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A new family of symmetric and generating functions of binary products of (p,q)-numbers at consecutive and nonconsecutive terms

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Abstract. In this study, by making use of the symmetrizing operator $\delta_{a_1a_2}^{k+1}$ we introduce a new theorem. By using this theorem we give a new class of generating functions of the products of (p,q)-Fibonacci numbers, (p,q)-Lucas numbers, (p,q)-Pell numbers, (p,q)-Pell Lucas numbers, (p,q)-Jacobsthal numbers and (p,q)-Jacobsthal Lucas numbers at consecutive and nonconsecutive terms and the products of these (p,q)-numbers with Mersenne numbers at consecutive and nonconsecutive terms.

1. Introduction

The concept of Mersenne numbers was originally introduced by Marin Mersenne, these numbers are defined as:

$$M_n := \begin{cases} 0, & \text{if } n=0 \\ 1, & \text{if } n=1 \\ 3M_{n-1}-2M_{n-2}, & \text{if } n\geq 2 \end{cases}, \tag{1}$$

or

$$\{M_n\}_{n\in\mathbb{N}} = \{0,\ 1,\ 3,\ 7,\ 15,\ 31,\ 63,\ 127,\ 255,\ 511,\ 1023,\ 2047,\ 4095...\}.$$

From the definition of Mersenne numbers, we have that the characteristic equation of (1) is in the form

$$z^2-3z+2=0, \tag{2}$$

the roots of equation (2) are $a_1=2$ and $a_2=1$ and we easily get the Binet's formula

$$M_n = 2^n - 1.$$

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The generating function of Mersenne numbers is given by

$$M_n = \frac{z}{1-3z+2z^2}, \text{ with } M_n = S_{n-1}(a_1 + [-a_2]).$$

For more informations of Mersenne numbers see [18].

The authors in [12] defined and studied the generalized (p,q)-numbers. Considering this sequence, they gave the Binet's formulas and generating functions of (p,q)-Fibonacci numbers, (p,q)-Lucas numbers, (p,q)-Pell numbers, (p,q)-Pell Lucas numbers, (p,q)-Jacobsthal numbers and (p,q)-Jacobsthal Lucas numbers.

The generalized (p,q)-numbers $\left\{W_{p,q,n}\right\}_{n\in\mathbb{N}}$ is given by the following recurrence relation:

$$W_{p,q,n} = apW_{p,q,n-1} + bqW_{p,q,n-2},\ n \geq 2,$$

with $W_{p,q,0} = \alpha$, $W_{p,q,1} = \beta p + \gamma$ and $\{a,b,\alpha,\beta,\gamma\} \in \mathbb{C}$. The special cases of the numbers $W_{p,q,n}$ are listed as follows:

1° For $a = b = \gamma = 1$ and $\alpha = \beta = 0$ we get the (p,q)-Fibonacci numbers $F_{p,q,n}$ (see [5]).
2° For $a = b = \beta = 1$, $\alpha = 2$ and $\gamma = 0$ it yields (p,q)-Lucas numbers $L_{p,q,n}$ (see [4]).
3° For $a = \gamma = 1$, $\alpha = \beta = 0$ and $b = 2$ it reduces to the (p,q)-Jacobsthal numbers $J_{p,q,n}$ (see [19]).
4° For $a = \beta = 1$, $\alpha = b = 2$ and $\gamma = 0$ we get the (p,q)-Jacobsthal Lucas numbers $j_{p,q,n}$ (see [19]).
5° For $b = \gamma = 1$, $\alpha = \beta = 0$ and $a = 2$ it yields the (p,q)-Pell numbers $P_{p,q,n}$ (see [7]).
6° For $a = \alpha = \beta = 2$, $b = 1$ and $\gamma = 0$ it reduces to the (p,q)-Pell Lucas numbers $Q_{p,q,n}$ (see [7]).

The Binet's formula for generalized (p,q)-numbers is given by

$$W_{p,q,n} = \frac{Ax_1^n - Bx_2^n}{x_1 - x_2},$$

with $A = \beta p + \gamma - \alpha x_2$ and $B = \beta p + \gamma - \alpha x_1$, where $x_1 = \frac{ap+\sqrt{a^2p^2+4bq}}{2}$ and $x_2 = \frac{ap-\sqrt{a^2p^2+4bq}}{2}$ are roots of the characteristic equation $x^2 - apx - bq = 0$. We note that

$$x_1 + x_2 = ap,\ x_1x_2 = -bq \text{ and } x_1 - x_2 = \sqrt{a^2p^2 + 4bq}.$$

The special cases of the Binet's formula for generalized (p,q) -numbers are listed in the table below:

a	b	α	β	γ	Roots (x_1 and x_2)	Binet's formula $\left(W_{p,q,n}\right)$
1	1	0	0	1	$x_1 = \frac{p+\sqrt{p^2+4q}}{2}, x_2 = \frac{p-\sqrt{p^2+4q}}{2}$	$F_{p,q,n} = \frac{x_1^n-x_2^n}{x_1-x_2}$
1	1	2	1	0	$x_1 = \frac{p+\sqrt{p^2+4q}}{2}, x_2 = \frac{p-\sqrt{p^2+4q}}{2}$	$L_{p,q,n} = x_1^n + x_2^n$
1	2	0	0	1	$x_1 = \frac{p+\sqrt{p^2+8q}}{2}, x_2 = \frac{p-\sqrt{p^2+8q}}{2}$	$J_{p,q,n} = \frac{x_1^n-x_2^n}{x_1-x_2}$
1	2	2	1	0	$x_1 = \frac{p+\sqrt{p^2+8q}}{2}, x_2 = \frac{p-\sqrt{p^2+8q}}{2}$	$j_{p,q,n} = x_1^n + x_2^n$
2	1	0	0	1	$x_1 = p + \sqrt{p^2+q}, x_2 = p - \sqrt{p^2+q}$	$P_{p,q,n} = \frac{x_1^n-x_2^n}{x_1-x_2}$
2	1	2	2	0	$x_1 = p + \sqrt{p^2+q}, x_2 = p - \sqrt{p^2+q}$	$Q_{p,q,n} = x_1^n + x_2^n$

Table 1: Binet's formulas for some (p,q)-numbers.

Generating functions are one of the most surprising and useful inventions in mathematics. Roughly speaking generating functions transforms problems about functions. Sometimes we can find nice generating functions for more complicated sequences. For example, the generating function of generalized (p,q)-numbers $\left(W_{p,q,n}\right)$ is $\frac{\alpha+(p(\beta-a\alpha)+\gamma)z}{1-apz-bqz^2}$. Similarly, the generating function for Mersenne numbers (M_n) is $\frac{z}{1-3z+2z^2}$.

The authors in [10], introduced the following generating functions: $\sum_{n=0}^{\infty} F_{p,q,n}^2 z^n$, $\sum_{n=0}^{\infty} L_{p,q,n}^2 z^n$, $\sum_{n=0}^{\infty} P_{p,q,n}^2 z^n$, $\sum_{n=0}^{\infty} Q_{p,q,n}^2 z^n$, $\sum_{n=0}^{\infty} J_{p,q,n}^2 z^n$, $\sum_{n=0}^{\infty} j_{p,q,n}^2 z^n$, $\sum_{n=0}^{\infty} F_{p,q,n} M_n z^n$, $\sum_{n=0}^{\infty} L_{p,q,n} M_n z^n$, $\sum_{n=0}^{\infty} P_{p,q,n} M_n z^n$, $\sum_{n=0}^{\infty} Q_{p,q,n} M_n z^n$, $\sum_{n=0}^{\infty} J_{p,q,n} M_n z^n$ and

$\sum\limits_{n=0}^{\infty} j_{p,q,n}M_n z^n$. In this work, we investigate the generating functions of the products of these numbers at consecutive and nonconsecutive terms.

The rest of the paper is organized as follows. In the next section we present some backgrounds about the symmetric functions. In section 3, we prove our main result which relates the symmetric function defined in the previous section with the symmetrizing operator $\delta^{k+1}_{a_1a_2}$. The new generating functions of the products of (p,q)-Fibonacci, (p,q)-Lucas, (p,q)-Pell, (p,q)-Pell Lucas, (p,q) -Jacobsthal and (p,q)-Jacobsthal Lucas numbers at consecutive and nonconsecutive terms are derived in section 4. By making use of the operator defined in this paper the new generating functions of the products of (p,q)-numbers with Mersenne numbers at consecutive and nonconsecutive terms are given in section 5.

2. Some preliminary properties

This section is devoted to recalling some preliminary facts and results on the symmetric functions. Let us now start at the following definition.

Definition 2.1. *[9] Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables the k-th complete homogeneous symmetric function $h_k(a_1, a_2, ..., a_n)$ is defined by*

$$h_k(a_1, a_2, ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1}a_2^{i_2}...a_n^{i_n} \qquad (k \geq 0),$$

with $i_1, i_2, ..., i_n \geq 0$.

Remark 2.2. *Set $h_0(a_1, a_2, ..., a_n) = 1$, by usual convention. For $k < 0$, we set $h_k(a_1, a_2, ..., a_n) = 0$.*

Definition 2.3. *[2] Let A and P be any two alphabets. We define $S_n(A - P)$ by the following form*

$$\frac{\prod\limits_{p\in P}(1-pz)}{\prod\limits_{a\in A}(1-az)} = \sum_{n=0}^{\infty} S_n(A-P)z^n, \tag{3}$$

with the condition $S_n(A - P) = 0$ for $n < 0$.

