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On the approximation of several modified Bernstein-type operators

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Abstract. In this paper, we first propose a new class of modified Bernstein Durrmeyer operators, which are independent of one endpoint value of any continuous function. We investigate their approximation rate, and obtain Voronovaskaja's asymptotic estimation. Then we further introduce two classes of positive linear Bernstein-type operators, to study their approximation performance in both qualitative and quantitative ways. We compare the three mentioned operators, to explore their unique properties such as linearity, positivity, genuineness, and approximation performance both analytically and empirically.

1. Introduction

Among all the proofs of the famous Weierstrass Theorem, which states approximating continuous functions with polynomial functions, the proof Bernstein [4] provided is the one having attracted a lot of attention. The well-known Bernstein operator can be written in (1):

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \tag{1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \cdots, n; \quad n = 1, 2, \cdots. \tag{2}$$

Then, the discussion of the linear positive operator (1) and its generalizations has become one of the hottest topics in approximation theory (see [1–3, 7–25], for example).

Singular functions play a key role in various applications (e.g., the applications in electromagnetic field, wave computation, spherical harmonics, etc.) , and their approximation is crucial for solving applied mathematics problems efficiently. Many scholars have conducted in-depth research on the use of generalized Bernstein operators to solve the problem of approximation to functions with singularities (see

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[1, 13, 14, 16, 20–23, 25]). In 2004, Vecchia, Mastroianni, and Szabados[20] introduced a new Bernstein-type operator as defined in (3).

$$B_n^*(f,x) = \left[2f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right)\right]p_{n,0}(x) + \left[2f\left(1-\frac{1}{n}\right) - f\left(1-\frac{2}{n}\right)\right]p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x) f\left(\frac{k}{n}\right). \tag{3}$$

Comparing with the original Bernstein operator (1), we observe that operator (3) is independent of the function values at the endpoints 0 and 1, therefore, such a structure prepares for studying the properties of approximation of functions that are singular at the endpoints.

Recently, based on the genuine Bernstein-Durrmeyer(BD) operator constructed in [10], Zhang and Yu introduced a modified operator (4) (see [24]).

$$\begin{aligned} U_n^*(f,x) = &\left[2f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right)\right]p_{n,0}(x) + \left[2f\left(1-\frac{1}{n}\right) - f\left(1-\frac{2}{n}\right)\right]p_{n,n}(x) \\ &+ (n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 p_{n-2,k-1}(t) f(t)\mathrm{d}t. \end{aligned} \tag{4}$$

In addition to being independent of the values of $f(0)$ and $f(1)$, the operator reproduces linear functions, and also has good convergence properties. The specific conclusions from [10] are listed as Theorems 1.1 and 1.2.

Theorem 1.1. *Let $0 \le \lambda \le 1$ be a fixed number. For any $f \in C[0,1]$, define $U_n^*(f,x)$ as in (4), then we have*

$$\left|U_n^*(f,x) - f(x)\right| \le C\omega_{\varphi^\lambda}^2\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right),$$

where $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$, $\varphi(x) = \sqrt{x(1-x)}$, and $\omega_{\varphi^\lambda}^2(f,t)$, the Ditzian-Totik modulus of the second order, is given by

$$\omega_{\varphi^\lambda}^2(f,t) = \sup_{h\in(0,t]} \sup_{x\pm h\varphi^\lambda(x)\in[0,1]} \left|f\left(x-h\varphi^\lambda(x)\right) - 2f(x) + f\left(x+h\varphi^\lambda(x)\right)\right|.$$

Theorem 1.2. *For any $f \in C^2[0,1]$, define $U_n^*(f,x)$ as in (4), there exists a positive constant C only depending on λ such that*

$$\left|U_n^*(f,x) - \left(\frac{\varphi^2(x)}{n+2} - \frac{1}{n^2}((1-x)^n + x^n)\right)f(x) - f''(x)\right| \le C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}^2\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right).$$

Our first focus is studying the approximation to functions with potential singularity at only one endpoint. Therefore, we propose a modified BD operator defined in (5).

$$\widetilde{U}_n(f,x) = \left[2f\left(1-\frac{1}{n}\right) - f\left(1-\frac{2}{n}\right)\right]p_{n,n}(x) + n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)f(t)\mathrm{d}t. \tag{5}$$

Obviously, $\widetilde{U}_n(f,x)$ is a linear operator and is independent of the value $f(1)$. We will show the properties of $\widetilde{U}_n(f,x)$ and will investigate its approximation rate for continuous functions.

Furthermore, we will introduce two classes of positive linear Bernstein-type operators in Sections 4 as a comparison to the BD operator (5). This allows us to explore the properties and approximation performance of all the operators proposed in this paper from different angles.

Throughout the paper, C denotes either a positive absolute constant or a positive constant, and may depend on some parameters but not on f, x and n. Their values may be varied in different situations. In addition, $A \sim B$ refers that there is a positive constant C such that $C^{-1}A \le B \le CA$.

The paper is organized as follows. Section 2 presents the lemmas and their proofs, which prepare the proof of the approximation theorem in Section 3; Section 4 proposes two positive Bernstein-type operators and their approximation performance; The analytical and numerical comparison of all the operators proposed are presented in Section 5. Finally, Section 6 summarizes the research and future research directions.

2. Auxiliary Lemmas

To prepare the proof of the approximation theorems for the proposed operator $\widetilde{U}_n(f,x)$, we first need to prove the auxiliary theories - Lemmas 2.1, 2.2, 2.3, and 2.4.

Lemma 2.1. *Let $p_{n,k}$ be defined by* (2)*, then the following equations hold [24].*

$$\sum_{k=0}^{n} p_{n,k}(x) = 1, \quad \sum_{k=0}^{n} \frac{k}{n} p_{n,k}(x) = x, \quad \sum_{k=0}^{n} \frac{k^2}{n^2} p_{n,k}(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x,$$
$$\sum_{k=0}^{n} \frac{k^3}{n^3} p_{n,k}(x) = \frac{(n-1)(n-2)}{n^2}x^3 + \frac{3(n-1)}{n^2}x^2 + \frac{1}{n^2}x,$$
$$\sum_{k=0}^{n} \frac{k^4}{n^4} p_{n,k}(x) = \frac{(n-1)(n-2)(n-3)}{n^3}x^4 + \frac{6(n-1)(n-2)}{n^3}x^3 + \frac{7(n-1)}{n^3}x^2 + \frac{1}{n^3}x.$$

Lemma 2.2. *It holds that*

$$\widetilde{U}_n(1,x) = 1, \quad \widetilde{U}_n(t,x) = \frac{n}{n+1}x + \frac{1}{n+1},$$
$$\widetilde{U}_n(t^2,x) = \frac{n(n-1)}{(n+1)(n+2)}x^2 + \frac{4n}{(n+1)(n+2)}x + \frac{2}{(n+1)(n+2)} - \frac{2}{n^2}x^n.$$

Proof. For $k = 1, 2, \cdots$, we have

$$\begin{aligned}\int_0^1 p_{n-1,k}(t) \cdot t^j \mathrm{d}t &= \int_0^1 \binom{n-1}{k} t^{k+j}(1-t)^{n-k-1}\mathrm{d}t \\ &= \frac{(n-1)!}{k!\,(n-k-1)!}\frac{(k+j)!\,(n-k-1)!}{(n+j)!} \\ &= \frac{1}{n} \cdot \frac{n!\,(k+j)!}{k!\,(n+j)!}, \quad j = 0, 1, 2, \cdots. \end{aligned} \tag{6}$$

Therefore, by Lemma 2.1 we have

$$\widetilde{U}_n(1,x) = p_{n,n}(x) + n\sum_{k=0}^{n-1} p_{n,k}(x) \int_0^1 p_{n-1,k}(t)\mathrm{d}t = p_{nn}(x) + \sum_{k=0}^{n-1} p_{n,k}(x) = 1,$$

$$\begin{aligned}\widetilde{U}_n(t,x) &= p_{n,n}(x) + n\sum_{k=0}^{n-1} p_{n,k}(x) \int_0^1 p_{n-1,k}(t) \cdot t\mathrm{d}t \\ &= p_{n,n}(x) + \sum_{k=0}^{n-1} p_{n,k}(x)\frac{k+1}{n+1} \\ &= p_{n,n}(x) + \frac{n}{n+1}\sum_{k=0}^{n-1} \frac{k}{n}p_{n,k}(x) + \frac{1}{n+1}\sum_{k=0}^{n-1} p_{n,k}(x) = \frac{n}{n+1}x + \frac{1}{n+1},\end{aligned}$$

$$\begin{aligned}\widetilde{U}_n(t^2,x) &= \left(1-\frac{2}{n^2}\right)p_{n,n}(x) + n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\cdot t^2\mathrm{d}t\\ &= \left(1-\frac{2}{n^2}\right)p_{n,n}(x) + \sum_{k=0}^{n-1} p_{n,k}(x)\cdot\frac{(k+1)(k+2)}{(n+1)(n+2)}\\ &= \frac{n^2}{(n+1)(n+2)}\sum_{k=0}^{n}\frac{k^2}{n^2}p_{n,k}(x) + \frac{3n}{(n+1)(n+2)}\sum_{k=0}^{n}\frac{k}{n}p_{n,k}(x) +\\ &\quad \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}p_{n,k}(x) - \frac{2}{n^2}p_{n,n}(x)\\ &= \frac{n(n-1)}{(n+1)(n+2)}x^2 + \frac{4n}{(n+1)(n+2)}x + \frac{2}{(n+1)(n+2)} - \frac{2}{n^2}x^n.\end{aligned}$$

