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Tempered (κ, ϖ)-Hilfer hybrid pantograph fractional boundary value problems with retarded and advanced arguments

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Abstract. Our research primarily focuses on utilizing tempered (κ, ϖ)-fractional operators to explore the existence, uniqueness, and κ-Mittag-Leffler-Ulam-Hyers stability of a specific class of hybrid Pantograph boundary value problems involving both retarded and advanced arguments. Our approach is based on Banach's fixed-point theorem, along with a generalized form of the well-known Gronwall inequality. Furthermore, we include an illustrative example to demonstrate the practical implications of our findings.

1. Introduction

Fractional calculus, which extends differentiation and integration to non-integer orders, has garnered significant attention in both theoretical and practical contexts across various research domains. Its adaptability has made it an essential tool in the field. Recently, there has been a marked increase in research on fractional calculus, exploring a wide range of outcomes under different conditions and types of fractional differential equations and inclusions. For more detailed information on fractional calculus, readers may refer to the works cited [1–3, 3, 5, 13, 14, 20, 21, 23, 48, 50].

In [18], Diaz introduced new definitions for the specialized functions κ-gamma and κ-beta. For further information, readers may consult additional sources such as [16, 33, 34]. Sousa *et al.* introduced the ϖ-Hilfer fractional derivative in their work [53], highlighting important properties of this fractional operator. For more detailed insights and results based on this operator, references such as [8, 51, 52] are recommended. Additionally, in [46], a new extension of the Hilfer fractional derivative, termed the κ-generalized ϖ-Hilfer fractional derivative, was proposed. For further reading on this fractional derivative, please refer to [24, 40–45].

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Tempered fractional calculus has recently gained prominence as a key subclass of fractional calculus operators. This subclass generalizes various forms of fractional calculus by incorporating analytic kernels, broadening its applications to model the transition between normal and anomalous diffusion. Initially introduced by Buschman in [15], the definitions of fractional integration with weak singular and exponential kernels have since been expanded. For more detailed discussions, see [9, 27, 30, 31, 35, 38, 49]. While the Caputo tempered fractional derivative has been relatively underexplored, it presents significant potential in the field. This study aims to thoroughly examine its properties and applications within this specialized mathematical framework, advancing the understanding of fractional calculus. Kucche *et al.* made a significant contribution in [29] by introducing a novel framework for computing tempered fractional integrals and derivatives, along with a comprehensive set of properties and results. In a follow-up study [26], the authors extended this theory to functions, proposing the tempered Hilfer-type operator. Building on earlier research and utilizing the κ-gamma, κ-beta, and κ-Mittag-Leffler functions from [18], a new definition of the tempered (κ, ψ)-Hilfer fractional operator was introduced in [39].

Unlike Lyapunov and exponential stability analysis, which address the stability of a dynamical system or equilibrium point, Ulam-Hyers stability analysis focuses on the behavior of a function under perturbations. The authors of [6, 7, 43, 46] have investigated the Ulam stability of fractional differential problems under various conditions. Significant attention has been given to the stability of different types of functional equations, particularly Ulam-Hyers and Ulam-Hyers-Rassias stability. This emphasis is highlighted in the book by Benchohra *et al.* [13], as well as in research by Luo *et al.* [28] and Rus [37], who explored the stability of operatorial equations using the Ulam-Hyers framework.

In [55], Zhao *et al.* examined fractional hybrid differential equations involving the Riemann-Liouville fractional differential operator defined as:

$$\begin{cases} {}^{RL}\mathbb{D}^{\alpha}_{0^+}\left(\dfrac{x(t)}{\mathrm{g}(t, x(t))}\right) = f(t, x(t)), \quad t \in [0, T], \\ x(0) = 0, \end{cases}$$

where ${}^{RL}\mathbb{D}^{\alpha}_{0^+}$ denotes the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$. The functions g and f are continuous, with g mapping $[0, T] \times \mathbb{R}$ to $\mathbb{R} \setminus \{0\}$ and f mapping $[0, T] \times \mathbb{R}$ to $\mathbb{R}$.

The pantograph equation is a flexible differential equation utilized in various fields such as electrodynamics, astrophysics, and cellular growth modeling. Its wide range of applications has sparked a recent increase in studies on the fractional-order pantograph equation by numerous researchers, as seen in [17, 22, 54].

Additionally, Balachandran *et al.* [11] established the existence and uniqueness of solutions for a fractional pantograph equation characterized by the Caputo fractional derivative

$$\begin{cases} {}^{C}\mathbb{D}^{\alpha}_{0^+}x(t) = f(t, x(t), x(\mathrm{y}t)), \ t \in [0, T], \ 0 < \mathrm{y} \leq 1, \\ x(0) = x_0, \end{cases}$$

where ${}^{C}\mathbb{D}^{\alpha}_{0^+}$ and ${}^{C}\mathbb{D}^{\beta}_{0^+}$ denote the Caputo fractional derivative of order $\alpha \in (0, 1)$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

In [24], the authors considered the initial value problem with nonlinear implicit ĸ-generalized ϖ-Hilfer type fractional differential equation:

$$\begin{cases} \left({}^{H}_{\kappa}\mathcal{D}^{\varkappa_1, \varkappa_2; \varpi}_{a+} x\right)(t) = \Psi\left(t, x(t), \left({}^{H}_{\kappa}\mathcal{D}^{\varkappa_1, \varkappa_2; \varpi}_{a+} x\right)(t)\right), \ \ t \in (a, b], \\ \\ \left(\mathcal{J}^{\kappa(1-\varkappa_4), \kappa; \varpi}_{a+} x\right)(a^+) = c_0, \end{cases}$$

where ${}_{\kappa}^{H}\mathcal{D}_{a+}^{\varkappa_1,\zeta;\varpi}$, $\mathcal{J}_{a+}^{\kappa(1-\varkappa_4),\kappa;\varpi}$ are the (κ,ϖ)-Hilfer fractional derivative of order $\varkappa_1 \in (0,\kappa)$ and type $\zeta \in [0,1]$, and (κ,ϖ)-fractional integral of order $\kappa(1-\varkappa_4)$, where $\kappa > 0$, $\Psi \in C([a,b]\times\mathbb{R}^2,\mathbb{R})$ and $c_0 \in \mathbb{R}$.

Motivated by above-mentioned results, in this paper, we establish existence and uniqueness results to the following tempered (κ,ϖ)-Hilfer hybrid boundary value problem with nonlinear implicit fractional differential equation:

$$ {}_{\kappa}^{TH}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}\left(\frac{x(t)}{\Theta(t,x(t))}\right) = \Psi\left(t, x^t(\cdot), x(\mu t), {}_{\kappa}^{TH}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}\left(\frac{x(t)}{\Theta(t,x(t))}\right)\right), \ t\in(0,b], \tag{1} $$

$$ \wp_1\left({}_{\varkappa_3}^{T}\mathcal{J}_{0+}^{\kappa(1-\varkappa_4),\kappa;\varpi}\frac{x(s)}{\Theta(s,x(s))}\right)(0^+) + \wp_2\left({}_{\varkappa_3}^{T}\mathcal{J}_{0+}^{\kappa(1-\varkappa_4),\kappa;\varpi}\frac{x(s)}{\Theta(s,x(s))}\right)(b) = \wp_3, \tag{2} $$

$$ x(t) = \vartheta(t), \ t\in[-\lambda,0], \ \lambda>0, \tag{3} $$

$$ x(t) = \tilde{\vartheta}(t), \ t\in\left[b, b+\tilde{\lambda}\right], \ \tilde{\lambda}>0, \tag{4} $$