Equation (3) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(A-P)z^n = \left(\sum_{n=0}^{\infty} S_n(A)z^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)z^n\right),$$

where

$$S_n(A-P) = \sum_{j=0}^{n} S_{n-j}(-P)S_j(A).$$

Definition 2.4. *[8] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows*

$$\partial_{a_ia_{i+1}}(f) = \frac{f(a_1,\cdots,a_i,a_{i+1},\cdots,a_n) - f(a_1,\cdots,a_{i-1},a_{i+1},a_i,a_{i+2},\cdots,a_n)}{a_i - a_{i+1}}.$$

Definition 2.5. *[17] Let n be positive integer and $A = \{a_1, a_2\}$ are set of given variables. Then, the n-th symmetric function $S_n(a_1 + a_2)$ is defined by*

$$S_n(A) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

with

$$\begin{aligned} S_0(A) &= S_0(a_1+a_2)=1, \\ S_1(A) &= S_1(a_1+a_2)=a_1+a_2, \\ S_2(A) &= S_2(a_1+a_2)=a_1^2+a_1a_2+a_2^2, \\ &\vdots \end{aligned}$$

Proposition 2.6. *[12] For $n \in \mathbb{N}$, we have the symmetric functions for some (p,q)-numbers as*

$$\begin{aligned} F_{p,q,n} &= S_{n-1}(a_1+[-a_2]) \text{ and } L_{p,q,n} = 2S_n(a_1+[-a_2]) - pS_{n-1}(a_1+[-a_2]), \text{ with } a_{1,2} = \frac{p \pm \sqrt{p^2+4q}}{2}, \\ J_{p,q,n} &= S_{n-1}(a_1+[-a_2]) \text{ and } j_{p,q,n} = 2S_n(a_1+[-a_2]) - pS_{n-1}(a_1+[-a_2]), \text{ with } a_{1,2} = \frac{p \pm \sqrt{p^2+8q}}{2}, \\ P_{p,q,n} &= S_{n-1}(a_1+[-a_2]) \text{ and } Q_{p,q,n} = 2S_n(a_1+[-a_2]) - 2pS_{n-1}(a_1+[-a_2]), \text{ with } a_{1,2} = p \pm \sqrt{p^2+q}. \end{aligned}$$

Definition 2.7. *[3] Given an alphabet $A = \{a_1, a_2\}$, the symmetrizing operator $\delta^k_{a_1a_2}$ is defined by*

$$\delta^k_{a_1a_2} f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \text{ for all } k \in \mathbb{N}_0 := \{\mathbb{N} \cup \{0\}\} = \{0,1,2,\dots\}. \quad (4)$$

If $f(a_1) = a_1^{n+1}$, the operator (4) gives us

$$\begin{aligned} \delta^k_{a_1a_2}(a_1^{n+1}) &= \frac{a_1^{k+n+1} - a_2^{k+n+1}}{a_1 - a_2} \\ &= S_{k+n}(a_1+a_2). \end{aligned}$$

Proposition 2.8. *[1] Let $A = \{a_1, a_2\}$ an alphabet, we define the operator $\delta^k_{a_1a_2}$ as follows:*

$$\delta^k_{a_1a_2}(f) = S_{k-1}(a_1+a_2)\, f(a_1) + a_2^k \partial_{a_1a_2}(f), \text{ for all } k \in \mathbb{N}_0.$$

3. The main results

In this part, we are now in a position to provide new theorem by using the symmetrizing operator $\delta^{k+1}_{a_1a_2}$.

Theorem 3.1. *Given two alphabets $A = \{a_1, a_2\}$ and $E = \{e_1, e_2, \dots\}$, we have*

$$\sum_{n=0}^{\infty} S_{n+k}(A)\, S_n(E)\, z^n = \frac{S_k(A) \sum\limits_{n=0}^{\infty} S_n(-E)\, a_2^n z^n - a_2^{k+1} \sum\limits_{n=0}^{\infty} S_n(-E)\, S_{n-1}(A)\, z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)\, a_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\, a_2^n z^n\right)}, \quad (5)$$

for all $n, k \in \mathbb{N}_0$.

Proof. By applying the operator $\delta^{k+1}_{a_1a_2}$ to the series $f(a_1z) = \sum\limits_{n=0}^{\infty} S_n(E)\, a_1^n z^n$, the left-hand side of the formula (5) can be written as:

$$\delta^{k+1}_{a_1a_2} f(a_1z) = \frac{a_1^{k+1} \sum\limits_{n=0}^{\infty} S_n(E)\, a_1^n z^n - a_2^{k+1} \sum\limits_{n=0}^{\infty} S_n(E)\, a_2^n z^n}{a_1 - a_2}$$

$$= \sum_{n=0}^{\infty} S_n(E)\left(\frac{a_1^{n+k+1}-a_2^{n+k+1}}{a_1-a_2}\right)z^n$$
$$= \sum_{n=0}^{\infty} S_{n+k}(A)\, S_n(E)\, z^n.$$

By applying the operator $\delta_{a_1a_2}^{k+1}$ to the series $f(a_1z) = \frac{1}{\sum\limits_{n=0}^{\infty} S_n(-E)a_1^n z^n}$, the right-hand side of the formula (5) can be expressed as:

$$\begin{aligned}
\partial_{a_1a_2} f(a_1z) &= \frac{\frac{1}{\sum\limits_{n=0}^{\infty} S_n(-E)a_1^nz^n} - \frac{1}{\sum\limits_{n=0}^{\infty} S_n(-E)a_2^nz^n}}{a_1-a_2} \\
&= \frac{\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^nz^n - \sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^nz^n}{(a_1-a_2)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^nz^n\right)} \\
&= \frac{-\sum\limits_{n=0}^{\infty} S_n(-E)\,S_{n-1}(A)\,z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^nz^n\right)}.
\end{aligned}$$

By proposition 2.8, it follows that

$$\begin{aligned}
\delta_{a_1a_2}^{k+1} f(a_1z) &= \frac{S_k(A)}{\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^nz^n} - a_2^{k+1}\frac{\sum\limits_{n=0}^{\infty} S_n(-E)\,S_{n-1}(A)\,z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^nz^n\right)} \\
&= \frac{S_k(A)\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^nz^n - a_2^{k+1}\sum\limits_{n=0}^{\infty} S_n(-E)\,S_{n-1}(A)\,z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^nz^n\right)}. \\
&= \frac{\sum\limits_{n=0}^{\infty} S_n(-E)\left(S_k(A)\,a_2^n - a_2^{k+1}S_{n-1}(A)\right)z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-E)\,a_2^nz^n\right)}.
\end{aligned}$$

Thus, this completes the proof. □

- For $A = \{a_1, a_2\}$, $E = \{e_1, e_2\}$ and $k \in \{1,\ 2,\ 3\}$ in theorem 3.1 we deduce the following lemmas.

Lemma 3.2. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, then*

$$\sum_{n=0}^{\infty} S_{n+1}(a_1+a_2)S_n(e_1+e_2)z^n = \frac{(a_1+a_2) - a_1a_2\,(e_1+e_2)\,z}{\prod\limits_{e\in E}(1-ea_1z)\prod\limits_{e\in E}(1-ea_2z)}. \tag{6}$$

Proof. By applying the operator $\delta_{a_1a_2}^2$ to the series $f(a_1z) = \sum\limits_{n=0}^{\infty} S_n(E)\,a_1^nz^n$, we prove this result by the same method given in theorem 3.1. □

Lemma 3.3. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, then*

$$\sum_{n=0}^{\infty} S_{n+2}(a_1 + a_2) S_n(e_1 + e_2) z^n = \frac{(a_1 + a_2)^2 - a_1a_2 - a_1a_2 (e_1 + e_2)(a_1 + a_2) z + e_1e_2 (a_1a_2)^2 z^2}{\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 - ea_2z)}. \quad (7)$$

Proof. By applying the operator $\delta^3_{a_1a_2}$ to the series $f(a_1z) = \sum_{n=0}^{\infty} S_n(E) a_1^n z^n$,the result can be proved by the same method given in theorem 3.1. □

Note that, based on the relationship (7), we get

$$\sum_{n=0}^{\infty} S_{n+1}(a_1 + a_2) S_{n-1}(e_1 + e_2) z^n = \frac{\left((a_1 + a_2)^2 - a_1a_2\right) z - a_1a_2 (e_1 + e_2)(a_1 + a_2) z^2 + e_1e_2 (a_1a_2)^2 z^3}{\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 - ea_2z)}. \quad (8)$$

Lemma 3.4. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, then*

$$\sum_{n=0}^{\infty} S_{n+3}(a_1 + a_2) S_n(e_1 + e_2) z^n \quad (9)$$

$$= \frac{(a_1 + a_2)\left((a_1 + a_2)^2 - 2a_1a_2\right) - a_1a_2 (e_1 + e_2)\left((a_1 + a_2)^2 - a_1a_2\right) z - e_1e_2 (a_1a_2)^2 (a_1 + a_2) z^2}{\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 - ea_2z)}.$$

Proof. By applying the operator $\delta^4_{a_1a_2}$ to the series $f(a_1z) = \sum_{n=0}^{\infty} S_n(E) a_1^n z^n$, we prove this result by the same method given in theorem 3.1. □

From relationship (9), we obtain

$$\sum_{n=0}^{\infty} S_{n+2}(a_1 + a_2) S_{n-1}(e_1 + e_2) z^n \quad (10)$$

$$= \frac{(a_1 + a_2)\left((a_1 + a_2)^2 - 2a_1a_2\right) z - a_1a_2 (e_1 + e_2)\left((a_1 + a_2)^2 - a_1a_2\right) z^2 - e_1e_2 (a_1a_2)^2 (a_1 + a_2) z^3}{\prod_{e\in E}(1 - ea_1z)\prod_{e\in E}(1 - ea_2z)}.$$

4. Generating functions of the products of (p, q)-numbers at consecutive and nonconsecutive terms

In this part, we are now in a position to provide theorems. Also we derive the new generating functions of the products of (p, q) numbers at consecutive and nonconsecutive indices.