□

Lemma 2.3. *It holds that*

$$n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\,|t-x|^3\,\mathrm{d}t \le C\left(\frac{\delta(x)}{\sqrt{n}}\right)^3.$$

Proof. By Cauchy's inequality, (6), and Lemma 2.1, we have

$$\begin{aligned}& n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\,|t-x|^3\,\mathrm{d}t\\ \le& n\sum_{k=0}^{n-1} p_{n,k}(x)\left(\int_0^1 p_{n-1,k}(t)\,|t-x|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^1 p_{n-1,k}(t)\,|t-x|^4\,\mathrm{d}t\right)^{\frac{1}{2}}\\ \le& \left(n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\,|t-x|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\left(n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\,|t-x|^4\,\mathrm{d}t\right)^{\frac{1}{2}}\\ \le& \left(-\frac{2(n-1)}{(n+1)(n+2)}x^2 + \frac{2(n-2)}{(n+1)(n+2)}x + \frac{2}{(n+1)(n+2)} - x^n(1-x)^2\right)^{\frac{1}{2}}\\ & \left(\frac{12(n^2-13n+2)}{(n+1)(n+2)(n+3)(n+4)}x^4 - \frac{24(n^2-15n+4)}{(n+1)(n+2)(n+3)(n+4)}x^3 +\right.\\ & \left.\frac{12(n^2-23n+12)}{(n+1)(n+2)(n+3)(n+4)}x^2 + \frac{24(3n-4)}{(n+1)(n+2)(n+3)(n+4)}x - x^n(1-x)^4\right)^{\frac{1}{2}}\\ \le& C\left(\frac{\varphi^2(x)}{n} + \frac{1}{n^2}\right)^{\frac{1}{2}}\left(\frac{1}{n^2}\left(\varphi^2(x)+\frac{1}{n}\right)^2\right)^{\frac{1}{2}} \le C\frac{\delta_n^2(x)}{n}\frac{\delta_n(x)}{\sqrt{n}}.\end{aligned}$$

□

Lemma 2.4. *For any $t, x \in [0,1]$, and for any $u \in [t,x]$ or $u \in [x,t]$, it holds that*

$$\frac{|t-u|}{\delta_n^{2\lambda}(u)} \le \frac{|t-x|}{\delta_n^{2\lambda}(x)}.$$

Proof. We may assume that $t \le x$. For any $u \in [t,x]$, let $u = t + \theta(t-x)$, $0 \le \theta \le 1$, by the convexity of $\delta_n^{2\lambda}(x)$, we have

$$\frac{|t-u|}{\delta_n^{2\lambda}(u)} \le \frac{\theta|t-x|}{(1-\theta)\delta_n^{2\lambda}(t)+\theta\delta_n^{2\lambda}(x)} \le \frac{|t-x|}{\delta_n^{2\lambda}(x)}.$$

□

3. Approximation Theorems of $\widetilde{U}_n(f,x)$

In this section, the direct approximation theorem of $\widetilde{U}_n(f,x)$ and the Voronovskaja's asymptotic estimate are established and proved. Theorem 3.1 obtains the approximation rate of $\widetilde{U}_n(f,x)$ with respect to the second-order modulus plus an extra term C/n, while Theorem 3.3 studies the asymptotic behavior of this operator.

Theorem 3.1. *Let $0 \le \lambda \le 1$ be a fixed number, for any $f \in C[0,1]$, there is a positive constant only depending on λ such that*

$$\left|\widetilde{U}_n(f,x) - f(x)\right| \le C\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right) + \frac{C}{n}. \tag{7}$$

Proof. For any $f \in C[0,1]$ (we may assume f is not a constant function, since otherwise the conclusion is trivial), define

$$K^*_{\varphi^\lambda}(f,t^2) := \inf_{g\in D^2_\lambda}\left\{\left\|f-g\right\| + t^2\left\|\varphi^{2\lambda}g''\right\| + t^{\frac{2}{1-\frac{\lambda}{2}}}\left\|g''\right\|\right\},$$

where $D^2_\lambda := \left\{f : f \in C[0,1], f' \in A.C._{loc}, \|\varphi^{2\lambda}f''\| < \infty, \|f''\| < \infty\right\}$. It is well known that $K^*_{\varphi^\lambda}\left(f,t^2\right) \sim \omega^2_{\varphi^\lambda}(f,t)$[7]. Therefore, for any n, λ and x, we may choose $g = g_{n,x,\lambda} \in D^2_\lambda$ such that

$$\left\|f-g\right\| \le C\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right), \tag{8}$$

$$\frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\varphi^{2\lambda}g''\right\| \le C\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right), \tag{9}$$

$$\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\left\|g''\right\| \le C\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{10}$$

By the definition of $\widetilde{U}_n(f,x)$, it is easy to find that

$$\begin{aligned}\left|\widetilde{U}_n(f,x) - f(x)\right| &\le \left|\widetilde{U}_n(f-g,x)\right| + \left|f(x)-g(x)\right| + \left|\widetilde{U}_n(g,x) - g(x)\right| \\ &\le 4\|f-g\| + \left|\widetilde{U}_n(g,x) - g(x)\right| \\ &\le 4\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right) + \left|\widetilde{U}_n(g,x) - g(x)\right|. \end{aligned} \tag{11}$$

Here, the second inequality used (8) and the fact that $\|\widetilde{U}_n(f)\| \le 3\|f\|$.

Applying Taylor's expansion to g,

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u,$$

$$2g\left(1-\frac{1}{n}\right) - g\left(1-\frac{2}{n}\right) = g(x) - (x-1)g'(x) - \int_x^{\frac{n-1}{n}} 2(u - \frac{n-1}{n})g''(u)\mathrm{d}u + \int_x^{\frac{n-2}{n}} (u - \frac{n-2}{n})g''(u)\mathrm{d}u,$$

we have

$$
\begin{aligned}
\widetilde{U}_n(g,x) =& \left(g(x) - (x-1)g'(x) - \int_x^{\frac{n-1}{n}} 2(u - \frac{n-1}{n})g''(u)\mathrm{d}u + \int_x^{\frac{n-2}{n}} (u - \frac{n-2}{n})g''(u)\mathrm{d}u \right) p_{n,n}(x) \\
&+ n\sum_{k=0}^{n-1} p_{n,k}(x) \int_0^1 p_{n-1,k}(t) \left(g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u \right) \mathrm{d}t \\
=& \left(g(x) - (x-1)g'(x) - \int_x^{\frac{n-1}{n}} 2(u - \frac{n-1}{n})g''(u)\mathrm{d}u + \int_x^{\frac{n-2}{n}} (u - \frac{n-2}{n})g''(u)\mathrm{d}u \right) p_{nn}(x) \\
&+ g(x)\sum_{k=0}^{n-1} p_{n,k}(x) + g'(x)\left(\sum_{k=0}^{n-1} \frac{k+1}{n+1} p_{n,k}(x) - x\sum_{k=0}^{n-1} p_{n,k}(x) \right) \\
&+ n\sum_{k=0}^{n-1} p_{n,k}(x) \int_0^1 p_{n-1,k}(t) \int_x^t (t-u)g''(u)\mathrm{d}u\mathrm{d}t \\
=& g(x) + \frac{1}{n+1}(1-x)g'(x) + n\sum_{k=0}^{n-1} p_{n,k}(x) \int_0^1 p_{n-1,k}(t) \int_x^t (t-u)g''(u)\mathrm{d}u\mathrm{d}t \\
&+ \left(-\int_x^{\frac{n-1}{n}} 2(u - \frac{n-1}{n})g''(u)\mathrm{d}u + \int_x^{\frac{n-2}{n}} (u - \frac{n-2}{n})g''(u)\mathrm{d}u \right) p_{n,n}(x).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\left|\widetilde{U}_n(g,x) - g(x)\right| \leq& \frac{1}{n+1}\left|(1-x)g'(x)\right| + n\left| \sum_{k=0}^{n-1} p_{n,k}(x) \int_0^1 p_{n-1,k}(t) \left| \int_x^t (t-u)g''(u)\mathrm{d}u \right| \mathrm{d}t \right| \\
&+ 2\left| \int_x^{\frac{n-1}{n}} (u - \frac{n-1}{n})g''(u)\mathrm{d}u \right| p_{n,n}(x) + \left| \int_x^{\frac{n-2}{n}} (u - \frac{n-2}{n})g''(u)\mathrm{d}u \right| p_{n,n}(x) \\
:=& I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Since $g \in D_\lambda^2$, it is obvious that

