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Dynamical behavior of solution of twenty-fourth order rational difference equation

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Abstract. This paper examines discrete-time systems, which are sometimes used to explain nonlinear natural phenomena in the sciences. Specifically, we investigate the boundedness, oscillation, stability, and exact solutions of nonlinear difference equations. We obtain these solutions using the standard iteration method and test the stability of equilibrium points using well-known theorems. We also provide numerical examples to validate our theoretical work and implement the numerical component using Wolfram Mathematica. The method presented can be easily applied to other rational recursive problems.

In this paper, we explore the dynamics of adhering to rational difference formula

$$x_{n+1} = \frac{x_{n-23}}{\pm 1 \pm x_{n-5}x_{n-11}x_{n-17}x_{n-23}},$$

where the initials are arbitrary nonzero real numbers.

1. Introduction

Differential equations are commonly employed to describe the evolution of various natural phenomena over time. However, discrete time steps, modeled using difference equations, find application in addressing real-life problems. Recursive equations play a crucial and potent role in mathematics, effectively exploring applications across engineering, physics, biology, economics, and more. Notably, they have been instrumental in modeling diverse phenomena, including population size, the Fibonacci sequence, drug distribution in the blood system, information transmission, commodity pricing, and the growth of annual plants [11].

Furthermore, scholars have utilized difference equations to numerically solve certain differential equations. The discretization of a given differential equation results in a difference equation, exemplified by the derivation of the Runge-Kutta scheme from a first-order differential equation. This prompts inquiries into the convergence of the difference scheme to the differential equation solution and the correspondence between the properties of their solutions.

Addressing these concerns, [27] focuses on conserving a solution bounded across the entire axis during the transition from differential to difference equations and vice versa. Similarly, [8] explores the preservation

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of the oscillatory property of solutions to second-order equations. The advent of technology has spurred the application of recurrence equations as approximations to partial differential equations. Fractional order difference equations frequently find use in investigating nonlinear phenomena in the sciences.

Numerous researchers have extensively delved into properties of recursive expressions, investigating stability, periodicity, boundedness, and solutions of recursive equations. Noteworthy works include Alayachi et al., examined the periodicity and global attractivity in solving sixth order difference equations, obtaining precise solutions with the aid of the Fibonacci sequence. [4]. Sanbo and Elsayed present findings on the stability and solutions of a fifth-order recursive sequence in [34]. Almatrafi and Alzubaidi [5] have examined the dynamic behaviors of an eighth-degree difference equation and presented the results 2D graphics. Ahmed et al. [3] contribute by discovering new solutions and conducting a dynamical analysis for certain nonlinear fifteenth-order difference relations. Further discussions on nonlinear recursive problems can be found in [1–42].

This article is motivated by the study of eighteenth-order difference equations presented in [31]. We are now delving into more intricate rational difference equations of the twenty-fourth order. Hence, the objective of this study is to examine various dynamical properties including equilibrium points, local and global behaviors, boundedness, and analytical solutions of the nonlinear recursive sequences,

$$x_{n+1} = \frac{x_{n-23}}{\pm 1 \pm x_{n-5}x_{n-11}x_{n-17}x_{n-23}}. \tag{1}$$

Here, the initial values $x_{-23}, x_{-22}, x_{-21}, \ldots, x_{-2}, x_{-1}, x_0$, are arbitrary non-zero real numbers. In this work, we also illustrate some 2D figures with the help of Wolfram Mathematica to validate the obtained results.

Assume I is an interval of real numbers, and let $f : I^{k+1} \to I$ be a continuously differentiable function. For any set of initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, the following difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots, \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [29]. An equilibrium point $\overline{x} \in I$ is defined for equation (2) if

$$\overline{x} = f(, \overline{x}, \overline{x}, \ldots, \overline{x}).$$

In other words, $x_n = \overline{x}$ for $n \geq 0$ constitutes a solution to equation (2), or equivalently, $\overline{x}$ is a fixed point of f.

2. Preliminaries

Definition 2.1. (Stability)

1. The equilibrium point $\overline{x}$ of equation (2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq k.$$

2. The equilibrium point $\overline{x}$ of equation (2) is called locally asymptotically stable if $\overline{x}$ is a locally stable solution of equation (2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

3. The equilibrium point $\overline{x}$ of equation (2) is called a global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

4. The equilibrium point $\overline{x}$ of equation (2) is called a global asymptotically stable if $\overline{x}$ is locally stable and $\overline{x}$ is also a global attractor of equation (2).
5. The equilibrium point $\overline{x}$ of equation (2) is called unstable if $\overline{x}$ is not locally stable. The linearized equation of equation (2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem 2.2. (see[28]) *Assume that* $p, q \in \mathbb{R}$ *and* $k \in \mathbb{N}_0$. *Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n \in \mathbb{N}_0.$$

Remark 2.3. Theorem 2.2 can be easily extended to general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n \in \mathbb{N}_0, \tag{3}$$

where, $p_1, p_2, ..., p_k \in \mathbb{R}$ and $k \in \mathbb{N}$. Then (3) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Definition 2.4. The equilibrium point $\overline{x}$ is said to be hyperbolic if $|f(\overline{x})| \neq 1$. If $|f(\overline{x})| = 1$, x is non hyperbolic.

3. On the difference equation $x_{n+1} = \frac{x_{n-23}}{1+x_{n-5}x_{n-11}x_{n-17}x_{n-23}}$

In this section, we give a specific form of the solutions of the difference equation below, provided that the initial conditions are arbitrary real numbers.

$$x_{n+1} = \frac{x_{n-23}}{1 + x_{n-5}x_{n-11}x_{n-17}x_{n-23}}, \tag{4}$$

where,

$$\begin{array}{llllll} x_{-23} = A_{23}, & x_{-22} = A_{22}, & x_{-21} = A_{21}, & x_{-20} = A_{20}, & x_{-19} = A_{19}, & x_{-18} = A_{18}, \\ x_{-17} = A_{17}, & x_{-16} = A_{16}, & x_{-15} = A_{15}, & x_{-14} = A_{14}, & x_{-13} = A_{13}, & x_{-12} = A_{12}, \\ x_{-11} = A_{11}, & x_{-10} = A_{10}, & x_{-9} = A_9, & x_{-8} = A_8, & x_{-7} = A_7, & x_{-6} = A_6, \\ x_{-5} = A_5, & x_{-4} = A_4, & x_{-3} = A_3, & x_{-2} = A_2, & x_{-1} = A_1, & x_0 = A_0. \end{array} \tag{5}$$