- In the following, we replacing a_2 by $(-a_2)$ and e_2 by $(-e_2)$ in (6), (7), (8) and (10), we obtain:

$$\sum_{n=0}^{\infty} S_{n+1}(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n = \frac{(a_1 - a_2) + a_1a_2 (e_1 - e_2) z}{(1 - a_1e_1z)(1 + a_2e_1z)(1 + a_1e_2z)(1 - a_2e_2z)}, \quad (11)$$

$$\sum_{n=0}^{\infty} S_{n+2}(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n = \frac{(a_1 - a_2)^2 + a_1a_2 + a_1a_2 (e_1 - e_2)(a_1 - a_2) z - e_1e_2 (a_1a_2)^2 z^2}{(1 - a_1e_1z)(1 + a_2e_1z)(1 + a_1e_2z)(1 - a_2e_2z)}, \quad (12)$$

$$\sum_{n=0}^{\infty} S_{n+1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n = \frac{\left((a_1 - a_2)^2 + a_1a_2\right) z + a_1a_2 (e_1 - e_2)(a_1 - a_2) z^2 - e_1e_2 (a_1a_2)^2 z^3}{(1 - a_1e_1z)(1 + a_2e_1z)(1 + a_1e_2z)(1 - a_2e_2z)}, \quad (13)$$

$$\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \tag{14}$$
$$= \frac{(a_1-a_2)\left((a_1-a_2)^2+2a_1a_2\right)z+a_1a_2(e_1-e_2)\left((a_1-a_2)^2+a_1a_2\right)z^2+e_1e_2(a_1a_2)^2(a_1-a_2)z^3}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)},$$

respectively. This part consists of three cases.

Case 1. *Put $a_1-a_2=e_1-e_2=p$ and $a_1a_2=e_1e_2=q$ in the relationships* (11), (12), (13) *and* (14), *we get*

$$\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{pz+pqz}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}, \tag{15}$$

$$\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{p^2+q+p^2qz-q^3z^2}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}, \tag{16}$$

$$\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{\left(p^2+q\right)z+p^2qz^2-q^3z^3}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}, \tag{17}$$

$$\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{p\left(p^2+2q\right)z+pq\left(p^2+q\right)z^2-pq^3z^3}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}, \tag{18}$$

respectively, and we deduce the following proposition and theorems.

Proposition 4.1. *For $n\in\mathbb{N}$, the new generating function of the product of (p,q) -Fibonacci numbers $\left(F_{p,q,n+2}F_{p,q,n}\right)$ is given by*

$$\sum_{n=0}^{\infty} F_{p,q,n+2}F_{p,q,n}z^n = \frac{\left(p^2+q\right)z+p^2qz^2-q^3z^3}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}, \tag{19}$$

with $F_{p,q,n+2}F_{p,q,n} = S_{n+1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])$.

Theorem 4.2. *Let n be a natural number. Then we have the following new generating function for $\left(F_{p,q,n+1}F_{p,q,n}\right)$ as:*

$$\sum_{n=0}^{\infty} F_{p,q,n+1}F_{p,q,n}z^n = \frac{pz+pqz^2}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}. \tag{20}$$

Proof. By (1°) in introduction, we get

$$F_{p,q,n+2} = pF_{p,q,n+1}+qF_{p,q,n}.$$

Then, we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_{p,q,n+2}F_{p,q,n}z^n &= \sum_{n=0}^{\infty}\left(pF_{p,q,n+1}+qF_{p,q,n}\right)F_{p,q,n}z^n\\ &= p\sum_{n=0}^{\infty}F_{p,q,n+1}F_{p,q,n}z^n+q\sum_{n=0}^{\infty}F_{p,q,n}^2z^n,\end{aligned}$$

since

$$\sum_{n=0}^{\infty} F_{p,q,n}^2z^n = \frac{z-q^2z^3}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}, \text{ (see [10]).}$$

We have

$$\begin{aligned}\sum_{n=0}^{\infty} F_{p,q,n+1}F_{p,q,n}z^n &= \frac{1}{p}\left[\sum_{n=0}^{\infty} F_{p,q,n+2}F_{p,q,n}z^n - q\sum_{n=0}^{\infty} F_{p,q,n}^2 z^n\right] \\ &= \frac{1}{p}\left[\frac{\left(p^2+q\right)z + p^2qz^2 - q^3z^3}{1-p^2z-2q\left(p^2+q\right)z^2-p^2q^2z^3+q^4z^4}\right] \\ &\quad -\frac{q}{p}\left[\frac{z-q^2z^3}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}\right] \\ &= \frac{1}{p}\left[\frac{p^2z+p^2qz^2}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}\right],\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} F_{p,q,n+1}F_{p,q,n}z^n = \frac{pz+pqz^2}{1-p^2z-2q\left(p^2+q\right)z^2-p^2q^2z^3+q^4z^4}.$$

Thus, this completes the proof. □

Theorem 4.3. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Lucas numbers $\left(L_{p,q,n+2}L_{p,q,n}\right)$ is given by*

$$\sum_{n=0}^{\infty} L_{p,q,n+2}L_{p,q,n}z^n = \frac{2\left(p^2+2q\right) - p^2\left(p^2+q\right)z - q\left(p^4+2p^2q+4q^2\right)z^2 + p^2q^3z^3}{1-p^2z-2q\left(p^2+q\right)z^2-p^2q^2z^3+q^4z^4}. \tag{21}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n+2}L_{p,q,n}z^n &= \sum_{n=0}^{\infty}\begin{pmatrix} (2S_{n+2}(a_1+[-a_2]) - pS_{n+1}(a_1+[-a_2])) \\ \times(2S_n(e_1+[-e_2]) - pS_{n-1}(e_1+[-e_2]))\end{pmatrix} z^n \\ &= 4\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\ &\quad -2p\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\ &\quad -2p\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\ &\quad +p^2\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.\end{aligned}$$

Using the relationships (15), (16), (17) and (18), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n+2}L_{p,q,n}z^n &= \frac{4\left(p^2+q+qp^2z-q^3z^2\right)}{1-p^2z-2q\left(p^2+q\right)z^2-p^2q^2z^3+q^4z^4} \\ &\quad -\frac{2p\left(p\left(p^2+2q\right)z + pq\left(p^2+q\right)z^2 - pq^3z^3\right)}{1-p^2z-2q\left(p^2+q\right)z^2-p^2q^2z^3+q^4z^4} \\ &\quad -\frac{2p\left(p+pqz\right)}{1-p^2z-2q\left(p^2+q\right)z^2-p^2q^2z^3+q^4z^4}\end{aligned}$$

$$+\frac{p^2\left(\left(p^2+q\right)z+qp^2z^2-q^3z^3\right)}{1-p^2z-2q\left(p^2+q\right)z^2-p^2q^2z^3+q^4z^4}$$
$$=\frac{2\left(p^2+2q\right)-p^2\left(p^2+q\right)z-q\left(p^4+2p^2q+4q^2\right)z^2+p^2q^3z^3}{1-p^2z-2q\left(p^2+q\right)z^2-p^2q^2z^3+q^4z^4}.$$

So, the proof is completed. □

Theorem 4.4. *Let n be a natural number. Then we have the following new generating function for* $\left(L_{p,q,n+1}L_{p,q,n}\right)$ *as:*

$$\sum_{n=0}^{\infty}L_{p,q,n+1}L_{p,q,n}z^n=\frac{2p+p\left(2q-p^2\right)z+pq\left(2q-p^2\right)z^2+2pq^3z^3}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}. \tag{22}$$

Proof. By (2°) in introduction, we get

$$L_{p,q,n+2}=pL_{p,q,n+1}+qL_{p,q,n}.$$

We know that

$$\begin{aligned}\sum_{n=0}^{\infty}L_{p,q,n+2}L_{p,q,n}z^n &= \sum_{n=0}^{\infty}\left(pL_{p,q,n+1}+qL_{p,q,n}\right)L_{p,q,n}z^n\\ &= p\sum_{n=0}^{\infty}L_{p,q,n+1}L_{p,q,n}z^n+q\sum_{n=0}^{\infty}L_{p,q,n}^2z^n.\end{aligned}$$

On the other hand, we have

$$\sum_{n=0}^{\infty}L_{p,q,n}^2z^n=\frac{4-3p^2z-4q\left(q+p^2\right)z^2-p^2q^2z^3}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4},\ \text{(see [10])}.$$

From which it follows

$$\begin{aligned}\sum_{n=0}^{\infty}L_{p,q,n+1}L_{p,q,n}z^n &= \frac{1}{p}\left[\sum_{n=0}^{\infty}L_{p,q,n+2}L_{p,q,n}z^n-q\sum_{n=0}^{\infty}L_{p,q,n}^2z^n\right]\\ &= \frac{1}{p}\left[\frac{2p^2+p^2\left(2q-p^2\right)z+p^2q\left(2q-p^2\right)z^2+2p^2q^3z^3}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}\right],\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty}L_{p,q,n+1}L_{p,q,n}z^n=\frac{2p+p\left(2q-p^2\right)z+pq\left(2q-p^2\right)z^2+2pq^3z^3}{1-p^2z-2q(p^2+q)z^2-p^2q^2z^3+q^4z^4}.$$