$$
I_1 = \frac{\left|(1-x)g'(x)\right|}{n+1} \leq \frac{C}{n}. \tag{12}
$$

Applying Lemma 2.4 and Lemma 2.1, we obtain

$$
\begin{aligned}
I_2 \leq& \left\|g''\delta_n^{2\lambda}\right\| n\left| \sum_{k=0}^{n-1} p_{n,k}(x) \int_0^1 p_{n-1,k}(t) \int_x^t \frac{|u-t|}{\delta_n^{2\lambda}(u)} \mathrm{d}u\mathrm{d}t \right| \\
\leq& \left\|g''\delta_n^{2\lambda}\right\| n\left| \sum_{k=0}^{n-1} p_{n,k}(x) \int_0^1 p_{n-1,k}(t) \frac{(t-x)^2}{\delta_n^{2\lambda}(x)} \mathrm{d}t \right| \\
\leq& \frac{\left\|g''\delta_n^{2\lambda}\right\|}{\delta_n^{2\lambda}(x)} n\left| \sum_{k=0}^{n-1} p_{n,k}(x) \int_0^1 p_{n-1,k}(t)(t^2 - 2xt + x^2)\mathrm{d}t \right| \\
\leq& \frac{\left\|g''\delta_n^{2\lambda}\right\|}{\delta_n^{2\lambda}(x)} \left(\frac{n(n-1)x^2}{(n+1)(n+2)} + \frac{4nx+2}{(n+1)(n+2)} - x^n - 2x(\frac{nx+1}{n+1} - x^n) + x^2(1-x^n) \right) \\
\leq& \frac{\left\|g''\delta_n^{2\lambda}\right\|}{\delta_n^{2\lambda}(x)} \left(-\frac{2(n-1)}{(n+1)(n+2)}x^2 + \frac{2(n-2)}{(n+1)(n+2)}x + \frac{2}{(n+1)(n+2)} \right)
\end{aligned}
$$

$$\leq \frac{\left\|g''\delta_n^{2\lambda}\right\|}{\delta_n^{2\lambda}(x)}\left(\frac{2}{n}\left(x-x^2\right)+\frac{2}{(n+1)(n+2)}\right)$$
$$\leq C\frac{\left\|g''\delta_n^{2\lambda}\right\|}{\delta_n^{2\lambda}(x)}\left(\frac{\varphi^2(x)}{n}+\frac{1}{n^2}\right). \tag{13}$$

By Lemma 2.4, we have

$$I_3 \leq C\left\|g''\delta_n^{2\lambda}\right\|\left|\int_x^{\frac{n-1}{n}}\frac{\left|u-\frac{n-1}{n}\right|}{\delta_n^{2\lambda}(u)}\mathrm{d}u\right|p_{n,n}(x) \leq C\frac{\left\|g''\delta_n^{2\lambda}\right\|}{\delta_n^{2\lambda}(x)}\left(x-\frac{n-1}{n}\right)^2x^n \leq C\frac{\left\|g''\delta_n^{2\lambda}\right\|}{\delta_n^{2\lambda}(x)}\frac{1}{n^2}. \tag{14}$$

Similarly,

$$I_4 \leq C\frac{\left\|g''\delta_n^{2\lambda}\right\|}{\delta_n^{2\lambda}(x)}\frac{1}{n^2}. \tag{15}$$

By (9) and (10), it follows from (12)-(15) that

$$\begin{aligned}\left|\widetilde{U}_n(g,x)-g(x)\right| &\leq C\frac{\left\|g''\delta_n^{2\lambda}\right\|}{\delta_n^{2\lambda}(x)}\left(\frac{\varphi^2(x)}{n}+\frac{1}{n^2}\right)+\frac{C}{n} \leq C\frac{\left\|g''\delta_n^{2\lambda}\right\|}{\delta_n^{2\lambda}(x)}\frac{1}{n}\left(\varphi^2(x)+\frac{1}{\sqrt{n}}\right)^2+\frac{C}{n}\\ &\leq C\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\varphi^{2\lambda}g''\right\|+\frac{\delta_n^{2(1-\lambda)}(x)}{n}\left(\frac{1}{\sqrt{n}}\right)^{2\lambda}\left\|g''\right\|\right)+\frac{C}{n}\\ &\leq C\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\varphi^{2\lambda}g''\right\|+\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\left\|g''\right\|\right)+\frac{C}{n}\\ &\leq C\omega_{\varphi^\lambda}^2\left(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)+\frac{C}{n}.\end{aligned} \tag{16}$$

By (11) and (16), we obtain

$$\left|\widetilde{U}_n(f,x)-f(x)\right| \leq C\omega_{\varphi^\lambda}^2\left(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)+\frac{C}{n},$$

which completes the proof of Theorem 3.1. □

Remark 3.2. *$\widetilde{U}_n(f,x)$ is a linear operator, however, it can not reproduce affine functions (Lemma 2.2). Therefore, although we use the second order modulus to evaluate the approximation rate in Theorem 3.1, the term $\frac{C}{n}$ in addition to $C\omega_{\varphi^\lambda}^2\left(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)$ on the right-hand side of (7) cannot be cancelled.*

On the other hand, since $\widetilde{U}_n(f,x)$ is independent of the value $f(1)$, it has the potential advantage to approximate functions with singularity at one endpoint, and the results in this paper also provide a foundation for the subsequent work on weighted approximation in $C_\omega = \{f, f \in C[0,1)\}$.

Next, we give a Voronovskaja type theorem of $\widetilde{U}_n(f,x)$ to study its asymptotic behaviour.

Theorem 3.3. *For any $f \in C^2[0,1]$, we have*

$$\left|\widetilde{U}_n(f,x)-f(x)-\frac{1-x}{n+1}f'(x)-\left(\frac{1-n}{(n+1)(n+2)}x^2+\frac{n-2}{(n+1)(n+2)}x+\frac{1}{(n+1)(n+2)}-\frac{x^n}{n^2}\right)f''(x)\right|$$
$$\leq C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}\left(f'',\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right),$$

where $\omega_{\varphi^\lambda}(f,t)$ is the Ditzian-Totik modulus, defined by

$$\omega_{\varphi^\lambda}(f,t) = \sup_{h\in(0,t]} \sup_{x,\, x+h\varphi^\lambda(x)\in[0,1]} \left| f\left(x + h\varphi^\lambda(x)\right) - f(x)\right|. \tag{17}$$

Proof. For any $f \in C^2[0,1]$, define

$$K^*_{\varphi^\lambda}(f,t) := \inf_{g\in D_\lambda} \left\{ \left\| f'' - g \right\| + t \left\| \varphi^\lambda g' \right\| + t^{\frac{1}{1-\frac{\lambda}{2}}} \left\| g' \right\| \right\}.$$

where $D_\lambda := \left\{ f : f \in A.C._{loc}, \|\varphi^\lambda f'\| < \infty, \|f'\| < \infty \right\}$. It is well known that $K^*_{\varphi^\lambda}(f,t) \sim \omega_{\varphi^\lambda}(f,t)$[6]. Therefore, for $f \in C^2[0,1]$, we may choose $g = g_{n,x,\lambda}(t) \in D_\lambda$ such that

$$\left\| f'' - g \right\| \le C\omega_{\varphi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \right), \tag{18}$$

$$\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \left\| \varphi^\lambda g' \right\| \le C\omega_{\varphi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \right), \tag{19}$$

$$\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \right)^{\frac{1}{1-\frac{\lambda}{2}}} \left\| g' \right\| \le C\omega_{\varphi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}} \right). \tag{20}$$