Theorem 3.1. *Let $\{x_n\}_{n=-23}^{\infty}$ be a solution of (4). Then,*

$$x_{24n+1} = \frac{A_{23}\prod_{i=0}^{n-1}(1+4iA_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n}(1+(4i+1)A_5A_{11}A_{17}A_{23})}, \quad x_{24n+2} = \frac{A_{22}\prod_{i=0}^{n-1}(1+4iA_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n}(1+(4i+1)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n+3} = \frac{A_{21}\prod_{i=0}^{n-1}(1+4iA_3A_9A_{15}A_{21})}{\prod_{i=0}^{n}(1+(4i+1)A_3A_9A_{15}A_{21})}, \quad x_{24n+4} = \frac{A_{20}\prod_{i=0}^{n-1}(1+4iA_2A_8A_{14}A_{20})}{\prod_{i=0}^{n}(1+(4i+1)A_2A_8A_{14}A_{20})},$$

$$x_{24n+5} = \frac{A_{19}\prod_{i=0}^{n-1}(1+4iA_1A_7A_{13}A_{19})}{\prod_{i=0}^{n}(1+(4i+1)A_1A_7A_{13}A_{19})}, \quad x_{24n+6} = \frac{A_{18}\prod_{i=0}^{n-1}(1+4i)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n}(1+(4i+1)A_0A_6A_{12}A_{18})},$$

$$x_{24n+7} = \frac{A_{17}\prod_{i=0}^{n}(1+(4i+1)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n}(1+(4i+2)A_5A_{11}A_{17}A_{23})}, \quad x_{24n+8} = \frac{A_{16}\prod_{i=0}^{n}(1+(4i+1)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n}(1+(4i+2)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n+9} = \frac{A_{15}\prod_{i=0}^{n}(1+(4i+1)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n}(1+(4i+2)A_3A_9A_{15}A_{21})}, \quad x_{24n+10} = \frac{A_{14}\prod_{i=0}^{n}(1+(4i+1)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n}(1+(4i+2)A_2A_8A_{14}A_{20})},$$

$$x_{24n+11} = \frac{A_{13}\prod_{i=0}^{n}(1+(4i+1)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{t}(1+(4i+2)A_1A_7A_{13}A_{19})}, \quad x_{24n+12} = \frac{A_{12}\prod_{i=0}^{n}(1+(4i+1)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n}(1+(4i+2)A_0A_6A_{12}A_{18})},$$

$$x_{24n+13} = \frac{A_{11}\prod_{i=0}^{n}(1+(4i+2)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n}(1+(4i+3)A_5A_{11}A_{17}A_{23})}, \quad x_{24n+14} = \frac{A_{10}\prod_{i=0}^{n}(1+(4i+2)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n}(1+(4i+3)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n+15} = \frac{A_9\prod_{i=0}^{n}(1+(4i+2)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n}(1+(4i+3)A_3A_9A_{15}A_{21})}, \quad x_{24n+16} = \frac{A_8\prod_{i=0}^{n}(1+(4i+2)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n}(1+(4i+3)A_2A_8A_{14}A_{20})},$$

$$x_{24n+17} = \frac{A_7\prod_{i=0}^{n}(1+(4i+2)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{n}(1+(4i+3)A_1A_7A_{13}A_{19})}, \quad x_{24n+18} = \frac{A_6\prod_{i=0}^{n}(1+(4i+2)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n}(1+(4i+3)A_0A_6A_{12}A_{18})},$$

$$x_{24n+19} = \frac{A_5\prod_{i=0}^{n}(1+(4i+3)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n}(1+(4i+4)A_5A_{11}A_{17}A_{23})}, \quad x_{24n+20} = \frac{A_4\prod_{i=0}^{n}(1+(4i+3)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n}(1+(4i+4)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n+21} = \frac{A_3\prod_{i=0}^{n}(1+(4i+3)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n}(1+(4i+4)A_3A_9A_{15}A_{21})}, \quad x_{24n+22} = \frac{A_2\prod_{i=0}^{n}(1+(4i+3)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n}(1+(4i+4)A_2A_8A_{14}A_{20})},$$

$$x_{24n+23} = \frac{A_1\prod_{i=0}^{n}(1+(4i+3)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{n}(1+(4i+4)A_1A_7A_{13}A_{19})}, \quad x_{24n+24} = \frac{A_0\prod_{i=0}^{n}(1+(4i+3)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n}(1+(4i+4)A_0A_6A_{12}A_{18})},$$

where, $x_0, \ldots, x_{-23}$ defines as in (5).

Proof. The proof of each formula are carried out in similar way. So, we will demonstrate proof using one of the formula. We will employ the mathematical induction method. Suppose that $n > 0$ and that our assumption holds for $n = 1$. That is,

$$x_{24n-23} = \frac{A_{23}\prod_{i=0}^{n-2}(1+4iA_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1+(4i+1)A_5A_{11}A_{17}A_{23})}, \quad x_{24n-22} = \frac{A_{22}\prod_{i=0}^{n-2}(1+4iA_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1+(4i+1)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n-21} = \frac{A_{21}\prod_{i=0}^{n-2}(1+4iA_3A_9A_{15}A_{21})}{\prod_{i=0}^{n-1}(1+(4i+1)A_3A_9A_{15}A_{21})}, \quad x_{24n-20} = \frac{A_{20}\prod_{i=0}^{n-2}(1+4iA_2A_8A_{14}A_{20})}{\prod_{i=0}^{n-1}(1+(4i+1)A_2A_8A_{14}A_{20})},$$

$$x_{24n-19} = \frac{A_{19}\prod_{i=0}^{n-2}(1+4iA_1A_7A_{13}A_{19})}{\prod_{i=0}^{n-1}(1+(4i+1)A_1A_7A_{13}A_{19})}, \quad x_{24n-18} = \frac{A_{18}\prod_{i=0}^{n-2}(1+4i)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n-1}(1+(4i+1)A_0A_6A_{12}A_{18})},$$

$$x_{24n-17} = \frac{A_{17}\prod_{i=0}^{n-1}(1+(4i+1)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1+(4i+2)A_5A_{11}A_{17}A_{23})}, \quad x_{24n-16} = \frac{A_{16}\prod_{i=0}^{n-1}(1+(4i+1)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1+(4i+2)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n-15}=\frac{A_{15}\prod_{i=0}^{n-1}(1+(4i+1)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n-1}(1+(4i+2)A_3A_9A_{15}A_{21})},\quad x_{24n-14}=\frac{A_{14}\prod_{i=0}^{n-1}(1+(4i+1)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n-1}(1+(4i+2)A_2A_8A_{14}A_{20})},$$