where ${}_{\kappa}^{TH}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}$, ${}_{\varkappa_3}^{T}\mathcal{J}_{0+}^{\kappa(1-\varkappa_4),\kappa;\varpi}$ are the tempered (κ,ϖ)-Hilfer fractional derivative of order $\varkappa_1 \in (0,\kappa)$, $\varkappa_2 \in [0,1]$ and index $\varkappa_3 \in \mathbb{R}$, and tempered (κ,ϖ)-fractional integral of order $\kappa(1-\varkappa_4)$ and index $\varkappa_3$ defined in Section 2 respectively, where $\varkappa_4 = \frac{1}{\kappa}(\varkappa_2(\kappa-\varkappa_1)+\varkappa_1)$, $\kappa>0$, $\wp_1,\wp_2,\wp_3\in\mathbb{R}$, where $\wp_1+\wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}\neq 0$, $\mu\in(0,1)$, $\vartheta\in C([a-\lambda,a],\mathbb{R})$, $\tilde{\vartheta}\in C\left(\left[b,b+\tilde{\lambda}\right],\mathbb{R}\right)$, $\Psi:[0,b]\times C_{\varkappa_4;\varpi}([-\lambda,\lambda],\mathbb{R})\times\mathbb{R}\longrightarrow\mathbb{R}$ and $\Theta:[0,b]\times\mathbb{R}\longrightarrow\mathbb{R}\backslash\{0\}$ are given appropriate functions specified later. For each function x defined on $\left[a-\lambda, b+\tilde{\lambda}\right]$ and for any $t\in(a,b]$, we denote by x^t the element defined by

$$ x^t(\tau) = x(t+\tau), \ \tau\in\left[-\lambda,\tilde{\lambda}\right]. $$

The paper is organized as follows: Section 2 introduces the necessary elements. Section 3 presents existence and uniqueness results for the problem (1)-(4), based on Banach's fixed point theorem. Section 4 provides definitions of κ-Mittag-Leffler-Ulam-Hyers stability, along with related remarks, and includes the proof of the stability result for problem (1)-(4). The final section offers an illustrative example that effectively demonstrates the practical applicability of the main results.

2. Preliminaries

Let $0<0<b<\infty$, $F=[0,b]$, $\varkappa_1\in(0,\kappa)$, $\varkappa_2\in[0,1]$, $\varkappa_3\in\mathbb{R}$, $\kappa>0$ and $\varkappa_4=\frac{1}{\kappa}(\varkappa_2(\kappa-\varkappa_1)+\varkappa_1)$. By $C(F,\mathbb{R})$ we denote the Banach space of all continuous functions from F into $\mathbb{R}$ with the norm

$$ \|x\|_\infty = \sup\{|x(t)| : t\in F\}. $$

Let $C = C([-\lambda,0],\mathbb{R})$ and $\tilde{C} = C\left(\left[b,b+\tilde{\lambda}\right],\mathbb{R}\right)$ be the spaces endowed, respectively, with the norms

$$ \|x\|_C = \sup\{|x(t)| : t\in[-\lambda,0]\}, $$

and

$$ \|x\|_{\tilde{C}} = \sup\left\{|x(t)| : t\in\left[b,b+\tilde{\lambda}\right]\right\}. $$

$AC^\jmath(F,\mathbb{R})$, $C^\jmath(F,\mathbb{R})$ be the spaces of continuous functions, $\jmath$-times absolutely continuous and $\jmath$-times continuously differentiable functions on F, respectively.
Consider the weighted Banach space

$$ C_{\varkappa_4;\varpi}(F) = \left\{x:(0,b]\to\mathbb{R} : t\to\Pi_{\varkappa_4}^{\varpi}(t,0)x(t)\in C(F,\mathbb{R})\right\}, $$

where $\Pi^{\varpi}_{\varkappa_4}(t,0) = (\varpi(t) - \varpi(0))^{1-\varkappa_4}$, with the norm

$$\|x\|_{C_{\varkappa_4;\varpi}} = \sup_{t\in[0,b]} \left|\Pi^{\varpi}_{\varkappa_4}(t,0)x(t)\right|,$$

and

$$C^{J}_{\varkappa_4;\varpi}(F) = \left\{x \in C^{J-1}(F,\mathbb{R}) : x^{(J)} \in C_{\varkappa_4;\varpi}(F)\right\}, J \in \mathbb{N},$$
$$C^{0}_{\varkappa_4;\varpi}(F) = C_{\varkappa_4;\varpi}(F),$$

with the norm

$$\|x\|_{C^{J}_{\varkappa_4;\varpi}} = \sum_{i=0}^{J-1} \|x^{(i)}\|_\infty + \|x^{(J)}\|_{C_{\varkappa_4;\varpi}}.$$

Consider the space

$$\begin{aligned} C_{\varkappa_4;\varpi}([-\lambda,\lambda],\mathbb{R}) = \Big\{ & x : \left[-\lambda,\tilde{\lambda}\right] \to \mathbb{R} : \tau \to \Pi^{\varpi}_{\varkappa_4}(t,0)x(\tau) \in C([\tilde{a},\tilde{b}],\mathbb{R}) \text{ for each } t \in F, \\ & \tau \to x(\tau) \in C([-\lambda,\tilde{a}],\mathbb{R}) \text{ and } \tau \to x(\tau) \in C([\tilde{b},\tilde{\lambda}],\mathbb{R}) \\ & \text{for each } t \in F, \text{ where } \tilde{a} = -t > -\lambda \text{ and } \tilde{b} = b - t < \tilde{\lambda} \Big\}, \end{aligned}$$

with the norm

$$\|x^t\|_{[-\lambda,\tilde{\lambda}]} = \max\left\{ \sup_{\tau\in[\tilde{a},\tilde{b}]} \left|\Pi^{\varpi}_{\varkappa_4}(t,0)x^t(\tau)\right|, \sup_{\tau\in[-\lambda,\tilde{a}]} \left|x^t(\tau)\right|, \sup_{\tau\in[\tilde{b},\tilde{\lambda}]} \left|x^t(\tau)\right| \right\}.$$

Next, we consider the Banach space

$$\mathbb{F} = \left\{x : \left[-\lambda, b+\tilde{\lambda}\right] \to \mathbb{R} : \ x|_{[-\lambda,0]} \in C, \ x|_{[b,b+\tilde{\lambda}]} \in \tilde{C} \text{ and } \ x|_{(0,b]} \in C_{\varkappa_4;\varpi}(F)\right\}$$

with the norm

$$\|x\|_{\mathbb{F}} = \max\left\{\|x\|_C, \|x\|_{\tilde{C}}, \|x\|_{C_{\varkappa_4;\varpi}}\right\}.$$

Consider the space $X^p_{\varpi}(0,b)$, $(c \in \mathbb{R},\ 1 \le p \le \infty)$ of those real-valued Lebesgue measurable functions $\widehat{\Psi}$ on $[0,b]$ for which $\|\widehat{\Psi}\|_{X^p_{\varpi}} < \infty$, where the norm is defined by

$$\|\widehat{\Psi}\|_{X^p_{\varpi}} = \left(\int_0^b \varpi'(t)|\widehat{\Psi}(t)|^p dt\right)^{\frac{1}{p}},$$

where ϖ is an increasing and positive function on $[0,b]$ such that ϖ' is continuous on $[0,b]$ with $\varpi(0) = 0$. In particular, when $\varpi(x) = x$, the space $X^p_{\varpi}(0,b)$ coincides with the $L_p(0,b)$ space.

In what follows, and to keep it concise, we will take into account the following:

$$\widehat{\varkappa_3} := \max_{(t,\gamma)\in[0,b]\times[0,t]} e^{-\varkappa_3\left(\varpi(t)-\varpi(\gamma)\right)} = \begin{cases} 1, & \text{if } \varkappa_3 \ge 0, \\ e^{-\varkappa_3(\varpi(b)-\varpi(0))}, & \text{if } \varkappa_3 < 0. \end{cases}$$

Definition 2.1 ([18]). *The* κ*-gamma function is defined by*

$$\Gamma_\kappa(\varsigma) = \int_0^\infty t^{\varsigma-1} e^{-\frac{t^\kappa}{\kappa}} dt, \varsigma > 0.$$

