume $U = V_1 \cap V_2$, where $V_1 \in \tau_1$ and $V_2 \in \tau_2$. Since $(G, \oplus, \tau_1)$ and $(G, \oplus, \tau_2)$ are paratopological gyrogroups, there are $W_1, H_1 \in \tau_1$ and $W_2, H_2 \in \tau_2$ such that $x \in W_1 \cap W_2$, $y \in H_1 \cap H_2$, $W_1 \oplus H_1 \subset V_1$ and $W_2 \oplus H_2 \subset V_2$. Thus,

$(W_1 \cap W_2) \oplus (H_1 \cap H_2) \subset V_1 \cap V_2$. Clearly, $x \in W_1 \cap W_2 \in \tau_1$ and $y \in H_1 \cap H_2 \in \tau_2$. This means that the binary operation $\oplus$ is continuous at the unit in $(G, \oplus, \tau_1 \vee \tau_2)$.

Secondly, we shall prove that $\mathcal{B}_1 \vee \mathcal{B}_2$ is a neighborhood base at the unit. Clearly, $\mathcal{B}_1 \vee \mathcal{B}_2$ is a neighborhood at the unit, because $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases at the unit in (G, τ_1) and (G, τ_2) respectively. Take any neighborhood U of the unit in $(G, \tau_1 \vee \tau_2)$. Clearly, we can assume $U = V_1 \cap V_2$, where $V_1 \in \tau_1$ and $V_2 \in \tau_2$. Then one can find $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ such that $B_1 \subset V_1$ and $B_2 \subset V_2$. Thus, $B_1 \cap B_2 \in \mathcal{B}_1 \vee \mathcal{B}_2$ and $B_1 \cap B_2 \subset U$. This shows that $\mathcal{B}_1 \vee \mathcal{B}_2$ is a neighborhood base for $(G, \tau_1 \vee \tau_2)$. □

Proposition 2.22. *Let G be a paratopological gyrocommutative gyrogroup with topology τ and a neighborhood base $\mathcal{U}$ at 0. One defines the conjugate topology τ^{-1} on G by $\tau^{-1} = \{\ominus U : U \in \tau\}$. Then $G' = (G, \tau^{-1})$ is also a paratopological gyrocommutative gyrogroup with neighborhood base $\ominus\mathcal{U}$ at 0, and the inversion $x \to \ominus x$ is a homeomorphism of G onto G'. The upper bound $\tau^* = \tau \vee \tau^{-1}$ is a topological gyrocommutative gyrogroup topology on G with neighborhood base $\mathcal{U} \vee (\ominus\mathcal{U}) = \{U \cap (\ominus U) : U \in \mathcal{U}\}$ at 0, and we call $G^* = (G, \tau^*)$ the topological gyrocommutative gyrogroup associated to G.*

Proof. We first prove that $G' = (G, \tau^{-1})$ is a paratopological gyrocommutative gyrogroup. Let $G = (G, \tau)$. For every points $x, y \in G$ and every open neighborhood $\ominus U \subset G'$ of the point $x \oplus y$, that is $(\ominus x) \oplus (\ominus y) = \ominus(x \oplus y) \in U$ by Theorem 2.5 (1), there exist open neighborhoods $V, W \subset G$ of the points $\ominus x, \ominus y$ respectively such that $V \oplus W \subset U$. By Theorem 2.5 (1) we can get $(\ominus V) \oplus (\ominus W) = \ominus(V \oplus W) \subset \ominus U$. Thus, for every $x \oplus y \in \ominus U$ there exist $x \in \ominus V$ and $y \in \ominus W$ such that $(\ominus V) \oplus (\ominus W) \subset \ominus U$. It is obvious the inversion $x \to \ominus x$ is a homeomorphism of G onto G'. By Proposition 2.21, $G^* = (G, \tau^*)$ is a paratopological gyrocommutative gyrogroup.

Next we verify that $G^* = (G, \tau^*)$ is a topological gyrocommutative gyrogroup. Let $\mathcal{U}$ be an open neighborhood base at 0 of G. Then $\mathcal{U} \vee (\ominus\mathcal{U})$ is a family of subsets containing 0 of G. Since the identity functions $id : (G, \tau) \to (G, \tau)$ and $id' : (G, \tau^{-1}) \to (G, \tau^{-1})$ are continuous, we take any $x \in G$ and any $U \in \mathcal{U}$, then there are $U_1, U_2 \in \mathcal{U}$ such that $\ominus x \oplus U_1 \subset \ominus x \oplus U$ and $\ominus x \oplus (\ominus U_2)) \subset \ominus x \oplus (\ominus U)$. We have $U_3 \in \mathcal{U}$ such that $U_3 \subset U_1 \cap U_2$. So we can get

$$\begin{aligned} \ominus x \oplus U_3 \cap \ominus x \oplus (\ominus U_3) &\subset \ominus x \oplus U \cap \ominus x \oplus (\ominus U) \\ &= \ominus x \oplus ((\ominus U) \cap U). \end{aligned}$$

Since $\ominus x \oplus U_3 \cap \ominus x \oplus (\ominus U_3) = \ominus(x \oplus (\ominus U_3)) \cap \ominus(x \oplus U_3) = \ominus(x \oplus (U_3 \cap (\ominus U_3)))$, which means $\ominus(x \oplus (U_3 \cap (\ominus U_3))) \subset \ominus x \oplus ((\ominus U) \cap U)$. Thus we have proved that the inverse operation $\ominus$ is continuous. □

Remark 2.23. Can the "gyrocommutative" be removed from this proposition?

Definition 2.24. Let $\mathcal{P}$ be a (topological) property. A paratopological gyrogroup H is called totally $\mathcal{P}$ if the associated topological group H^* has property $\mathcal{P}$.

Lemma 2.25. *Let H be a T_0 paratopological gyrocommutative gyrogroup. Then considered with the topology induced from $H \times H'$, the diagonal $\triangle = \{(x, x) : x \in H\}$ is a Hausdorff topological gyrocommutative gyrogroup topologically isomorphic to the gyrogroup H^*. In addition, if H is a T_1 space, then $\triangle$ is closed in $H \times H'$.*

Proof. To see that $\triangle$ with the topology $\tau \times \tau^{-1}$ is a topological gyrogroup it suffices to pay attention to the fact that if $(g_\alpha)_{\alpha \in D}$ is a net converging to g in τ, then the net $(\ominus g_\alpha)_{\alpha \in D}$ converges to $\ominus g$ in τ^{-1}. As a consequence of the fact that T_0 topological gyrogroups are Hausdorff, the property of Hausdorffness then follows. It should be noted that the projection map from $\triangle$ onto $H^* = (H, \tau \vee \tau^{-1})$, represented by $\pi(x, x) = x$, is a topological isomorphism.

