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New iterative criterions for testing the positive definiteness of multivariate homogeneous forms

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Abstract. The positive definite homogeneous multivariate forms play an important role in the automatic control and medical imaging, and the definiteness of the forms can be identified by special structured tensors. In this paper, we first state the equivalence between the positive definite multivariate forms and the corresponding tensors and account for the links between the positive definite tensors with $\mathcal{H}$-tensors. Then based on diagonal dominance, some iterative criterions are presented to test $\mathcal{H}$-tensors. Furthermore, we establish new iterative schemes for testing the positive definite multivariate homogeneous forms. The efficiency and validity of new methods are illustrated by numerical examples.

1. Introduction

For a positive integer n, let $N = \{1, 2, \cdots, n\}$. A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a multidimensional array with n^m entries, where $i_t = 1, \cdots, n$ for $t = 1, \cdots, m$ [4, 14]. Let $C^{[m,n]}(R^{[m,n]})$ be the set of mth-order n-dimensional complex (real) tensors. For $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in C^{[m,n]}$, $\mathcal{A}$ is called symmetric [6] if $a_{i_1 i_2 \cdots i_m} = a_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_m)}$ for any permutation σ on N. Tensor $\mathcal{I} = (\delta_{i_1 i_2 \cdots i_m})$ is called the unit tensor [21], where

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

For $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$, if there exist a complex number λ and a nonzero complex vector $x = (x_1, x_2, \cdots, x_n)^T$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathcal{A}$, and x is called an eigenvector corresponding to λ of $\mathcal{A}$ [5, 7, 14, 15]. $\mathcal{A}x^{m-1}$ and $\lambda x^{[m-1]}$ are vectors, whose ith components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \qquad (x^{[m-1]})_i = x_i^{m-1}.$$

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In particular, if λ and x are restricted to the real field, then we call λ an H-eigenvalue of $\mathcal{A}$ and x an H-eigenvector corresponding to λ of $\mathcal{A}$ [14].

An mth-degree homogeneous polynomial of n variables $f(x)$ can be denoted as

$$f(x) = \sum_{i_1,i_2,\ldots,i_m \in N} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \tag{1}$$

where $x \in R^n$. The homogeneous polynomial $f(x)$ in (1) is equivalent to the tensor product of a symmetric tensor $\mathcal{A}$ and x^m denoted as

$$f(x) \equiv \mathcal{A}x^m = \sum_{i_1,i_2,\ldots,i_m \in N} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \tag{2}$$

where $x = (x_1, x_2, \ldots, x_n) \in R^n$ [12]. When m is even, $f(x)$ is called positive definite if

$$f(x) > 0, \quad \text{for any } x \in R^n, \quad x \neq 0.$$

The symmetric tensor $\mathcal{A}$ is called positive definite if $f(x)$ is positive definite [12].

The positive definiteness of multivariate polynomial $f(x)$ plays an important role in automatic control, medical imaging, and the stability study of nonlinear autonomous systems and so on [13, 22]. However, for $n > 3$ and $m > 4$, it is not easy to identify the positive definiteness of such a multivariate form. For solving this problem, Qi [14] pointed out that $f(x)$ defined by (2) is positive definite if and only if the real symmetric tensor $\mathcal{A}$ is positive definite, and provided an eigenvalue method to verify the positive definiteness of $\mathcal{A}$ when m is even (see Theorem 1).

Theorem 1. [14] *Let $\mathcal{A}$ be an even-order real symmetric tensor, then $\mathcal{A}$ is positive definite if and only if all of its H-eigenvalues are positive.*

Although from Theorem 1 we can verify the positive definiteness of an even-order symmetric tensor $\mathcal{A}$ (the positive definiteness of the mth-degree homogeneous polynomial $f(x)$) by computing the H-eigenvalues of $\mathcal{A}$, it is difficult to compute all these H-eigenvalues when m and n are large. Recently, by introducing the definition of $\mathcal{H}$-tensor [2, 8], Li et al. [8] provided a practical sufficient condition for identifying the positive definiteness of an even-order symmetric tensor (see Theorem 2).

Theorem 2. [7] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an even-order real symmetric tensor of order m dimension n with $a_{k\cdots k} > 0$ for all $k \in N$. If $\mathcal{A}$ is an $\mathcal{H}$-tensor, then $\mathcal{A}$ is positive definite.*

For the latter, it is increasingly recognized that $\mathcal{H}$-tensors have important applications in the study of the multilinear systems, the higher-order Markov chains and the symmetric multiplayer games. Subsequently, with the help of generalized diagonally dominant tensor, various criterions for $\mathcal{H}$-tensors are established [1, 3, 9–11, 16–20, 23], which only depend on the entries of tensors and are very effective to identify whether a given tensor is an $\mathcal{H}$-tensor or not. For example, the following results are presented.

Theorem 3. [9] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in C^{[m,n]}$. If*

$$|a_{ii\cdots i}| > \sum_{\substack{i_2,i_3,\ldots,i_m \in N^{m-1} \setminus \Lambda_1^{m-1} \\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in \Lambda_1^{m-1}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \frac{R_j(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2\cdots i_m}|, \quad \forall i \in \Lambda_2,$$

then $\mathcal{A}$ is an $\mathcal{H}$-tensor, where

$$R_i(\mathcal{A}) = \sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}|, \quad \Lambda_1 = \{i \in N : |a_{ii\cdots i}| > R_i(\mathcal{A})\}, \quad \Lambda_2 = \{i \in N : |a_{ii\cdots i}| \leq R_i(\mathcal{A})\}.$$

Theorem 4. [1] *Let* $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in C^{[m,n]}$. *If for* $\forall i \in N_2$,

$$|a_{ii\cdots i}| > \frac{R_i(\mathcal{A})}{R_i(\mathcal{A}) - |a_{ii\cdots i}|}\left(q\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}|a_{ii_2\cdots i_m}| + q\sum_{\substack{i_2i_3\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}| + \sum_{i_2i_3\cdots i_m\in N_3^{m-1}}\max_{j\in\{i_2,i_3,\dots,i_m\}}\frac{tP_j(\mathcal{A})}{|a_{jj\cdots j}|}|a_{ii_2\cdots i_m}|\right),$$

and for $i \in N_1$, $|a_{ii\cdots i}| \neq \sum_{\substack{i_2i_3\cdots i_m\in N_0^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|$, *then* $\mathcal{A}$ *is an* $\mathcal{H}$*-tensor, where* $R_i(\mathcal{A}) = \sum_{\substack{i_2,\dots,i_m\in N\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|$,

$$N_1 = \{i \in N : 0 < |a_{ii\cdots i}| = R_i(\mathcal{A})\},\quad N_2 = \{i \in N : 0 < |a_{ii\cdots i}| < R_i(\mathcal{A})\},\quad N_3 = \{i \in N : |a_{ii\cdots i}| > R_i(\mathcal{A})\}.$$

This article is organized as follows: In Sect. 2, some definitions and notations are collected. In Sect. 3, some iterative criterions for testing $\mathcal{H}$-tensors are obtained, which extend the corresponding conclusions in [1, 9, 16]. As applications, some sufficient conditions for the positive definiteness of even-order homogeneous multivariate forms are provided in Sect. 4. Numerical examples are given to verify the corresponding results.