When $\kappa \to 1$ *then* $\Gamma_\kappa(\varsigma) \to \Gamma(\varsigma)$, *we have also some useful following relations* $\Gamma_\kappa(\varsigma) = \kappa^{\frac{\varsigma}{\kappa}-1}\Gamma\left(\frac{\varsigma}{\kappa}\right)$, $\Gamma_\kappa(\varsigma+\kappa) = \varsigma\Gamma_\kappa(\varsigma)$ *and* $\Gamma_\kappa(\kappa) = \Gamma(1) = 1$. *Furthermore* κ*-beta function is defined as follows*

$$B_\kappa(\varsigma,\widehat{\varsigma}) = \frac{1}{\kappa}\int_0^1 t^{\frac{\varsigma}{\kappa}-1}(1-t)^{\frac{\widehat{\varsigma}}{\kappa}-1}dt$$

so that $B_\kappa(\varsigma,\widehat{\varsigma}) = \frac{1}{\kappa}B\left(\frac{\varsigma}{\kappa},\frac{\widehat{\varsigma}}{\kappa}\right)$ *and* $B_\kappa(\varsigma,\widehat{\varsigma}) = \frac{\Gamma_\kappa(\varsigma)\Gamma_\kappa(\widehat{\varsigma})}{\Gamma_\kappa(\varsigma+\widehat{\varsigma})}$. *The* κ*-Mittag-Leffler function is given by*

$$\mathbb{E}_\kappa^{\varsigma,\widehat{\varsigma}}(x) = \sum_{i=0}^{\infty}\frac{x^i}{\Gamma_\kappa(\varsigma i+\widehat{\varsigma})}, \varsigma,\widehat{\varsigma} > 0,$$

then, we can have

$$\mathbb{E}_\kappa^{\varsigma}(x) = \mathbb{E}_\kappa^{\varsigma,\kappa}(x) = \sum_{i=0}^{\infty}\frac{x^i}{\Gamma_\kappa(\varsigma i+\kappa)}, \varsigma > 0.$$

Definition 2.2 (The (κ,ϖ)-tempered fractional Integral [39]). *Let* $\widehat{\Psi} \in X_\varpi^p(0,b)$ *and* $[0,b]$ *be a finite or infinite interval on the real axis* $\mathbb{R}$, $\varpi(t) > 0$ *be an increasing function on* $(0,b]$ *and* $\varpi'(t) > 0$ *be continuous on* $(0,b)$, $\varkappa_3 \in \mathbb{R}$, $\kappa > 0$ *and* $\varkappa_1 > 0$. *The* (κ,ϖ)*-tempered fractional integral operators of a function* $\widehat{\Psi}$ *of order* $\varkappa_1$ *and index* $\varkappa_3$ *are defined by*

$$\begin{aligned}
{}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{\varkappa_1,\kappa;\varpi}\widehat{\Psi}(t) &= e^{-\varkappa_3\varpi(t)}\mathcal{J}_{0+}^{\varkappa_1,\kappa;\varpi}\left(\widehat{\Psi}(t)e^{\varkappa_3\varpi(t)}\right) = \int_0^t \bar{\Pi}_{\varkappa_1}^{\kappa,\varpi}(t,\gamma)e^{-\varkappa_3\left(\varpi(t)-\varpi(\gamma)\right)}\varpi'(\gamma)\widehat{\Psi}(\gamma)d\gamma,\\
{}^{T}_{\varkappa_3}\mathcal{J}_{b-}^{\varkappa_1,\kappa;\varpi}\widehat{\Psi}(t) &= e^{\varkappa_3\varpi(t)}\mathcal{J}_{b-}^{\varkappa_1,\kappa;\varpi}\left(\widehat{\Psi}(t)e^{-\varkappa_3\varpi(t)}\right) = \int_t^b \bar{\Pi}_{\varkappa_1}^{\kappa,\varpi}(\gamma,t)e^{-\varkappa_3\left(\varpi(\gamma)-\varpi(t)\right)}\varpi'(\gamma)\widehat{\Psi}(\gamma)d\gamma,
\end{aligned}$$

with $\bar{\Pi}_{\varkappa_1}^{\kappa,\varpi}(t,\gamma) = \dfrac{(\varpi(t)-\varpi(\gamma))^{\frac{\varkappa_1}{\kappa}-1}}{\kappa\Gamma_\kappa(\varkappa_1)}$. *Now, the* (κ,ϖ)*-tempered fractional integral* ${}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{\varkappa_1,\kappa;\varpi}$ *reduces to the* ϖ*-tempered fractional integral* ${}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{\varkappa_1;\varpi}$ *if* $\kappa = 1$.

Definition 2.3 (The tempered (κ,ϖ)-Hilfer Derivative [39]). *Let* $\jmath - 1 < \dfrac{\varkappa_1}{\kappa} \leq \jmath$ *with* $\jmath \in \mathbb{N}$, $\varkappa_3 \in \mathbb{R}$, $\kappa > 0$, $F = [0,b]$ *an interval such that* $-\infty \leq 0 < b \leq \infty$ *and* $\widehat{\Psi},\varpi \in C^\jmath([0,b],\mathbb{R})$ *two functions such that* ϖ *is increasing and* $\varpi'(t) \neq 0$, *for all* $t \in F$. *The tempered* (κ,ϖ)*-Hilfer fractional derivatives (left-sided and right-sided)* ${}^{TH}_{\kappa}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}(\cdot)$ *and* ${}^{TH}_{\kappa}\mathcal{D}_{b-}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}(\cdot)$ *of a function* $\widehat{\Psi}$ *of order* $\varkappa_1$, *index* $\varkappa_3$ *and type* $0 \leq \varkappa_2 \leq 1$, *are defined by*

$$\begin{aligned}
{}^{TH}_{\kappa}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}\widehat{\Psi}(t) &= \left({}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{\varkappa_2(\kappa\jmath-\varkappa_1),\kappa;\varpi}\left(\frac{1}{\varpi'(t)}\frac{d}{dt}+\varkappa_3\right)^{\jmath}\left(\kappa^{\jmath}\,{}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{(1-\varkappa_2)(\kappa\jmath-\varkappa_1),\kappa;\varpi}\widehat{\Psi}\right)\right)(t)\\
&= \left({}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{\varkappa_2(\kappa\jmath-\varkappa_1),\kappa;\varpi}\mho_{\varpi}^{\jmath}\left(\kappa^{\jmath}\,{}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{(1-\varkappa_2)(\kappa\jmath-\varkappa_1),\kappa;\varpi}\widehat{\Psi}\right)\right)(t)\\
&= e^{-\varkappa_3\varpi(t)}\times{}^{H}_{\kappa}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2;\varpi}\left(\widehat{\Psi}(t)e^{\varkappa_3\varpi(t)}\right)
\end{aligned}$$

and

$$\begin{aligned}
{}^{TH}_{\kappa}\mathcal{D}_{b-}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}\widehat{\Psi}(t) &= \left({}^{T}_{\varkappa_3}\mathcal{J}_{b-}^{\varkappa_2(\kappa\jmath-\varkappa_1),\kappa;\varpi}\left(-\frac{1}{\varpi'(t)}\frac{d}{dt}+\varkappa_3\right)^{\jmath}\left(\kappa^{\jmath}\,{}^{T}_{\varkappa_3}\mathcal{J}_{b-}^{(1-\varkappa_2)(\kappa\jmath-\varkappa_1),\kappa;\varpi}\widehat{\Psi}\right)\right)(t)\\
&= \left({}^{T}_{\varkappa_3}\mathcal{J}_{b-}^{\varkappa_2(\kappa\jmath-\varkappa_1),\kappa;\varpi}(-1)^{\jmath}\mho_{\varpi}^{\jmath}\left(\kappa^{\jmath}\,{}^{T}_{\varkappa_3}\mathcal{J}_{b-}^{(1-\varkappa_2)(\kappa\jmath-\varkappa_1),\kappa;\varpi}\widehat{\Psi}\right)\right)(t)\\
&= e^{\varkappa_3\varpi(t)}\times{}^{H}_{\kappa}\mathcal{D}_{b-}^{\varkappa_1,\varkappa_2;\varpi}\left(\widehat{\Psi}(t)e^{-\varkappa_3\varpi(t)}\right),
\end{aligned}$$

where $\mho_{\varpi}^{\jmath} = \left(\dfrac{1}{\varpi'(t)}\dfrac{d}{dt}+\varkappa_3\right)^{\jmath}$. *The tempered* (κ,ϖ)*-Hilfer fractional derivative* ${}^{TH}_{\kappa}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}$ *reduces to the tempered* ϖ*-Hilfer fractional derivative* ${}^{TH}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}$ *if* $\kappa = 1$.