Next we shall show that $\triangle$ is closed in $H \times H'$ in case that H is a T_1-space. For consider $(x, y) \in H \times H'$ with $x \neq y$. Since H is a T_1 paratopological gyrogroup and $\ominus x \oplus y \neq 0$, we can choose open neighborhoods U and V of $\ominus x$ and y, respectively, such that $0 \notin U \oplus V$. Then $(\ominus U \times V) \cap \triangle = \emptyset$. Thus, $(x, y) \notin cl_{(H \times H')} \triangle$. □

Proposition 2.26. *Let H be a Hausdorff paratopological gyrocommutative gyrogroup. The following statements are true:*

(a) If the space H^2 is Lindelöf, then H is totally Lindelöf.
(b) If H is σ-compact, then so is H^*.
(c) If H is a Lindelöf Σ-space, so is H^*.
(d) If H has a countable network, so has H^*.
(e) If H is second countable, then so is H^*.
(f) If H is first countable, then so is H^*.
(g) If H is a P-space, so is H^*.

Proof. Based on Lemma 2.25, it is confirmed that the diagonal $\triangle$ in the product space $H \times H'$ is closed and is topologically isomorphic to the gyrogroup H^*. This immediately establishes item (a) based on the provided Definition 2.24. The attributes of being σ-compact or a Lindelöf Σ-space are properties that are both preserved within finite products and inherited by closed subsets, so (b) and (c) of the proposition are direct implications from Lemma 2.25. This rationale is similarly applied to draw conclusions for items (d), (e), (f), and (g). □

Lemma 2.27. *Let G be a paratopological gyrogroup. Then for every $g \in G$ and an open neighborhood V at 0 in G, there exists an open neighborhood O at 0 in G such that $O \oplus (O \oplus g) \subset V \oplus g$.*

Proof. For G is a paratopological gyrogroup, then $op_3 : G \times G \times G \to G$ defined by $op_3(x, y, g) = x \oplus (y \oplus g)$ is continuous. Since $0 \oplus (0 \oplus g) = g$, there exists an open neighborhood O at 0 in G such that $op_3(O) = O \oplus (O \oplus g) \subset V \oplus g$. □

Lemma 2.28. ([17]) *Let the neighborhood base $\mathcal{U}$ at 0 of G witness that G is a strongly paratopological gyrogroup. Then we have $a \boxplus U \subset a \oplus U$ and $a \boxminus U \subset a \ominus U$ for each $a \in G$ and $U \in \mathcal{U}$.*

Lemma 2.29. ([17]) *Let the neighborhood base $\mathcal{U}$ at 0 of G witness that G is a strongly paratopological gyrogroup. Then we have $(a \oplus U) \oplus W = a \oplus (U \oplus W)$ for each $a \in G$ and $U, W \in \mathcal{U}$.*

Lemma 2.30. ([17]) *Let the neighborhood base $\mathcal{U}$ at 0 of G witness that G is a strongly paratopological gyrogroup. If $U \oplus V \subset W$, then $\ominus V \ominus U \subset \ominus W$, for each $W, U, V \in \mathcal{U}$.*

Lemma 2.31. ([17]) *Let the neighborhood base $\mathcal{U}$ at 0 of G witness that G is a strongly paratopological gyrogroup. Then for each $U_1, U_2 \in \mathcal{U}$ we have $U_1 \oplus U_2 \in \mathcal{U}$.*

Proposition 2.32. ([17]) *Let $(G, \tau, \oplus)$ be a paratopological gyrogroup, let F be a compact subset of G, and let O be an open subset of G such that $F \subset O$. Then there exists an open neighborhood V of the identity element 0 such that $F \oplus V \subset O$ and $V \oplus F \subset O$.*

Proposition 2.33. *For every disjoint compact subsets K_1, K_2 of a Hausdorff paratopological gyrogroup G, there exists a neighborhood U of the unit such that $(U \oplus K_1) \cap (U \oplus K_2) = \emptyset$.*

Proof. By [11, Theorem 3.1.6], for every disjoint compact subsets K_1, K_2 of a Hausdorff space G, there exist open sets $U_1, V_1 \subset G$ such that $K_1 \subset U_1$, $K_2 \subset V_1$ and $U_1 \cap V_1 = \emptyset$. Then by Proposition 2.32 there exists an open neighborhood U of the identity element 0 such that $U \oplus K_1 \subset U_1$ and $U \oplus K_2 \subset V_1$. And also $(U \oplus K_1) \cap (U \oplus K_2) = \emptyset$. □

Let $(G, \tau, \oplus)$ be a paratopological gyrogroup and let H be a L-subgyrogroup of G. It follows from [25, Theorem 20] that $G/H = \{a \oplus H : a \in G\}$ is a partition of G. We denote by π the mapping $a \mapsto a \oplus H$ from G onto G/H. Clearly, for each $a \in G$, we have $\pi^{-1}(\pi(a)) = a \oplus H$. Denote by $\tau(G)$ the topology of G. In the left cosets G/H of the gyrogroup G, we define a topology $\widetilde{\tau} = \tau(G/H)$ of subsets as follows:

$$\widetilde{\tau} = \tau(G/H) = \{O \subset G/H : \pi^{-1}(O) \in \tau(G)\}.$$

Proposition 2.34. *Let $(G, \tau, \oplus)$ be a paratopological gyrogroup and let H be a L-subgyrogroup of G. Then the canonical quotient mapping π from G to its quotient topology on G/H is an open and continuous mapping, and the family $\{\pi(x \oplus U) : U \in \tau, 0 \in U)\}$ is a local base of the space G/H at the point $x \oplus H \in G/H$. Moreover, if the subgyrogroup H is a closed strongly L-subgyrogroup, G/H is a homogeneous T_1-space.*

Proof. The continuity of the map π is obvious. If $U \subset G$ is an open set then $\pi^{-1}(\pi(U)) = U \oplus H$ and hence $\pi(U)$ is open.