$$x_{24n-13}=\frac{A_{13}\prod_{i=0}^{n-1}(1+(4i+1)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{n-1}(1+(4i+2)A_1A_7A_{13}A_{19})},\quad x_{24n-12}=\frac{A_{12}\prod_{i=0}^{n-1}(1+(4i+1)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n-1}(1+(4i+2)A_0A_6A_{12}A_{18})},$$

$$x_{24n-11}=\frac{A_{11}\prod_{i=0}^{n-1}(1+(4i+2)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1+(4i+3)A_5A_{11}A_{17}A_{23})},\quad x_{24n-10}=\frac{A_{10}\prod_{i=0}^{n-1}(1+(4i+2)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1+(4i+3)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n-9}=\frac{A_9\prod_{i=0}^{n-1}(1+(4i+2)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n-1}(1+(4i+3)A_3A_9A_{15}A_{21})},\quad x_{24n-8}=\frac{A_8\prod_{i=0}^{n-1}(1+(4i+2)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n-1}(1+(4i+3)A_2A_8A_{14}A_{20})},$$

$$x_{24n-7}=\frac{A_7\prod_{i=0}^{n-1}(1+(4i+2)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{n-1}(1+(4i+3)A_1A_7A_{13}A_{19})},\quad x_{24n-8}=\frac{A_6\prod_{i=0}^{n-1}(1+(4i+2)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n-1}(1+(4i+3)A_0A_6A_{12}A_{18})},$$

$$x_{24n-5}=\frac{A_5\prod_{i=0}^{n-1}(1+(4i+3)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1+(4i+4)A_5A_{11}A_{17}A_{23})},\quad x_{24n-4}=\frac{A_4\prod_{i=0}^{n-1}(1+(4i+3)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1+(4i+4)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n-3}=\frac{A_3\prod_{i=0}^{n-1}(1+(4i+3)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n-1}(1+(4i+4)A_3A_9A_{15}A_{21})},\quad x_{24n-2}=\frac{A_2\prod_{i=0}^{n-1}(1+(4i+3)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n-1}(1+(4i+4)A_2A_8A_{14}A_{20})},$$

$$x_{24n-1}=\frac{A_1\prod_{i=0}^{n-1}(1+(4i+3)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{n-1}(1+(4i+4)A_1A_7A_{13}A_{19})},\quad x_{24n}=\frac{A_0\prod_{i=0}^{n-1}(1+(4i+3)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n-1}(1+(4i+4)A_0A_6A_{12}A_{18})}.$$

Now, using the main (4), one has

$$\begin{aligned}x_{24n+1}&=\frac{x_{24n-23}}{1+x_{24n-5}x_{24n-11}x_{24n-17}x_{24n-23}},\\&=\frac{\frac{A_{23}\prod_{i=0}^{n-2}(1+4iA_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1+(4i+1)A_5A_{11}A_{17}A_{23})}}{1+\frac{A_5\prod_{i=0}^{n-1}(1+(4i+3)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1+(4i+4)A_5A_{11}A_{17}A_{23})}\frac{A_{11}\prod_{i=0}^{n-1}(1+(4i+2)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1+(4i+3)A_5A_{11}A_{17}A_{23})}\frac{A_{17}\prod_{i=0}^{n-1}(1+(4i+1)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1+(4i+2)A_5A_{11}A_{17}A_{23})}\frac{A_{23}\prod_{i=0}^{n-2}(1+4iA_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1+(4i+1)A_5A_{11}A_{17}A_{23})}}.\end{aligned}$$

Hence, we have

$$x_{24n+1}=\frac{A_{23}\prod_{i=0}^{n-1}(1+4iA_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n}(1+(4i+1)A_5A_{11}A_{17}A_{23})}.$$

Similarly,

$$\begin{aligned}x_{24n+2}&=\frac{x_{24n-22}}{1+x_{24n-4}x_{24n-10}x_{24n-16}x_{24n-22}},\\&=\frac{\frac{A_{22}\prod_{i=0}^{n-2}(1+4iA_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1+(4i+1)A_4A_{10}A_{16}A_{22})}}{1+\frac{A_4\prod_{i=0}^{n-1}(1+(4i+3)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1+(4i+4)A_4A_{10}A_{16}A_{22})}\frac{A_{10}\prod_{i=0}^{n-1}(1+(4i+2)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1+(4i+3)A_4A_{10}A_{16}A_{22})}\frac{A_{16}\prod_{i=0}^{n-1}(1+(4i+1)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1+(4i+2)A_4A_{10}A_{16}A_{22})}\frac{A_{22}\prod_{i=0}^{n-2}(1+4iA_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1+(4i+1)A_4A_{10}A_{16}A_{22})}}.\end{aligned}$$

Therefore, we have

$$x_{24n+2}=\frac{A_{22}\prod_{i=0}^{n-1}(1+4iA_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n}(1+(4i+1)A_4A_{10}A_{16}A_{22})}.$$

Other relations can also be obtained in a similar way, and thus the proof is complete. □

Theorem 3.2. *The equation (4) has unique equilibrium point which is the number zero and this equilibrium is not locally asymptotically stable. Also $\overline{x}$ is non hyperbolic.*

Proof. For the equilibriums of equation (4), we have

$$\overline{x} = \frac{\overline{x}}{1+\overline{x}^4},$$

then

$$\overline{x} + \overline{x}^5 = \overline{x}, \quad \overline{x}^5 = 0.$$

Thus the equilibrium point of (4), is $\overline{x} = 0$.

Let $f : (0,\infty)^4 \to (0,\infty)$ be the function defined by

$$f(l,o,t,w) = \frac{l}{1+lotw}.$$

Therefore it follows that,

$$f_l(l,o,t,w) = \frac{1}{(1+lotw)^2}, \qquad f_o(l,o,t,w) = \frac{-l^2tw}{(1+lotw)^2},$$

$$f_t(l,o,t,w) = \frac{-l^2ow}{(1+lotw)^2}, \qquad f_w(l,o,t,w) = \frac{-l^2ot}{(1+lotw)^2}.$$

We see that

$$f_l(\overline{x},\overline{x},\overline{x},\overline{x}) = 1, \qquad f_o(\overline{x},\overline{x},\overline{x},\overline{x}) = 0, \qquad f_t(\overline{x},\overline{x},\overline{x},\overline{x}) = 0, \qquad f_w(\overline{x},\overline{x},\overline{x},\overline{x}) = 0.$$

The proof now follows by using Theorem 3.1. □

4. On the difference equation $x_{n+1} = \frac{x_{n-23}}{1-x_{n-5}x_{n-11}x_{n-17}x_{n-23}}$

In this part, we give a specific form of the solutions of the difference equation below, provided that the initial conditions are arbitrary real numbers,

$$x_{n+1} = \frac{x_{n-23}}{1-x_{n-5}x_{n-11}x_{n-17}x_{n-23}}, \tag{6}$$

where, $x_0,\ldots,x_{-23}$ defines as in (5).