2. Preliminaries

Now some definitions, notations and lemmas are presented, which will be used in the sequel.

Definition 1. [8] Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in C^{[m,n]}$. $\mathcal{A}$ is called an $\mathcal{H}$-tensor if there is a positive vector $x = (x_1, x_2, \dots, x_n)^T \in R^n$ such that

$$|a_{ii\cdots i}|\, x_i^{m-1} > \sum_{\substack{i_2,\dots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}\left|a_{ii_2\cdots i_m}\right| x_{i_2}\cdots x_{i_m},\quad \forall i \in N.$$

Definition 2. [8] Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in C^{[m,n]}$. $\mathcal{A}$ is called a diagonally dominant tensor if

$$|a_{ii\cdots i}| \geq \sum_{\substack{i_2,\dots,i_m\in N\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|,\quad \forall i \in N. \tag{3}$$

We call $\mathcal{A}$ a strictly diagonally dominant tensor if all strict inequalities in (3) hold.

Definition 3. [21] Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in C^{[m,n]}$. $\mathcal{A}$ is called reducible, if there exists a nonempty proper index subset $I \subset N$ such that

$$a_{i_1i_2\cdots i_m} = 0,\quad \forall i_1 \in I,\quad \forall i_2, \dots, i_m \notin I.$$

$\mathcal{A}$ is irreducible if $\mathcal{A}$ is not reducible.

Definition 4. [4] Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in C^{[m,n]}$, and $X = diag(x_1, x_2, \dots, x_n)$. Denote

$$\mathcal{B} = (b_{i_1\cdots i_m}) = \mathcal{A}X^{m-1},\ \ b_{i_1i_2\cdots i_m} = a_{i_1i_2\cdots i_m}x_{i_2}x_{i_3}\cdots x_{i_m},\ \ i_j \in N,\ \ j \in N,$$

we call $\mathcal{B}$ the product of the tensor $\mathcal{A}$ and the matrix X.

Definition 5. [17] Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in C^{[m,n]}$. For $i, j \in N\,(i \neq j)$, if there exist indices $k_1, k_2, \dots, k_r$ satisfying

$$\sum_{\substack{i_2\cdots i_m\in N,\\ \delta_{k_si_2\cdots i_m}=0,\\ k_{s+1}\in\{i_2\cdots i_m\}}}\left|a_{ii_2\cdots i_m}\right| \neq 0,\quad s = 0, 1, \cdots, r,$$

where $k_0 = i, k_{r+1} = j$, we call that there is a nonzero elements chain from i to j.

Let S be a nonempty subset of N and let $N \setminus S$ be the complement of S in N. Given $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in C^{[m,n]}$, for $i, k \in N$, denote as

$$
\begin{aligned}
R_i(\mathcal{A}) &= \sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| = \sum_{i_2,\ldots,i_m \in N} |a_{ii_2\cdots i_m}| - |a_{ii\cdots i}|, \\
N_1(\mathcal{A}) &= \{i \in N : |a_{ii\cdots i}| \le \frac{1}{2} R_i(\mathcal{A})\}, \quad N_2(\mathcal{A}) = \{i \in N : \frac{1}{2} R_i(\mathcal{A}) < |a_{ii\cdots i}| < R_i(\mathcal{A})\}, \\
N_3(\mathcal{A}) &= \{i \in N : |a_{ii\cdots i}| = R_i(\mathcal{A})\}, \quad N_4(\mathcal{A}) = \{i \in N : |a_{ii\cdots i}| > R_i(\mathcal{A})\}, \\
S^{m-1} &= \left\{ i_2 i_3 \cdots i_m : i_j \in S, j = 2, 3, \cdots, m \right\}, \\
N^{m-1} \setminus S^{m-1} &= \left\{ i_2 i_3 \cdots i_m : i_2 i_3 \cdots i_m \in N^{m-1} \text{ and } i_2 i_3 \cdots i_m \notin S^{m-1} \right\}, \\
N_0^{m-1} &= N^{m-1} \setminus \left(N_1^{m-1} \cup N_2^{m-1} \cup N_4^{m-1} \right), \\
y_i &= \begin{cases} \frac{|a_{ii\cdots i}|}{R_i(\mathcal{A})}, & i \in N_1, \\ \frac{R_i(\mathcal{A}) - |a_{ii\cdots i}|}{R_i(\mathcal{A})}, & i \in N_2. \end{cases}, \quad s_i = \frac{|a_{ii\cdots i}|}{R_i(\mathcal{A}) - |a_{ii\cdots i}|}, \quad t_i = \frac{R_i(\mathcal{A}) + |a_{ii\cdots i}|}{2 R_i(\mathcal{A})}, \\
r &= \max_{i \in N_4} \left\{ \frac{\sum\limits_{i_2 i_3 \cdots i_m \in N_1^{m-1}} |a_{ii_2\cdots i_m}| + \sum\limits_{i_2 i_3 \cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}| + \sum\limits_{i_2 i_3 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}|}{|a_{ii\cdots i}| - \sum\limits_{\substack{i_2 i_3 \cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|} \right\}, \\
R_i^{(1)}(\mathcal{A}) &= \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|, \\
R_i^{(2)}(\mathcal{A}) &= \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + r \sum_{\substack{i_2 i_3 \cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|, \\
R_i^{(3)}(\mathcal{A}) &= \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \{y_j\} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \{y_j\} |a_{ii_2\cdots i_m}| \\
&\quad + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ \frac{R_j^{(2)}(\mathcal{A})}{|a_{jj\cdots j}|} \right\} |a_{ii_2\cdots i_m}|, \\
&\;\;\vdots \\
R_i^{(k+1)}(\mathcal{A}) &= \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \{y_j\} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \{y_j\} |a_{ii_2\cdots i_m}| \\
&\quad + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ \frac{R_j^{(k)}(\mathcal{A})}{|a_{jj\cdots j}|} \right\} |a_{ii_2\cdots i_m}|.
\end{aligned}
$$

Lemma 1. [8] *If $\mathcal{A} \in C^{[m,n]}$ is a strictly diagonally dominant tensor, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.*

Lemma 2. [8] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in C^{[m,n]}$. If $\mathcal{A}$ is an $\mathcal{H}$-tensor, then $N_4(\mathcal{A}) \neq \emptyset$.*

By Lemma 1, if $N_1(\mathcal{A}) \cup N_2(\mathcal{A}) \cup N_3(\mathcal{A}) = \emptyset$, then $\mathcal{A}$ is an $\mathcal{H}$-tensor. By Lemma 2, if $\mathcal{A}$ is an $\mathcal{H}$-tensor, then $N_4(\mathcal{A}) \neq \emptyset$. Hence, we always assume that $N_1(\mathcal{A}) \cup N_2(\mathcal{A}) \cup N_3(\mathcal{A}) \neq \emptyset$, $N_4(\mathcal{A}) \neq \emptyset$. In addition, we also assume that $\mathcal{A}$ satisfies: $a_{ii\cdots i} \neq 0$, $R_i(\mathcal{A}) \neq 0$, $\forall i \in N$.