Let us now prove the homogeneity of G/H. For any $a \in G$, define a mapping h_a of G/H to itself by the rule $h_a(x \oplus H) = a \oplus (x \oplus H)$. Since $a \oplus (x \oplus H) = (a \oplus x) \oplus H \in G/H$ by Proposition 2.12, this definition is correct. Since G is a gyrogroup, the mapping h_a is evidently a bijection of G/H onto G/H. In fact, h_a is a homeomorphism. This can be seen from the following argument.

Take any $x \oplus H \in G/H$ and any open neighbourhood U of 0. Then $\pi((x \oplus U) \oplus H)$ is a basic neighbourhood of $x \oplus H$ in G/H. Similarly, the set $\pi(a \oplus ((x \oplus U) \oplus H))$ is a basic neighbourhood of $a \oplus (x \oplus H)$ in G/H. Since, obviously, $h_a(\pi((x \oplus U) \oplus H)) = \pi(a \oplus ((x \oplus U) \oplus H))$, it easily follows that h_a is a homeomorphism. Now, for any given $x \oplus H$ and $y \oplus H$ in G/H, we can take $a = y \boxminus x$. Then $h_a(x \oplus H) = a \oplus (x \oplus H) = (a \oplus x) \oplus H = ((y \boxminus x) \oplus x) \oplus H = y \oplus H$, by (5) in Theorem 2.3 and Proposition 2.12. Hence, the quotient space G/H is homogeneous. It is a T_1-space, since all cosets $x \oplus H$ are closed in G and the mapping π is quotient. □

Proposition 2.35. *Let $(G, \tau, \oplus)$ be a paratopological gyrogroup and let H be an invariant subgyrogroup of G, then the operation $(x \oplus H) \oplus (y \oplus H) = (x \oplus y) \oplus H$ in G/H is continuous and $(G/H, \widetilde{\tau})$ is a paratopological gyrogroup.*

Proof. If $\widetilde{U}$ is a neighborhood of the point $\widetilde{c} = \widetilde{a} \oplus \widetilde{b}$ in $(G/H, \widetilde{\tau})$, then $c = a \oplus b$ for some representatives a, b, c from the classes $\widetilde{a}, \widetilde{b}, \widetilde{c}$ respectively. For a neighborhood $U = \pi^{-1}(\widetilde{U}) \ni c$ there exist neighborhoods $V_1(a)$ and $V_2(b)$ such that $V_1(a) \oplus V_2(b) \subset U$. Thus $\pi(V_1(a)) \oplus \pi(V_2(b)) \subset \pi(U) = \widetilde{U}$ and G/H is a paratopological gyrogroup. □

Proposition 2.36. *Let $(G, \tau, \oplus)$ be a paratopological gyrogroup and let H be a strongly L-subgyrogroup of G. If H is a compact subgyrogroup of G, then the quotient mapping π of G onto the quotient space G/H is perfect. If the space (G, τ) is Hausdorff then the space $(G/H, \widetilde{\tau})$ is Hausdorff. If the space (G, τ) is regular then the space $(G/H, \widetilde{\tau})$ is regular.*

Proof. Let F be a closed subset of the gyrogroup G. Let $\widetilde{x} \in G/H \setminus \pi(F)$. Consider an arbitrary point $x \in \pi^{-1}(\widetilde{x})$. Then $(x \oplus H) \cap F = \emptyset$. By Proposition 2.32 there exists an open neighborhood U of the unit such that $(U \oplus (x \oplus H)) \cap F = \emptyset$. By the Definition of strongly L-subgyrogroup, we can get $U \oplus (x \oplus H) = (U \oplus x) \oplus \bigcup_{u \in U} \mathrm{gyr}[u, x]H = (U \oplus x) \oplus H$ and $\pi(U \oplus x) = \pi((U \oplus x) \oplus H)$. Then $\widetilde{x} \in \pi(U \oplus x)$ and $\pi(U \oplus x) = \pi(U \oplus (x \oplus H)) \cap \pi(F) = \emptyset$. So the map π is closed. Furthermore, if $y \in G/H$ and $\pi(x) = y$ for some $x \in G$, we obtain that $\pi^{-1}(y) = x \oplus H$ is a compact subset of G. Hence, the fibers of π are compact. Thus π is perfect. Hence, utilizing Theorem 3.7.20 as referenced in Engelking [11], we deduce that if the space (G, τ) is Hausdorff (or regular), then the quotient space $(G/H, \widetilde{\tau})$ will also be Hausdorff (or regular). □

Corollary 2.37. *Let $(G, \tau, \oplus)$ be a paratopological gyrogroup and let H be a compact strongly L-subgyrogroup of G. If F is a closed subset of G, then $F \oplus H$ is a closed subset of G.*

Proof. Let $\pi : G \to G/H$ be the standard projection. Then $F \oplus H = \pi^{-1}(\pi(F))$ is a closed subset of G. □

3. The three-space property for compact type sets

Let $\mathcal{P}$ be a (topological, algebraic, or a mixed nature) property. We call a property $\mathcal{P}$ *three-space property* in paratopological groups [27] if the quotient paratopological group G/H and a closed invariant subgroup H of a paratopological group G both have $\mathcal{P}$, then G enjoys $\mathcal{P}$, too.

Then we give the definitions of three-space property in paratopological gyrogroups.

Definition 3.1. We call a topological-algebraic property $\mathcal{P}$ three-space property in paratopological gyrogroups if a paratopological quotient gyrogroup G/H and a closed invariant subgyrogroup H of G both have $\mathcal{P}$, then G also has $\mathcal{P}$.

A topological property $\mathcal{P}$ is called an *inverse fiber property* [9] if (*) $f : X \to Y$ is a continuous and surjective mapping such that both the space Y and the fibers of f have $\mathcal{P}$, then X also has $\mathcal{P}$. If the conclusion in (*) holds under the additional assumption that the domain X is compact (countably compact), we say that $\mathcal{P}$ is an *inverse fiber property for compact (countably compact) sets*.