Theorem 4.1. *Let $\{x_n\}_{n=-23}^{\infty}$ be a solution of equation (6). Then,*

$$x_{24n+1} = \frac{A_{23}\prod_{i=0}^{n-1}(1-4iA_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n}(1-(4i+1)A_5A_{11}A_{17}A_{23})}, \qquad x_{24n+2} = \frac{A_{22}\prod_{i=0}^{n-1}(1-4iA_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n}(1-(4i+1)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n+3} = \frac{A_{21}\prod_{i=0}^{n-1}(1-4iA_3A_9A_{15}A_{21})}{\prod_{i=0}^{n}(1-(4i+1)A_3A_9A_{15}A_{21})}, \qquad x_{24n+4} = \frac{A_{20}\prod_{i=0}^{n-1}(1-4iA_2A_8A_{14}A_{20})}{\prod_{i=0}^{n}(1-(4i+1)A_2A_8A_{14}A_{20})},$$

$$x_{24t+5} = \frac{A_{19}\prod_{i=0}^{n-1}(1-4iA_1A_7A_{13}A_{19})}{\prod_{i=0}^{n}(1-(4i+1)A_1A_7A_{13}A_{19})}, \qquad x_{24n+6} = \frac{A_{18}\prod_{i=0}^{n-1}(1-4i)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n}(1-(4i+1)A_0A_6A_{12}A_{18})},$$

$$x_{24n+7} = \frac{A_{17}\prod_{i=0}^{n}(1-(4i+1)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n}(1-(4i+2)A_5A_{11}A_{17}A_{23})}, \qquad x_{24n+8} = \frac{A_{16}\prod_{i=0}^{n}(1-(4i+1)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n}(1-(4i+2)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n+9} = \frac{A_{15}\prod_{i=0}^{n}(1-(4i+1)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n}(1-(4i+2)A_3A_9A_{15}A_{21})}, \qquad x_{24n+10} = \frac{A_{14}\prod_{i=0}^{n}(1-(4i+1)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n}(1-(4i+2)A_2A_8A_{14}A_{20})},$$

$$x_{24n+11} = \frac{A_{13}\prod_{i=0}^{n}(1-(4i+1)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{t}(1-(4i+2)A_1A_7A_{13}A_{19})}, \qquad x_{24n+12} = \frac{A_{12}\prod_{i=0}^{n}(1-(4i+1)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n}(1-(4i+2)A_0A_6A_{12}A_{18})},$$

$$x_{24n+13} = \frac{A_{11}\prod_{i=0}^{n}(1-(4i+2)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n}(1-(4i+3)A_5A_{11}A_{17}A_{23})}, \qquad x_{24n+14} = \frac{A_{10}\prod_{i=0}^{n}(1-(4i+2)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n}(1-(4i+3)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n+15} = \frac{A_9 \prod_{i=0}^{n}(1-(4i+2)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n}(1-(4i+3)A_3A_9A_{15}A_{21})}, \quad x_{24n+16} = \frac{A_8 \prod_{i=0}^{n}(1-(4i+2)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n}(1-(4i+3)A_2A_8A_{14}A_{20})},$$

$$x_{24n+17} = \frac{A_7 \prod_{i=0}^{n}(1-(4i+2)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{n}(1-(4i+3)A_1A_7A_{13}A_{19})}, \quad x_{24n+18} = \frac{A_6 \prod_{i=0}^{n}(1-(4i+2)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n}(1-(4i+3)A_0A_6A_{12}A_{18})},$$

$$x_{24n+19} = \frac{A_5 \prod_{i=0}^{n}(1-(4i+3)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n}(1-(4i+4)A_5A_{11}A_{17}A_{23})}, \quad x_{24n+20} = \frac{A_4 \prod_{i=0}^{n}(1-(4i+3)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n}(1-(4i+4)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n+21} = \frac{A_3 \prod_{i=0}^{n}(1-(4i+3)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n}(1-(4i+4)A_3A_9A_{15}A_{21})}, \quad x_{24n+22} = \frac{A_2 \prod_{i=0}^{n}(1-(4i+3)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n}(1-(4i+4)A_2A_8A_{14}A_{20})},$$

$$x_{24n+23} = \frac{A_1 \prod_{i=0}^{n}(1-(4i+3)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{n}(1-(4i+4)A_1A_7A_{13}A_{19})}, \quad x_{24n+24} = \frac{A_0 \prod_{i=0}^{n}(1-(4i+3)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n}(1-(4i+4)A_0A_6A_{12}A_{18})},$$

holds.

Proof. Suppose that $n > 0$ and that our assumption holds for $n = 1$. That is,

$$x_{24n-23} = \frac{A_{23} \prod_{i=0}^{n-2}(1-4iA_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1-(4i+1)A_5A_{11}A_{17}A_{23})}, \quad x_{24n-22} = \frac{A_{22} \prod_{i=0}^{n-2}(1-4iA_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1-(4i+1)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n-21} = \frac{A_{21} \prod_{i=0}^{n-2}(1-4iA_3A_9A_{15}A_{21})}{\prod_{i=0}^{n-1}(1-(4i+1)A_3A_9A_{15}A_{21})}, \quad x_{24n-20} = \frac{A_{20} \prod_{i=0}^{n-2}(1-4iA_2A_8A_{14}A_{20})}{\prod_{i=0}^{n-1}(1-(4i+1)A_2A_8A_{14}A_{20})},$$

$$x_{24n-19} = \frac{A_{19} \prod_{i=0}^{n-2}(1-4iA_1A_7A_{13}A_{19})}{\prod_{i=0}^{n-1}(1-(4i+1)A_1A_7A_{13}A_{19})}, \quad x_{24n-18} = \frac{A_{18} \prod_{i=0}^{n-2}(1-4i)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n-1}(1-(4i+1)A_0A_6A_{12}A_{18})},$$

$$x_{24n-17} = \frac{A_{17} \prod_{i=0}^{n-1}(1-(4i+1)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1-(4i+2)A_5A_{11}A_{17}A_{23})}, \quad x_{24n-16} = \frac{A_{16} \prod_{i=0}^{n-1}(1-(4i+1)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1-(4i+2)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n-15} = \frac{A_{15} \prod_{i=0}^{n-1}(1-(4i+1)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n-1}(1-(4i+2)A_3A_9A_{15}A_{21})}, \quad x_{24n-14} = \frac{A_{14} \prod_{i=0}^{n-1}(1-(4i+1)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n-1}(1-(4i+2)A_2A_8A_{14}A_{20})},$$