Lemma 3. [8] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in C^{[m,n]}$. If $\mathcal{A}$ is irreducible,*

$$|a_{i\cdots i}| \geq R_i(\mathcal{A}), \qquad \forall i \in N,$$

and strictly inequality holds for at least one i, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Lemma 4. [4] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in C^{[m,n]}$. If there exists a positive diagonal matrix X such that $\mathcal{A}X^{m-1}$ is an $\mathcal{H}$-tensor, then $\mathcal{A}$ is an $\mathcal{H}$-tensor.*

Lemma 5. [13] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in C^{[m,n]}$. If $\mathcal{A}$ satisfies the following conditions:*
(i) $|a_{ii\cdots i}| \geq R_i(\mathcal{A}), \forall i \in N$.
(ii) $N_4(\mathcal{A}) = \{i \in N : |a_{ii\cdots i}| > R_i(\mathcal{A})\} \neq \emptyset$.
(iii) For any $i \notin N_4(\mathcal{A})$, there exists a nonzero elements chain form i to j such that $j \in N_4(\mathcal{A})$.
Then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

3. New criteria for identifying $\mathcal{H}$-tensors

In this section, we provide some new criterions for testing $\mathcal{H}$-tensors.

Theorem 5. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in C^{[m,n]}$. If there exists $k \geq 1$ such that $\mathcal{A}$ satisfies the following conditions:*
(i) For $\forall i \in N_1(\mathcal{A})$,

$$\begin{aligned}|a_{ii\cdots i}|y_i \quad > \quad & \sum_{\substack{i_2 i_3 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m = 0}}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \{y_j\} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \{y_j\} |a_{ii_2\cdots i_m}| \\ & + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \frac{R_j^{(k)}(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2\cdots i_m}|. \end{aligned} \tag{4}$$

(ii) For $\forall i \in N_2(\mathcal{A})$,

$$\begin{aligned}|a_{ii\cdots i}|y_i \quad > \quad & \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \{y_j\} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m = 0}}} \max_{j\in\{i_2,i_3,\cdots,i_m\}} \{y_j\} |a_{ii_2\cdots i_m}| \\ & + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \frac{R_j^{(k)}(\mathcal{A})}{|a_{jj\cdots j}|} |a_{ii_2\cdots i_m}|. \end{aligned} \tag{5}$$

(iii) For $\forall i \in N_3(\mathcal{A})$,

$$\sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} |a_{ii_2\cdots i_m}| \neq 0. \tag{6}$$

Then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof. From the definition of r, $0 \leq r < 1$. According to the definition of $R_i^{(k)}(\mathcal{A})$, for $\forall i \in N_4(\mathcal{A})$, we have

$$R_i^{(2)}(\mathcal{A}) \leq R_i^{(1)}(\mathcal{A}) = R_i(\mathcal{A}),$$

and

$$R_i^{(3)}(\mathcal{A}) \quad = \quad \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \{y_j\} |a_{ii_2\cdots i_m}| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \{y_j\} |a_{ii_2\cdots i_m}|$$

$$
\begin{aligned}
&+\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2i_3\cdots i_m\in N_4^{m-1}\\ \delta_{ii_2\cdots i_m=0}}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\frac{R_j^{(2)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right| \\
\leq\ &\sum_{i_2i_3\cdots i_m\in N_1^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_2^{m-1}}\left|a_{ii_2\cdots i_m}\right| \\
&+\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2i_3\cdots i_m\in N_4^{m-1}\\ \delta_{ii_2\cdots i_m=0}}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\frac{R_j^{(2)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right| \\
\leq\ &\sum_{i_2i_3\cdots i_m\in N_1^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_2^{m-1}}\left|a_{ii_2\cdots i_m}\right| \\
&+\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2i_3\cdots i_m\in N_4^{m-1}\\ \delta_{ii_2\cdots i_m=0}}} r\left|a_{ii_2\cdots i_m}\right| \\
=\ & R_i^{(2)}(\mathcal{A})\leq R_i^{(1)}(\mathcal{A})=R_i(\mathcal{A}).
\end{aligned}
$$

Similar to the above proof, we can prove that $R_i^{(k+1)}(\mathcal{A})\leq R_i^{(k)}(\mathcal{A})$. From the definitions of y_i and $R_i^{(k)}(\mathcal{A})$, it holds that

$$
0<y_i<1,\ \ \forall i\in N_1(\mathcal{A})\cup N_2(\mathcal{A});\qquad 0\leq\frac{R_i^{(k)}(\mathcal{A})}{|a_{ii\cdots i}|}<1,\ \ \forall i\in N_4(\mathcal{A}).
$$

Hence, there exists a sufficient small number $\varepsilon>0$ such that the following conclusions all hold.

(a) For any $i\in N_1(\mathcal{A})$, we get

$$
\begin{aligned}
|a_{ii\cdots i}|y_i\ >\ &\sum_{\substack{i_2i_3\cdots i_m\in N_1^{m-1}\\ \delta_{ii_2\cdots i_m=0}}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_2^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right| \\
&+\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_4^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\frac{R_j^{(k)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right|+\varepsilon\sum_{i_2i_3\cdots i_m\in N_4^{m-1}}\left|a_{ii_2\cdots i_m}\right|. \qquad (7)
\end{aligned}
$$

(b) For any $i\in N_2(\mathcal{A})$, we have

$$
\begin{aligned}
|a_{ii\cdots i}|y_i\ >\ &\sum_{i_2i_3\cdots i_m\in N_1^{m-1}}\max_{j\in\{i_2,i_3,\cdots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2i_3\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m=0}}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right| \\
&+\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_4^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\frac{R_j^{(k)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right|+\varepsilon\sum_{i_2i_3\cdots i_m\in N_4^{m-1}}\left|a_{ii_2\cdots i_m}\right|. \qquad (8)
\end{aligned}
$$

(c) For any $i\in N_4(\mathcal{A})$, it holds that

$$
0<\frac{R_i^{(k)}(\mathcal{A})}{|a_{ii\cdots i}|}+\varepsilon<1. \qquad (9)
$$

Let the matrix $X=diag(x_1,x_2,\ldots,x_n)$, where

$$
x_i=\begin{cases}(y_i)^{\frac{1}{m-1}}, & i\in N_1\cup N_2,\\ 1, & i\in N_3,\\ \left(\frac{R_i^{(k)}(\mathcal{A})}{|a_{ii\cdots i}|}+\varepsilon\right)^{\frac{1}{m-1}}, & i\in N_4.\end{cases}
$$

As $\varepsilon \neq +\infty$, $x_i \neq +\infty$, which implies that X is a diagonal matrix with positive entries. Let $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1}$. Next, we will prove that $\mathcal{B}$ is strictly diagonally dominant.