Lemma 3.2. *The first axiom of countability is an inverse fiber property for compact, countably compact [9, Proposition 2.8] and sequentially compact [18, Lemma 2.5] sets.*

Let $\mathcal{P}$ be a topological property. A space X is called $\mathcal{P}$*-compact* (resp., $\mathcal{P}$*-closed*) if every subset of X with the property $\mathcal{P}$ is compact (resp., closed). A space X is called *locally* $\mathcal{P}$ if for every $x \in X$ there exists a neighborhood U of the point x with property $\mathcal{P}$ [18].

Lemma 3.3. ([18]) *Suppose that $\mathcal{P}$ is a topological property preserved by continuous mappings and also inherited by closed sets. Then the property of being $\mathcal{P}$-closed (resp., $\mathcal{P}$-compact) is a regular-inverse fiber property (resp., inverse fiber property).*

Proposition 3.4. *If $\mathcal{P}$ is an inverse fiber property, then it is a three-space property in paratopological gyrogroups.*

Proof. We assume that H is a closed invariant subgyrogroup of a paratopological gyrogroup G. We assume further that both gyrogroups H and G/H have an inverse fiber property $\mathcal{P}$. Let $\pi : G \to G/N$ be the quotient homomorphism. If $y \in G/H$, we can find $x \in G$ such that $\pi(x) = y$. Then $\pi^{-1}(y) = x \oplus H$ is homeomorphic with H, so the fiber $\pi^{-1}(y)$ has $\mathcal{P}$ for all $y \in G/H$. It follows from the inverse fiber property of $\mathcal{P}$ that G also has $\mathcal{P}$. □

In the proof of [6, Theorem 3.3], the Right Gyroassociative Law of subgyrogroup H is mainly utilized, and therefore, according to Theorem 2.9, we have the following conclusion for an invariant (normal) subgyrogroup H. Since its proof is similar to that of [6, Theorem 3.3], the proof has been omitted.

Theorem 3.5. *Let G be a topological gyrogroup, H be a closed invariant subgyrogroup of G and P be a closed symmetric subset of G such that P contains an open neighborhood of 0 in G, and $\overline{P \oplus (P \oplus P)} \cap H$ is compact. Then the restriction f of π to P is a perfect mapping from P onto the subspace $\pi(P)$ of G/H, where $\pi : G \to G/H$ is the natural quotient mapping from G onto the topological quotient gyrogroup G/H.*

Proposition 3.6. *Let G be a topological gyrogroup. If H is a closed invariant subgyrogroup of G, then the topological quotient gyrogroup G/H is a homogenous space.*

Proof. The proof process is similar to Proposition 2.34, as an invariant subgyrogroup is a strongly L-subgyrogroup, and hence, the conclusion can be reached. □

Theorem 3.7. *Let G be a topological gyrogroup, H be a locally compact closed invariant subgyrogroup of G. Then there exists an open neighborhood U of the identity element 0 such that $\pi(\overline{U})$ is closed in G/H and the restriction of π to $\overline{U}$ is a perfect mapping from $\overline{U}$ onto the subspace $\pi(\overline{U})$, where $\pi : G \to G/H$ is the natural quotient mapping from G onto the topological quotient gyrogroup G/H.*

Proof. The proof process is similar to that of [6, Theorem 3.4], and the conclusion can be derived based on Theorem 3.5 and Proposition 3.6. □

Corollary 3.8. *Assume that $\mathcal{P}$ is a topological property preserved by preimages of spaces under perfect mappings (in the class of completely regular spaces) and also inherited by regular closed sets. Assume further that $(G, \tau, \oplus)$ is a topological gyrogroup, H is a closed invariant subgyrogroup of G, and the topological quotient gyrogroup G/H has the property $\mathcal{P}$. Then there exists an open neighborhood U of the identity element 0 such that $\overline{U}$ has the property $\mathcal{P}$.*

Corollary 3.9. *Let $(G, \tau, \oplus)$ be a topological gyrogroup, and let H be a locally compact closed invariant subgyrogroup of G. If the topological quotient gyrogroup G/H is locally compact, then G also possesses this property.*

Theorem 3.10. *Local compactness is a three-space property in the class of Hausdorff strongly paratopological gyrogroups.*

Proof. Let G be a strongly paratopological gyrogroup and let N be a closed invariant subgyrogroup of G. Suppose that both N and the paratopological quotient gyrogroup G/N are locally compact. It suffices to prove that G is locally compact. From the fact that every locally compact strongly paratopological gyrogroup is a strongly topological gyrogroup [17, Theorem 3.5], it follows that both N and G/N are strongly topological gyrogroups. Note that being a strongly topological gyrogroup is a three-space property in the class of strongly paratopological gyrogroups [17, Theorem 3.6], so that G is a strongly topological gyrogroup. In addition, local compactness is identified as a three-space property within the class of topological gyrogroups, as stated in Corollary 3.9. Consequently, G is locally compact. □

Theorem 3.11. *The following are three space properties in the class of Hausdorff paratopological gyrogroups:*

(a) *compactness;*
(b) *connectedness;*
(c) *every compact set being first-countable;*
(d) *every countably compact set being first-countable;*
(e) *every sequentially compact set being first-countable;*
(f) *every countably compact set being compact;*
(g) *every sequentially compact set being compact.*

Proof. Let G be a paratopological gyrogroup and let N be a closed invariant subgyrogroup of G.

(a) Suppose that both N and the paratopological quotient gyrogroup G/N are compact. By Proposition 2.36, for the Hausdorff space G the quotient mapping π of G onto the quotient space G/N is perfect. It is known that compactness of Hausdorff spaces which is both invariants and inverse invariants under perfect mappings. So it suffices to prove that G is compact.

(b) Suppose that both N and the paratopological quotient gyrogroup G/N are connected. It suffices to prove that G is connected. Suppose by contradiction that G is not connected. Thus there exist two non-empty disjoint open sets U, V in G such that $G = V \cup U$. The space N being connected implies that $x \oplus N$ is also connected for each $x \in G$, so that either $x \oplus N \subset U$ or $x \oplus N \subset V$. Let $\pi : G \to G/N$ be the canonical quotient mapping. Thus one readily verifies that $\pi(U)$ and $\pi(V)$ are two non-empty disjoint open sets in G/N such that $G/N = \pi(V) \cup \pi(U)$, which implies that G/N is not connected. This contradiction completes the proof.

By Lemma 3.2, we have that every (countably, sequentially) compact subset satisfying the first axiom of countability is an inverse fiber property and (c)-(e) hold by Proposition 3.4. The statements (f)-(g) directly follow from Lemma 3.3 and Proposition 3.4. □

A topological space X is called a *k-space* if X is a Hausdorff space and X is an image of a locally compact space under a quotient mapping [11].