$$x_{24n-13} = \frac{A_{13} \prod_{i=0}^{n-1}(1-(4i+1)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{n-1}(1-(4i+2)A_1A_7A_{13}A_{19})}, \quad x_{24n-12} = \frac{A_{12} \prod_{i=0}^{n-1}(1-(4i+1)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n-1}(1-(4i+2)A_0A_6A_{12}A_{18})},$$

$$x_{24n-11} = \frac{A_{11} \prod_{i=0}^{n-1}(1-(4i+2)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1-(4i+3)A_5A_{11}A_{17}A_{23})}, \quad x_{24n-10} = \frac{A_{10} \prod_{i=0}^{n-1}(1-(4i+2)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1-(4i+3)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n-9} = \frac{A_9 \prod_{i=0}^{n-1}(1-(4i+2)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n-1}(1-(4i+3)A_3A_9A_{15}A_{21})}, \quad x_{24n-8} = \frac{A_8 \prod_{i=0}^{n-1}(1-(4i+2)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n-1}(1-(4i+3)A_2A_8A_{14}A_{20})},$$

$$x_{24n-7} = \frac{A_7 \prod_{i=0}^{n-1}(1-(4i+2)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{n-1}(1-(4i+3)A_1A_7A_{13}A_{19})}, \quad x_{24n-6} = \frac{A_6 \prod_{i=0}^{n-1}(1-(4i+2)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n-1}(1-(4i+3)A_0A_6A_{12}A_{18})},$$

$$x_{24n-5} = \frac{A_5 \prod_{i=0}^{n-1}(1-(4i+3)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1-(4i+4)A_5A_{11}A_{17}A_{23})}, \quad x_{24n-4} = \frac{A_4 \prod_{i=0}^{n-1}(1-(4i+3)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1-(4i+4)A_4A_{10}A_{16}A_{22})},$$

$$x_{24n-3} = \frac{A_3 \prod_{i=0}^{n-1}(1-(4i+3)A_3A_9A_{15}A_{21})}{\prod_{i=0}^{n-1}(1-(4i+4)A_3A_9A_{15}A_{21})}, \quad x_{24n-2} = \frac{A_2 \prod_{i=0}^{n-1}(1-(4i+3)A_2A_8A_{14}A_{20})}{\prod_{i=0}^{n-1}(1-(4i+4)A_2A_8A_{14}A_{20})},$$

$$x_{24n-1} = \frac{A_1 \prod_{i=0}^{n-1}(1-(4i+3)A_1A_7A_{13}A_{19})}{\prod_{i=0}^{n-1}(1-(4i+4)A_1A_7A_{13}A_{19})}, \quad x_{24n} = \frac{A_0 \prod_{i=0}^{n-1}(1-(4i+3)A_0A_6A_{12}A_{18})}{\prod_{i=0}^{n-1}(1-(4i+4)A_0A_6A_{12}A_{18})}.$$

Now, using the main equation (6), one has

$$x_{24n+1} = \frac{x_{24n-23}}{1 - x_{24n-5}x_{24n-11}x_{24n-17}x_{24n-23}},$$

$$= \frac{\frac{A_{23}\prod_{i=0}^{n-2}(1-4iA_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1-(4i+1)A_5A_{11}A_{17}A_{23})}}{1 - \frac{A_5\prod_{i=0}^{n-1}(1-(4i+3)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1-(4i+4)A_5A_{11}A_{17}A_{23})}\frac{A_{11}\prod_{i=0}^{n-1}(1-(4i+2)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1-(4i+3)A_5A_{11}A_{17}A_{23})}\frac{A_{17}\prod_{i=0}^{n-1}(1-(4i+1)A_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1-(4i+2)A_5A_{11}A_{17}A_{23})}\frac{A_{23}\prod_{i=0}^{n-2}(1-4iA_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n-1}(1-(4i+1)A_5A_{11}A_{17}A_{23})}}.$$

Hence, we have

$$x_{24n+1} = \frac{A_{23}\prod_{i=0}^{n-1}(1 - 4iA_5A_{11}A_{17}A_{23})}{\prod_{i=0}^{n}(1 - (4i+1)A_5A_{11}A_{17}A_{23})}.$$

Similarly,

$$x_{24n+2} = \frac{x_{24n-22}}{1 + x_{24n-4}x_{24n-10}x_{24n-16}x_{24n-22}},$$

$$= \frac{\frac{A_{22}\prod_{i=0}^{n-2}(1-4iA_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1-(4i+1)A_4A_{10}A_{16}A_{22})}}{1 - \frac{A_4\prod_{i=0}^{n-1}(1-(4i+3)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1-(4i+4)A_4A_{10}A_{16}A_{22})}\frac{A_{10}\prod_{i=0}^{n-1}(1-(4i+2)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1-(4i+3)A_4A_{10}A_{16}A_{22})}\frac{A_{16}\prod_{i=0}^{n-1}(1-(4i+1)A_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1-(4i+2)A_4A_{10}A_{16}A_{22})}\frac{A_{22}\prod_{i=0}^{n-2}(1-4iA_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n-1}(1-(4i+1)A_4A_{10}A_{16}A_{22})}}.$$

Therefore, we have

$$x_{24n+2} = \frac{A_{22}\prod_{i=0}^{n-1}(1 - 4iA_4A_{10}A_{16}A_{22})}{\prod_{i=0}^{n}(1 - (4i+1)A_4A_{10}A_{16}A_{22})}.$$

Similarly, it is easily obtained in other relationships. □

Theorem 4.2. *The equation (6) has a unique equilibrium point $\overline{x} = 0$, which is not locally asymptotically stable.*

Proof. The proof is the same as the proof of Theorem 3.2 and hence is omitted. □

5. On the difference equation $x_{n+1} = \frac{x_{n-23}}{-1+x_{n-5}x_{n-11}x_{n-17}x_{n-23}}$

In this case, we give a specific form of the solutions of the difference equation below, provided that the initial conditions are arbitrary real numbers,

$$x_{n+1} = \frac{x_{n-23}}{-1 + x_{n-5}x_{n-11}x_{n-17}x_{n-23}} \tag{7}$$

where, $x_0, \ldots, x_{-23}$ defines as in (5) with $x_{-5}x_{-11}x_{-17}x_{-23} \neq 1$, $x_{-4}x_{-10}x_{-16}x_{-22} \neq 1$, $x_{-3}x_{-9}x_{-15}x_{-21} \neq 1$, $x_{-2}x_{-8}x_{-14}x_{-20} \neq 1$,$x_{-1}x_{-7}x_{-13}x_{-19} \neq 1$, $x_0x_{-6}x_{-12}x_{-18} \neq 1$.