Theorem 2.4 ([39]). *Let $\widehat{\Psi} : [0,b] \to \mathbb{R}$ be an integrable function, and take $\varkappa_1 > 0$, $\varkappa_3 \in \mathbb{R}$ and $\kappa > 0$. Then ${}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}\widehat{\Psi}$ exists for all $t \in [0,b]$.*

Theorem 2.5 ([39]). *Let $\widehat{\Psi} \in X^p_{\varpi}(0,b)$ and take $\varkappa_1 > 0$, $\varkappa_3 \in \mathbb{R}$ and $\kappa > 0$. Then ${}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}\widehat{\Psi} \in C([0,b],\mathbb{R})$.*

Lemma 2.6 ([39]). *Let $\varkappa_1 > 0$, $\varkappa_2 > 0$, $\varkappa_3 \in \mathbb{R}$ and $\kappa > 0$. Then, we have the following semigroup property given by*

$${}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}\,{}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_2,\kappa;\varpi}_{0+}\Psi(t) = {}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1+\varkappa_2,\kappa;\varpi}_{0+}\Psi(t) = {}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_2,\kappa;\varpi}_{0+}\,{}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}\Psi(t)$$

and

$${}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{b-}\,{}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_2,\kappa;\varpi}_{b-}\Psi(t) = {}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1+\varkappa_2,\kappa;\varpi}_{b-}\Psi(t) = {}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_2,\kappa;\varpi}_{b-}\,{}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{b-}\Psi(t).$$

Lemma 2.7 ([46]). *Let $\varkappa_1, \varkappa_2 > 0$, and $\kappa > 0$. Then, we have*

$$\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}\bar{\Pi}^{\kappa,\varpi}_{\varkappa_2}(t,0) = \bar{\Pi}^{\kappa,\varpi}_{\varkappa_1+\varkappa_2}(t,0)$$

and

$$\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{b-}\bar{\Pi}^{\kappa,\varpi}_{\varkappa_2}(b,t) = \bar{\Pi}^{\kappa,\varpi}_{\varkappa_1+\varkappa_2}(b,t).$$

Lemma 2.8 ([39]). *Let $\varkappa_1, \varkappa_2 > 0$, $\varkappa_3 \in \mathbb{R}$ and $\kappa > 0$. Then, we have*

$${}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}e^{-\varkappa_3(\varpi(t)-\varpi(0))}\bar{\Pi}^{\kappa,\varpi}_{\varkappa_2}(t,0) = e^{-\varkappa_3(\varpi(t)-\varpi(0))}\,\bar{\Pi}^{\kappa,\varpi}_{\varkappa_1+\varkappa_2}(t,0)$$

and

$${}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{b-}e^{\varkappa_3(\varpi(t)-\varpi(0))}\bar{\Pi}^{\kappa,\varpi}_{\varkappa_2}(b,t) = e^{\varkappa_3(\varpi(t)-\varpi(0))}\,\bar{\Pi}^{\kappa,\varpi}_{\varkappa_1+\varkappa_2}(b,t).$$

Theorem 2.9 ([39]). *Let $0 < 0 < b < \infty, \varkappa_1 > 0, 0 \le \varkappa_4 < 1$, $\varkappa_3 \in \mathbb{R}$, $\kappa > 0$ and $x \in C_{\varkappa_4;\varpi}(F)$. If $\dfrac{\varkappa_1}{\kappa} > 1 - \varkappa_4$, then*

$$\left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}x\right)(0) = \lim_{t\to 0^+}\left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}x\right)(t) = 0.$$

Lemma 2.10 ([39]). *Let $t > 0$, $\varkappa_1 > 0, 0 \le \varkappa_2 \le 1, \varkappa_3 \in \mathbb{R}, \kappa > 0$. Then for $0 < \varkappa_4 < 1; \varkappa_4 = \frac{1}{\kappa}(\varkappa_2(\kappa - \varkappa_1) + \varkappa_1)$, we have*

$$\left[{}^{TH}_{\kappa}\mathcal{D}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}_{0+}\left(\Pi^{\varpi}_{\varkappa_4}(\gamma,0)\right)^{-1}e^{-\varkappa_3(\varpi(\gamma)-\varpi(0))}\right](t) = 0.$$

Theorem 2.11 ([39]). *If $\Psi \in C^{\jmath}_{\varkappa_4;\varpi}[0,b]$, $\jmath - 1 < \dfrac{\varkappa_1}{\kappa} < \jmath$, $0 \le \varkappa_2 \le 1$, $\varkappa_3 \in \mathbb{R}$, where $\jmath \in \mathbb{N}$ and $\kappa > 0$, then*

$$\begin{aligned}&\left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}\,{}^{TH}_{\kappa}\mathcal{D}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}_{0+}\Psi\right)(t)\\ &\quad= \Psi(t) - e^{-\varkappa_3\varpi(t)}\sum_{i=1}^{\jmath}\frac{(\varpi(t)-\varpi(0))^{\varkappa_4-i}}{\kappa^{i-\jmath}\Gamma_\kappa(\kappa(\varkappa_4-i+1))}\left\{t^{\jmath-i}_{\varpi}\left(\mathcal{J}^{(1-\varkappa_2)(\kappa\jmath-\varkappa_1),\kappa;\varpi}_{0+}\Psi(0)e^{\varkappa_3\varpi(0)}\right)\right\},\end{aligned}$$

where

$$\varkappa_4 = \frac{1}{\kappa}\left(\varkappa_2(\kappa\jmath - \varkappa_1) + \varkappa_1\right).$$

In particular, if $\jmath = 1$, we have

$$\left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}\,{}^{TH}_{\kappa}\mathcal{D}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}_{0+}\Psi\right)(t) = \Psi(t) - e^{-\varkappa_3(\varpi(t)-\varpi(0))}\frac{(\varpi(t)-\varpi(0))^{\varkappa_4-1}}{\Gamma_\kappa(\varkappa_2(\kappa-\varkappa_1)+\varkappa_1)}\,{}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+}\Psi(0).$$

Lemma 2.12 ([39]). *Let $\varkappa_1 > 0, 0 \le \varkappa_2 \le 1$, $\varkappa_3 \in \mathbb{R}$, and $x \in C^1_{\varkappa_4;\varpi}(F)$, where $\kappa > 0$, then for $t \in (0,b]$, we have*

$$\left({}^{TH}_{\kappa}\mathcal{D}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}_{0+}\,{}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}x\right)(t) = x(t).$$

Lemma 2.13 ([47]). *Let $\varkappa_1, \kappa > 0$ and $\varkappa_3 \in \mathbb{R}$. Then, we have*

$$ {}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+} e^{-\varkappa_3(\varpi(t)-\varpi(0))}\mathbb{E}^{\varkappa_1}_{\kappa}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right) = e^{-\varkappa_3(\varpi(t)-\varpi(0))}\left[\mathbb{E}^{\varkappa_1}_{\kappa}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)-1\right], $$

and

$$ {}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}\mathbb{E}^{\varkappa_1}_{\kappa}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right) \leq \widehat{\varkappa_3}\left[\mathbb{E}^{\varkappa_1}_{\kappa}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)-1\right]. $$

Theorem 2.14 ([47]). *Let x, y be two integrable functions and $\mathfrak{x}$ continuous, with domain $[0,b]$. Let $\varpi \in C^1[0,b]$ an increasing function such that $\varpi'(t) \neq 0$, $t \in [0,b]$, $\varkappa_1 > 0$, $\kappa > 0$ and $\varkappa_3 \in \mathbb{R}$. Assume that:*

1. *x and y are nonnegative;*
2. *$\mathfrak{x}$ is nonnegative and nondecreasing.*

If

$$ x(t) \leq y(t) + \mathfrak{x}(t)\Gamma_\kappa(\varkappa_1)\int_0^t \varpi'(\gamma)e^{-\varkappa_3\left(\varpi(t)-\varpi(\gamma)\right)}\bar{\Pi}^{\kappa,\varpi}_{\varkappa_1}(t,\gamma)x(\gamma)\,d\gamma, $$

then

$$ x(t) \leq y(t) + \int_0^t \sum_{i=1}^{\infty}\left[\widehat{\varkappa_3}\mathfrak{x}(t)\Gamma_\kappa(\varkappa_1)\right]^i \varpi'(\gamma)\bar{\Pi}^{\kappa,\varpi}_{i\varkappa_1}(t,\gamma)y(\gamma)\,d\gamma, \tag{5} $$

for all $t \in [0,b]$, where

$$ \widehat{\varkappa_3} := \max_{(t,\gamma)\in[0,b]\times[0,t]} e^{-\varkappa_3\left(\varpi(t)-\varpi(\gamma)\right)} = \begin{cases} 1, & \text{if } \varkappa_3 \geq 0,\\ e^{-\varkappa_3(\varpi(b)-\varpi(0))}, & \text{if } \varkappa_3 < 0.\end{cases} $$