Let $f : X \to Y$ be a mapping. The mapping f is called *k-gentle* if for each compact subset F of X the image $f(F)$ is also compact. A paratopological gyrogroup G is called *k-gentle* if the inverse mapping $x \to \ominus x$ is *k-gentle*.

Lemma 3.12. *The mapping $g : X \to \ominus X$ defined by $g(x) = \ominus x$ is continuous for every k-subspace X of G, where G is a k-gentle paratopological gyrogroup.*

Proof. Let D be a compact subspace of X. Given that G is a k-gentle paratopological gyrogroup, it is simple to verify that the restriction $g_{|D} : D \to \ominus D$ $(= g(D))$ is continuous. It is well known that a mapping f of a k-space F to a topological space Y is continuous if and only if for every compact subspace $Z \subset F$ the restriction $f_{|Z} : Z \to Y$ is continuous [11, Theorem 3.3.21], so that g is continuous. □

We call a topological space X having a *G_δ-diagonal* [14] if the diagonal $\Delta = \{(x,x) : x \in X\}$ of $X \times X$ is a G_δ-set in $X \times X$.

Proposition 3.13. *The following conditions are equivalent for a k-gentle paratopological gyrogroup G.*

(a) all compact (resp., sequentially compact, countably compact) subspaces of G are first-countable;
(b) all compact (resp., sequentially compact, countably compact) subspaces of G are metrizable.

Proof. We only need to prove $(a) \Rightarrow (b)$. Suppose that X is an arbitrary non-empty compact (resp., sequentially compact, countably compact) subset of G. By our assumption, X is first-countable, and thus is a k-space; it follows from Lemma 3.12 that the mapping $g : X \to \ominus X$ defined by $g(x) = \ominus x$ is continuous, because G is a k-gentle paratopological gyrogroup. Therefore, the space $\ominus X$ is compact (resp., sequentially compact, countably compact) as the continuous image of X, so that $\ominus X$ is first-countable by our assumption. Observe that the compactness, sequential compactness and countable compactness are preserved by the finite Cartesian product in the first-countable spaces, so that $\ominus X \oplus X$ is a compact (resp., sequentially compact, countably compact) subspace as the continuous image of $(\ominus X) \times X$, and thus is first-countable by our assumption. Define a mapping $\varphi : G \times G \to G$ as $\varphi(x, y) = (\ominus x) \oplus y$ for every $x, y \in G$. It is obvious $\varphi|_{X\times X}$ is continuous and the identity $0 \in \ominus X \oplus X$ with a local countable base in $\ominus X \oplus X$, we have that $\Delta = \varphi^{-1}(0)$ is a G_δ-set in $X \times X$, i.e., X has a G_δ-diagonal, so that it follows from the fact that every Hausdorff countably compact space with a G_δ-diagonal is metrizable [14, Theorem 2.14] that X is metrizable, because both compactness and sequential compactness imply countable compactness. □

Theorem 3.14. *Metrizability of compact (resp., sequentially compact, countably compact) subsets is a three-space property in the class of k-gentle paratopological gyrogroups.*

Proof. Let G be a k-gentle paratopological gyrogroup and let H be a closed invariant subgyrogroup of G such that all compact (resp., sequentially compact, countably compact) subsets in both H and the quotient paratopological gyrogroup G/H are metrizable. From Theorem 3.11 it follows that all compact (resp., sequentially compact, countably compact) subsets of G are first-countable, so the statement directly follows from Proposition 3.13. □

Clearly, every topological gyrogroup is a k-gentle paratopological gyrogroup, so we have the following:

Corollary 3.15. *Metrizability of compact (resp., sequentially compact, countably compact) subsets is a three space property in the class of topological gyrogroups.*

Let X be a topological space. A subset A of X is called *sequentially closed* if no sequence of points of A converges to a point not in A. X is called *sequential* [13] if each sequentially closed subset of X is closed. A space X is called *Fréchet at a point* $x \in X$ if $x \in \overline{A} \subset X$ there is a sequence $\{x_n\}_{n\in\omega}$ in A such that $\{x_n\}_{n\in\omega}$ converges to x in X. A space X is called *Fréchet* [13] if it is Fréchet at every point $x \in X$. A space X is called *strongly Fréchet at a point* $x \in X$ if whenever $\{A_n\}_{n\in\omega}$ is a decreasing sequence of subsets in X and $x \in \bigcap_{n\in\omega} \overline{A_n}$, there exists $x_n \in A_n$ for each $n \in \omega$ such that the sequence $x_n \to x$. A space X is called *strongly Fréchet* [23] if it is strongly Fréchet at every point $x \in X$. Fréchet spaces (resp., strongly Fréchet spaces) are also called Fréchet-Urysohn spaces (resp., strongly Fréchet-Urysohn spaces).

Lemma 3.16. *If every compact (resp., countably compact, sequentially compact) subspace of a k-gentle paratopological gyrogroup G is Fréchet, then every compact (resp., countably compact, sequentially compact) subspace of G is strongly Fréchet.*

Proof. Let G be a paratopological gyrogroup and all compact (resp., countably compact, sequentially compact) subspaces of G be Fréchet, and A a compact (resp., countably compact, sequentially compact) subset of G. By our assumption, A is Fréchet, and thus is closed in G by [18, Lemma 2.4]. Suppose that $\{A_n\}_{n\in\omega}$ is a decreasing sequence of subsets in A with $a \in \bigcap_{n\in\omega} \overline{A_n}$. We can assume that a is an accumulation point of A. By the Fréchet property of A one can find a sequence $\{a_n\}_{n\in\omega}$ in $A \setminus \{a\}$ converging to a. Put

$$B = (\ominus a) \oplus A, \text{ and } B_n = (\ominus a) \oplus A_n,\ b_n = (\ominus a) \oplus a_n \text{ for all } n \in \omega.$$