This completes the proof. □

Corollary 4.5. *Putting* $p=k$ *and* $q=1$ *in Eqs. (19)-(22) and (10) gives the following new generating functions:*

$$\begin{aligned}\sum_{n=0}^{\infty}F_{k,n+2}F_{k,n}z^n &= \frac{\left(k^2+1\right)z+k^2z^2-z^3}{1-k^2z-2\left(k^2+1\right)z^2-k^2z^3+z^4}.\\ \sum_{n=0}^{\infty}F_{k,n+1}F_{k,n}z^n &= \frac{kz+kz^2}{1-k^2z-2\left(k^2+1\right)z^2-k^2z^3+z^4}.\end{aligned}$$

$$\sum_{n=0}^{\infty} L_{k,n+2}L_{k,n}z^n = \frac{2\left(k^2+2\right)-k^2\left(k^2+1\right)z-\left(k^4+2k^2+4\right)z^2+k^2z^3}{1-k^2z-2\left(k^2+1\right)z^2-k^2z^3+z^4}.$$

$$\sum_{n=0}^{\infty} L_{k,n+1}L_{k,n}z^n = \frac{2k+k\left(2-k^2\right)z+k\left(2-k^2\right)z^2+2kz^3}{1-k^2z-2\left(k^2+1\right)z^2-k^2z^3+z^4}.$$

- Put $k=1$ in the Corollary 4.6, we obtain the following table:

Coefficient of z^n	**Generating function**
$F_{n+2}F_n$	$\frac{2z+z^2-z^3}{1-z-4z^2-z^3+z^4}$
$F_{n+1}F_n$	$\frac{z+z^2}{1-z-4z^2-z^3+z^4}$
$L_{n+2}L_n$	$\frac{6-2z-7z^2+z^3}{1-z-4z^2-z^3+z^4}$
$L_{n+1}L_n$	$\frac{2+z+z^2+2z^3}{1-z-4z^2-z^3+z^4}$

Table 2: The new generating functions of the products of some numbers.

Case 2. *Put $a_1-a_2=e_1-e_2=p$ and $a_1a_2=e_1e_2=2q$ in the relationships* (11), (12), (13) *and* (14), *we obtain*

$$\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{p+2pqz}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4}, \tag{23}$$

$$\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{p^2+2q+2p^2qz-8q^3z^2}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4}, \tag{24}$$

$$\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{\left(p^2+2q\right)z+2p^2qz^2-8q^3z^3}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4}, \tag{25}$$

$$\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{p\left(p^2+4q\right)z+2pq\left(p^2+2q\right)z^2-8pq^3z^3}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4}, \tag{26}$$

respectively, thus we get the following proposition and theorems.

Proposition 4.6. *For $n\in\mathbb{N}$, the new generating function of the product of (p,q) -Jacobsthal numbers $\left(J_{p,q,n+2}J_{p,q,n}\right)$ is given by*

$$\sum_{n=0}^{\infty} J_{p,q,n+2}J_{p,q,n}z^n = \frac{\left(p^2+2q\right)z+2p^2qz^2-8q^3z^3}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4}, \tag{27}$$

with $J_{p,q,n+2}J_{p,q,n}=S_{n+1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])$.

Theorem 4.7. *Let n be a natural number. Then we have the following new generating function for $\left(J_{p,q,n+1}J_{p,q,n}\right)$ as:*

$$\sum_{n=0}^{\infty} J_{p,q,n+1}J_{p,q,n}z^n = \frac{pz+2pqz^2}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4}. \tag{28}$$

Proof. By (3°) in introduction, we get

$$J_{p,q,n+2}=pJ_{p,q,n+1}+2qJ_{p,q,n}.$$

Then, we have

$$\begin{aligned}\sum_{n=0}^{\infty} J_{p,q,n+2}J_{p,q,n}z^n &= \sum_{n=0}^{\infty}\left(pJ_{p,q,n+1}+2qJ_{p,q,n}\right)J_{p,q,n}z^n\\ &= p\sum_{n=0}^{\infty} J_{p,q,n+1}J_{p,q,n}z^n+2q\sum_{n=0}^{\infty} J_{p,q,n}^2 z^n,\end{aligned}$$

since

$$\sum_{n=0}^{\infty} J_{p,q,n}^2 z^n=\frac{z-4q^2z^3}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4},\ \text{(see [10])}.$$

Then, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} J_{p,q,n+1}J_{p,q,n}z^n &= \frac{1}{p}\left[\sum_{n=0}^{\infty} J_{p,q,n+2}J_{p,q,n}z^n-2q\sum_{n=0}^{\infty} J_{p,q,n}^2 z^n\right]\\ &= \frac{1}{p}\left[\frac{p^2z+2p^2qz^2}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}\right],\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} J_{p,q,n+1}J_{p,q,n}z^n=\frac{pz+2pqz^2}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4}.$$

Thus, this completes the proof. □

Theorem 4.8. *For $n\in\mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal Lucas numbers $\left(j_{p,q,n+2}\,j_{p,q,n}\right)$ is given by*

$$\sum_{n=0}^{\infty} j_{p,q,n+2}j_{p,q,n}z^n=\frac{2\left(p^2+4q\right)-p^2\left(p^2+2q\right)z-2q\left(p^4+4p^2q+16q^2\right)z^2+8p^2q^3z^3}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4}. \tag{29}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} j_{p,q,n+2}j_{p,q,n}z^n &= \sum_{n=0}^{\infty}\begin{pmatrix}(2S_{n+2}(a_1+[-a_2])-pS_{n+1}(a_1+[-a_2]))\\ \times(2S_n(e_1+[-e_2])-pS_{n-1}(e_1+[-e_2]))\end{pmatrix}z^n\\ &= 4\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_n(e_1+[-e_2])z^n\\ &\quad -2p\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n\\ &\quad -2p\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n\\ &\quad +p^2\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.\end{aligned}$$

Using the relationships (23), (24), (25) and (26), we obtain

$$\sum_{n=0}^{\infty} j_{p,q,n+2}j_{p,q,n}z^n = \frac{4\left(p^2+2q+2qp^2z-8q^3z^2\right)}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4}$$

$$
\begin{aligned}
& -\frac{2p\left(p\left(p^2+4q\right)z+2pq\left(p^2+2q\right)z^2-8pq^3z^3\right)}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4} \\
& -\frac{2p\left(p+2pqz\right)}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4} \\
& +\frac{p^2\left(\left(p^2+2q\right)z+2qp^2z^2-8q^3z^3\right)}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4} \\
= & \frac{2\left(p^2+4q\right)-p^2\left(p^2+2q\right)z-2q\left(p^4+4p^2q+16q^2\right)z^2+8p^2q^3z^3}{1-p^2z-4q\left(p^2+2q\right)z^2-4p^2q^2z^3+16q^4z^4}.
\end{aligned}
$$

So, the proof is completed. □

Theorem 4.9. *Let n be a natural number. Then we have the following new generating function for $\left(j_{p,q,n+1}j_{p,q,n}\right)$ as:*

$$
\sum_{n=0}^{\infty} j_{p,q,n+1}j_{p,q,n}z^n = \frac{2p+p\left(4q-p^2\right)z+2pq\left(4q-p^2\right)z^2+16pq^3z^3}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}. \tag{30}
$$

Proof. By (4°) in introduction, we get

$$
j_{p,q,n+2} = pj_{p,q,n+1} + 2qj_{p,q,n}.
$$

We know that

$$
\begin{aligned}
\sum_{n=0}^{\infty} j_{p,q,n+2}j_{p,q,n}z^n &= \sum_{n=0}^{\infty}\left(pj_{p,q,n+1}+2qj_{p,q,n}\right)j_{p,q,n}z^n \\
&= p\sum_{n=0}^{\infty} j_{p,q,n+1}j_{p,q,n}z^n + 2q\sum_{n=0}^{\infty} j_{p,q,n}^2 z^n.
\end{aligned}
$$

On the other hand, we have

$$
\sum_{n=0}^{\infty} j_{p,q,n}^2 z^n = \frac{4-3p^2z-8q\left(p^2+2q\right)z^2-4p^2q^2z^3}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}, \text{ (see [10]).}
$$

From which it follows

$$
\begin{aligned}
\sum_{n=0}^{\infty} j_{p,q,n+1}j_{p,q,n}z^n &= \frac{1}{p}\left[\sum_{n=0}^{\infty} j_{p,q,n+2}j_{p,q,n}z^n - 2q\sum_{n=0}^{\infty} j_{p,q,n}^2 z^n\right] \\
&= \frac{1}{p}\left[\frac{2p^2+p^2\left(4q-p^2\right)z+2p^2q\left(4q-p^2\right)z^2+16p^2q^3z^3}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}\right],
\end{aligned}
$$

therefore

$$
\sum_{n=0}^{\infty} j_{p,q,n+1}j_{p,q,n}z^n = \frac{2p+p\left(4q-p^2\right)z+2pq\left(4q-p^2\right)z^2+16pq^3z^3}{1-p^2z-4q(p^2+2q)z^2-4p^2q^2z^3+16q^4z^4}.
$$

This completes the proof. □

Corollary 4.10. *Setting $p = k$ and $q = 1$ in Eqs. (27)-(30) gives the following new generating functions:*

$$\sum_{n=0}^{\infty} J_{k,n+2}J_{k,n}z^n = \frac{\left(k^2+2\right)z + 2k^2z^2 - 8z^3}{1 - k^2z - 4\left(k^2+2\right)z^2 - 4k^2z^3 + 16z^4}.$$

$$\sum_{n=0}^{\infty} J_{k,n+1}J_{k,n}z^n = \frac{kz + 2kz^2}{1 - k^2z - 4\left(k^2+2\right)z^2 - 4k^2z^3 + 16z^4}.$$

$$\sum_{n=0}^{\infty} j_{k,n+2}j_{k,n}z^n = \frac{2\left(k^2+4\right) - k^2\left(k^2+2\right)z - 2\left(k^4+4k^2+16\right)z^2 + 8k^2z^3}{1 - k^2z - 4\left(k^2+2\right)z^2 - 4k^2z^3 + 16z^4}.$$

$$\sum_{n=0}^{\infty} j_{k,n+1}j_{k,n}z^n = \frac{2k + k\left(4-k^2\right)z + 2k\left(4-k^2\right)z^2 + 16kz^3}{1 - k^2z - 4\left(k^2+2\right)z^2 - 4k^2z^3 + 16z^4}.$$

- Put $k = 1$ in the Corollary 4.12, we obtain the following table:

Coefficient of z^n	**Generating function**
$J_{n+2}J_n$	$\frac{3z+2z^2-8z^3}{1-z-12z^2-4z^3+16z^4}$
$J_{n+1}J_n$	$\frac{z+2z^2}{1-z-12z^2-4z^3+16z^4}$
$j_{n+2}j_n$	$\frac{10-3z-42z^2+8z^3}{1-z-12z^2-4z^3+16z^4}$
$j_{n+1}j_n$	$\frac{2+3z+6z^2+16z^3}{1-z-12z^2-4z^3+16z^4}$

Table 3: The new generating functions of the products of some numbers.