Theorem 5.1. *Let $\{x_n\}_{n=-23}^{\infty}$ be a solution of (7). Then,*

$$x_{24n+1} = \frac{A_{23}}{(-1 + A_5A_{11}A_{17}A_{23})^{n+1}}, \qquad x_{24n+2} = \frac{A_{22}}{(-1 + A_4A_{10}A_{16}A_{22})^{n+1}},$$

$$x_{24n+3} =, \frac{A_{21}}{(-1 + A_3A_9A_{15}A_{21})^{n+1}}, \qquad x_{24n+4} = \frac{A_{20}}{(-1 + A_2A_8A_{14}A_{20})^{n+1}},$$

$$x_{24n+5} = \frac{A_{19}}{(-1 + A_1A_7A_{13}A_{19})^{n+1}}, \qquad x_{24n+6} = \frac{A_{18}}{(-1 + A_0A_6A_{12}A_{18})^{n+1}},$$

$$x_{24n+7} = A_{17}(-1 + A_5A_{11}A_{17}A_{23})^{n+1}, \qquad x_{24n+8} = A_{16}(-1 + A_4A_{10}A_{16}A_{22})^{n+1},$$

$$x_{24n+9} = A_{15}(-1 + A_3A_9A_{15}A_{21})^{n+1}, \qquad x_{24n+10} = A_{14}(-1 + A_2A_8A_{14}A_{20})^{n+1},$$

$$x_{24n+11} = A_{13}(-1 + A_1A_7A_{13}A_{19})^{n+1}, \qquad x_{24n+12} = A_{12}(-1 + A_0A_6A_{12}A_{18})^{n+1},$$

$$x_{24n+13} = \frac{A_{11}}{(-1 + A_5A_{11}A_{17}A_{23})^{n+1}}, \qquad x_{24n+14} = \frac{A_{10}}{(-1 + A_4A_{10}A_{16}A_{22})^{n+1}},$$
$$x_{24n+15} =, \frac{A_{11}}{(-1 + A_3A_9A_{15}A_{21})^{n+1}}, \qquad x_{24n+16} = \frac{A_8}{(-1 + A_2A_8A_{14}A_{20})^{n+1}},$$
$$x_{24n+17} = \frac{A_7}{(-1 + A_1A_7A_{13}A_{19})^{n+1}}, \qquad x_{24n+18} = \frac{A_6}{(-1 + A_0A_6A_{12}A_{18})^{n+1}},$$
$$x_{24n+19} = A_5(-1 + A_5A_{11}A_{17}A_{23})^{n+1}, \qquad x_{24n+20} = A_4(-1 + A_4A_{10}A_{16}A_{22})^{n+1},$$
$$x_{24n+21} = A_3(-1 + A_3A_9A_{15}A_{21})^{n+1}, \qquad x_{24n+22} = A_2(-1 + A_2A_8A_{14}A_{20})^{n+1},$$
$$x_{24n+23} = A_1(-1 + A_1A_7A_{13}A_{19})^{n+1}, \qquad x_{24n+24} = A_0(-1 + A_0A_6A_{12}A_{18})^{n+1}.$$

Proof. Suppose,

$$x_{24n-23} = \frac{A_{23}}{(-1 + A_5A_{11}A_{17}A_{23})^{n}}, \qquad x_{24n-22} = \frac{A_{22}}{(-1 + A_4A_{10}A_{16}A_{22})^{n}},$$
$$x_{24n-21} =, \frac{A_{21}}{(-1 + A_3A_9A_{15}A_{21})^{n}}, \qquad x_{24n-20} = \frac{A_{20}}{(-1 + A_2A_8A_{14}A_{20})^{n}},$$
$$x_{24n-19} = \frac{A_{19}}{(-1 + A_1A_7A_{13}A_{19})^{n}}, \qquad x_{24n-18} = \frac{A_{18}}{(-1 + A_0A_6A_{12}A_{18})^{n}},$$
$$x_{24n-17} = A_{17}(-1 + A_5A_{11}A_{17}A_{23})^{n}, \qquad x_{24n-16} = A_{16}(-1 + A_4A_{10}A_{16}A_{22})^{n},$$
$$x_{24n-15} = A_{15}(-1 + A_3A_9A_{15}A_{21})^{n}, \qquad x_{24n-14} = A_{14}(-1 + A_2A_8A_{14}A_{20})^{n},$$
$$x_{24n-13} = A_{13}(-1 + A_1A_7A_{13}A_{19})^{n}, \qquad x_{24n-12} = A_{12}(-1 + A_0A_6A_{12}A_{18})^{n},$$
$$x_{24n-11} = \frac{A_{11}}{(-1 + A_5A_{11}A_{17}A_{23})^{n}}, \qquad x_{24n-10} = \frac{A_{10}}{(-1 + A_4A_{10}A_{16}A_{22})^{n}},$$
$$x_{24n-9} =, \frac{A_{11}}{(-1 + A_3A_9A_{15}A_{21})^{n}}, \qquad x_{24n-8} = \frac{A_8}{(-1 + A_2A_8A_{14}A_{20})^{n}},$$
$$x_{24n-7} = \frac{A_7}{(-1 + A_1A_7A_{13}A_{19})^{n}}, \qquad x_{24n-6} = \frac{A_6}{(-1 + A_0A_6A_{12}A_{18})^{n}},$$
$$x_{24n-5} = A_5(-1 + A_5A_{11}A_{17}A_{23})^{n}, \qquad x_{24n-4} = A_4(-1 + A_4A_{10}A_{16}A_{22})^{n},$$
$$x_{24n-3} = A_3(-1 + A_3A_9A_{15}A_{21})^{n}, \qquad x_{24n-2} = A_2(-1 + A_2A_8A_{14}A_{20})^{n},$$
$$x_{24n-1} = A_1(-1 + A_1A_7A_{13}A_{19})^{n}, \qquad x_{24n} = A_0(-1 + A_0A_6A_{12}A_{18})^{n}.$$

Now, it follows from equation (7) that

$$\begin{aligned} x_{24n+1} &= \frac{x_{24n-23}}{-1 + x_{24n-5}x_{24n-11}x_{24n-17}x_{24n-23}}, \\ &= \frac{\frac{A_{23}}{(-1+A_5A_{11}A_{17}A_{23})^n},}{-1 + A_5(-1 + A_5A_{11}A_{17}A_{23})^n \frac{A_{11}}{(-1+A_5A_{11}A_{17}A_{23})^n} A_{17}(-1 + A_5A_{11}A_{17}A_{23})^n \frac{A_{23}}{(-1+A_5A_{11}A_{17}A_{23})^n}}. \end{aligned}$$