(1) For any $i \in N_1(\mathcal{A})$, by Inequality (7), we have

$$\begin{aligned}
R_i(\mathcal{B}) &= \sum_{\substack{i_2 i_3 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| (y_{i_2})^{\frac{1}{m-1}} \cdots (y_{i_m})^{\frac{1}{m-1}} + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| (y_{i_2})^{\frac{1}{m-1}} \cdots (y_{i_m})^{\frac{1}{m-1}} \\
&\quad + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| x_{i_2} \cdots x_{i_m} + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{R_{i_2}^{(k)}(\mathcal{A})}{\left|a_{i_2 i_2 \cdots i_2}\right|} + \varepsilon \right)^{\frac{1}{m-1}} \cdots \left(\frac{R_{i_m}^{(k)}(\mathcal{A})}{\left|a_{i_m i_m \cdots i_m}\right|} + \varepsilon \right)^{\frac{1}{m-1}} \\
&\leq \sum_{\substack{i_2 i_3 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \left\{ y_j \right\} \left|a_{ii_2\cdots i_m}\right| \\
&\quad + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ \frac{R_j^{(k)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|} + \varepsilon \right\} \left|a_{ii_2\cdots i_m}\right| \\
&< \left|a_{ii\cdots i}\right| y_i = \left|b_{ii\cdots i}\right|.
\end{aligned} \tag{10}$$

(2) For any $i \in N_2(\mathcal{A})$, by Inequality (8), it holds that

$$\begin{aligned}
R_i(\mathcal{B}) &= \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \left|a_{ii_2\cdots i_m}\right| (y_{i_2})^{\frac{1}{m-1}} \cdots (y_{i_m})^{\frac{1}{m-1}} + \sum_{\substack{i_2 i_3 \cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| (y_{i_2})^{\frac{1}{m-1}} \cdots (y_{i_m})^{\frac{1}{m-1}} \\
&\quad + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| x_{i_2} \cdots x_{i_m} + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{R_{i_2}^{(k)}(\mathcal{A})}{\left|a_{i_2 i_2 \cdots i_2}\right|} + \varepsilon \right)^{\frac{1}{m-1}} \cdots \left(\frac{R_{i_m}^{(k)}(\mathcal{A})}{\left|a_{i_m i_m \cdots i_m}\right|} + \varepsilon \right)^{\frac{1}{m-1}} \\
&\leq \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left|a_{ii_2\cdots i_m}\right| \\
&\quad + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ \frac{R_j^{(k)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|} + \varepsilon \right\} \left|a_{ii_2\cdots i_m}\right| \\
&< \left|a_{ii\cdots i}\right| y_i = \left|b_{ii\cdots i}\right|.
\end{aligned} \tag{11}$$

(3) For any $i \in N_3(\mathcal{A})$, by Inequality (9), we get

$$\begin{aligned}
R_i(\mathcal{B}) &= \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \left|a_{ii_2\cdots i_m}\right| (y_{i_2})^{\frac{1}{m-1}} \cdots (y_{i_m})^{\frac{1}{m-1}} + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| (y_{i_2})^{\frac{1}{m-1}} \cdots (y_{i_m})^{\frac{1}{m-1}} \\
&\quad + \sum_{\substack{i_2 i_3 \cdots i_m \in N_0^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| x_{i_2} \cdots x_{i_m} + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{R_{i_2}^{(k)}(\mathcal{A})}{\left|a_{i_2 i_2 \cdots i_2}\right|} + \varepsilon \right)^{\frac{1}{m-1}} \cdots \left(\frac{R_{i_m}^{(k)}(\mathcal{A})}{\left|a_{i_m i_m \cdots i_m}\right|} + \varepsilon \right)^{\frac{1}{m-1}} \\
&\leq \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left|a_{ii_2\cdots i_m}\right| \\
&\quad + \sum_{\substack{i_2 i_3 \cdots i_m \in N_0^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ \frac{R_j^{(k)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|} + \varepsilon \right\} \left|a_{ii_2\cdots i_m}\right|
\end{aligned}$$

$$< \quad R_i(\mathcal{A}) = |a_{ii\cdots i}| = |b_{ii\cdots i}|. \tag{12}$$

(4) For any $i \in N_4(\mathcal{A})$, when $k = 1$, by Inequality (9), we have

$$\begin{aligned}
R_i(\mathcal{B}) &= \sum_{i_2i_3\cdots i_m \in N_1^{m-1}} \left|a_{ii_2\cdots i_m}\right| (y_{i_2})^{\frac{1}{m-1}} \cdots (y_{i_m})^{\frac{1}{m-1}} + \sum_{i_2i_3\cdots i_m \in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| (y_{i_2})^{\frac{1}{m-1}} \cdots (y_{i_m})^{\frac{1}{m-1}} \\
&\quad + \sum_{i_2i_3\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| x_{i_2} \cdots x_{i_m} + \sum_{\substack{i_2i_3\cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m=0}}} \left|a_{ii_2\cdots i_m}\right| \left(\frac{R_{i_2}^{(1)}(\mathcal{A})}{\left|a_{i_2i_2\cdots i_2}\right|} + \varepsilon \right)^{\frac{1}{m-1}} \cdots \left(\frac{R_{i_m}^{(1)}(\mathcal{A})}{\left|a_{i_mi_m\cdots i_m}\right|} + \varepsilon \right)^{\frac{1}{m-1}} \\
&\le \sum_{i_2i_3\cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2i_3\cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2,i_3,\cdots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| \\
&\quad + \sum_{i_2i_3\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2i_3\cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m=0}}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \left\{ \frac{R_j^{(1)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|} + \varepsilon \right\} \left|a_{ii_2\cdots i_m}\right| \\
&= \sum_{i_2i_3\cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2i_3\cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2,i_3,\cdots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| \\
&\quad + \sum_{i_2i_3\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2i_3\cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m=0}}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \frac{R_j^{(1)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|} \left|a_{ii_2\cdots i_m}\right| + \varepsilon \sum_{\substack{i_2i_3\cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m=0}}} \left|a_{ii_2\cdots i_m}\right| \\
&< \sum_{i_2i_3\cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2,i_3,\cdots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2i_3\cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| \\
&\quad + \sum_{i_2i_3\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2i_3\cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m=0}}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \frac{R_j^{(1)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|} \left|a_{ii_2\cdots i_m}\right| + \varepsilon \left|a_{ii\cdots i}\right| \\
&\le R_i(\mathcal{A}) + \varepsilon \left|a_{ii\cdots i}\right| = \left|a_{ii\cdots i}\right| \left(\frac{R_i^{(1)}(\mathcal{A})}{\left|a_{ii\cdots i}\right|} + \varepsilon \right) = \left|b_{ii\cdots i}\right|.
\end{aligned} \tag{13}$$