Case 3. *Put $a_1 - a_2 = e_1 - e_2 = 2p$ and $a_1a_2 = e_1e_2 = q$ in the relationships* (11), (12), (13) *and* (14), *we obtain*

$$\sum_{n=0}^{\infty} S_{n+1}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n = \frac{2p + 2pqz}{1 - 4p^2z - 2q\left(4p^2+q\right)z^2 - 4p^2q^2z^3 + q^4z^4}, \tag{31}$$

$$\sum_{n=0}^{\infty} S_{n+2}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n = \frac{4p^2 + q + 4p^2qz - q^3z^2}{1 - 4p^2z - 2q\left(4p^2+q\right)z^2 - 4p^2q^2z^3 + q^4z^4}, \tag{32}$$

$$\sum_{n=0}^{\infty} S_{n+1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n = \frac{\left(4p^2+q\right)z + 4p^2qz^2 - q^3z^3}{1 - 4p^2z - 2q\left(4p^2+q\right)z^2 - 4p^2q^2z^3 + q^4z^4}, \tag{33}$$

$$\sum_{n=0}^{\infty} S_{n+2}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n = \frac{4p\left(2p^2+q\right)z + 2pq\left(4p^2+q\right)z^2 - 2pq^3z^3}{1 - 4p^2z - 2q\left(4p^2+q\right)z^2 - 4p^2q^2z^3 + q^4z^4}, \tag{34}$$

respectively, thus we get the following proposition and theorems.

Proposition 4.11. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell numbers $\left(P_{p,q,n+2}P_{p,q,n}\right)$ is given by*

$$\sum_{n=0}^{\infty} P_{p,q,n+2}P_{p,q,n}z^n = \frac{\left(4p^2+q\right)z + 4p^2qz^2 - q^3z^3}{1 - 4p^2z - 2q\left(4p^2+q\right)z^2 - 4p^2q^2z^3 + q^4z^4}, \tag{35}$$

with $P_{p,q,n+2}P_{p,q,n} = S_{n+1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])$.

Theorem 4.12. *Let n be a natural number. Then we have the following new generating function for* $\left(P_{p,q,n+1}P_{p,q,n}\right)$ *as:*

$$\sum_{n=0}^{\infty} P_{p,q,n+1}P_{p,q,n}z^n = \frac{2pz+2pqz^2}{1-4p^2z-2q\left(4p^2+q\right)z^2-4p^2q^2z^3+q^4z^4}. \tag{36}$$

Proof. By (5°) in introduction, we get

$$P_{p,q,n+2} = 2pP_{p,q,n+1} + qP_{p,q,n}.$$

Then, we have

$$\begin{aligned}\sum_{n=0}^{\infty} P_{p,q,n+2}P_{p,q,n}z^n &= \sum_{n=0}^{\infty}\left(2pP_{p,q,n+1}+qP_{p,q,n}\right)P_{p,q,n}z^n \\ &= 2p\sum_{n=0}^{\infty} P_{p,q,n+1}P_{p,q,n}z^n + q\sum_{n=0}^{\infty} P_{p,q,n}^2 z^n,\end{aligned}$$

since

$$\sum_{n=0}^{\infty} P_{p,q,n}^2 z^n = \frac{z-q^2z^3}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4}, \text{ (see [10]).}$$

Then, we get

$$\begin{aligned}\sum_{n=0}^{\infty} P_{p,q,n+1}P_{p,q,n}z^n &= \frac{1}{2p}\left[\sum_{n=0}^{\infty} P_{p,q,n+2}P_{p,q,n}z^n - q\sum_{n=0}^{\infty} P_{p,q,n}^2 z^n\right] \\ &= \frac{1}{2p}\left[\frac{4p^2z+4p^2qz^2}{1-4p^2z-2q\left(4p^2+q\right)z^2-4p^2q^2z^3+q^4z^4}\right],\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} P_{p,q,n+1}P_{p,q,n}z^n = \frac{2pz+2pqz^2}{1-4p^2z-2q\left(4p^2+q\right)z^2-4p^2q^2z^3+q^4z^4}.$$

Thus, this completes the proof. □

Theorem 4.13. *For* $n \in \mathbb{N}$*, the new generating function of the product of* (p,q)*-Pell Lucas numbers* $\left(Q_{p,q,n+2}Q_{p,q,n}\right)$ *is given by*

$$\sum_{n=0}^{\infty} Q_{p,q,n+2}Q_{p,q,n}z^n = \frac{4\left(2p^2+q\right)-4p^2\left(4p^2+q\right)z-4q\left(4p^4+2p^2q+q^2\right)z^2+4p^2q^3z^3}{1-4p^2z-2q\left(4p^2+q\right)z^2-4p^2q^2z^3+q^4z^4}. \tag{37}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{p,q,n+2}Q_{p,q,n}z^n &= \sum_{n=0}^{\infty}\begin{pmatrix}(2S_{n+2}(a_1+[-a_2])-2pS_{n+1}(a_1+[-a_2])) \\ \times(2S_n(e_1+[-e_2])-2pS_{n-1}(e_1+[-e_2]))\end{pmatrix}z^n \\ &= 4\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\ &\quad -4p\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n\end{aligned}$$

$$-4p\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n$$
$$+4p^2\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.$$

Using the relationships (31) , (32) , (33) and (34), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{p,q,n+2}Q_{p,q,n}z^n &= \frac{4\left(4p^2+q+4qp^2z-q^3z^2\right)}{1-4p^2z-2q\left(4p^2+q\right)z^2-4p^2q^2z^3+q^4z^4}\\ &-\frac{4p\left(4p\left(2p^2+q\right)z+2pq\left(4p^2+q\right)z^2-2pq^3z^3\right)}{1-4p^2z-2q\left(4p^2+q\right)z^2-4p^2q^2z^3+q^4z^4}\\ &-\frac{4p\left(2p+2pqz\right)}{1-4p^2z-2q\left(4p^2+q\right)z^2-4p^2q^2z^3+q^4z^4}\\ &+\frac{4p^2\left(\left(4p^2+q\right)z+4qp^2z^2-q^3z^3\right)}{1-4p^2z-2q\left(4p^2+q\right)z^2-4p^2q^2z^3+q^4z^4}\\ &= \frac{4\left(2p^2+q\right)-4p^2\left(4p^2+q\right)z-4q\left(4p^4+2p^2q+q^2\right)z^2+4p^2q^3z^3}{1-4p^2z-2q\left(4p^2+q\right)z^2-4p^2q^2z^3+q^4z^4}.\end{aligned}$$

So, the proof is completed. □

Theorem 4.14. *Let n be a natural number. Then we have the following new generating function for $\left(Q_{p,q,n+1}Q_{p,q,n}\right)$ as:*

$$\sum_{n=0}^{\infty} Q_{p,q,n+1}Q_{p,q,n}z^n = \frac{4p+4p\left(q-2p^2\right)z+4pq\left(q-2p^2\right)z^2+4pq^3z^3}{1-4p^2z-2q\left(4p^2+q\right)z^2-4p^2q^2z^3+q^4z^4}. \tag{38}$$

Proof. By (6°) in introduction, we get

$$Q_{p,q,n+2} = 2pQ_{p,q,n+1} + qQ_{p,q,n}.$$

We know that

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{p,q,n+2}Q_{p,q,n}z^n &= \sum_{n=0}^{\infty}\left(2pQ_{p,q,n+1}+qQ_{p,q,n}\right)Q_{p,q,n}z^n\\ &= 2p\sum_{n=0}^{\infty} Q_{p,q,n+1}Q_{p,q,n}z^n + q\sum_{n=0}^{\infty} Q^2_{p,q,n}z^n.\end{aligned}$$

On the other hand, we have

$$\sum_{n=0}^{\infty} Q^2_{p,q,n}z^n = \frac{4-12p^2z-4q\left(4p^2+q\right)z^2-4p^2q^2z^3}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4}, \text{ (see [10]).}$$

From which it follows

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{p,q,n+1}Q_{p,q,n}z^n &= \frac{1}{2p}\left[\sum_{n=0}^{\infty} Q_{p,q,n+2}Q_{p,q,n}z^n - q\sum_{n=0}^{\infty} Q^2_{p,q,n}z^n\right]\\ &= \frac{1}{2p}\left[\frac{8p^2+8p^2\left(q-2p^2\right)z+8p^2q\left(q-2p^2\right)z^2+8p^2q^3z^3}{1-4p^2z-2q(4p^2+q)z^2-4p^2q^2z^3+q^4z^4}\right],\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} Q_{p,q,n+1}Q_{p,q,n}z^n = \frac{4p + 4p\left(q - 2p^2\right)z + 4pq\left(q - 2p^2\right)z^2 + 4pq^3z^3}{1 - 4p^2z - 2q(4p^2 + q)z^2 - 4p^2q^2z^3 + q^4z^4}.$$