Direct calculations yield that

$$\begin{aligned}
\widetilde{U}_n(f,x) &= f(x) + \frac{1-x}{n+1} f'(x) + \left(\frac{(1-n)x^2}{(n+1)(n+2)} + \frac{(n-2)x}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} - \frac{x^n}{n^2} \right) f''(x) \\
&+ n \sum_{k=0}^{n-1} p_{n,k}(x) \int_0^1 p_{n-1,k}(t) \int_x^t (t-u)\left(f''(u) - f''(x)\right) \mathrm{d}u\mathrm{d}t \\
&+ \left(\int_x^{\frac{n-1}{n}} 2\left(\frac{n-1}{n} - u\right)\left(f''(u) - f''(x)\right) \mathrm{d}u + \int_x^{\frac{n-2}{n}} \left(\frac{n-2}{n} - u\right)\left(f''(u) - f''(x)\right) \mathrm{d}u \right) p_{n,n}(x).
\end{aligned}$$

Then,

$$\begin{aligned}
&\left| \widetilde{U}_n(f,x) - f(x) - \frac{1-x}{n+1} f'(x) - \left(\frac{(1-n)x^2}{(n+1)(n+2)} + \frac{(n-2)x}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} - \frac{x^n}{n^2} \right) f''(x) \right| \\
&\le n \left| \sum_{k=0}^{n-1} p_{n,k}(x) \int_0^1 p_{n-1,k}(t) \int_x^t |t-u| \left(f''(u) - f''(x)\right) \mathrm{d}u\mathrm{d}t \right| \\
&+ 2 \left| \int_x^{\frac{n-1}{n}} \left| \frac{n-1}{n} - u \right| \left(f''(u) - f''(x)\right) \mathrm{d}u \right| p_{n,n}(x) \\
&+ \left| \int_x^{\frac{n-2}{n}} \left| \frac{n-2}{n} - u \right| \left(f''(u) - f''(x)\right) \mathrm{d}u \right| p_{n,n}(x) \\
&:= M_1 + M_2 + M_3.
\end{aligned}$$

For M_1,

$$M_1 \le n\left|\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\int_x^t |t-u|\,(f''(u)-g(u))\,\mathrm{d}u\mathrm{d}t\right|$$
$$+\, n\left|\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\int_x^t |t-u|\,(g(x)-f''(x))\,\mathrm{d}u\mathrm{d}t\right| \tag{21}$$
$$+\, n\left|\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\int_x^t |t-u|\,(g(u)-g(x))\,\mathrm{d}u\mathrm{d}t\right| := \sum_{i=1}^{3} M_{1i}.$$

By Lemma 2.1 and (6), we have

$$\begin{aligned} M_{11} &\le \|f''-g\|\left|n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\int_x^t |t-u|\mathrm{d}u\mathrm{d}t\right| \\ &\le C\|f''-g\|\left|n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)(t-x)^2\mathrm{d}t\right| \\ &\le C\|f''-g\|\left(\frac{\varphi^2(x)}{n}+\frac{1}{n^2}\right) \le C\|f''-g\|\,\frac{\delta_n^2(x)}{n} \\ &\le C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}\left(f'',\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \end{aligned} \tag{22}$$

Similarly,

$$M_{12} \le C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}\left(f'',\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{23}$$

For M_{13}, we have

$$\begin{aligned} M_{13} &\le n\left|\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\int_x^t |t-u|\left|\int_x^u g'(v)\mathrm{d}v\right|\mathrm{d}u\mathrm{d}t\right| \\ &\le Cn\left|\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)(t-x)^2\left|\int_x^t g'(v)\mathrm{d}v\right|\mathrm{d}t\right|. \end{aligned}$$

When $x\in\left[0,\frac{1}{n}\right]\cup\left[\frac{n-1}{n},1\right]$, $\delta_n(x)\sim\frac{1}{\sqrt{n}}$ and $\delta_n(x)\le C\delta_n(t),\ t\in[0,1]$. By Lemma 2.3, (19), and (20),

$$\begin{aligned} M_{13} &\le C\|g'\delta_n^\lambda\|\, n\left|\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)(t-x)^2\left|\int_x^t \frac{1}{\delta_n^\lambda(v)}\mathrm{d}v\right|\mathrm{d}t\right| \\ &\le C\frac{\|g'\delta_n^\lambda\|}{\delta_n^\lambda(x)}n\left|\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)|t-x|^3\mathrm{d}t\right| \\ &\le C\frac{\|g'\delta_n^\lambda\|}{\delta_n^\lambda(x)}\frac{\delta_n^2(x)}{n}\frac{\delta_n(x)}{\sqrt{n}} \\ &\le C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\varphi^\lambda g'\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|\right) \end{aligned}$$

$$\leq C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{24}$$

When $x \in \left(\frac{1}{n}, \frac{n-1}{n}\right)$, we have $\delta_n(x) \sim \varphi(x)$. Thus, using Lemma 2.3 and (19),

$$\begin{aligned}
M_{13} &\leq C\left\|\varphi^\lambda g'\right\| n \left|\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)(t-x)^2\left|\int_x^t \frac{1}{\varphi_n^\lambda(v)}\mathrm{d}v\right|\mathrm{d}t\right| \\
&\leq C\frac{\left\|\varphi^\lambda g'\right\|}{\varphi_n^\lambda(x)} n \left|\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)|t-x|^3\mathrm{d}t\right| \\
&\leq C\frac{\delta_n^2(x)}{n}\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left\|\varphi^\lambda g'\right\| \\
&\leq C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right).
\end{aligned} \tag{25}$$

Combining (21)-(25), we obtain

$$M_1 \leq C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{26}$$

For M_2, we have

$$\begin{aligned}
M_2 &\leq C\left|\int_x^{\frac{n-1}{n}}\left|\frac{n-1}{n}-u\right|\left(f''(u)-g(u)\right)\mathrm{d}u\right| p_{n,n}(x) \\
&+ C\left|\int_x^{\frac{n-1}{n}}\left|\frac{n-1}{n}-u\right|\left(g(x)-f''(x)\right)\mathrm{d}u\right| p_{n,n}(x) \\
&+ C\left|\int_x^{\frac{n-1}{n}}\left|\frac{n-1}{n}-u\right|\left(g(u)-g(x)\right)\mathrm{d}u\right| p_{n,n}(x) := \sum_{i=1}^{3} M_{2i}.
\end{aligned} \tag{27}$$

Thus,

$$M_{21} \leq C\left\|f''-g\right\|\left(x-\frac{n-1}{n}\right)^2 p_{n,n}(x) \leq C\left\|f''-g\right\|\frac{1}{n^2} \leq C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{28}$$

Similarly,

$$M_{22} \leq C\left\|f''-g\right\|\frac{1}{n^2} \leq C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{29}$$

$$M_{23} \leq C\left(x-\frac{n-1}{n}\right)^2 p_{n,n}(x)\left|\int_x^{\frac{n-1}{n}} g'(v)\mathrm{d}v\right| \leq C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{30}$$

Combining (27)-(30), we have

$$M_2 \leq C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{31}$$

Similar calculation leads to

$$M_3 \leq C\frac{\delta_n^2(x)}{n}\omega_{\varphi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{32}$$

All together, (26), (31), and (32) give the result of Theorem 3.3. □

4. Approximation by Two More Bernstein-type Operators

As introduced in Section 1, one key feature of $\widetilde{U}_n(f,x)$ is that its structure does not depend on the value $f(1)$. This means the weighted approximation of $\widetilde{U}_n(f,x)$ to functions with singularity at the endpoint 1 can be further investigated. Meanwhile we are also curious about the scenario where $f(x)$ does have a value at the endpoint 1. To accommodate the aforementioned scenario, we construct one new BD operator defined in (33).

$$V_n(f,x) = f(1)p_{n,n}(x) + n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)f(t)\mathrm{d}t. \tag{33}$$

Notice that $V_n(f,x)$ is a positive linear operator.

Another generalization of Bernstein operator is called Bernstein Kantorovich (BK) operator. Recall it is defined by

$$K_n(f,x) = (n+1)\sum_{k=0}^{n}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\mathrm{d}t\ p_{n,k}(x). \tag{34}$$

It was proved in [5] that:

$$\|K_n(f) - f\|_{L_P} \le C\omega_2(f, 1/\sqrt{n}),\ \ p \ge 1, f \in L_p[0,1]. \tag{35}$$

Based on the modified BK operator [25], we propose another new operator $\widetilde{K}_n(f,x)$ as a comparison to $V_n(f,x)$, as well as to $\widetilde{U}_n(f,x)$ detailed in Section 3.

$$\widetilde{K}_n(f,x) = f(1)p_{n,n}(x) + (n+1)\sum_{k=0}^{n-1}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(a_{nk}t)\mathrm{d}t\ p_{n,k}(x), \tag{36}$$

where the scaling factor $a_{nk} := \dfrac{n+1}{n}\cdot\dfrac{2k}{2k+1}, k = 0,1,\cdots;\ n = 1,2,\cdots$.