And if y is a nondecreasing function on $[0,b]$, then we have

$$ x(t) \leq y(t)\,\mathbb{E}^{\varkappa_1}_{\kappa}\left(\widehat{\varkappa_3}\mathfrak{x}(t)\Gamma_\kappa(\varkappa_1)(\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right). $$

3. Existence of Solutions

We consider the following fractional differential equation

$$ {}^{TH}_{\kappa}\mathcal{D}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}_{0+}\left(\frac{x(t)}{\widehat{\Theta}(t)}\right) = \chi(t), \quad t \in (0,b], \tag{6} $$

where $0 < \varkappa_1 < \kappa, 0 \leq \varkappa_2 \leq 1, \varkappa_3 \in \mathbb{R}$ with the conditions

$$ \wp_1\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+}\frac{x(s)}{\widehat{\Theta}(s)}\right)(0^+) + \wp_2\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+}\frac{x(s)}{\widehat{\Theta}(s)}\right)(b) = \wp_3, \tag{7} $$

$$ x(t) = \vartheta(t), \quad t \in [-\lambda,0],\ \lambda > 0, \tag{8} $$

$$ x(t) = \tilde{\vartheta}(t), \quad t \in \left[b, b+\tilde{\lambda}\right],\ \tilde{\lambda} > 0, \tag{9} $$

where $\varkappa_4 = \frac{\varkappa_2(\kappa-\varkappa_1)+\varkappa_1}{\kappa}$, $\kappa > 0$, $\chi(\cdot) \in C(F,\mathbb{R})$, $\widehat{\Theta}(\cdot) \in C(F,\mathbb{R}\backslash\{0\})$, $\vartheta(\cdot) \in C$, $\tilde{\vartheta}(\cdot) \in \tilde{C}$, $\wp_1,\wp_2,\wp_3 \in \mathbb{R}$, where $\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))} \neq 0$.

The following theorem shows that the problem (6)-(9) have a unique solution.

Theorem 3.1. *Let* $0 < \varkappa_1 < \kappa, 0 \le \varkappa_2 \le 1, \varkappa_3 \in \mathbb{R}$, $\kappa > 0$, $\chi(\cdot) \in C(F, \mathbb{R})$, $\widehat{\Theta}(\cdot) \in C(F, \mathbb{R}\backslash\{0\})$. *The problem* (6)-(9) *has a unique solution given by:*

$$x(t) = \frac{\widehat{\Theta}(t) e^{-\varkappa_3(\varpi(t)-\varpi(0))}}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)} \left[\frac{\wp_3 - \wp_2 \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi \right)(b)}{\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}} \right] + \widehat{\Theta}(t) \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}\chi \right)(t). \tag{10}$$

Proof. Assume x satisfies (6)-(9). By applying ${}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}(\cdot)$ on (6) and using Theorem 2.11, we obtain

$$\frac{x(t)}{\widehat{\Theta}(t)} = \frac{{}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \dfrac{x(0)}{\widehat{\Theta}(0)}}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)} e^{-\varkappa_3(\varpi(t)-\varpi(0))} + \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}\chi \right)(t). \tag{11}$$

Applying $\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+}(\cdot)$ on both sides of (11), using Lemma 2.6, Lemma 2.8 and taking $t = b$, we have

$$\begin{aligned} \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \frac{x(s)}{\widehat{\Theta}(s)} \right)(b) = {} & e^{-\varkappa_3(\varpi(b)-\varpi(0))} {}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \frac{x(0)}{\widehat{\Theta}(0)} \\ & + \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi \right)(b). \end{aligned} \tag{12}$$

Multiplying both sides of (12) by $\wp_2$, we get

$$\begin{aligned} \wp_2 \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \frac{x(s)}{\widehat{\Theta}(s)} \right)(b) = {} & \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))} {}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \frac{x(0)}{\widehat{\Theta}(0)} \\ & + \wp_2 \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi \right)(b). \end{aligned}$$

Using condition (7), we obtain

$$\wp_2 \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \frac{x(s)}{\widehat{\Theta}(s)} \right)(b) = \wp_3 - \wp_1 \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \frac{x(s)}{\widehat{\Theta}(s)} \right)(0^+).$$

Thus,

$$\begin{aligned} \wp_3 - \wp_1 \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \frac{x(s)}{\widehat{\Theta}(s)} \right)(0^+) = {} & \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))} {}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \frac{x(0)}{\widehat{\Theta}(0)} \\ & + \wp_2 \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi \right)(b). \end{aligned}$$

Then,

$$\begin{aligned} \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \frac{x(s)}{\widehat{\Theta}(s)} \right)(0^+) = {} & \frac{\wp_3}{\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}} \\ & - \frac{\wp_2}{\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}} \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi \right)(b). \end{aligned} \tag{13}$$

Substituting (13) into (11), we obtain (10).

Reciprocally, applying ${}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+}$ on both sides of (10) and using Lemma 2.8 and Lemma 2.6, we get

$$\begin{aligned} \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \frac{x(s)}{\widehat{\Theta}(s)} \right)(t) = {} & \left[\frac{\wp_3 - \wp_2 \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi \right)(b)}{\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}} \right] e^{-\varkappa_3(\varpi(t)-\varpi(0))} \\ & + \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi \right)(t), \ t \in (0, b]. \end{aligned} \tag{14}$$

Next, taking the limit $t \to 0^+$ of (14) and using Theorem 2.9, with $\kappa(1-\varkappa_4) < \kappa(1-\varkappa_4) + \varkappa_1$, we obtain

$$\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+}\frac{x(s)}{\widetilde{\Theta}(s)}\right)(0^+) = \frac{\wp_3 - \wp_2\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi\right)(b)}{\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}}. \tag{15}$$

Now, taking $t = b$ in (14), to get

$$\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+}\frac{x(s)}{\widetilde{\Theta}(s)}\right)(b) = \left[\frac{\wp_3 - \wp_2\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi\right)(b)}{\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}}\right]e^{-\varkappa_3(\varpi(b)-\varpi(0))} + \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi\right)(b). \tag{16}$$

From (15) and (16), we obtain

$$\begin{aligned}
&\wp_1\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+}\frac{x(s)}{\widetilde{\Theta}(s)}\right)(0^+) + \wp_2\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+}\frac{x(s)}{\widetilde{\Theta}(s)}\right)(b)\\
&= \left[\frac{\wp_2\wp_3 - \wp_2^2\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi\right)(b)}{\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}}\right]e^{-\varkappa_3(\varpi(b)-\varpi(0))} + \wp_2\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi\right)(b)\\
&\quad + \frac{\wp_1\wp_3 - \wp_1\wp_2\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi\right)(b)}{\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}}\\
&= \wp_3,
\end{aligned}$$

which proves that (7) is verified. Apply ${}^{TH}_{\kappa}\mathcal{D}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}_{0+}(\cdot)$ on both sides of (10). Then, from Lemma 2.10 and Lemma 2.12 we obtain equation (6). □

Lemma 3.2. *Let* $\varkappa_4 = \dfrac{\varkappa_2(\kappa-\varkappa_1)+\varkappa_1}{\kappa}$ *where* $0 < \varkappa_1 < \kappa$ *and* $0 \le \varkappa_2 \le 1, \varkappa_3 \in \mathbb{R}$, $\vartheta(\cdot) \in C$, $\tilde{\vartheta}(\cdot) \in \tilde{C}$, *let* $\Psi : F \times C_{\varkappa_4;\varpi}([-\lambda,\lambda],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ *and* $\Theta : F \times \mathbb{R} \longrightarrow \mathbb{R}\backslash\{0\}$ *be continuous functions. Then, the problem* (1)-(4) *is equivalent to the following integral equation:*

$$x(t) = \begin{cases}\Theta(t,x(t))\left(\frac{e^{-\varkappa_3(\varpi(t)-\varpi(0))}}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)}\left[\frac{\wp_3 - \wp_2\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\chi\right)(b)}{\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}}\right] + \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+}\chi\right)(t)\right),\\[2mm] \vartheta(t), \quad t\in[-\lambda,0],\\[2mm] \tilde{\vartheta}(t), \quad t\in\left[b,b+\tilde{\lambda}\right],\end{cases} \tag{17}$$

where χ *be a function satisfying:*

$$\chi(t) = \Psi(t, x^t(\cdot), x(\mu t), \chi(t)).$$

The hypotheses:

(*Ax1*) The functions $\Psi : F\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\Theta : F\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ are continuous.