(5) For any $i \in N_4(\mathcal{A})$, when $k = 2$, by Inequality (9), we get

$$\begin{aligned}
R_i(\mathcal{B}) &\le \sum_{i_2i_3\cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2i_3\cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2,i_3,\cdots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| \\
&\quad + \sum_{i_2i_3\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2i_3\cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m=0}}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \left\{ \frac{R_j^{(2)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|} + \varepsilon \right\} \left|a_{ii_2\cdots i_m}\right| \\
&< \sum_{i_2i_3\cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2i_3\cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| \\
&\quad + \sum_{i_2i_3\cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2i_3\cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m=0}}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \frac{R_j^{(2)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|} \left|a_{ii_2\cdots i_m}\right| + \varepsilon \left|a_{ii\cdots i}\right| \\
&\le \sum_{i_2i_3\cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2i_3\cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right|
\end{aligned}$$

$$+\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+r\sum_{\substack{i_2i_3\cdots i_m\in N_4^{m-1}\\ \delta_{ii_2\cdots i_m=0}}}\left|a_{ii_2\cdots i_m}\right|+\varepsilon\left|a_{ii\cdots i}\right|$$

$$= R_i^{(2)}(\mathcal{A})+\varepsilon\left|a_{ii\cdots i}\right|=\left|a_{ii\cdots i}\right|\left(\frac{R_i^{(2)}(\mathcal{A})}{\left|a_{ii\cdots i}\right|}+\varepsilon\right)=\left|b_{ii\cdots i}\right|. \quad (14)$$

(6) For any $i\in N_4(\mathcal{A})$, when $k\geq 3$, by Inequality (9), we get

$$\begin{aligned}
R_i(\mathcal{B}) &\leq \sum_{i_2i_3\cdots i_m\in N_1^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_2^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right| \\
&\quad+\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2i_3\cdots i_m\in N_4^{m-1}\\ \delta_{ii_2\cdots i_m=0}}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\left\{\frac{R_j^{(k)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|}+\varepsilon\right\}\left|a_{ii_2\cdots i_m}\right| \\
&< \sum_{i_2i_3\cdots i_m\in N_1^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_2^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right| \\
&\quad+\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2i_3\cdots i_m\in N_4^{m-1}\\ \delta_{ii_2\cdots i_m=0}}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\frac{R_j^{(k)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\left|a_{ii_2\cdots i_m}\right|+\varepsilon\left|a_{ii\cdots i}\right| \\
&= R_i^{(k+1)}(\mathcal{A})+\varepsilon\left|a_{ii\cdots i}\right|\leq R_i^{(k)}(\mathcal{A})+\varepsilon\left|a_{ii\cdots i}\right| \\
&= \left|a_{ii\cdots i}\right|\left(\frac{R_i^{(k)}(\mathcal{A})}{\left|a_{ii\cdots i}\right|}+\varepsilon\right)=\left|b_{ii\cdots i}\right|. \quad (15)
\end{aligned}$$

Therefore, from Inequalities (10)-(15), we obtain that $|b_{ii\cdots i}|>R_i(\mathcal{B})$ for all $i\in N$, that is, $\mathcal{B}$ is a strictly diagonally dominant tensor. By Lemma 1, $\mathcal{B}$ is an $\mathcal{H}$-tensor. Further, by Lemma 4, $\mathcal{A}$ is also an $\mathcal{H}$-tensor. □

Remark 1. Let tensor $\mathcal{A}$ satisfy all the conditions of Theorem 5. For $\forall i\in N_4(\mathcal{A})$, if $\mathcal{A}$ also satisfies:

$$\sum_{i_2i_3\cdots i_m\in N_1^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_2^{m-1}}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|=0,$$

then

$$R_i^{(2)}(\mathcal{A})=R_i^{(3)}(\mathcal{A})=\cdots=R_i^{(k)}(\mathcal{A})=0,\ \ \forall i\in N_4(\mathcal{A}).$$

In this case, the condition (i) of Theorem 5 is equivalent to

$$|a_{ii\cdots i}|y_i > \sum_{\substack{i_2i_3\cdots i_m\in N_1^{m-1}\\ \delta_{ii_2\cdots i_m=0}}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_2^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|.$$

Meanwhile, the condition (ii) of Theorem 5 is equivalent to

$$|a_{ii\cdots i}|y_i > \sum_{i_2i_3\cdots i_m\in N_1^{m-1}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right|+\sum_{\substack{i_2i_3\cdots i_m\in N_2^{m-1}\\ \delta_{ii_2\cdots i_m=0}}}\max_{j\in\{i_2,i_3,\ldots,i_m\}}\left\{y_j\right\}\left|a_{ii_2\cdots i_m}\right|+\sum_{i_2i_3\cdots i_m\in N_0^{m-1}}\left|a_{ii_2\cdots i_m}\right|.$$

So the value of $|a_{ii_2\cdots i_m}|\left(\forall i\in N_1(\mathcal{A})\cup N_2(\mathcal{A}), i_2i_3\cdots i_m\in N_4^{m-1}\right)$ doesn't affect $\mathcal{A}\in\mathcal{H}$ as long as we don't change

$$\sum_{i_2i_3\cdots i_m\in N_4^{m-1}}|a_{ii_2\cdots i_m}|\left(\forall i\in N_1(\mathcal{A})\cup N_2(\mathcal{A})\right).$$

There are other sufficient conditions that have the property, and the following result further illustrates this phenomenon.

Theorem 6. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in C^{[m,n]}$. *If* $\mathcal{A}$ *satisfies the following conditions:*
(i) *For* $\forall i \in N_1(\mathcal{A})$,

$$|a_{ii\cdots i}|s_i > \sum_{\substack{i_2 i_3 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{s_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{t_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right|.$$

(ii) *For* $\forall i \in N_2(\mathcal{A})$,

$$|a_{ii\cdots i}|t_i > \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{s_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{t_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right|.$$

(iii) *For* $\forall i \in N_3(\mathcal{A})$,

$$|a_{ii\cdots i}| > \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{s_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{t_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_0^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right|.$$

(iv) *For* $\forall i \in N_4(\mathcal{A})$,

$$\sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_3^{m-1}} \left|a_{ii_2\cdots i_m}\right| = 0.$$

Then $\mathcal{A}$ *is an* $\mathcal{H}$*-tensor.*

Proof. According to the above conditions, there exists a sufficient small number $\varepsilon > 0$ such taht the following conclusions hold.
(i) For any $i \in N_1(\mathcal{A})$, it holds that

$$|a_{ii\cdots i}|s_i > \sum_{\substack{i_2 i_3 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{s_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{t_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \varepsilon \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \left|a_{ii_2\cdots i_m}\right|. \tag{16}$$

(ii) For any $i \in N_2(\mathcal{A})$, we have

$$|a_{ii\cdots i}|t_i > \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{s_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{t_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \varepsilon \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \left|a_{ii_2\cdots i_m}\right|. \tag{17}$$

(iii) For any $i \in N_3(\mathcal{A})$, we get

$$|a_{ii\cdots i}| > \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{s_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{t_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_0^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \left|a_{ii_2\cdots i_m}\right| + \varepsilon \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \left|a_{ii_2\cdots i_m}\right|. \tag{18}$$

Let the matrix $X = diag(x_1, x_2, \dots, x_n)$, where

$$x_i = \begin{cases} (s_i)^{\frac{1}{m-1}}, & \forall i \in N_1, \\ (t_i)^{\frac{1}{m-1}}, & \forall i \in N_2, \\ 1, & \forall i \in N_3, \\ (\varepsilon)^{\frac{1}{m-1}}, & \forall i \in N_4. \end{cases}$$

As $\varepsilon \neq +\infty$, $x_i \neq +\infty$, which implies that X is a diagonal matrix with positive entries. Let $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1}$. Next, we will prove that $\mathcal{B}$ is strictly diagonally dominant.