Obviously, $\widetilde{K}_n(f,x)$ is also a positive operator, with the property that it reproduces affine functions (proved in Theorem 4.5). So far, we have proposed three variants of Bernstein-type operators, which will be compared based on their different properties and demonstrated in Section 5.

In Sections 4.1 and 4.2, the unique properties and approximation performance of $V_n(f,x)$ and $\widetilde{K}_n(f,x)$ will be presented.

4.1. Properties and Approximation performance of $V_n(f,x)$

We start with Lemma 4.1.

Lemma 4.1. *For any $x \in [0,1]$, $0 \le \lambda \le 1$, it holds that*

$$\int_x^1 \frac{1}{\delta_n^\lambda(u)}\mathrm{d}u \le C\frac{1-x}{\delta_n^\lambda(x)},$$

where $\delta_n(x) := \dfrac{1}{\sqrt{n}} + \varphi(x)$.

Proof. When $x \in \left[0, \frac{1}{n}\right] \cup \left[\frac{n-1}{n}, 1\right]$, $\delta_n(x) \sim \frac{1}{\sqrt{n}}$.

$$\int_x^1 \frac{1}{\delta_n^\lambda(u)}\mathrm{d}u \le \int_x^1 \frac{1}{n^{-\frac{\lambda}{2}}}\mathrm{d}u = \frac{1-x}{\left(\frac{1}{\sqrt{n}}\right)^\lambda} \le C\frac{1-x}{\delta_n^\lambda(x)}.$$

When $x \in \left(\frac{1}{n}, \frac{n-1}{n}\right)$,

$$\begin{aligned}\int_x^1 \frac{1}{\delta_n(u)}\mathrm{d}u &= \int_x^1 \frac{1}{\sqrt{u(1-u)} + \sqrt{\frac{1}{n}}}\mathrm{d}u \le \int_x^1 \frac{1}{\sqrt{u+\frac{1}{n}}}\mathrm{d}u + \int_x^1 \frac{1}{\sqrt{1-u}}\mathrm{d}u \\ &= 2\left(\sqrt{1+\frac{1}{n}} - \sqrt{x+\frac{1}{n}} + \sqrt{1-x}\right) = 2\left(\frac{1-x}{\sqrt{1+\frac{1}{n}} + \sqrt{x+\frac{1}{n}}} + \frac{1-x}{\sqrt{1-x}}\right) \\ &\le C\frac{1-x}{\delta_n(x)}.\end{aligned} \tag{37}$$

Thus, by Hölder inequality and (37), we have

$$\int_x^1 \frac{1}{\delta_n^\lambda(u)}\mathrm{d}u \le \left(\int_x^1 \frac{1}{\delta_n(u)}\mathrm{d}u\right)^\lambda (1-x)^{1-\lambda} \le \left(C\frac{1-x}{\delta_n(x)}\right)^\lambda (1-x)^{1-\lambda} \le C\frac{1-x}{\delta_n^\lambda(x)}.$$

□

Next, we will investigate the approximation properties of $V_n(f,x)$. Theorem 4.2 is about the uniform convergence of $V_n(f,x)$ to $f \in C[0,1]$, which qualitatively shows the approximation of $V_n(f,x)$ to a continuous function $f(x)$. Theorem 4.3 shows an important property of $V_n(f,x)$, i.e., the unique relationship between the derivative of $V_n(f,x)$ and the smoothness of function $f(x)$.

Theorem 4.2. *For a sequence of positive linear operators $V_n(f)$ in $C[0,1]$, it holds that $V_n(f,x)$ uniformly convergence to $f(x)$.*

Proof.

$$V_n(1,x) = 1, \quad V_n(t,x) = \frac{n}{n+1}x + \frac{1}{n+1} \to x, \ \ n \to \infty,$$

$$V_n(t^2,x) = \frac{n(n-1)}{(n+1)(n+2)}x^2 + \frac{4n}{(n+1)(n+2)}x + \frac{2}{(n+1)(n+2)} \to x^2, \ \ n \to \infty,$$

by Korovkin's theorem, for every $f \in C[0,1]$, $V_n(f,x)$ uniformly converges to $f(x)$. □

Theorem 4.3. *Let $V_n(f,x)$ be defined by (33), assume $f'(x), f''(x)$ exist, then it holds that*

$$V_n'(f,x) = n\sum_{k=0}^{n-1} p_{n-1,k}(x)\int_0^1 p_{n,k+1}(t)f'(t)\mathrm{d}t,$$

$$V_n''(f,x) = \frac{n(n-1)}{n+1}\sum_{k=0}^{n-2} p_{n-2,k}(x)\int_0^1 p_{n+1,k+2}(t)f''(t)\mathrm{d}t.$$

Proof. Denote

$$V_n(f,x) := \sum_{k=0}^{n} v_k^{(n)}(f) p_{n,k}(x),$$

where

$$v_k^{(n)}(f) = n\int_0^1 p_{n-1,k}(t) f(t)\mathrm{d}t,\ 0 \le k \le n-1,$$

and

$$v_n^{(n)}(f) = f(1).$$

Let

$$\Delta^1 v_k^{(n)}(f) = v_{k+1}^{(n)}(f) - v_k^{(n)}(f),$$
$$\Delta^2 v_k^{(n)}(f) = \Delta^1\left(\Delta^1 v_k^{(n)}(f)\right).$$

Using the above notations, we calculate the derivative of $V_n(f,x)$ and obtain,

$$V_n'(f,x) = n\sum_{k=0}^{n-1} \Delta^1 v_k^{(n)}(f) p_{n-1,k}(x), \tag{38}$$

$$V_n''(f,x) = n(n-1)\sum_{k=0}^{n-2} \Delta^2 v_k^{(n)}(f) p_{n-2,k}(x). \tag{39}$$

Direct calculations lead to:

$$\Delta^1 v_k^{(n)}(f) = \int_0^1 p_{n,k+1}(t) f'(t)\mathrm{d}t. \tag{40}$$

$$\Delta^2 v_k^{(n)}(f) = \frac{1}{n+1}\int_0^1 p_{n+1,k+2}(t) f''(t)\mathrm{d}t. \tag{41}$$

Actually, when $0 \le k \le n-3$,

$$\begin{aligned}\Delta^2 v_k^{(n)}(f) &= n\int_0^1 \left(p_{n-1,k+2}(t) - 2p_{n-1,k+1}(t) + p_{n-1,k}(t)\right) f(t)\mathrm{d}t \\ &= \frac{1}{n+1}\int_0^1 n\left(p''_{n+1,k+2}(t)\right) f(t)\mathrm{d}t = \frac{1}{n+1}\int_0^1 p_{n+1,k+2}(t) f''(t)\mathrm{d}t.\end{aligned}$$

When $k = n-2$,

$$\begin{aligned}\Delta^2 v_k^{(n)}(f) &= \Delta^2 v_{n-2}^{(n)}(f) = \Delta^1\left(v_{n-1}^{(n)}(f) - v_{n-2}^{(n)}(f)\right) = v_n^{(n)}(f) - 2v_{n-1}^{(n)}(f) + v_{n-2}^{(n)}(f) \\ &= f(1) - 2n\int_0^1 p_{n-1,n-1}(t) f(t)\mathrm{d}t + \int_0^1 n p_{n-1,n-2}(t) f(t)\mathrm{d}t \\ &= f(1) - n\int_0^1 p_{n-1,n-1}(t) f(t)\mathrm{d}t + n\left(\int_0^1 p_{n-1,n-2}(t)\mathrm{d}t - \int_0^1 p_{n-1,n-1}(t)\mathrm{d}t\right) f(t) \\ &= f(1) - \int_0^1 n t^{n-1} f(t)\mathrm{d}t + \int_0^1 p'_{n-1,n-1}(t) f(t)\mathrm{d}t \\ &= \int_0^1 \left(p_{n,n}(t) - p_{n,n-1}(t)\right) f'(t)\mathrm{d}t \\ &= -\frac{1}{n+1}\int_0^1 p'_{n+1,n}(t) f'(t)\mathrm{d}t = \frac{1}{n+1}\int_0^1 p_{n+1,n}(t) f''(t)\mathrm{d}t.\end{aligned}$$

Here we only prove (41), since the proof of (40) is similar and simpler with only fewer steps.