Then, we have

$$x_{24n+1} = \frac{A_{23}}{(-1 + A_5A_{11}A_{17}A_{23})^{n+1}}.$$

Other relation can be given by the same way. □

Theorem 5.2. *The equation (7) has three equilibrium points which are* $0, \pm\sqrt[4]{2}$*, and these equilibrium points are not locally asymptotically stable.*

Proof. The proof is the same as the proof of Theorem 3.2 and hence is omitted. □

6. On the difference equation $x_{n+1} = \frac{x_{n-23}}{-1-x_{n-5}x_{n-11}x_{n-17}x_{n-23}}$

In this section, we give a specific form of the solutions of the difference equation below, provided that the initial conditions are arbitrary real numbers,

$$x_{n+1} = \frac{x_{n-23}}{-1 - x_{n-5}x_{n-11}x_{n-17}x_{n-23}}, \tag{8}$$

where, $x_0, \dots, x_{-23}$ defines as in (5) with $x_{-5}x_{-11}x_{-17}x_{-23} \neq -1$, $x_{-4}x_{-10}x_{-16}x_{-22} \neq -1$, $x_{-3}x_{-9}x_{-15}x_{-21} \neq -1$, $x_{-2}x_{-8}x_{-14}x_{-20} \neq -1$,$x_{-1}x_{-7}x_{-13}x_{-19} \neq -1$, $x_0x_{-6}x_{-12}x_{-18} \neq -1$.

Theorem 6.1. *Let $\{x_n\}_{n=-23}^{\infty}$ be a solution of (8). Then,*

$$x_{24n+1} = \frac{A_{23}}{(-1 - A_5A_{11}A_{17}A_{23})^{n+1}}, \qquad x_{24n+2} = \frac{A_{22}}{(-1 - A_4A_{10}A_{16}A_{22})^{n+1}},$$
$$x_{24n+3} =, \frac{A_{21}}{(-1 - A_3A_9A_{15}A_{21})^{n+1}}, \qquad x_{24n+4} = \frac{A_{20}}{(-1 - A_2A_8A_{14}A_{20})^{n+1}},$$
$$x_{24n+5} = \frac{A_{19}}{(-1 - A_1A_7A_{13}A_{19})^{n+1}}, \qquad x_{24n+6} = \frac{A_{18}}{(-1 - A_0A_6A_{12}A_{18})^{n+1}},$$
$$x_{24n+7} = A_{17}(-1 - A_5A_{11}A_{17}A_{23})^{n+1}, \qquad x_{24n+8} = A_{16}(-1 - A_4A_{10}A_{16}A_{22})^{n+1},$$
$$x_{24n+9} = A_{15}(-1 - A_3A_9A_{15}A_{21})^{n+1}, \qquad x_{24n+10} = A_{14}(-1 - A_2A_8A_{14}A_{20})^{n+1},$$
$$x_{24n+11} = A_{13}(-1 - A_1A_7A_{13}A_{19})^{n+1}, \qquad x_{24n+12} = A_{12}(-1 - A_0A_6A_{12}A_{18})^{n+1},$$
$$x_{24n+13} = \frac{A_{11}}{(-1 - A_5A_{11}A_{17}A_{23})^{n+1}}, \qquad x_{24n+14} = \frac{A_{10}}{(-1 - A_4A_{10}A_{16}A_{22})^{n+1}},$$
$$x_{24n+15} =, \frac{A_{11}}{(-1 - A_3A_9A_{15}A_{21})^{n+1}}, \qquad x_{24n+16} = \frac{A_8}{(-1 - A_2A_8A_{14}A_{20})^{n+1}},$$
$$x_{24n+17} = \frac{A_7}{(-1 - A_1A_7A_{13}A_{19})^{n+1}}, \qquad x_{24n+18} = \frac{A_6}{(-1 - A_0A_6A_{12}A_{18})^{n+1}},$$
$$x_{24n+19} = A_5(-1 - A_5A_{11}A_{17}A_{23})^{n+1}, \qquad x_{24n+20} = A_4(-1 - A_4A_{10}A_{16}A_{22})^{n+1},$$
$$x_{24n+21} = A_3(-1 - A_3A_9A_{15}A_{21})^{n+1}, \qquad x_{24n+22} = A_2(-1 - A_2A_8A_{14}A_{20})^{n+1},$$
$$x_{24n+23} = A_1(-1 - A_1A_7A_{13}A_{19})^{n+1}, \qquad x_{24n+24} = A_0(-1 - A_0A_6A_{12}A_{18})^{n+1}.$$

holds.

Proof. Suppose

$$x_{24n-23} = \frac{A_{23}}{(-1 - A_5A_{11}A_{17}A_{23})^{n}}, \qquad x_{24n-22} = \frac{A_{22}}{(-1 - A_4A_{10}A_{16}A_{22})^{n}},$$
$$x_{24n-21} =, \frac{A_{21}}{(-1 - A_3A_9A_{15}A_{21})^{n}}, \qquad x_{24n-20} = \frac{A_{20}}{(-1 - A_2A_8A_{14}A_{20})^{n}},$$
$$x_{24n-19} = \frac{A_{19}}{(-1 - A_1A_7A_{13}A_{19})^{n}}, \qquad x_{24n-18} = \frac{A_{18}}{(-1 - A_0A_6A_{12}A_{18})^{n}},$$
$$x_{24n-17} = A_{17}(-1 - A_5A_{11}A_{17}A_{23})^{n}, \qquad x_{24n-16} = A_{16}(-1 - A_4A_{10}A_{16}A_{22})^{n+1},$$
$$x_{24n-15} = A_{15}(-1 - A_3A_9A_{15}A_{21})^{n}, \qquad x_{24n-14} = A_{14}(-1 - A_2A_8A_{14}A_{20})^{n},$$
$$x_{24n-13} = A_{13}(-1 - A_1A_7A_{13}A_{19})^{n}, \qquad x_{24n-12} = A_{12}(-1 - A_0A_6A_{12}A_{18})^{n},$$
$$x_{24n-11} = \frac{A_{11}}{(-1 - A_5A_{11}A_{17}A_{23})^{n}}, \qquad x_{24n-10} = \frac{A_{10}}{(-1 - A_4A_{10}A_{16}A_{22})^{n}},$$
$$x_{24n*9} =, \frac{A_{11}}{(-1 - A_3A_9A_{15}A_{21})^{n}}, \qquad x_{24n-8} = \frac{A_8}{(-1 - A_2A_8A_{14}A_{20})^{n}},$$