Clearly the set $B = (\ominus a) \oplus A$ is closed, compact (resp., countably compact, sequentially compact) and the sequence $\{b_n\}_{n\in\omega}$ converges to 0, where 0 is the identity of G, so that we have, for each $n \in \omega$, the closure of the set $B_n = (\ominus a) \oplus A_n$ being included in B, $0 \in \overline{B_n}$ and $b_n = (\ominus a) \oplus a_n \in B \setminus \{0\}$; since G is a Hausdorff paratopological gyrogroup, for each $n \in \omega$ we can find an open set V_n containing 0 such that $V_n \cap (b_n \oplus V_n) = \emptyset$. By $0 \in \overline{B_n}$ we have $0 \in \overline{B_n \cap V_n}$. Put $C_n = b_n \oplus (B_n \cap V_n)$ for every $n \in \omega$. Since $0 \in \overline{B_n \cap V_n}$, we have $b_n \in \overline{C_n}$. Moreover, $0 \notin \overline{C_n}$, because $V_n \cap C_n \subset V_n \cap (b_n \oplus V_n) = \emptyset$, for each $n \in \omega$. Put

$$D = \bigcup\{C_n : n \in \omega\}, \text{ and } S = \{0\} \cup \{b_n : n \in \omega\}.$$

Then $D \subset \bigcup_{n\in\omega} b_n \oplus B_n \subset S \oplus B$.

Next, we shall show that the subspace $S \oplus B$ of G is closed and Fréchet. Obviously, S is compact and sequentially compact. Moreover, given that a convergent sequence is compact and metrizable, and considering that $b_n \to 0$, it follows that S is compact and metrizable. Observe that the Cartesian product of two compact (resp., sequentially compact, countably compact) spaces, if one of which is first-countable, is compact (resp., sequentially compact, countably compact), so that the Cartesian product $S \times B$ of the spaces S and B is compact (resp., countably compact, sequentially compact), because S is compact and metrizable. Since the multiplication in G is jointly continuous and the subset $S \oplus B$ of G is the continuous image of the subset $S \times B$ of $G \times G$ under the multiplication mapping, $S \oplus B$ is compact (resp., sequentially compact). Thus $S \oplus B$ is closed and Fréchet by our assumption.

By $b_n \in \overline{C_n}$ for each $n \in \omega$ and $b_n \to 0$, we have $0 \in \overline{D} \subset S \oplus B$, so that one can find a sequence $\{d_k\}_{k\in\omega}$ in D converging to 0 by the Fréchet property of $S \oplus B$; in addition, as we already know $0 \notin \overline{C_n}$ for each $n \in \omega$, the set C_n contains only finitely many terms of the sequence $\{d_k\}_{k\in\omega}$; thus we can assume that there is a subsequence $\{C_{n_k}\}_{k\in\omega}$ of the sequence $\{C_n\}_{n\in\omega}$ such that $d_k \in C_{n_k}$ for each $k \in \omega$. By $C_{n_k} \subset b_{n_k} \oplus B_{n_k} = b_{n_k} \oplus ((\ominus a) \oplus A_{n_k})$, for each $k \in \omega$ we have $d_k = b_{n_k} \oplus ((\ominus a) \oplus x_{n_k})$ for some $x_{n_k} \in A_{n_k}$; one readily check that $\ominus b_{n_k} \to 0$, because $b_{n_k} \to 0$ and G is a k-gentle paratopological gyrogroup, so that $x_{n_k} = a \oplus (\ominus b_{n_k} \oplus d_k) \to a$ when $k \to \infty$. Take $x_n = x_{n_k}$ when $n_{k-1} < n < n_k$, then $x_n \in A_n$ for each $n \in \omega$ and $x_n \to a$. Hence, A is strongly Fréchet. □

Lemma 3.17. ([1]) *Suppose that X is a regular space, and that $f : X \to Y$ is a closed mapping. Suppose also that $b \in X$ is a G_δ-point in the space $F = f^{-1}(f(b))$ (i.e., the singleton $\{b\}$ is a G_δ-set in the space F) and F is Fréchet at b (resp., strictly Fréchet at b). If the space Y is strongly Fréchet, then X is Fréchet at b.*

Lemma 3.18. ([18]) *If all countably compact (resp., sequentially compact) subsets of a topological space X are sequential, then all countably compact (resp., sequentially compact) subsets of X are closed.*

Theorem 3.19. *Let G be a regular k-gentle paratopological gyrogroup and H a closed L-subgyrogroup of G such that all compact (resp., countably compact, sequentially compact) subsets of the paratopological gyrogroup H are first-countable. If the quotient space G/H has the following property, then so does the paratopological gyrogroup G.*

(a) all compact (resp., countably compact, sequentially compact) subsets are strongly Fréchet.

Proof. Let C be a compact (resp., countably compact, sequentially compact) subset of the paratopological gyrogroup G.

We first prove C is closed. Since every first-countable space is a strongly Fréchet space, as noted in [19], we can get all compact (resp., countably compact, sequentially compact) subsets of the paratopological gyrogroup H are strongly Fréchet. So they are sequential. Following Lemma 3.18, it is concluded that all compact (resp., countably compact, sequentially compact) subsets of H are closed. Since the quotient space G/H has the property (a), we can also get all compact (resp., countably compact, sequentially compact) subsets of G/H are closed by Lemma 3.18. Let $\pi : G \to G/H$ be the quotient homomorphism. If $y \in G/H$, take an element $x \in G$ with $\pi(x) = y$. Then $\pi^{-1}(y) = x \oplus H$ is a homeomorphic copy of H, so the fiber $\pi^{-1}(y)$ has the property that all compact (resp., countably compact, sequentially compact) subsets of $\pi^{-1}(y)$ are closed for each $y \in G/H$. Since being compact (resp., countably compact, sequentially compact) is a topological property preserved by continuous mappings and inherited by closed sets, it follows from Lemma 3.3 that being compact (resp., countably compact, sequentially compact)-closed is a regular-inverse fiber property. So we can get the set C is closed within G.

Put $f = \pi|_C : C \to \pi(C)$. Then $\pi(C)$ is compact (resp., countably compact, sequentially compact). Moreover, f is a closed mapping by [18, Lemma 2.4], and $f^{-1}(f(b)) = \pi^{-1}(\pi(b)) \cap C = (b \oplus H) \cap C$ is first-countable for each $b \in C$. We complete the proof by Lemmas 3.16 and 3.17. □

4. Quotient with respect to second-countable topological invariant subgyrogroups

In this section, we investigate the extensions of paratopological gyrocommutative gyrogroups. Let $\mathcal{P}$ be a family of subsets of a topological space X. $\mathcal{P}$ is called a *network* [11] for X if whenever $x \in U$ with U open in X, then there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.