(1) For any $i \in N_1(\mathcal{A})$, by Inequality (16), then

$$\begin{aligned} R_i(\mathcal{B}) &= \sum_{\substack{i_2 i_3 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m = 0}}} \left|a_{ii_2\cdots i_m}\right| (s_{i_2})^{\frac{1}{m-1}} \cdots (s_{i_m})^{\frac{1}{m-1}} + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \left|a_{ii_2\cdots i_m}\right| (t_{i_2})^{\frac{1}{m-1}} \cdots (t_{i_m})^{\frac{1}{m-1}} \\ &\quad + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| x_{i_2} \cdots x_{i_m} + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \left|a_{ii_2\cdots i_m}\right| (\varepsilon)^{\frac{1}{m-1}} \cdots (\varepsilon)^{\frac{1}{m-1}} \\ &\leq \sum_{\substack{i_2 i_3 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m = 0}}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \left\{s_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{t_j\right\} \left|a_{ii_2\cdots i_m}\right| \\ &\quad + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \varepsilon \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \left|a_{ii_2\cdots i_m}\right| \\ &< \left|a_{ii\cdots i}\right| s_i = \left|b_{ii\cdots i}\right|. \end{aligned} \tag{19}$$

(2) For any $i \in N_2(\mathcal{A})$, by Inequality (17), we have

$$\begin{aligned} R_i(\mathcal{B}) &\leq \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{s_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m = 0}}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{t_j\right\} \left|a_{ii_2\cdots i_m}\right| \\ &\quad + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \varepsilon \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \left|a_{ii_2\cdots i_m}\right| \\ &< \left|a_{ii\cdots i}\right| t_i = \left|b_{ii\cdots i}\right|. \end{aligned} \tag{20}$$

(3) For any $i \in N_3(\mathcal{A})$, by Inequality (18), we get

$$\begin{aligned} R_i(\mathcal{B}) &\leq \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \left\{s_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{t_j\right\} \left|a_{ii_2\cdots i_m}\right| \\ &\quad + \sum_{\substack{i_2 i_3 \cdots i_m \in N_0^{m-1} \\ \delta_{ii_2\cdots i_m = 0}}} \left|a_{ii_2\cdots i_m}\right| + \varepsilon \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \left|a_{ii_2\cdots i_m}\right| \\ &< \left|a_{ii\cdots i}\right| = \left|b_{ii\cdots i}\right|. \end{aligned} \tag{21}$$

(4) For any $i \in N_4(\mathcal{A})$, it holds that

$$\begin{aligned} R_i(\mathcal{B}) &\leq \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| \\ &\quad + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \varepsilon \sum_{\substack{i_2 i_3 \cdots i_m \in N_4^{m-1} \\ \delta_{ii_2\cdots i_m = 0}}} \left|a_{ii_2\cdots i_m}\right| \end{aligned}$$

$$= \varepsilon \sum_{\substack{i_2 i_3 \cdots i_m \in N_4^{m-1} \\ \delta_{ii_2 \cdots i_m = 0}}} \left| a_{ii_2 \cdots i_m} \right| < \varepsilon \left| a_{ii \cdots i} \right| = \left| b_{ii \cdots i} \right|. \tag{22}$$

Therefore, from Inequalities (19 – 22), we conclude that $|b_{ii\cdots i}| > R_i(\mathcal{B})$ for all $i \in N$, that is, $\mathcal{B}$ is strictly diagonally dominant tensor. From Lemma 1, $\mathcal{B}$ is an $\mathcal{H}$-tensor. Further, by Lemma 4, $\mathcal{A}$ is an $\mathcal{H}$-tensor. □

Let

$$J_1(\mathcal{A}) = \left\{ i \in N_1 : |a_{ii\cdots i}|\, y_i > \sum_{\substack{i_2 i_3 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2 \cdots i_m = 0}}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left| a_{ii_2 \cdots i_m} \right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left| a_{ii_2 \cdots i_m} \right| \right.$$

$$\left. + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left| a_{ii_2 \cdots i_m} \right| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \frac{R_j^{(k)}(A)}{\left| a_{jj \cdots j} \right|} \left| a_{ii_2 \cdots i_m} \right| \right\},$$

$$J_2(\mathcal{A}) = \left\{ i \in N_2 : |a_{ii\cdots i}|\, y_i > \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left| a_{ii_2 \cdots i_m} \right| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_2^{m-1} \\ \delta_{ii_2 \cdots i_m = 0}}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left| a_{ii_2 \cdots i_m} \right| \right.$$

$$\left. + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left| a_{ii_2 \cdots i_m} \right| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \frac{R_j^{(k)}(A)}{\left| a_{jj \cdots j} \right|} \left| a_{ii_2 \cdots i_m} \right| \right\},$$

$$J_3(\mathcal{A}) = \left\{ i \in N_3 : \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \left| a_{ii_2 \cdots i_m} \right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \left| a_{ii_2 \cdots i_m} \right| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \left| a_{ii_2 \cdots i_m} \right| \neq 0 \right\},$$

and $\Omega(\mathcal{A}) \equiv J_1(\mathcal{A}) \cup J_2(\mathcal{A}) \cup J_3(\mathcal{A})$.

Theorem 7. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. If there exists $k \geq 1$ such that $\mathcal{A}$ satisfies the following conditions:*
(i) For $\forall i \in N_1(\mathcal{A})$,

$$|a_{ii\cdots i}| y_i \geq \sum_{\substack{i_2 i_3 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2 \cdots i_m = 0}}} \max_{j \in \{i_2, i_3, \cdots, i_m\}} \left\{ y_j \right\} \left| a_{ii_2 \cdots i_m} \right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left| a_{ii_2 \cdots i_m} \right|$$

$$+ \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left| a_{ii_2 \cdots i_m} \right| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \frac{R_j^{(k)}(\mathcal{A})}{\left| a_{jj \cdots j} \right|} \left| a_{ii_2 \cdots i_m} \right|.$$

(ii) For $\forall i \in N_2(\mathcal{A})$,

$$|a_{ii\cdots i}| y_i \geq \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left| a_{ii_2 \cdots i_m} \right| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_2^{m-1} \\ \delta_{ii_2 \cdots i_m = 0}}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \left\{ y_j \right\} \left| a_{ii_2 \cdots i_m} \right|$$

$$+ \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left| a_{ii_2 \cdots i_m} \right| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \max_{j \in \{i_2, i_3, \ldots, i_m\}} \frac{R_j^{(k)}(\mathcal{A})}{\left| a_{jj \cdots j} \right|} \left| a_{ii_2 \cdots i_m} \right|.$$

And for $\forall i \in N \backslash \Omega(\mathcal{A}) \neq \emptyset$, there exists a nonzero elements chain from i to j such that $j \in \Omega(\mathcal{A}) \neq \emptyset$. Then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Proof. Let the matrix $X = diag(x_1, x_2, \ldots, x_n)$, where

$$x_i = \begin{cases} (y_i)^{\frac{1}{m-1}}, & i \in N_1(\mathcal{A}) \cup N_2(\mathcal{A}), \\ 1, & i \in N_3(\mathcal{A}), \\ \left(\frac{R_i^{(k)}(\mathcal{A})}{|a_{ii\cdots i}|}\right)^{\frac{1}{m-1}}, & i \in N_4(\mathcal{A}). \end{cases}$$

which implies that X is a diagonal matrix with positive entries. Let $\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1}$. Similar to the proof of Theorem 1, we can prove that $|b_{ii\cdots i}| \geq R_i(\mathcal{B})(\forall i \in N)$, and there is at least one $i \in \Omega(\mathcal{A})$ such that $|b_{ii\cdots i}| > R_i(\mathcal{B})$.