This completes the proof. □

Corollary 4.15. *Taking $p = 1$ and $q = k$ in Eqs. (35)-(38) gives the following new generating functions:*

$$\begin{aligned}
\sum_{n=0}^{\infty} P_{k,n+2}P_{k,n}z^n &= \frac{(4 + k)z + 4kz^2 - k^3z^3}{1 - 4z - 2k(4 + k)z^2 - 4k^2z^3 + k^4z^4}.\\
\sum_{n=0}^{\infty} P_{k,n+1}P_{k,n}z^n &= \frac{2z + 2kz^2}{1 - 4z - 2k(4 + k)z^2 - 4k^2z^3 + k^4z^4}.\\
\sum_{n=0}^{\infty} Q_{k,n+2}Q_{k,n}z^n &= \frac{4(2 + k) - 4(4 + k)z - 4k\left(4 + 2k + k^2\right)z^2 + 4k^3z^3}{1 - 4z - 2k(4 + k)z^2 - 4k^2z^3 + k^4z^4}.\\
\sum_{n=0}^{\infty} Q_{k,n+1}Q_{k,n}z^n &= \frac{4 + 4(k - 2)z + 4k(k - 2)z^2 + 4k^3z^3}{1 - 4z - 2k(4 + k)z^2 - 4k^2z^3 + k^4z^4}.
\end{aligned}$$

- Put $k = 1$ in the Corollary 4.18, we obtain the following table:

Coefficient of z^n	**Generating function**
$P_{n+2}P_n$	$\frac{5z+4z^2-z^3}{1-4z-10z^2-4z^3+z^4}$
$P_{n+1}P_n$	$\frac{2z+2z^2}{1-4z-10z^2-4z^3+z^4}$
$Q_{n+2}Q_n$	$\frac{12-20z-28z^2+4z^3}{1-4z-10z^2-4z^3+z^4}$
$Q_{n+1}Q_n$	$\frac{4-4z-4z^2+4z^3}{1-4z-10z^2-4z^3+z^4}$

Table 4: The new generating functions of the products of some numbers.

5. Generating functions of the products of (p, q)-numbers with Mersenne numbers at consecutive and nonconsecutive terms

In this part, the new generating functions of the products of Mersenne numbers with (p, q)-Fibonacci and (p, q)-Lucas numbers, (p, q)-Jacobsthal and (p, q)-Jacobsthal Lucas numbers, (p, q)-Pell and (p, q)-Pell Lucas numbers at consecutive and nonconsecutive terms will be introduced.

This part consists of three cases.

Case 1. *put $a_1 - a_2 = p$, $a_1a_2 = q$, $e_1 - e_2 = 3$ and $e_1e_2 = -2$ in the relationships* (13) *and* (14) *, we obtain*

$$\sum_{n=0}^{\infty} S_{n+1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n = \frac{\left(p^2 + q\right)z + 3pqz^2 + 2q^2z^3}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4}, \tag{39}$$

$$\sum_{n=0}^{\infty} S_{n+2}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n = \frac{p\left(p^2 + 2q\right)z + 3q\left(p^2 + q\right)z^2 + 2pq^2z^3}{1 - 3pz - (5q - 2p^2)z^2 + 6pqz^3 + 4q^2z^4}, \tag{40}$$

respectively, and we deduce the following proposition and theorems.

Proposition 5.1. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q) -Fibonacci numbers with Mersenne numbers $\left(F_{p,q,n+2}M_n\right)$ is given by*

$$\sum_{n=0}^{\infty} F_{p,q,n+2}M_n z^n = \frac{\left(p^2+q\right)z + 3pqz^2 + 2q^2z^3}{1-3pz-(5q-2p^2)z^2+6pqz^3+4q^2z^4}. \tag{41}$$

with $F_{p,q,n+2}M_n = S_{n+1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])$.

Theorem 5.2. *Let n be a natural number. Then we have the following new generating function for $\left(F_{p,q,n+1}M_n\right)$ as:*

$$\sum_{n=0}^{\infty} F_{p,q,n+1}M_n z^n = \frac{pz+3qz^2}{1-3pz-(5q-2p^2)z^2+6pqz^3+4q^2z^4}. \tag{42}$$

Proof. By (1°) in introduction, we get

$$F_{p,q,n+2} = pF_{p,q,n+1} + qF_{p,q,n}.$$

Then, we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_{p,q,n+2}M_n z^n &= \sum_{n=0}^{\infty}\left(pF_{p,q,n+1}+qF_{p,q,n}\right)M_n z^n\\ &= p\sum_{n=0}^{\infty}F_{p,q,n+1}M_n z^n + q\sum_{n=0}^{\infty}F_{p,q,n}M_n z^n,\end{aligned}$$

since

$$\sum_{n=0}^{\infty} F_{p,q,n}M_n z^n = \frac{z+2qz^3}{1-3pz-(5q-2p^2)z^2+6pqz^3+4q^2z^4}, \text{ (see [10])}.$$

Then, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} F_{p,q,n+1}M_n z^n &= \frac{1}{p}\left[\sum_{n=0}^{\infty}F_{p,q,n+2}M_n z^n - q\sum_{n=0}^{\infty}F_{p,q,n}M_n z^n\right]\\ &= \frac{1}{p}\left[\frac{p^2z+3pqz^2}{1-3pz-(5q-2p^2)z^2+6pqz^3+4q^2z^4}\right],\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} F_{p,q,n+1}M_n z^n = \frac{pz+3qz^2}{1-3pz-(5q-2p^2)z^2+6pqz^3+4q^2z^4}.$$

Thus, this completes the proof. □

Theorem 5.3. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Lucas numbers with Mersenne numbers $\left(L_{p,q,n+2}M_n\right)$ is given by*

$$\sum_{n=0}^{\infty} L_{p,q,n+2}M_n z^n = \frac{p\left(p^2+3q\right)z + 3q\left(p^2+2q\right)z^2 + 2pq^2z^3}{1-3pz-(5q-2p^2)z^2+6pqz^3+4q^2z^4}. \tag{43}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n+2}M_nz^n &= \sum_{n=0}^{\infty}\left(2S_{n+2}(a_1+[-a_2]) - pS_{n+1}(a_1+[-a_2])\right)S_{n-1}(e_1+[-e_2])z^n\\ &= 2\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n\\ &\quad -p\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n.\end{aligned}$$

Using the relationships (39) and (40), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n+2}M_nz^n &= \frac{2\left(p\left(p^2+2q\right)z+3q\left(p^2+q\right)z^2+2pq^2z^3\right)}{1-3pz-\left(5q-2p^2\right)z^2+6pqz^3+4q^2z^4}\\ &\quad -\frac{p\left(\left(p^2+q\right)z+3pqz^2+2q^2z^3\right)}{1-3pz-\left(5q-2p^2\right)z^2+6pqz^3+4q^2z^4}\\ &= \frac{p\left(p^2+3q\right)z+3q\left(p^2+2q\right)z^2+2pq^2z^3}{1-3pz-\left(5q-2p^2\right)z^2+6pqz^3+4q^2z^4}.\end{aligned}$$

So, the proof is completed. □

Theorem 5.4. *Let n be a natural number. Then we have the following new generating function for* $\left(L_{p,q,n+1}M_n\right)$ *as:*

$$\sum_{n=0}^{\infty} L_{p,q,n+1}M_nz^n = \frac{\left(p^2+2q\right)z+3pqz^2+4q^2z^3}{1-3pz-\left(5q-2p^2\right)z^2+6pqz^3+4q^2z^4}. \tag{44}$$

Proof. By (2°) in introduction, we get

$$L_{p,q,n+2} = pL_{p,q,n+1} + qL_{p,q,n}.$$

We know that

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n+2}M_nz^n &= \sum_{n=0}^{\infty}\left(pL_{p,q,n+1}+qL_{p,q,n}\right)M_nz^n\\ &= p\sum_{n=0}^{\infty} L_{p,q,n+1}M_nz^n + q\sum_{n=0}^{\infty} L_{p,q,n}M_nz^n.\end{aligned}$$

On the other hand, we have

$$\sum_{n=0}^{\infty} L_{p,q,n}M_nz^n = \frac{pz+6qz^2-2pqz^3}{1-3pz-(5q-2p^2)z^2+6pqz^3+4q^2z^4}, \text{ (see [10]).}$$

From which it follows

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n+1}M_nz^n &= \frac{1}{p}\left[\sum_{n=0}^{\infty} L_{p,q,n+2}M_nz^n - q\sum_{n=0}^{\infty} L_{p,q,n}M_nz^n\right]\\ &= \frac{1}{p}\left[\frac{p\left(p^2+2q\right)z+3p^2qz^2+4pq^2z^3}{1-3pz-\left(5q-2p^2\right)z^2+6pqz^3+4q^4z^4}\right],\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} L_{p,q,n+1} M_n z^n = \frac{\left(p^2+2q\right)z + 3pqz^2 + 4q^2z^3}{1-3pz-(5q-2p^2)z^2+6pqz^3+4q^2z^4}.$$

This completes the proof. □

Corollary 5.5. *Putting $p = k$ and $q = 1$ in Eqs. (41)-(44) gives the following new generating functions:*

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{k,n+2} M_n z^n &= \frac{\left(k^2+1\right)z+3kz^2+2z^3}{1-3kz-(5-2k^2)z^2+6kz^3+4z^4}.\\
\sum_{n=0}^{\infty} F_{k,n+1} M_n z^n &= \frac{kz+3z^2}{1-3kz-(5-2k^2)z^2+6kz^3+4z^4}.\\
\sum_{n=0}^{\infty} L_{k,n+2} M_n z^n &= \frac{k\left(k^2+3\right)z+3\left(k^2+2\right)z^2+2kz^3}{1-3kz-(5-2k^2)z^2+6kz^3+4z^4}.\\
\sum_{n=0}^{\infty} L_{k,n+1} M_n z^n &= \frac{\left(k^2+2\right)z+3kz^2+4z^3}{1-3kz-(5-2k^2)z^2+6kz^3+4z^4}.
\end{aligned}$$

- Put $k = 1$ in the Corollary 5.5, we obtain the following table:

Coefficient of z^n	**Generating function**
$F_{n+2}M_n$	$\frac{2z+3z^2+2z^3}{1-3z-3z^2+6z^3+4z^4}$
$F_{n+1}M_n$	$\frac{z+3z^2}{1-3z-3z^2+6z^3+4z^4}$
$L_{n+2}M_n$	$\frac{4z+9z^2+2z^3}{1-3z-3z^2+6z^3+4z^4}$
$L_{n+1}M_n$	$\frac{3z+3z^2+4z^3}{1-3z-3z^2+6z^3+4z^4}$

Table 5: The new generating functions of the products of Mersenne numbers with numbers.