The authors have proved the following conclusions in [12, Theorem 2.6]

Theorem 4.1. *[12] Let H be an invariant topological subgroup of a paratopological group G. If H is second-countable and G/H has countable network, then G has countable network as well.*

And then we try to extend this result to paratopological gyrocommutative gyrogroups. The following results play an important role in the proof of Theorem 4.6.

Proposition 4.2. *Let G be a strongly paratopological gyrocommutative gyrogroup and let H be an invariant topological subgyrogroup of G. Then $(G/H)^* = G^*/H$.*

Proof. Let $\pi : G \to G/H$ be the canonical homomorphism and let the neighborhood base $\mathcal{U}$ at 0 of G witness that G is a strongly paratopological gyrogroup. The family $\{\pi(x \oplus U \cap x \ominus U) : x \in G, U \in \mathcal{U}\}$ is a base for G^*/H. Also, the family $\{\pi(x \oplus U) \cap \pi(x \ominus U) : x \in G, U \in \mathcal{U}\}$ is a base for $(G/H)^*$. Clearly, the topology on $(G/H)^*$ is weaker than the topology on G^*/H. Let us show the other inclusion. Take $x \in G$ and $U \in \mathcal{U}$. Since G is a paratopological gyrogroup, we can find $V \in \mathcal{U}$ such that $V \oplus V \subset U$. Then for G is a gyrocommutative gyrogroup we can get $\ominus V \oplus (\ominus V) \subset \ominus U$ by Theorem 2.5(1). Since H is a topological subgyrogroup of G, there exists $W \in \mathcal{U}$ with $W \subset V$ and $(W \oplus W) \cap H \subset \ominus V \cap H$. We claim that $\pi(x \oplus W) \cap \pi(x \ominus W) \subset \pi(x \oplus U \cap x \ominus U)$. Indeed, choose $y \in \pi(x \oplus W) \cap \pi(x \ominus W)$ and $a \in G$ such that $\pi(a) = y$. Then

$$\begin{aligned} a &\in a \oplus H = \pi^{-1}(y) \\ &\subset \pi^{-1}(\pi(x \oplus W) \cap \pi(x \ominus W)) \\ &= ((x \oplus W) \oplus H) \cap ((x \ominus W) \oplus H) \\ &= (x \oplus (W \oplus H)) \cap (x \oplus (\ominus W \oplus H)) && \text{by Theorem 2.9.} \end{aligned}$$

It follows that $\ominus x \oplus a \in (W \oplus H) \cap (\ominus W \oplus H)$. So there exist $w_1, w_2 \in W$ and $h_1', h_2' \in H$ such that $\ominus x \oplus a = w_1 \oplus h_1' = \ominus w_2 \oplus h_2'$. Since H is an invariant topological subgyrogroup of G, we have $w_1 \oplus H = H \oplus w_1$ and $\ominus w_2 \oplus H = H \oplus (\ominus w_2)$. There exist $h_1, h_2 \in H$ such that $w_1 \oplus h_1' = h_1 \oplus w_1$ and $\ominus w_2 \oplus h_2' = h_2 \oplus (\ominus w_2)$, which means $\ominus x \oplus a = h_1 \oplus w_1 = h_2 \oplus (\ominus w_2)$. Thus, we have $w_1 = \ominus h_1 \oplus (h_2 \ominus w_2) = (\ominus h_1 \oplus h_2) \ominus gyr[\ominus h_1, h_2](w_2)$, and according to Definition 2.15 and Lemma 2.28, we can conclude that $\ominus h_1 \oplus h_2 = w_1 \boxplus gyr[\ominus h_1, h_2](w_2) \in W \boxplus W \subset W \oplus W$.

So $\ominus h_1 \oplus h_2 \in (W \oplus W) \cap H \subset \ominus V \cap H \subset \ominus V$, whence $h_2 \in h_1 \oplus (\ominus V)$. Therefore,

$$\begin{aligned}
&\ominus x \oplus a \in (h_1 \oplus W) \cap (h_2 \oplus (\ominus W)) \\
&\subset (h_1 \oplus W) \cap ((h_1 \oplus (\ominus V)) \oplus (\ominus W)) \\
&\subset (h_1 \oplus W) \cap ((h_1 \oplus (\ominus V)) \oplus (\ominus V)) \\
&\subset (h_1 \oplus V) \cap (h_1 \oplus (\ominus V \oplus \bigcup_{v \in V} gyr[\ominus v, h_1](\ominus V))) \\
&\subset (h_1 \oplus V) \cap (h_1 \oplus (\ominus V \oplus (\ominus V))) \\
&\subset (h_1 \oplus U) \cap (h_1 \oplus (\ominus U)) \\
&= h_1 \oplus (U \cap (\ominus U)).
\end{aligned}$$

It follows that $a \in x \oplus (h_1 \oplus (U \cap (\ominus U)))$. Hence $y = \pi(a) \in \pi(x \oplus (h_1 \oplus (U \cap (\ominus U)))) = \pi(x) \oplus \pi(U \cap (\ominus U)) = \pi(x \oplus (U \cap (\ominus U))) = \pi((x \oplus U) \cap (x \ominus U))$. This finishes the proof. □

Lemma 4.3. *Suppose that $(G, \tau, \oplus)$ is a paratopological gyrogroup and H is a separable L-subgyrogroup of G. If Y is a separable subset of G/H, then $\pi^{-1}(Y)$ is also separable in G, where π is the natural mapping of G onto the quotient space G/H.*

Proof. Let B be a countable dense subset of Y. Since H is separable, there exists a countable, dense subset M_b of $\pi^{-1}(b)$ for each $b \in B$. Put $M = \bigcup_{b \in B} M_b$. Then M is a countable subset of $\pi^{-1}(Y)$, and is dense in $\pi^{-1}(B)$. Since π is an open mapping of G onto G/H, π restricting to $\pi^{-1}(Y)$ is also an open mapping of $\pi^{-1}(Y)$ onto Y. Therefore, $\overline{M} = \overline{\pi^{-1}(B)} = \pi^{-1}(\overline{B}) = \pi^{-1}(Y)$ by [2, Lemma 1.5.22]. Hence, M is dense in $\pi^{-1}(Y)$, i.e., $\pi^{-1}(Y)$ is separable. □

Here, we extend this result to the class of semitopological gyrogroups.