In addition, if $|b_{ii\cdots i}| = R_i(\mathcal{B})$, then $i \in N\backslash\Omega(\mathcal{A})$. Since $\mathcal{B}$ does not change the nonzero elements chain of $\mathcal{A}$, then $\mathcal{B}$ has a nonzero elements chain from i to j with $\left|b_{jj\cdots j}\right| > R_j(\mathcal{B})$. In conclusion, we get that $\mathcal{B}$ satisfies the conditions of Lemma 5, so $\mathcal{B}$ is an $\mathcal{H}$-tensor. By Lemma 4, $\mathcal{A}$ is also an $\mathcal{H}$-tensor. □

Corollary 1. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in C^{[m,n]}$ be irreducible. If there exists $k \geq 1$ such that $\mathcal{A}$ satisfies the following conditions:

(i) For $\forall i \in N_1(\mathcal{A})$,

$$\begin{aligned} |a_{ii\cdots i}|y_i \geq & \sum_{\substack{i_2 i_3 \cdots i_m \in N_1^{m-1} \\ \delta_{ii_2\cdots i_m=0}}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_2^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| \\ & + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{\frac{R_j^{(k)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\right\} \left|a_{ii_2\cdots i_m}\right|. \end{aligned}$$

(ii) For $\forall i \in N_2(\mathcal{A})$,

$$\begin{aligned} |a_{ii\cdots i}|y_i \geq & \sum_{i_2 i_3 \cdots i_m \in N_1^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| + \sum_{\substack{i_2 i_3 \cdots i_m \in N_2^{m-1} \\ \delta_{ii_2\cdots i_m=0}}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{y_j\right\} \left|a_{ii_2\cdots i_m}\right| \\ & + \sum_{i_2 i_3 \cdots i_m \in N_0^{m-1}} \left|a_{ii_2\cdots i_m}\right| + \sum_{i_2 i_3 \cdots i_m \in N_4^{m-1}} \max_{j\in\{i_2,i_3,\ldots,i_m\}} \left\{\frac{R_j^{(k)}(\mathcal{A})}{\left|a_{jj\cdots j}\right|}\right\} \left|a_{ii_2\cdots i_m}\right|. \end{aligned}$$

And $\Omega(\mathcal{A}) \neq \emptyset$. Then $\mathcal{A}$ is an $\mathcal{H}$-tensor.

Example 1. Consider a tensor $\mathcal{A} = (a_{i_1 i_2 i_3}) \in C^{[3,3]}$ defined as follows:

$$\mathcal{A} = [A(1,:,:), A(2,:,:), A(3,:,:)],$$

$$A(1,:,:) = \begin{pmatrix} 12 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 16 \end{pmatrix}, \quad A(2,:,:) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A(3,:,:) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

Then

$$|a_{111}| = 12, \quad R_1(\mathcal{A}) = 24, \quad |a_{222}| = 6, \quad R_2(\mathcal{A}) = 3, \quad |a_{333}| = 10, \; R_3(\mathcal{A}) = 2,$$

and $N_1(\mathcal{A}) = \{1\}, N_2(\mathcal{A}) = \emptyset, N_3(\mathcal{A}) = \emptyset, N_4(\mathcal{A}) = \{2,3\}$. By calculations, we have

$$\begin{aligned} r &= \frac{1}{4}, \quad y_1 = \frac{1}{2}, \quad R_2^{(1)}(\mathcal{A}) = 3, \quad R_3^{(1)}(\mathcal{A}) = 2, \quad R_2^{(2)}(\mathcal{A}) = \frac{3}{2}, \\ R_3^{(2)}(\mathcal{A}) &= \frac{5}{4}, \quad R_2^{(3)}(\mathcal{A}) = 1, \quad R_3^{(3)}(\mathcal{A}) = \frac{3}{4}, \quad \frac{R_2^{(3)}}{|a_{222}|} = \frac{1}{6}, \quad \frac{R_3^{(3)}}{|a_{333}|} = \frac{3}{40}. \end{aligned}$$

Since

$$\begin{aligned}
&\sum_{\substack{i_2i_3\in N_1^2\\ \delta_{1i_2i_3}=0}}\max_{j\in\{i_2,i_3\}}\{y_j\}\left|a_{1i_2i_3}\right|+\sum_{i_2i_3\in N_2^2}\max_{j\in\{i_2,i_3\}}\{y_j\}\left|a_{1i_2i_3}\right|+\sum_{i_2i_3\in N_0^2}\left|a_{1i_2i_3}\right|+\sum_{i_2i_3\in N_4^2}\max_{j\in\{i_2,i_3\}}\left\{\frac{R_j^{(3)}(\mathcal{A})}{\left|a_{jjj}\right|}\right\}\left|a_{1i_2i_3}\right|\\
=\ & 0+0+2+22\times\frac{1}{6}=\frac{17}{3}<12\times\frac{1}{2}=6=|a_{111}|\,y_1.
\end{aligned}$$

then $\mathcal{A}$ satisfies the conditions of Theorem 5, so $\mathcal{A}$ is an $\mathcal{H}$-tensor. But

$$|a_{111}|=12<24=R_1(\mathcal{A}),$$

and

$$\sum_{\substack{i_2i_3\in N^2/N_4^2\\ \delta_{1i_2i_3}=0}}\left|a_{1i_2i_3}\right|+\sum_{i_2i_3\in N_4^2}\max_{j\in\{i_2,i_3\}}\frac{R_j}{\left|a_{jjj}\right|}\left|a_{1i_2i_3}\right|=2+22\times\frac{1}{2}=13>12=|a_{111}|,$$

and

$$\begin{aligned}
&\frac{R_1(\mathcal{A})}{R_1(\mathcal{A})-|a_{111}|}\left[q\left(\sum_{i_2i_3\in N_0^2}|a_{1i_2i_3}|+\sum_{\substack{i_2i_3\in N_2^2\\ \delta_{1i_2i_3}=0}}|a_{1i_2i_3}|\right)+\sum_{i_2i_3\in N_3^2}\max_{j\in\{i_2,i_3\}}\frac{tP_j(\mathcal{A})}{|a_{jjj}|}|a_{1i_2i_3}|\right]\\
=\ & 2\times\left[\frac{1}{2}\times 2+\frac{1}{24}\times 22\right]=13>12==|a_{111}|.
\end{aligned}$$

Therefore, $\mathcal{A}$ does not satisfy the conditions of Theorem 3 in [16], the conditions of Theorem 3 (Theorem 1 in [9]), and the conditions of Theorem 4 (Theorem 2 in [1]), respectively.