Case 2. *put $a_1 - a_2 = p$, $a_1a_2 = 2q$, $e_1 - e_2 = 3$ and $e_1e_2 = -2$ in the relationships* (13) *and* (14) *, we obtain*

$$\sum_{n=0}^{\infty} S_{n+1}(a_1+[-a_2])\, S_{n-1}(e_1+[-e_2])\, z^n = \frac{\left(p^2+2q\right)z+6pqz^2+8q^2z^3}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4}, \tag{45}$$

$$\sum_{n=0}^{\infty} S_{n+2}(a_1+[-a_2])\, S_{n-1}(e_1+[-e_2])\, z^n = \frac{p\left(p^2+4q\right)z+6q\left(p^2+2q\right)z^2+8pq^2z^3}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4}, \tag{46}$$

respectively, thus we get the following proposition and theorems.

Proposition 5.6. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q) -Jacobsthal numbers with Mersenne numbers $\left(J_{p,q,n+2}M_n\right)$ is given by*

$$\sum_{n=0}^{\infty} J_{p,q,n+2} M_n z^n = \frac{\left(p^2+2q\right)z+6pqz^2+8q^2z^3}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4}, \tag{47}$$

with $J_{p,q,n+2}M_n = S_{n+1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])$.

Theorem 5.7. *Let n be a natural number. Then we have the following new generating function for $\left(J_{p,q,n+1}M_n\right)$ as:*

$$\sum_{n=0}^{\infty} J_{p,q,n+1}M_n z^n = \frac{pz + 6qz^2}{1 - 3pz - 2\left(5q - p^2\right)z^2 + 12pqz^3 + 16q^2z^4}. \tag{48}$$

Proof. By (3°) in introduction, we get

$$J_{p,q,n+2} = pJ_{p,q,n+1} + 2qJ_{p,q,n}.$$

Then, we have

$$\begin{aligned}\sum_{n=0}^{\infty} J_{p,q,n+2}M_n z^n &= \sum_{n=0}^{\infty}\left(pJ_{p,q,n+1} + 2qJ_{p,q,n}\right)M_n z^n \\ &= p\sum_{n=0}^{\infty} J_{p,q,n+1}M_n z^n + 2q\sum_{n=0}^{\infty} J_{p,q,n}M_n z^n,\end{aligned}$$

since

$$\sum_{n=0}^{\infty} J_{p,q,n}M_n z^n = \frac{z + 4qz^3}{1 - 3pz - 2(5q - p^2)z^2 + 12pqz^3 + 16q^2z^4}, \text{ (see [10]).}$$

Then, we get

$$\begin{aligned}\sum_{n=0}^{\infty} J_{p,q,n+1}M_n z^n &= \frac{1}{p}\left[\sum_{n=0}^{\infty} J_{p,q,n+2}M_n z^n - 2q\sum_{n=0}^{\infty} J_{p,q,n}M_n z^n\right] \\ &= \frac{1}{p}\left[\frac{p^2z + 6pqz^2}{1 - 3pz - 2\left(5q - p^2\right)z^2 + 12pqz^3 + 16q^2z^4}\right],\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} J_{p,q,n+1}M_n z^n = \frac{pz + 6qz^2}{1 - 3pz - 2\left(5q - p^2\right)z^2 + 12pqz^3 + 16q^2z^4}.$$

Thus, this completes the proof. □

Theorem 5.8. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q) -Jacobsthal Lucas numbers with Mersenne numbers $\left(j_{p,q,n+2}M_n\right)$ is given by*

$$\sum_{n=0}^{\infty} j_{p,q,n+2}M_n z^n = \frac{p\left(p^2 + 6q\right)z + 6q\left(p^2 + 4q\right)z^2 + 8pq^2z^3}{1 - 3pz - 2\left(5q - p^2\right)z^2 + 12pqz^3 + 16q^2z^4}. \tag{49}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} j_{p,q,n+2}M_n z^n &= \sum_{n=0}^{\infty}(2S_{n+2}(a_1 + [-a_2]) - pS_{n+1}(a_1 + [-a_2]))S_{n-1}(e_1 + [-e_2])z^n \\ &= 2\sum_{n=0}^{\infty} S_{n+2}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\ &\quad -p\sum_{n=0}^{\infty} S_{n+1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n.\end{aligned}$$

Using the relationships (45) and (46), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} j_{p,q,n+2}M_n z^n &= \frac{2\left(p\left(p^2+4q\right)z+6q\left(p^2+2q\right)z^2+8pq^2z^3\right)}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4} \\ &\quad -\frac{p\left(\left(p^2+2q\right)z+6pqz^2+8q^2z^3\right)}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4} \\ &= \frac{p\left(p^2+6q\right)z+6q\left(p^2+4q\right)z^2+8pq^2z^3}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4}.\end{aligned}$$

So, the proof is completed. □

Theorem 5.9. *Let n be a natural number. Then we have the following new generating function for $\left(j_{p,q,n+1}M_n\right)$ as:*

$$\sum_{n=0}^{\infty} j_{p,q,n+1}M_n z^n = \frac{\left(p^2+4q\right)z+6pqz^2+16q^2z^3}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4}. \tag{50}$$

Proof. By (4°) in introduction, we get

$$j_{p,q,n+2} = pj_{p,q,n+1} + 2qj_{p,q,n}.$$

We know that

$$\begin{aligned}\sum_{n=0}^{\infty} j_{p,q,n+2}M_n z^n &= \sum_{n=0}^{\infty}\left(pj_{p,q,n+1}+2qj_{p,q,n}\right)M_n z^n \\ &= p\sum_{n=0}^{\infty} j_{p,q,n+1}M_n z^n + 2q\sum_{n=0}^{\infty} j_{p,q,n}M_n z^n.\end{aligned}$$

On the other hand, we have

$$\sum_{n=0}^{\infty} j_{p,q,n}M_n z^n = \frac{pz+12qz^2-4pqz^3}{1-3pz-2(5q-p^2)z^2+12pqz^3+16q^2z^4}, \text{ (see [10])}.$$

From which it follows

$$\begin{aligned}\sum_{n=0}^{\infty} j_{p,q,n+1}M_n z^n &= \frac{1}{p}\left[\sum_{n=0}^{\infty} j_{p,q,n+2}M_n z^n - 2q\sum_{n=0}^{\infty} j_{p,q,n}M_n z^n\right] \\ &= \frac{1}{p}\left[\frac{p\left(p^2+4q\right)z+6p^2qz^2+16pq^2z^3}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4}\right],\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} j_{p,q,n+1}M_n z^n = \frac{\left(p^2+4q\right)z+6pqz^2+16q^2z^3}{1-3pz-2\left(5q-p^2\right)z^2+12pqz^3+16q^2z^4}.$$

This completes the proof. □

Corollary 5.10. *Setting $p = k$ and $q = 1$ in Eqs. (47)-(50) gives the following new generating functions:*

$$\sum_{n=0}^{\infty} J_{k,n+2} M_n z^n = \frac{\left(k^2+2\right) z + 6kz^2 + 8z^3}{1 - 3kz - 2\left(5-k^2\right) z^2 + 12kz^3 + 16z^4}.$$

$$\sum_{n=0}^{\infty} J_{k,n+1} M_n z^n = \frac{kz + 6z^2}{1 - 3kz - 2\left(5-k^2\right) z^2 + 12kz^3 + 16z^4}.$$

$$\sum_{n=0}^{\infty} j_{k,n+2} M_n z^n = \frac{k\left(k^2+6\right) z + 6\left(k^2+4\right) z^2 + 8kz^3}{1 - 3kz - 2\left(5-k^2\right) z^2 + 12kz^3 + 16z^4}.$$

$$\sum_{n=0}^{\infty} j_{k,n+1} M_n z^n = \frac{\left(k^2+4\right) z + 6kz^2 + 16z^3}{1 - 3kz - 2\left(5-k^2\right) z^2 + 12kz^3 + 16z^4}.$$

- Put $k = 1$ in the Corollary 5.10, we obtain the following table:

Coefficient of z^n	**Generating function**
$J_{n+2}M_n$	$\frac{3z+6z^2+8z^3}{1-3z-8z^2+12z^3+16z^4}$
$J_{n+1}M_n$	$\frac{z+6z^2}{1-3z-8z^2+12z^3+16z^4}$
$j_{n+2}M_n$	$\frac{7z+30z^2+8z^3}{1-3z-8z^2+12z^3+16z^4}$
$j_{n+1}M_n$	$\frac{5z+6z^2+16z^3}{1-3z-8z^2+12z^3+16z^4}$

Table 6: The new generating functions of the products of Mersenne numbers with numbers.