After substituting (40) in (38), (41) in (39), we complete the proof of Theorem 4.3. □

Furthermore, we estimate the approximation rate of $V_n(f,x)$, as a quantitative result, and obtain Theorem 4.4.

Theorem 4.4. *Let $0 \le \lambda \le 1$ be a fixed number, for any $f \in C[0,1]$, there is a positive constant C only depending on λ such that*

$$\left|V_n(f,x) - f(x)\right| \le C\omega_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right).$$

Proof. For any $f \in C[0,1]$, define

$$K_{\varphi^\lambda}(f,t) := \inf_{g\in W_\lambda}\left\{\left\|f-g\right\| + t\left\|\varphi^\lambda g'\right\| + t^{\frac{1}{1-\frac{\lambda}{2}}}\left\|g'\right\|\right\}.$$

where

$$W_\lambda = \left\{f : f \in A.C._{loc}, \|\varphi^\lambda f'\| < \infty, \|f'\| < \infty\right\}.$$

By [6] (Theorem 3.12), we have

$$K_{\varphi^\lambda}(f,t) \sim \omega_{\varphi^\lambda}(f,t), \quad 0 \le \lambda \le 1.$$

Therefore, we may choose $g = g_{n,x,\lambda} \in W_\lambda$ such that

$$\left\|f-g\right\| \le C\omega_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right), \tag{42}$$

$$\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left\|\varphi^\lambda g'\right\| \le C\omega_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right), \tag{43}$$

$$\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\left\|g'\right\| \le C\omega_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{44}$$

Using (6), we have $\left|V_n(f-g,x)\right| \le C\|f-g\|$, then by (42),

$$\begin{aligned}\left|V_n(f,x) - f(x)\right| &\le \left|V_n(f-g,x)\right| + \left|f(x)-g(x)\right| + \left|V_n(g,x)-g(x)\right| \\ &\le C\|f-g\| + \left|V_n(g,x)-g(x)\right| \\ &\le C\omega_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right) + \left|V_n(g,x)-g(x)\right|. \end{aligned}\tag{45}$$

By using Taylor's expansion,

$$g(1) = g(x) + \int_x^1 g'(u)\mathrm{d}u.$$

Thus,

$$\begin{aligned}\left|V_n(g,x) - g(x)\right| &= \left|g(1)\cdot p_{n,n}(x) + n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)g(t)\mathrm{d}t - g(x)\right| \\ &\le \left|\int_x^1 g'(u)\mathrm{d}u\right| p_{n,n}(x) + \left|n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\left(g(t)-g(x)\right)\mathrm{d}t\right| \\ &:= I_1 + I_2.\end{aligned}$$

It's easy to check that by Lemma 4.1,

$$I_1 \le \left\|g'\delta_n^\lambda\right\| \left|\int_x^1 \frac{1}{\delta_n^\lambda(u)}\mathrm{d}u\right| x^n \le \frac{\left\|g'\delta_n^\lambda\right\|}{\delta_n^\lambda(x)}(1-x)x^n \le C\frac{\left\|g'\delta_n^\lambda\right\|}{\delta_n^\lambda(x)}\frac{1}{n}. \tag{46}$$

Using Hölder inequality, we have

$$\begin{aligned}
I_2 &\le \left| n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\int_x^t \frac{g'(u)\delta_n^\lambda(u)}{\delta_n^\lambda(u)}\mathrm{d}u\mathrm{d}t\right| \\
&\le \frac{\left\|g'\delta_n^\lambda\right\|}{\delta_n^\lambda(x)}\cdot\left| n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\,|t-x|\,\mathrm{d}t\right| \\
&\le \frac{\left\|g'\delta_n^\lambda\right\|}{\delta_n^\lambda(x)}\left(n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\mathrm{d}t\right)^{\frac{1}{2}}\left(n\sum_{k=0}^{n-1} p_{n,k}(x)\int_0^1 p_{n-1,k}(t)\,(t-x)^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le \frac{\left\|g'\delta_n^\lambda\right\|}{\delta_n^\lambda(x)}\left(\sum_{k=0}^{n-1} p_{n,k}(x)\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n-1}\frac{(k+1)(k+2)}{(n+1)(n+2)}p_{n,k}(x) - 2x\sum_{k=0}^{n-1}\frac{k+1}{n+1}p_{n,k}(x) + x^2\sum_{k=0}^{n-1}p_{n,k}(x)\right)^{\frac{1}{2}} \\
&\le \frac{\left\|g'\delta_n^\lambda\right\|}{\delta_n^\lambda(x)}(1-x^n)^{\frac{1}{2}}\left(\frac{-2(n-1)}{(n+1)(n+2)}x^2 + \frac{2(n-2)}{(n+1)(n+2)}x + \frac{2}{(n+1)(n+2)} - x^n(x-1)^2\right)^{\frac{1}{2}} \\
&\le C\frac{\left\|g'\delta_n^\lambda\right\|}{\delta_n^\lambda(x)}\left(\frac{\varphi^2(x)}{n} + \frac{1}{n^2}\right)^{\frac{1}{2}}.
\end{aligned} \tag{47}$$

By (43), (44), it follows from (46) and (47) that

$$\begin{aligned}
\left|V_n(g,x) - g(x)\right| &\le C\frac{\left\|g'\delta_n^\lambda\right\|}{\delta_n^\lambda(x)}\left(\frac{\varphi^2(x)}{n} + \frac{1}{n^2}\right)^{\frac{1}{2}} \\
&\le C\frac{\left\|g'\delta_n^\lambda\right\|}{\delta_n^\lambda(x)}\left(\frac{1}{n}\left(\varphi + \frac{1}{\sqrt{n}}\right)^2\right)^{\frac{1}{2}} \le C\frac{\left\|g'\delta_n^\lambda\right\|}{\delta_n^\lambda(x)}\frac{\delta_n(x)}{\sqrt{n}} \\
&\le C\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left\|\varphi^\lambda g'\right\| + \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\right)^\lambda \left\|g'\right\|\right) \\
&\le C\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left\|\varphi^\lambda g'\right\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\left\|g'\right\|\right) \\
&\le C\omega_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right).
\end{aligned} \tag{48}$$

Combining (45) and (48) ends the proof of Theorem 4.4. □

4.2. *Approximation performance of $\widetilde{K}_n(f,x)$*

In this section, we focus on the approximation performance of $\widetilde{K}_n(f,x)$, which results in Theorems 4.5 and 4.6.

Theorem 4.5. *It holds that*

$$\widetilde{K}_n(1,x) = 1, \qquad \widetilde{K}_n(t,x) = x.$$

Theorem 4.6. *Let $0 \le \lambda \le 1$ be a fixed number. For any $f \in C[0,1]$, there is a positive constant only depending on λ such that*

$$\left|\widetilde{K}_n(f,x) - f(x)\right| \le C\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right).$$

Theorem 4.5 shows that $\widetilde{K}_n(f,x)$ reproduces affine functions, and its proof is straightforward. The proof of Theorem 4.6 is similar to that of Theorem 3.1 (but simpler), which is omitted here.

Remark 4.7. *Direct calculation leads to*

$$\widetilde{K}_n(t^2,x) = x^2 + \frac{1}{n}(x-x^2) + \frac{1}{12n^2}\sum_{k=0}^{n}\frac{k^2}{\left(k+\frac{1}{2}\right)^2}p_{nk}(x) - \frac{1}{12n^2+12n+3}x^n,$$

and the third term on the right-hand side can be bounded by:

$$\frac{1}{48n^2}\left(1-(1-x)^n\right) \le \frac{1}{12n^2}\sum_{k=0}^{n}\frac{k^2}{\left(k+\frac{1}{2}\right)^2}p_{nk}(x) \le \frac{1}{12n^2},$$

then, it is easy to check that for $x \in [0,1]$, when $n \to \infty$,

$$\widetilde{K}_n(t^2,x) \to x^2 \tag{49}$$

uniformly. Therefore, applying Theorem 4.5, (49) and Korovkin's theorem, we obtain the uniform convergence of $\widetilde{K}_n(f,x)$ to every $f \in C[0,1]$.