4. Applications

In this section, based on the criteria of $\mathcal{H}$-tensors in Section 3, we present new criteria for identifying the positive definiteness of an even-order real symmetric tensor.

From Theorem 2, we obtain easily the following result.

Theorem 8. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in C^{[m,n]}$ be an even-order real symmetric tensor with $a_{kk\cdots k}>0$ for all $k\in N$. If $\mathcal{A}$ satisfies one of the following conditions, then $\mathcal{A}$ is positive definite,*

(*i*) *all the conditions of Theorem* 5;

(*ii*) *all the conditions of Theorem* 6;

(*iii*) *all the conditions of Theorem* 7.

(*iiii*) *all the conditions of Corollary* 1.

Example 2. Consider the following 4th-degree homogeneous polynomial

$$f(x)=\mathcal{A}x^4=11x_1^4+15x_2^4+1000x_3^4+900x_4^4+80x_1^3x_4-80x_2x_3^3-400x_3x_4^3+4x_1^3x_2,$$

where $x=(x_1,x_2,x_3)^T$ and $\mathcal{A}=(a_{i_1i_2i_3i_4})$ is a real symmetric tensor of order 4 dimension 3 with elements defined as follows:

$$\begin{aligned}
a_{1111}&=11,\quad a_{2222}=15,\quad a_{3333}=1000,\quad a_{4444}=900,\\
a_{1444}&=a_{4144}=a_{4414}=a_{4441}=20,
\end{aligned}$$

$$\begin{aligned} a_{2333} &= a_{3233} = a_{3323} = a_{3332} = -20, \\ a_{3444} &= a_{4434} = a_{4344} = a_{4443} = -100, \\ a_{1112} &= a_{1121} = a_{1211} = a_{2111} = 1, \end{aligned}$$

and other $a_{i_1 i_2 i_3 i_4} = 0$. It can be verified that $\mathcal{A}$ satisfies all the conditions of Theorem 5 under $k = 2$. Since

$$\begin{aligned} |a_{1111}| &= 11, \quad |a_{2222}| = 15, \quad |a_{3333}| = 1000, \quad |a_{4444}| = 900, \\ R_1(\mathcal{A}) &= 23, \quad R_2(\mathcal{A}) = 21, \quad R_3(\mathcal{A}) = 160, \quad R_4(\mathcal{A}) = 360, \end{aligned}$$

then $N_1(\mathcal{A}) = \{1\}$, $N_2(\mathcal{A}) = \{2\}$, $N_3(\mathcal{A}) = \emptyset$, $N_4(\mathcal{A}) = \{3, 4\}$. By calculations, we have

$$\begin{aligned} y_1 &= \frac{11}{23}, \quad y_2 = \frac{2}{7}, \quad r = \frac{1}{10}, \quad R_3^{(1)}(\mathcal{A}) = 160, \quad R_4^{(1)}(\mathcal{A}) = 360, \\ R_3^{(2)}(\mathcal{A}) &= 70, \quad R_4^{(2)}(\mathcal{A}) = 90, \quad \frac{R_3^{(2)}(\mathcal{A})}{|a_{3333}|} = \frac{7}{100}, \quad \frac{R_4^{(2)}(\mathcal{A})}{|a_{4444}|} = \frac{1}{10}. \end{aligned}$$

For $i = 1$, we have

$$\begin{aligned} &\sum_{\substack{1 i_2 i_3 i_4 \in N_1^3 \\ \delta_{1 i_2 i_3 i_4} = 0}} \max_{j \in \{i_2, i_3, i_4\}} \{y_j\} \left|a_{1 i_2 i_3 i_4}\right| + \sum_{i_2 i_3 i_4 \in N_2^3} \max_{j \in \{i_2, i_3, i_4\}} \{y_j\} \left|a_{1 i_2 i_3 i_4}\right| + \sum_{i_2 i_3 i_4 \in N_0^3} \left|a_{1 i_2 i_3 i_4}\right| + \sum_{i_2 i_3 i_4 \in N_4^3} \max_{j \in \{i_2, i_3, i_4\}} \left\{ \frac{R_j^{(2)}(\mathcal{A})}{\left|a_{jjjj}\right|} \right\} \left|a_{1 i_2 i_3 i_4}\right| \\ &= 0 + 0 + 3 + 20 \times \frac{1}{10} = 5 \\ &< 11 \times \frac{11}{23} = \frac{121}{23} = |a_{1111}| \, y_1. \end{aligned}$$

For $i = 2$, we get

$$\begin{aligned} &\sum_{2 i_2 i_3 i_4 \in N_1^3} \max_{j \in \{i_2, i_3, i_4\}} \{y_j\} \left|a_{2 i_2 i_3 i_4}\right| + \sum_{\substack{i_2 i_3 i_4 \in N_2^3 \\ \delta_{2 i_2 i_3 i_4} = 0}} \max_{j \in \{i_2, i_3, i_4\}} \{y_j\} \left|a_{2 i_2 i_3 i_4}\right| + \sum_{i_2 i_3 i_4 \in N_0^3} \left|a_{2 i_2 i_3 i_4}\right| + \sum_{i_2 i_3 i_4 \in N_4^3} \max_{j \in \{i_2, i_3, i_4\}} \left\{ \frac{R_j^{(2)}(\mathcal{A})}{\left|a_{jjjj}\right|} \right\} \left|a_{2 i_2 i_3 i_4}\right| \\ &= 1 \times \frac{2}{7} + 0 + 0 + 20 \times \frac{1}{10} = \frac{16}{7} \\ &< 15 \times \frac{2}{7} = \frac{30}{7} = |a_{2222}| \, y_2. \end{aligned}$$

So $\mathcal{A}$ satisfies the conditions of Theorem 5, from Theorem 8, we have that $\mathcal{A}$ is positive definite, that is, $f(x)$ is positive definite.

But, for $i = 1$, it holds that

$$a_{1111} = 11 < 23 = R_1(\mathcal{A}),$$

and

$$\sum_{\substack{i_2 i_3 i_4 \in N^3 / N_4^3 \\ \delta_{1 i_2 i_3 i_4} = 0}} \left|a_{1 i_2 i_3 i_4}\right| + \sum_{i_2 i_3 i_4 \in N_4^3} \max_{j \in \{i_2, i_3, i_4\}} \frac{R_j(\mathcal{A})}{\left|a_{jjjj}\right|} \left|a_{1 i_2 i_3 i_4}\right| = 3 + 20 \times \frac{2}{5} = 11 = 11 = |a_{1111}|.$$

Hence, we cannot use Theorem 3 in [16] and Theorem 1 in [9] to identify the positive definiteness of $\mathcal{A}$, respectively.

5. Conclusions

In this paper, we present new iterative methods for identifying the $\mathcal{H}$-tensors, , which are also used to identify the positive definiteness of an even degree homogeneous polynomial $f(x) \equiv \mathcal{A}x^m$. These inequalities were expressed in terms of the entries of $\mathcal{A}$, so they can be checked easily.

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