Case 3. *put $a_1 - a_2 = 2p$, $a_1 a_2 = q$, $e_1 - e_2 = 3$ and $e_1 e_2 = -2$ in the relationships* (13) *and* (14) *, we obtain*

$$\sum_{n=0}^{\infty} S_{n+1}\left(a_1 + [-a_2]\right) S_{n-1}\left(e_1 + [-e_2]\right) z^n = \frac{\left(4p^2+q\right) z + 6pqz^2 + 2q^2z^3}{1 - 6pz - \left(5q - 8p^2\right) z^2 + 12pqz^3 + 4q^2z^4}, \tag{51}$$

$$\sum_{n=0}^{\infty} S_{n+2}\left(a_1 + [-a_2]\right) S_{n-1}\left(e_1 + [-e_2]\right) z^n = \frac{4p\left(2p^2+q\right) z + 3q\left(4p^2+q\right) z^2 + 4pq^2z^3}{1 - 6pz - \left(5q - 8p^2\right) z^2 + 12pqz^3 + 4q^2z^4}, \tag{52}$$

respectively, we deduce the following proposition and theorems.

Proposition 5.11. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Pell numbers with Mersenne numbers $\left(P_{p,q,n+2} M_n\right)$ is given by*

$$\sum_{n=0}^{\infty} P_{p,q,n+2} M_n z^n = \frac{\left(4p^2+q\right) z + 6pqz^2 + 2q^2z^3}{1 - 6pz - \left(5q - 8p^2\right) z^2 + 12pqz^3 + 4q^2z^4}, \tag{53}$$

with $P_{p,q,n+2} M_n = S_{n+1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2])$.

Theorem 5.12. *Let n be a natural number. Then we have the following new generating function for $\left(P_{p,q,n+1} M_n\right)$ as:*

$$\sum_{n=0}^{\infty} P_{p,q,n+1} M_n z^n = \frac{2pz + 3qz^2}{1 - 6pz - \left(5q - 8p^2\right) z^2 + 12pqz^3 + 4q^2z^4}. \tag{54}$$

Proof. By (5°) in introduction, we get

$$P_{p,q,n+2} = 2pP_{p,q,n+1} + qP_{p,q,n}.$$

Then, we have

$$\begin{aligned}\sum_{n=0}^{\infty} P_{p,q,n+2} M_n z^n &= \sum_{n=0}^{\infty} \left(2pP_{p,q,n+1} + qP_{p,q,n}\right) M_n z^n \\ &= 2p \sum_{n=0}^{\infty} P_{p,q,n+1} M_n z^n + q \sum_{n=0}^{\infty} P_{p,q,n} M_n z^n,\end{aligned}$$

since

$$\sum_{n=0}^{\infty} P_{p,q,n} M_n z^n = \frac{z + 2qz^3}{1 - 6pz - (5q - 8p^2)z^2 + 12pqz^3 + 4q^2z^4}, \text{ (see [10]).}$$

Then, we get

$$\begin{aligned}\sum_{n=0}^{\infty} P_{p,q,n+1} M_n z^n &= \frac{1}{2p} \left[\sum_{n=0}^{\infty} P_{p,q,n+2} M_n z^n - q \sum_{n=0}^{\infty} P_{p,q,n} M_n z^n\right] \\ &= \frac{1}{2p} \left[\frac{4p^2z + 6pqz^2}{1 - 6pz - (5q - 8p^2)\, z^2 + 12pqz^3 + 4q^2z^4}\right],\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} P_{p,q,n+1} M_n z^n = \frac{2pz + 3qz^2}{1 - 6pz - (5q - 8p^2)\, z^2 + 12pqz^3 + 4q^2z^4}.$$

Thus, this completes the proof. □

Theorem 5.13. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Pell Lucas numbers with Mersenne numbers $\left(Q_{p,q,n+2} M_n\right)$ is given by*

$$\sum_{n=0}^{\infty} Q_{p,q,n+2} M_n z^n = \frac{2p\left(4p^2 + 3q\right) z + 6q\left(2p^2 + q\right) z^2 + 4pq^2z^3}{1 - 6pz - (5q - 8p^2)\, z^2 + 12pqz^3 + 4q^2z^4}. \tag{55}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{p,q,n+2} M_n z^n &= \sum_{n=0}^{\infty} (2S_{n+2}(a_1 + [-a_2]) - 2pS_{n+1}(a_1 + [-a_2]))\, S_{n-1}(e_1 + [-e_2]) z^n \\ &= 2\sum_{n=0}^{\infty} S_{n+2}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\ &\quad -2p \sum_{n=0}^{\infty} S_{n+1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n.\end{aligned}$$

Using the relationships (51) and (52), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{p,q,n+2} M_n z^n &= \frac{2\left(4p\left(2p^2 + q\right) z + 3q\left(4p^2 + q\right) z^2 + 4pq^2z^3\right)}{1 - 6pz - (5q - 8p^2)\, z^2 + 12pqz^3 + 4q^2z^4} \\ &\quad - \frac{2p\left(\left(4p^2 + q\right) z + 6pqz^2 + 2q^2z^3\right)}{1 - 6pz - (5q - 8p^2)\, z^2 + 12pqz^3 + 4q^2z^4} \\ &= \frac{2p\left(4p^2 + 3q\right) z + 6q\left(2p^2 + q\right) z^2 + 4pq^2z^3}{1 - 6pz - (5q - 8p^2)\, z^2 + 12pqz^3 + 4q^2z^4}.\end{aligned}$$

So, the proof is completed. □

Theorem 5.14. *Let n be a natural number. Then we have the following new generating function for* $\left(Q_{p,q,n+1}M_n\right)$ *as:*

$$\sum_{n=0}^{\infty} Q_{p,q,n+1}M_n z^n = \frac{2\left(2p^2+q\right)z + 6pqz^2 + 4q^2z^3}{1-6pz-\left(5q-8p^2\right)z^2+12pqz^3+4q^2z^4}. \tag{56}$$

Proof. By (6°) in introduction, we get

$$Q_{p,q,n+2} = 2pQ_{p,q,n+1} + qQ_{p,q,n}.$$

We know that

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{p,q,n+2}M_n z^n &= \sum_{n=0}^{\infty}\left(2pQ_{p,q,n+1}+qQ_{p,q,n}\right)M_n z^n\\ &= 2p\sum_{n=0}^{\infty} Q_{p,q,n+1}M_n z^n + q\sum_{n=0}^{\infty} Q_{p,q,n}M_n z^n.\end{aligned}$$

On the other hand, we have

$$\sum_{n=0}^{\infty} Q_{p,q,n}M_n z^n = \frac{2pz+6qz^2-4pqz^3}{1-6pz-(5q-8p^2)z^2+12pqz^3+4q^2z^4}, \text{ (see [10]).}$$

From which it follows

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{p,q,n+1}M_n z^n &= \frac{1}{2p}\left[\sum_{n=0}^{\infty} Q_{p,q,n+2}M_n z^n - q\sum_{n=0}^{\infty} Q_{p,q,n}M_n z^n\right]\\ &= \frac{1}{2p}\left[\frac{4p\left(2p^2+q\right)z+12p^2qz^2+8pq^2z^3}{1-6pz-\left(5q-8p^2\right)z^2+12pqz^3+4q^2z^4}\right],\end{aligned}$$

therefore

$$\sum_{n=0}^{\infty} Q_{p,q,n+1}M_n z^n = \frac{2\left(2p^2+q\right)z + 6pqz^2 + 4q^2z^3}{1-6pz-\left(5q-8p^2\right)z^2+12pqz^3+4q^2z^4}.$$

This completes the proof. □

Corollary 5.15. *Taking* $p=1$ *and* $q=k$ *in Eqs. (53)-(56) gives the following new generating functions:*

$$\begin{aligned}\sum_{n=0}^{\infty} P_{k,n+2}M_n z^n &= \frac{(4+k)z+6kz^2+2k^2z^3}{1-6z-(5k-8)z^2+12kz^3+4k^2z^4}.\\ \sum_{n=0}^{\infty} P_{k,n+1}M_n z^n &= \frac{2z+3kz^2}{1-6z-(5k-8)z^2+12kz^3+4k^2z^4}.\\ \sum_{n=0}^{\infty} Q_{k,n+2}M_n z^n &= \frac{2(4+3k)z+6k(2+k)z^2+4k^2z^3}{1-6z-(5k-8)z^2+12kz^3+4k^2z^4}.\\ \sum_{n=0}^{\infty} Q_{k,n+1}M_n z^n &= \frac{2(2+k)z+6kz^2+4k^2z^3}{1-6z-(5k-8)z^2+12kz^3+4k^2z^4}.\end{aligned}$$

- Put $k=1$ in the Corollary 5.15, we obtain the following table:

Coefficient of z^n	Generating function
$P_{n+2}M_n$	$\frac{5z+6z^2+2z^3}{1-6z+3z^2+12z^3+4z^4}$
$P_{n+1}M_n$	$\frac{2z+3z^2}{1-6z+3z^2+12z^3+4z^4}$
$Q_{n+2}M_n$	$\frac{14z+18z^2+4z^3}{1-6z+3z^2+12z^3+4z^4}$
$Q_{n+1}M_n$	$\frac{6z+6z^2+4z^3}{1-6z+3z^2+12z^3+4z^4}$

Table 7: The new generating functions for the products of Mersenne numbers with some numbers.

6. Conclusion

In this paper, by making use of Theorem 3.1, we have derived some new generating functions of the products of (p,q)-Fibonacci numbers, (p,q)-Lucas numbers, (p,q) -Jacobsthal numbers, (p,q)-Jacobsthal Lucas numbers, (p,q)-Pell numbers and (p,q)-Pell Lucas numbers at consecutive and nonconsecutive terms and the products of Mersenne numbers with (p,q)-numbers at consecutive and nonconsecutive terms. The derived theorems and propositions are based on symmetric functions and products of these numbers.

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