(*Ax2*) There exist constants $\zeta_1, \zeta_3 > 0$ and $0 < \zeta_2 < 1$ such that

$$|\Psi(t,x_1,y_1,z_1) - \Psi(t,x_2,y_2,z_2)| \le \zeta_1\left(\|x_1-x_2\|_{[-\lambda,\tilde{\lambda}]} + \Pi^{\varpi}_{\varkappa_4}(t,0)|y_1-y_2|\right) + \zeta_2|z_1-z_2|,$$

$$|\Theta(t,y_1) - \Theta(t,y_2)| \le \zeta_3\Pi^{\varpi}_{\varkappa_4}(t,0)|y_1-y_2|,$$

for any $x_1, x_2 \in C_{\varkappa_4;\varpi}([-\lambda,\lambda],\mathbb{R})$, $y_1,y_2,z_1,z_2\in\mathbb{R}$ and $t\in F$.

$(Ax3)$ There exist $\overline{\Theta}, \overline{\Psi} > 0$ such that

$$|\Theta(t, y_1)| \leq \overline{\Theta} \text{ and } |\Psi(t, x_1, y_1, z_1)| \leq \overline{\Psi},$$

for all $t \in F$ and any $x_1 \in C_{\varkappa_4;\varpi}([-\lambda, \lambda], \mathbb{R})$, $y_1, z_1 \in \mathbb{R}$.

We are now in a position to state and prove our existence result for the problem (1)-(4) based on based on Banach's fixed point theorem [19].

Theorem 3.3. *Assume* (Ax1)-(Ax3) *hold. If*

$$\mathcal{L} = \left[\frac{|\wp_2|\widehat{\varkappa_3}^2 (\varpi(b) - \varpi(0))^{1-\varkappa_4+\frac{\varkappa_1}{\kappa}}}{\Gamma_\kappa(\kappa\varkappa_4)\Gamma_\kappa(2\kappa - \kappa\varkappa_4 + \varkappa_1) \left|\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|} + \frac{\widehat{\varkappa_3} (\varpi(b) - \varpi(0))^{1-\varkappa_4+\frac{\varkappa_1}{\kappa}}}{\Gamma_\kappa(\varkappa_1 + \kappa\varkappa_4)} \right] \left(\frac{2\overline{\Theta}\zeta_1}{1 - \zeta_2} + \overline{\Psi}\zeta_3 \right) < 1, \tag{18}$$

then the problem (1)-(4) *has a unique solution in* $\mathbb{F}$.

Proof. Transform problem (1)-(4) into a fixed point problem by considering the operator $\mathcal{T} : \mathbb{F} \to \mathbb{F}$ by

$$(\mathcal{T}x)(t) = \begin{cases} \Theta(t, x(t)) \left(\frac{e^{-\varkappa_3(\varpi(t)-\varpi(0))}}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)} \left[\frac{\wp_3 - \wp_2 \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+} \chi \right)(b)}{\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}} \right] \right. \\ \qquad \left. + \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+} \chi \right)(t) \right), \\ \vartheta(t), \quad t \in [-\lambda, 0], \\ \tilde{\vartheta}(t), \quad t \in \left[b, b + \tilde{\lambda}\right], \end{cases} \tag{19}$$

where χ be a function satisfying the functional equation

$$\chi(t) = \Psi(t, x^t(\cdot), x(\mu t), \chi(t)).$$

By Theorem 2.5, we have $\mathcal{T}x \in \mathbb{F}$. We show that the operator $\mathcal{T}$ has a unique fixed point in $\mathbb{F}$.

Let $x, y \in \mathbb{F}$. Then for any $t \in [-\lambda, 0] \cup \left[b, b + \tilde{\lambda}\right]$, we have

$$|\mathcal{T}x(t) - \mathcal{T}y(t)| = 0.$$

Thus,

$$\|\mathcal{T}x - \mathcal{T}y\|_C = \|\mathcal{T}x - \mathcal{T}y\|_{\tilde{C}} = 0. \tag{20}$$

For any for $t \in F$, we have

$$\begin{aligned} &|\mathcal{T}x(t) - \mathcal{T}y(t)| \\ &\leq \frac{|\Theta(t, x(t))| e^{-\varkappa_3(\varpi(t)-\varpi(0))}}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)} \left[\frac{|\wp_2| \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+} |\chi_1(\gamma) - \chi_2(\gamma)| \right)(b)}{\left|\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|} \right] \\ &\quad + |\Theta(t, x(t)) - \Theta(t, y(t))| \frac{e^{-\varkappa_3(\varpi(t)-\varpi(0))}}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)} \left[\frac{|\wp_2| \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+} |\chi_2(\gamma)| \right)(b)}{\left|\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|} \right] \end{aligned}$$

$$+ |\Theta(t,x(t))| \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0^+}|\chi_1(\gamma)-\chi_2(\gamma)|\right)(t)$$
$$+ |\Theta(t,x(t))-\Theta(t,y(t))| \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0^+}|\chi_2(\gamma)|\right)(t),$$

where χ_1 and χ_1 be functions satisfying the functional equations

$$\chi_1(t) = \Psi(t, x^t(\cdot), x(\mu t), \chi_1(t)),$$
$$\chi_2(t) = \Psi(t, y^t(\cdot), y(\mu t), \chi_2(t)).$$

By $(Ax2)$, we have

$$\begin{aligned} |\chi_1(t)-\chi_2(t)| &= |\Psi(t, x^t(\cdot), x(\mu t), \chi_1(t)) - \Psi(t, y^t(\cdot), y(\mu t), \chi_2(t))| \\ &\leq \zeta_1\left(\|x^t-y^t\|_{[-\lambda,\tilde{\lambda}]} + \Pi^{\varpi}_{\varkappa_4}(t,0)|x(\mu t)-y(\mu t)|\right) + \zeta_2|\chi_1(t)-\chi_2(t)|. \end{aligned}$$

Then,

$$|\chi_1(t)-\chi_2(t)| \leq \frac{\zeta_1}{1-\zeta_2}\left(\|x^t-y^t\|_{[-\lambda,\tilde{\lambda}]} + \Pi^{\varpi}_{\varkappa_4}(t,0)|x(\mu t)-y(\mu t)|\right).$$

Therefore, for each $t \in F$ we get

$$\begin{aligned} &|\mathcal{T}x(t)-\mathcal{T}y(t)| \\ &\leq \frac{\overline{\Theta}e^{-\varkappa_3(\varpi(t)-\varpi(0))}}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)}\left[\frac{\zeta_1|\wp_2|\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}\|x^\gamma-y^\gamma\|_{[-\lambda,\tilde{\lambda}]}+\Pi^{\varpi}_{\varkappa_4}(t,0)|x(\mu\gamma)-y(\mu\gamma)|\right)(b)}{(1-\zeta_2)\left|\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}\right] \\ &+ \zeta_3\frac{e^{-\varkappa_3(\varpi(t)-\varpi(0))}}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)}\left[\frac{|\wp_2|\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}|\chi_2(\gamma)|\right)(b)}{\left|\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}\right]\|x-y\|_{C_{\varkappa_4;\varpi}} \\ &+ \frac{\overline{\Theta}\zeta_1}{1-\zeta_2}\left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0^+}\|x^\gamma-y^\gamma\|_{[-\lambda,\tilde{\lambda}]}+\Pi^{\varpi}_{\varkappa_4}(t,0)|x(\mu\gamma)-y(\mu\gamma)|\right)(t) \\ &+ \zeta_3\|x-y\|_{C_{\varkappa_4;\varpi}}\left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0^+}|\chi_2(\gamma)|\right)(t). \end{aligned}$$