5. Analytical and Numerical Comparison of $\widetilde{U}_n(f,x)$, $V_n(f,x)$, and $\widetilde{K}_n(f,x)$

5.1. Analytical Comparison

Given the different structures of $\widetilde{U}_n(f,x)$, $V_n(f,x)$, and $\widetilde{K}_n(f,x)$, and their related approximation theorems and asymptotic behaviour, the specific properties of these three operators are summarized in Table 5.1.

	linearity	genuineness	positivity	approximation rate
$\widetilde{U}_n(f,x)$	✓			$C\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right) + \frac{C}{n}$
$V_n(f,x)$	✓		✓	$C\omega_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)$
$\widetilde{K}_n(f,x)$	✓	✓	✓	$C\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)$

Table 5.1. Properties and Approximation Rates of Three Constructed Operators

We highlight the similarity and difference among the operators below:

- All three operators are linear.
- It can be inferred from Theorem 4.5 that $\widetilde{K}_n$ reproduces affine functions.
- From the structures of $V_n(f,x)$ and $\widetilde{K}_n(f,x)$, it is clear that they are positive operators, and for every $f \in C[0,1]$, $V_n(f,x)$ and $\widetilde{K}_n(f,x)$ converge uniformly to $f(x)$ (Theorem 4.2, Remark 4.7).
- Comparison of the approximation rate of these operators:

- Comparing $\widetilde{U}_n$ and V_n based on Theorem 3.1 and Theorem 4.4, we observe that, for sufficiently large n, $\widetilde{U}_n$ generally owns a better approximation order than what the positive operator V_n can achieve.

- As for V_n and $\widetilde{K}_n$, it can be seen that $\widetilde{K}_n$ effectively balances the two properties - positivity and genuineness (reproducing linearity), and the genuineness of $\widetilde{K}_n$ leads to higher order of approximation rate. Comparing Theorem 4.4 and Theorem 4.6, we see that $\widetilde{K}_n$ achieves a better approximation rate than what V_n does.

- Comparing $\widetilde{U}_n$ and $\widetilde{K}_n$ based on Theorem 3.1 and Theorem 4.6, their approximation rates differ by the term $\frac{C}{n}$, which can not be cancelled.

5.2. *Numerical Comparison*

5.2.1. *Numerical Experiments of Three Positive Operators $V_n(f,x)$, $K_n(f,x)$ and $\widetilde{K}_n(f,x)$*

In this section, we dive in to investigate the approximation performance of the proposed operators on a sample function with singularity at the endpoint 1. The results are visualized to empirically verify the theoretical results of Theorem 4.4, Theorem 4.6 and (35). $V_n(f,x)$ and $\widetilde{K}_n(f,x)$ are structured in Section 4, while $K_n(f,x)$ is the original BK operator defined by (34).

Set $f(x) = (1-x)^2 \log(1-x)$. Figure 1 and Figure 2 show the approximation performance of three operators to function $f(x)$ on $(0,1)$ with $n = 50,\ 100$.

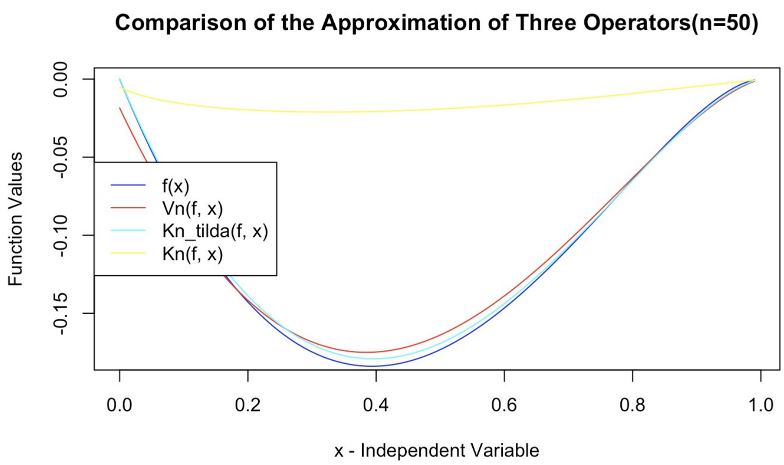

Fig. 1 Approximation by three positive operators to function $f(x) = (1-x)^2 \log(1-x)$.

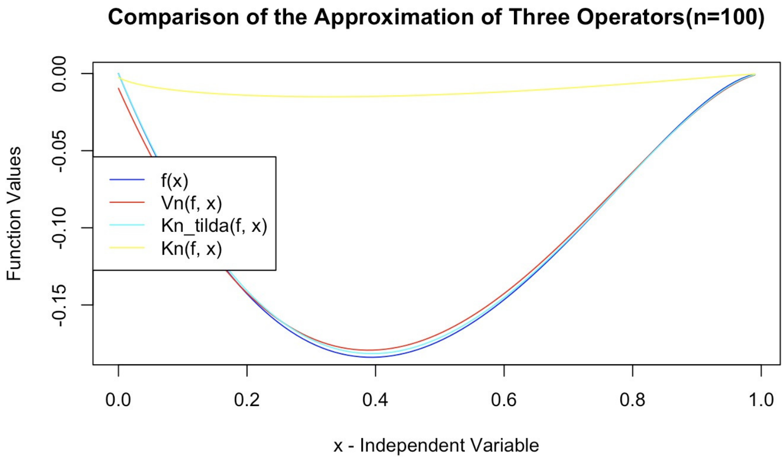

Fig. 2 Approximation by three positive operators to function $f(x) = (1-x)^2 \log(1-x)$.

As we can observe from Fig. 1 and Fig. 2, $\widetilde{K}_n$ has the best approximation performance, while the approximation performance of V_n is better than that of K_n. This observation is particularly clear where x approaches 1. The values of three operators and function $f(x)$ near the endpoint 1 are shown in the following figure.

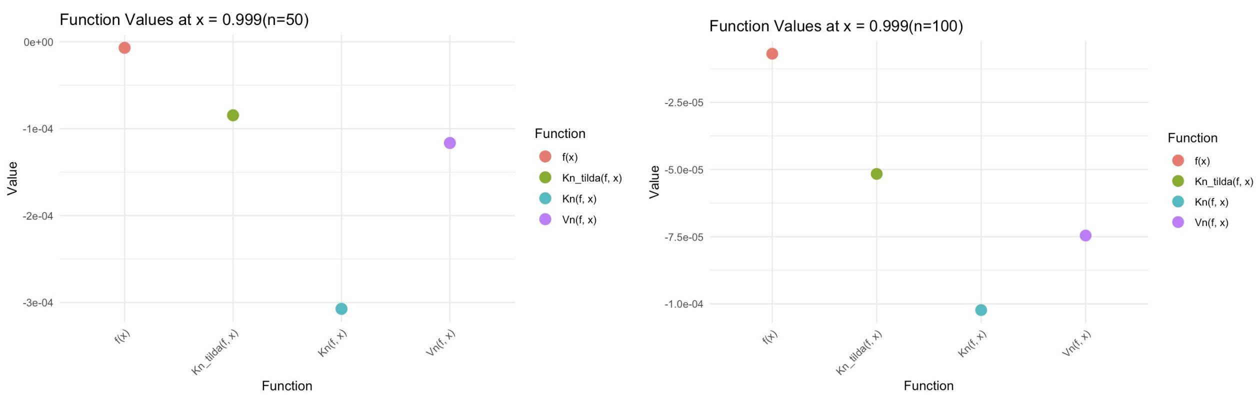

Fig. 3 The values of three positive operators and function $f(x) = (1-x)^2 \log(1-x)$ at $x = 0.999$.

5.2.2. *Numerical Experiments of Three Proposed Operators $\widetilde{U}_n$, $V_n(f,x)$, and $\widetilde{K}_n(f,x)$*

In this section, we compare the three newly proposed operators visually. As shown in Fig. 4-Fig. 5, $\widetilde{K}_n$ generally performs better than the other two operators when approaching the sample function, while it can be observed from Fig. 6 - Fig. 9 that $\widetilde{K}_n$ and $\widetilde{U}_n$ perform better than $V_n(f,x)$ near the endpoint 1, which is consistent with the theoretical results (Theorem 3.1, 4.6, and Theorem 4.4).

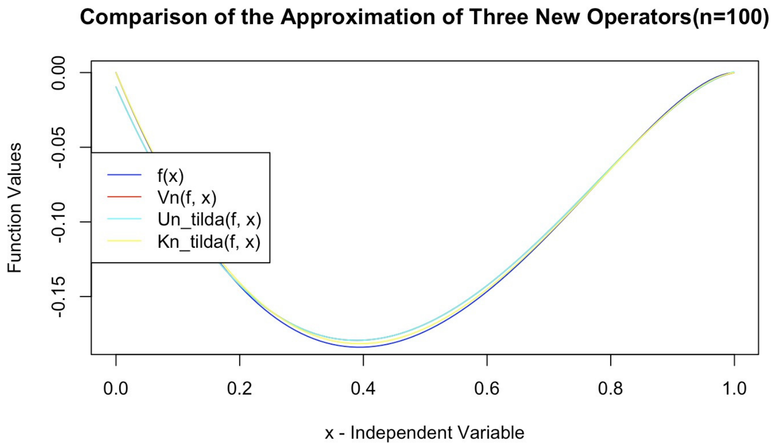

Fig. 4 Approximation by three new operators to function $f(x) = (1-x)^2\log(1-x)$.