Proposition 4.4. *Let H be a L-subgyrogroup of a semitopological gyrogroup G. If both H and G/H are separable, then so is G.*

Proof. Let $\pi : G \to G/H$ be the canonical quotient mapping. Since G/H is separable, there exists a countable subset $A \subset G$ such that $\pi(A)$ is dense in G/H. Also, we can find $B \subset H$ countable and dense in H. Let us show that $A \oplus B$ is dense in G. Take $g \in G$ and $U \in \mathcal{U}$. Hence $\pi(a) \in \pi(g \oplus U)$ for some $a \in A$. So $a \in a \oplus H \subset (g \oplus U) \oplus H$. Therefore, $\ominus a \oplus (g \oplus U) \cap H$ is a non-empty open set in H. By the choice of B, there exists $b \in B$ such that $b \in \ominus a \oplus (g \oplus U)$, equivalently, $a \oplus b \in g \oplus U$. This finishes the proof. □

Let X be a topological space. X is called *cosmic* if X is a regular space with a countable network [20]. In [8, Corollary 3.7], the authors showed that if G is a strongly topological gyrogroup with a symmetric neighborhood base $\mathcal{U}$ at 0 and H is a second-countable admissible subgyrogroup generated from $\mathcal{U}$. If the quotient space G/H is a cosmic space, G is also a cosmic space. In the proof of [8, Corollary 3.7], the Gyroassociative Law of subgyrogroup H is predominantly applied. Consequently, as per Theorem 2.9, we arrive at a particular conclusion regarding an invariant subgyrogroup H. Considering that the proof methodology mirrors that in [8, Corollary 3.7], the elaboration of the proof has been excluded.

Theorem 4.5. *If G is a strongly topological gyrogroup and H is a second-countable closed invariant subgyrogroup of G. If the quotient space G/H has countable network, G is also has countable network.*

Theorem 4.6. *Let G be a strongly paratopological gyrocommutative gyrogroup and H be a second-countable invariant topological subgyrogroup of G. If the paratopological gyrogroup G/H has a countable network, then so does G.*

Proof. Should G/H possess a countable network, the same property applies to $(G/H)^*$ as indicated in Proposition 2.26. Following Proposition 4.2, it's established that G^*/H also has a countable network. It's evident that the topological gyrogroup H adopts its topology from that of G^*. Given that H is second-countable, an application of Theorem 4.5 leads us to conclude that G^* is equipped with a countable network. Consequently, G has countable network as well. □

Problem 4.7. *Can the condition 'gyrocommutative' in Theorem 4.6 be omitted?*

Corollary 4.8. *Let G be a strongly paratopological gyrocommutative gyrogroup and H be a second-countable locally compact invariant subgyrogroup of G. If the paratopological gyrogroup G/H has a countable network, then so does G.*

Proof. Since every locally compact strongly paratopological gyrogroup is a topological gyrogroup [17, Theorem 3.5] the statement directly follows from Theorem 4.6. □

A paratopological gyrogroup G is *saturated* if for every non-empty open set U in G, its inverse $\ominus U$ has non-empty interior. Clearly, every topological gyrogroup is saturated.

Proposition 4.9. *Let the neighborhood base $\mathcal{U}$ at 0 of G witness that G is a strongly paratopological gyrogroup and H be an invariant subgyrogroup of G. Suppose that H is saturated and G/H is a topological gyrogroup. Then G is saturated.*

Proof. Let $U \in \mathcal{U}$. Since G is a paratopological gyrogroup, H is saturated, and G/H is a topological gyrogroup, there exist neighborhoods $W, V, O \in \mathcal{U}$, and $h \in \ominus O \cap H$ such that $W \subset V \subset O$, $O \oplus O \subset U$, $(h \oplus (V \oplus V)) \cap H = h \oplus ((V \oplus V) \cap H) \subset \ominus O \cap H$, and $W \subset H \ominus V$. We claim that $h \oplus W \subset \ominus U$. Let $x \in h \oplus W$. There are elements $w \in W$, $h_1 \in H$, and $v \in V$ such that $x = h \oplus w = h \oplus (h_1 \ominus v) = (h \oplus h_1) \ominus \mathrm{gyr}[h, h_1]v \in (h \oplus h_1) \ominus V$, by the definition of strongly paratopological gyrogroup. Put $h \oplus h_1 = h_2 \in H$, $v_1 \in V$. Thus $x = h \oplus w = h_2 \ominus v_1$. By Lemmas 2.28, 2.29 we get $h_2 = (h \oplus w) \boxplus v_1 \in ((h \oplus V) \oplus V) \cap H \subset (h \oplus (V \oplus V)) \cap H \subset \ominus O \cap H \subset \ominus O$. Therefore, $x = h_2 \ominus v_1 \in \ominus O \ominus O \subset \ominus U$. We just have shown that interior of $\ominus U$ is non-empty. Since U was arbitrary, we conclude that G is saturated. □

In the following, we consider the three-space property of countable tightness for paratopological gyrogroups.

Topological properties of Hausdorff spaces which are both invariants and inverse invariants under perfect mappings are called *perfect properties*. A class of all Hausdorff spaces that have some fixed perfect property is called a *perfect class of spaces*. It is known that classes of regular spaces, compact spaces, paracompact spaces and k-spaces are perfect (see [11]). Since the map $\pi : G \to G/H$ is perfect by Proposition 2.36, we can establish the following results.

Theorem 4.10. *Let H be a compact strongly L-subgyrogroup of a paratopological gyrogroup G and π the canonical quotient mapping from G to G/H. If G/H has a property $\mathcal{P}$, then G also has the property $\mathcal{P}$, where $\mathcal{P}$ is one of the following properties: (1) being regular spaces; (2) being k-spaces; (3) being paracompact.*

By Proposition 2.36 and [11, Exercises 3.12.8(d)], one can easily get the following theorem.

Theorem 4.11. *Let H be a compact strongly L-subgyrogroup of a paratopological gyrogroup G. If H and G/H have countable tightness, then G has countable tightness.*

Acknowledgments

We would like to extend our deepest gratitude to the referees for their insightful comments and suggestions. Their constructive feedback has significantly improved the quality and clarity of this manuscript.

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