Thus,

$$\begin{aligned} |\mathcal{T}x(t)-\mathcal{T}y(t)| &\leq \left[\frac{2\overline{\Theta}e^{-\varkappa_3(\varpi(t)-\varpi(0))}\zeta_1|\wp_2|\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}(1)\right)(b)}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)(1-\zeta_2)\left|\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}\right. \\ &\quad \left.+\frac{2\overline{\Theta}\zeta_1}{1-\zeta_2}\left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0^+}(1)\right)(t)\right]\|x-y\|_{\mathbb{F}} \\ &\quad +\left[\frac{\overline{\Psi}e^{-\varkappa_3(\varpi(t)-\varpi(0))}\zeta_3|\wp_2|\left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}(1)\right)(b)}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)\left|\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}\right. \\ &\quad \left.+\overline{\Psi}\zeta_3\left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0^+}(1)\right)(t)\right]\|x-y\|_{\mathbb{F}} \\ &\leq \left[\frac{2\overline{\Theta}e^{-\varkappa_3(\varpi(t)-\varpi(0))}\zeta_1|\wp_2|\left(\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}e^{-\varkappa_3(\varpi(t)-\varpi(\gamma))}\right)(b)}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)(1-\zeta_2)\left|\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}\right. \\ &\quad \left.+\frac{2\overline{\Theta}\zeta_1}{1-\zeta_2}\left(\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0^+}e^{-\varkappa_3(\varpi(t)-\varpi(\gamma))}\right)(t)\right]\|x-y\|_{\mathbb{F}} \\ &\quad +\left[\frac{\overline{\Psi}e^{-\varkappa_3(\varpi(t)-\varpi(0))}\zeta_3|\wp_2|\left(\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+}e^{-\varkappa_3(\varpi(t)-\varpi(\gamma))}\right)(b)}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_\kappa(\kappa\varkappa_4)\left|\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}\right. \end{aligned}$$

$$+\overline{\Psi}\zeta_3\left(\mathcal{J}_{0^+}^{\varkappa_1,\kappa;\varpi}e^{-\varkappa_3\left(\varpi(t)-\varpi(\gamma)\right)}\right)(t)\Bigg]\|x-y\|_{\mathbb{F}}.$$

By Lemma 2.7, we have

$$\begin{aligned}
|\mathcal{T}x(t)-\mathcal{T}y(t)| &\leq \Bigg[\frac{2\overline{\Theta}e^{-\varkappa_3(\varpi(t)-\varpi(0))}\zeta_1|\wp_2|\widehat{\varkappa_3}\left(\mathcal{J}_{0+}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}(1)\right)(b)}{\Pi_{\varkappa_4}^{\varpi}(t,0)\Gamma_\kappa(\kappa\varkappa_4)(1-\zeta_2)\left|\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}\\
&\quad+\frac{2\overline{\Theta}\zeta_1\widehat{\varkappa_3}}{1-\zeta_2}\left(\mathcal{J}_{0^+}^{\varkappa_1,\kappa;\varpi}(1)\right)(t)\Bigg]\|x-y\|_{\mathbb{F}}\\
&\quad+\Bigg[\frac{\overline{\Psi}e^{-\varkappa_3(\varpi(t)-\varpi(0))}\zeta_3|\wp_2|\widehat{\varkappa_3}\left(\mathcal{J}_{0+}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}(1)\right)(b)}{\Pi_{\varkappa_4}^{\varpi}(t,0)\Gamma_\kappa(\kappa\varkappa_4)\left|\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}\\
&\quad+\overline{\Psi}\zeta_3\widehat{\varkappa_3}\left(\mathcal{J}_{0^+}^{\varkappa_1,\kappa;\varpi}(1)\right)(t)\Bigg]\|x-y\|_{\mathbb{F}}\\
&\leq\Bigg[\frac{2\overline{\Theta}\zeta_1|\wp_2|\widehat{\varkappa_3}^2\left(\varpi(b)-\varpi(0)\right)^{1-\varkappa_4+\frac{\varkappa_1}{\kappa}}}{\Pi_{\varkappa_4}^{\varpi}(t,0)\Gamma_\kappa(\kappa\varkappa_4)\Gamma_\kappa(2\kappa-\kappa\varkappa_4+\varkappa_1)(1-\zeta_2)\left|\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}\\
&\quad+\frac{2\overline{\Theta}\zeta_1\widehat{\varkappa_3}\left(\varpi(t)-\varpi(0)\right)^{\frac{\varkappa_1}{\kappa}}}{\Gamma_\kappa(\varkappa_1+\kappa\varkappa_4)(1-\zeta_2)}\Bigg]\|x-y\|_{\mathbb{F}}\\
&\quad+\Bigg[\frac{\overline{\Psi}\zeta_3|\wp_2|\widehat{\varkappa_3}^2\left(\varpi(b)-\varpi(0)\right)^{1-\varkappa_4+\frac{\varkappa_1}{\kappa}}}{\Pi_{\varkappa_4}^{\varpi}(t,0)\Gamma_\kappa(\kappa\varkappa_4)\Gamma_\kappa(2\kappa-\kappa\varkappa_4+\varkappa_1)\left|\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}\\
&\quad+\frac{\overline{\Psi}\zeta_3\widehat{\varkappa_3}\left(\varpi(t)-\varpi(0)\right)^{\frac{\varkappa_1}{\kappa}}}{\Gamma_\kappa(\varkappa_1+\kappa\varkappa_4)}\Bigg]\|x-y\|_{\mathbb{F}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|\Pi_{\varkappa_4}^{\varpi}(t,0)\left(\mathcal{T}x(t)-\mathcal{T}y(t)\right)\right| &\leq\Bigg[\frac{|\wp_2|\widehat{\varkappa_3}^2\left(\varpi(b)-\varpi(0)\right)^{1-\varkappa_4+\frac{\varkappa_1}{\kappa}}}{\Gamma_\kappa(\kappa\varkappa_4)\Gamma_\kappa(2\kappa-\kappa\varkappa_4+\varkappa_1)\left|\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}\\
&\quad+\frac{\widehat{\varkappa_3}\left(\varpi(b)-\varpi(0)\right)^{1-\varkappa_4+\frac{\varkappa_1}{\kappa}}}{\Gamma_\kappa(\varkappa_1+\kappa\varkappa_4)}\Bigg]\left(\frac{2\overline{\Theta}\zeta_1}{1-\zeta_2}+\overline{\Psi}\zeta_3\right)\|x-y\|_{\mathbb{F}}.
\end{aligned}$$

And by (20), we have

$$\|\mathcal{T}x-\mathcal{T}y\|_{\mathbb{F}}\leq\mathcal{L}\|x-y\|_{\mathbb{F}}.$$

By (18), the operator $\mathcal{T}$ is a contraction on $\mathbb{F}$. Hence, by Banach's contraction principle, $\mathcal{T}$ has a unique fixed point $x\in\mathbb{F}$, which is a solution to our problem (1)-(4). □

4. κ-Mittag-Leffler-Ulam-Hyers stability

Let $x\in\mathbb{F}$, $\epsilon>0$. We consider the following inequality :

$$\begin{cases}\left|\left({}_{\kappa}^{TH}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}\dfrac{x(s)}{\Theta(s,x(s))}\right)(t)-\Psi\left(t,x^t(\cdot),x(\mu t),\left({}_{\kappa}^{TH}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}\dfrac{x(s)}{\Theta(s,x(s))}\right)(t)\right)\right|\\ \quad\leq\epsilon\mathbb{E}_{\kappa}^{\varkappa_1}\left(\left(\varpi(t)-\varpi(0)\right)^{\frac{\varkappa_1}{\kappa}}\right),\ t\in(0,b],\\ \\ |x(t)-\vartheta(t)|\leq\epsilon\Delta_1,\\ |x(t)-\tilde{\vartheta}(t)|\leq\epsilon\Delta_2.\end{cases}\tag{21}$$

Definition 4.1 ([43]). *Problem* (1)-(4) *is said to be* κ*-Mittag-Leffler-Ulam-Hyers (*κ*-M-L-U-H) stable with respect to* $\mathbb{E}_{\kappa}^{\varkappa_1}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)$ *if there exists* $a_{\mathbb{E}_{\kappa}^{\varkappa_1}}>0$ *where for each* $\epsilon>0$ *and for each solution* $x\in\mathbb{F}$ *of inequality (21) there exists a solution* $y\in\mathbb{F}$ *of* (1)-(4) *with*

$$|x(t)-y(t)|\leq a_{\mathbb{E}_{\kappa}^{\varkappa_1}}\epsilon\left[\mathbb{E}_{\kappa}^{\varkappa_1}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)+\Delta_1+\Delta_2\right],\qquad t\in F.$$