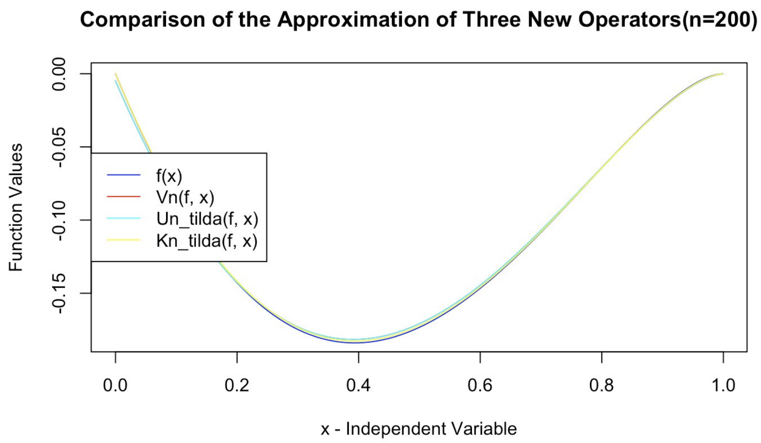

Fig. 5 Approximation by three new operators to function $f(x) = (1-x)^2\log(1-x)$.

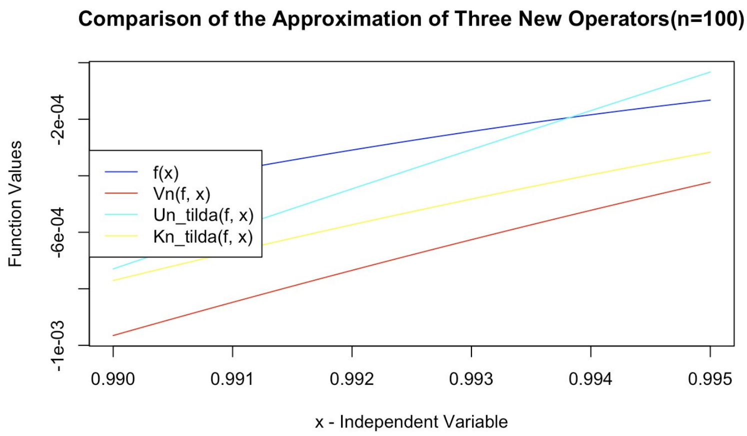

Fig. 6 Approximation by three new operators to function $f(x) = (1-x)^2\log(1-x)$, $(0.990 \le x \le 0.995)$.

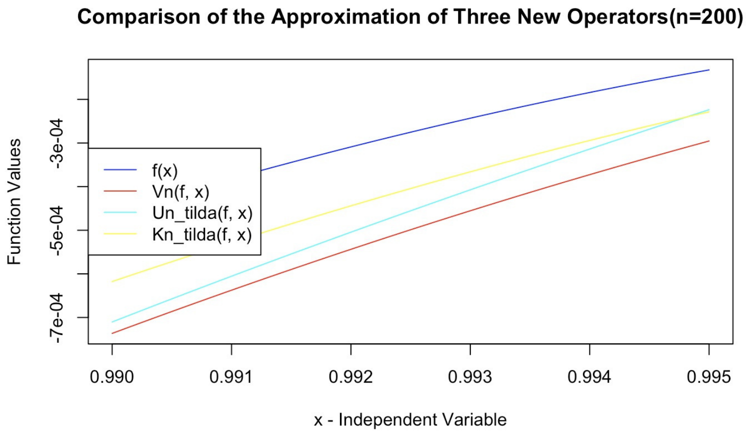

Fig. 7 Approximation by three new operators to function $f(x) = (1-x)^2 \log(1-x)$, $(0.990 \le x \le 0.995)$.

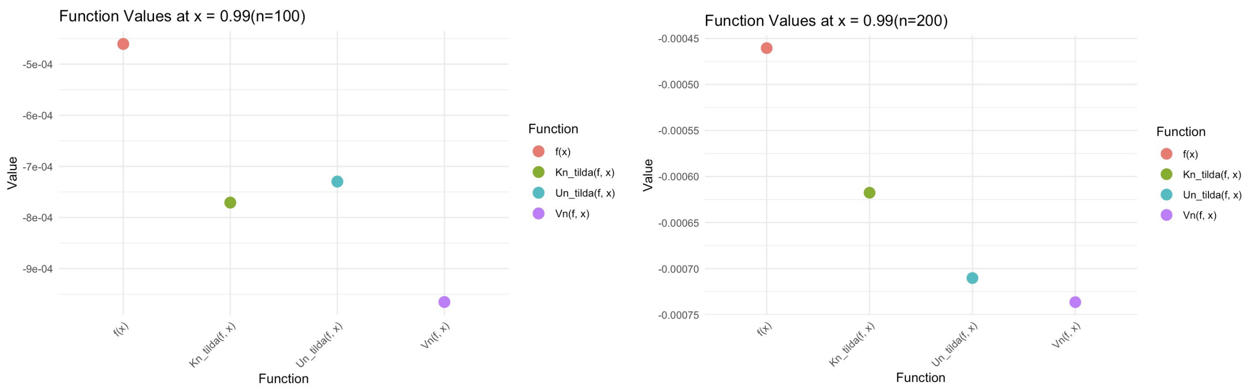

Fig. 8 The values of three new operators and function $f(x) = (1-x)^2 \log(1-x)$ at $x = 0.99$.

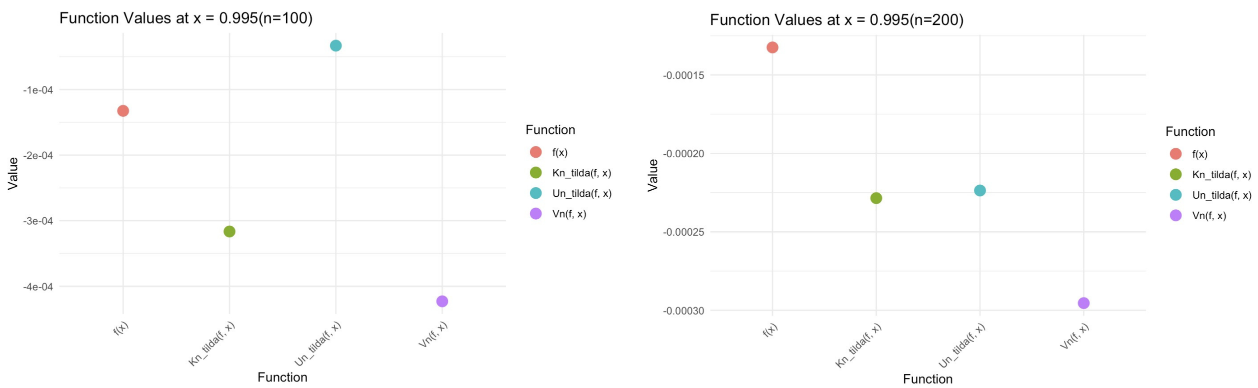

Fig. 9 The values of three new operators and function $f(x) = (1-x)^2 \log(1-x)$ at $x = 0.995$.

6. Conclusion and Future Research

In this paper, we construct three Bernstein-type operators $\widetilde{U}_n(f,x)$, $V_n(f,x)$, and $\widetilde{K}_n(f,x)$. We investigate the approximation rate of $\widetilde{U}_n(f,x)$ for continuous functions (Theorem 3.1), and obtain Voronovaskaja's asymptotic estimation (Theorem 3.3). We further study two positive linear Bernstein-type operators $V_n(f,x)$ and $\widetilde{K}_n(f,x)$, their approximation performance in both qualitative and quantitative ways (Theorem 4.2, Theorem 4.4, Theorem 4.6, Remark 4.7), and other related properties (Theorem 4.3, Theorem 4.5). In Section 5, we compare the three proposed operators theoretically from different perspectives, including the positivity, linearity, and their approximation rates. We discover that the approximation ability of $\widetilde{K}_n(f,x)$ outperforms those of the other two operators. These observations are verified by the numerical experiments of showing the approximation performance of three operators to some sample function.

The subsequent related research directions, including weighted approximation of $\widetilde{U}_n(f,x)$, the inverse theorems in approximation theory and so on, are within the scope of our further consideration.

Conflict of interest

The authors declare that they have no conflicts of interest.

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