$$x_{24n-7} = \frac{A_7}{(-1 - A_1A_7A_{13}A_{19})^n}, \qquad x_{24n-6} = \frac{A_6}{(-1 - A_0A_6A_{12}A_{18})^n},$$
$$x_{24n-5} = A_5(-1 - A_5A_{11}A_{17}A_{23})^n, \qquad x_{24n-4} = A_4(-1 - A_4A_{10}A_{16}A_{22})^n,$$
$$x_{24n-3} = A_3(-1 - A_3A_9A_{15}A_{21})^n, \qquad x_{24n-2} = A_2(-1 - A_2A_8A_{14}A_{20})^n,$$
$$x_{24n-1} = A_1(-1 - A_1A_7A_{13}A_{19})^n, \qquad x_{24n} = A_0(-1 - A_0A_6A_{12}A_{18})^n.$$

Now, it follows from equation (8) that,

$$\begin{aligned} x_{24n+1} &= \frac{x_{24n-23}}{-1 - x_{24n-5}x_{24n-11}x_{24n-17}x_{24n-23}}, \\ &= \frac{\frac{A_{23}}{(-1-A_5A_{11}A_{17}A_{23})^n},}{-1 - A_5(-1 - A_5A_{11}A_{17}A_{23})^n \frac{A_{11}}{(-1-A_5A_{11}A_{17}A_{23})^n} A_{17}(-1 - A_5A_{11}A_{17}A_{23})^n \frac{A_{23}}{(-1-A_5A_{11}A_{17}A_{23})^n}}. \end{aligned}$$

Then, we have

$$x_{24n+1} = \frac{A_{23}}{(-1 - A_5A_{11}A_{17}A_{23})^{n+1}}.$$

Other relations can be given by the same way. □

Theorem 6.2. *The equation (8) has three equilibrium point which are* $0, \pm\sqrt[4]{-2}$ *and this equilbrium points is not locally asymptotically stable.*

Proof. The proof is the same as the proof of Theorem 3.2 and hence is omitted. □

7. Numerical examples

We devote this section to verify the theoretical work obtained in this article.

Example 7.1. For Eq. 4 and 6 we consider following initial conditions.

$x_{-23} = 0.33$, $x_{-22} = 0.32$, $x_{-21} = 0.31$, $x_{-20} = 0.3$, $x_{-19} = 0.29$, $x_{-18} = 0.28$,
$x_{-17} = 0.27$, $x_{-16} = 0.26$, $x_{-15} = 0.25$, $x_{-14} = 0.24$, $x_{-13} = 0.23$, $x_{-12} = 0.235$,
$x_{-11} = 0.245$, $x_{-10} = 0.255$, $x_{-9} = 0.265$, $x_{-8} = 0.275$, $x_{-7} = 0.285$, $x_{-6} = 0.295$,
$x_{-5} = 0.305$, $x_{-4} = 0.315$, $x_{-3} = 0.325$, $x_{-2} = 0.335$, $x_{-1} = 0.345$, $x_0 = 0.355$.

Example 7.2. For Eq. 7 and 8 we consider following initial conditions.

$x_{-23} = 0.5$, $x_{-22} = 0.6$, $x_{-21} = 0.55$, $x_{-20} = 0.65$, $x_{-19} = 0.58$, $x_{-18} = 0.57$,
$x_{-17} = 0.56$, $x_{-16} = 0.49$, $x_{-15} = 0.48$, $x_{-14} = 0.29$, $x_{-13} = 0.63$, $x_{-12} = 0.62$,
$x_{-11} = 0.26$, $x_{-10} = 0.25$, $x_{-9} = 0.24$, $x_{-8} = 0.23$, $x_{-7} = 0.22$, $x_{-6} = 0.21$,
$x_{-5} = 0.2$, $x_{-4} = 0.185$, $x_{-3} = 0.295$, $x_{-2} = 0.293$, $x_{-1} = 0.435$, $x_0 = 0.475$.

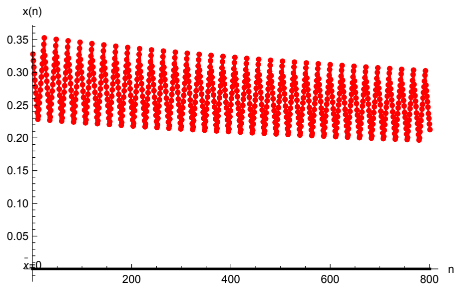

Figure 1: plot illustrates the stability of Eq. 4

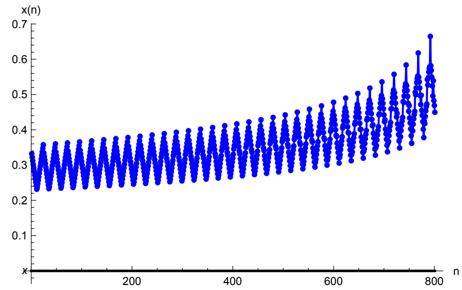

Figure 2: plot illustrates the stability of Eq. 6

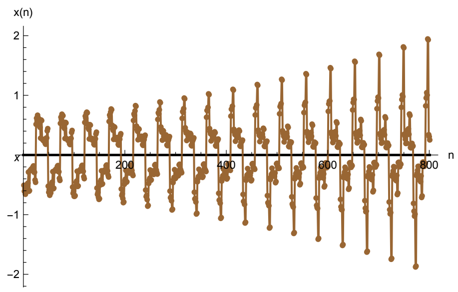

Figure 3: plot illustrates the stability of Eq. 7

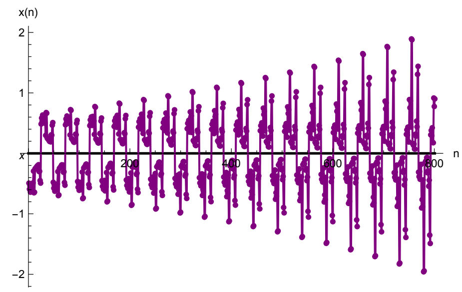

Figure 4: plot illustrates the stability of Eq. 8

8. Conclusion

We study the behavior of the difference equation

$$x_{n+1} = \frac{x_{n-23}}{\pm 1 \pm x_{n-5}x_{n-11}x_{n-17}x_{n-23}},$$

where the initials are positive real numbers. Local stability is discussed. Moreover,we get the solution of some special cases. Finally, some numerical examples are given.

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