Definition 4.2 ([43]). *Problem* (1)-(4) *is generalized* κ*-M-L-U-H stable with respect to* $\mathbb{E}_{\kappa}^{\varkappa_1}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)$ *if there exists* $v:C([0,\infty),[0,\infty))$ *with* $v(0)=0$ *such that for each* $\epsilon>0$ *and for each solution* $x\in\mathbb{F}$ *of inequality (21) there exists a solution* $y\in\mathbb{F}$ *of* (1)-(4) *with*

$$|x(t)-y(t)|\leq v(\epsilon)\left[\mathbb{E}_{\kappa}^{\varkappa_1}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)+\Delta_1+\Delta_2\right],\qquad t\in F.$$

Remark 4.3. *Its clear that : Definition 4.1* $\Longrightarrow$ *Definition 4.2.*

Remark 4.4. *A function* $x\in\mathbb{F}$ *is a solution of inequality (21) if and only if there exist* $\jmath\in C_{\varkappa_4,\kappa;\varpi}(F)$ *and constants* ν_1,ν_2 *such that for* $t\in(0,b]$, *we have*

1. $|\jmath(t)|\leq\epsilon\mathbb{E}_{\kappa}^{\varkappa_1}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)$, $|\nu_1|\leq\epsilon\Delta_1$ *and* $|\nu_2|\leq\epsilon\Delta_2$.
2. $\left({}^{TH}_{\kappa}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}\dfrac{x(s)}{\Theta(s,x(s))}\right)(t)=\Psi\left(t,x^t(\cdot),x(\mu t),\left({}^{TH}_{\kappa}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}\dfrac{x(s)}{\Theta(s,x(s))}\right)(t)\right)+\jmath(t).$
3. $x(t)=\vartheta(t)+\nu_1,\ t\in[-\lambda,0],\ \lambda>0.$
4. $x(t)=\tilde{\vartheta}(t)+\nu_2,\ t\in\left[b,b+\tilde{\lambda}\right],\ \tilde{\lambda}>0.$

Theorem 4.5. *Assume that the hypothesis* (Ax1)-(Ax3) *and the condition (18) hold. Then,* (1)-(4) *is* κ*-M-L-U-H stable with respect to* $\mathbb{E}_{\kappa}^{\varkappa_1}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)$ *and consequently generalized* κ*-M-L-U-H stable.*

Proof. Let $x\in\mathbb{F}$ be a solution if inequality (21), and let us assume that y is the unique solution of the problem

$$\begin{cases}\left({}^{TH}_{\kappa}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}\dfrac{y(s)}{\Theta(s,y(s))}\right)(t)=\Psi\left(t,y^t(\cdot),y(\mu t),\left({}^{TH}_{\kappa}\mathcal{D}_{0+}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}\dfrac{y(s)}{\Theta(s,y(s))}\right)(t)\right);\ t\in(0,b],\\[2ex] \wp_1\left({}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{\kappa(1-\varkappa_4),\kappa;\varpi}\dfrac{y(s)}{\Theta(s,y(s))}\right)(0^+)+\wp_2\left({}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{\kappa(1-\varkappa_4),\kappa;\varpi}\dfrac{y(s)}{\Theta(s,y(s))}\right)(b)=\wp_3,\\[2ex] \left({}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{\kappa(1-\varkappa_4),\kappa;\varpi}\dfrac{y(s)}{\Theta(s,y(s))}\right)(0^+)=\left({}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{\kappa(1-\varkappa_4),\kappa;\varpi}\dfrac{x(s)}{\Theta(s,x(s))}\right)(0^+),\\[2ex] y(t)=\vartheta(t),\ \ t\in[-\lambda,0],\ \lambda>0,\\ y(t)=\tilde{\vartheta}(t),\ \ t\in\left[b,b+\tilde{\lambda}\right],\ \tilde{\lambda}>0.\end{cases}$$

By Lemma 3.2, we obtain for each $t\in(0,b]$

$$y(t)=\begin{cases}\frac{\Theta(s,y(s))e^{-\varkappa_3(\varpi(t)-\varpi(0))}}{\Pi_{\varkappa_4}^{\varpi}(t,0)\Gamma_\kappa(\kappa\varkappa_4)}\left[\dfrac{\wp_3-\wp_2\left({}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}w\right)(b)}{\wp_1+\wp_2e^{-\varkappa_3(\varpi(b)-\varpi(0))}}\right]\\ \quad+\Theta(s,y(s))\left({}^{T}_{\varkappa_3}\mathcal{J}_{0+}^{\varkappa_1,\kappa;\varpi}w\right)(t),\\[2ex] y(t)=\vartheta(t),\ \ t\in[-\lambda,0],\\ y(t)=\tilde{\vartheta}(t),\ \ t\in\left[b,b+\tilde{\lambda}\right].\end{cases}$$

where $w \in \mathbb{F}$, be a function satisfying the functional equation

$$w(t) = \Psi(t, y^t(\cdot), y(\mu t), w(t)).$$

Since x is a solution of the inequality (21), by Remark 4.4, for $t \in (0, b]$ we have

$$\begin{cases} \left({}^{TH}_{\kappa}\mathcal{D}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}_{0+} \dfrac{x(s)}{\Theta(s, x(s))} \right)(t) = \Psi\left(t, x^t(\cdot), x(\mu t), \left({}^{TH}_{\kappa}\mathcal{D}^{\varkappa_1,\varkappa_2,\varkappa_3;\varpi}_{0+} \dfrac{x(s)}{\Theta(s, x(s))} \right)(t)\right) + \jmath(t), \\ x(t) = \vartheta(t) + \nu_1, \quad t \in [-\lambda, 0], \\ x(t) = \tilde{\vartheta}(t) + \nu_2, \quad t \in \left[b, b + \tilde{\lambda}\right]. \end{cases} \tag{22}$$

Clearly, the solution of (22) is given by

$$x(t) = \begin{cases} \Theta(t, x(t)) \dfrac{{}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4),\kappa;\varpi}_{0+} \dfrac{x(0)}{\Theta(t, x(0))}}{\Pi^{\varpi}_{\varkappa_4}(t,0)\Gamma_{\kappa}(\kappa\varkappa_4)} e^{-\varkappa_3(\varpi(t)-\varpi(0))} + \Theta(t, x(t)) \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+} (\chi + \jmath) \right)(t), \\ \vartheta(t) + \nu_1, \quad t \in [-\lambda, 0], \\ \tilde{\vartheta}(t) + \nu_2, \quad t \in \left[b, b + \tilde{\lambda}\right]. \end{cases}$$

where χ be a function satisfying the functional equation

$$\chi(t) = \Psi(t, x^t(\cdot), x(\mu t), \chi(t)).$$

Hence, for each $t \in (0, b]$, we have

$$\begin{aligned} &|x(t) - y(t)| \\ &\le |\Theta(t, x(t))| \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+} |\chi(\gamma) - w(\gamma)| \right)(t) + |\Theta(t, x(t))| \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+} \jmath \right)(t) \\ &\quad + |\Theta(t, x(t)) - \Theta(t, y(t))| \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+} |w(\gamma)| \right)(t) \\ &\quad + |\Theta(t, x(t)) - \Theta(t, y(t))| \frac{e^{-\varkappa_3(\varpi(t)-\varpi(0))}}{\Pi^{\varpi}_{\varkappa_4}(t, 0)\Gamma_{\kappa}(\kappa\varkappa_4)} \left[\frac{\wp_3 - \wp_2 \left({}^{T}_{\varkappa_3}\mathcal{J}^{\kappa(1-\varkappa_4)+\varkappa_1,\kappa;\varpi}_{0+} |w(\gamma)| \right)(b)}{\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}} \right] \\ &\le \epsilon \overline{\Theta}\, {}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+} \mathbb{E}^{\varkappa_1}_{\kappa} \left((\varpi(t) - \varpi(0))^{\frac{\varkappa_1}{\kappa}} \right) + \frac{(1 + \tilde{\zeta}_3)\tilde{\zeta}_1 \overline{\Theta}}{1 - \tilde{\zeta}_2} \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0^+} |x(\gamma) - y(\gamma)| \right)(t) \\ &\quad + \left[\frac{\overline{\Psi}\zeta_3 |\wp_2| \widehat{\varkappa_3}^2 (\varpi(b) - \varpi(0))^{\frac{\varkappa_1}{\kappa}}}{\Gamma_{\kappa}(\kappa\varkappa_4)\Gamma_{\kappa}(2\kappa - \kappa\varkappa_4 + \varkappa_1) \left| \wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))} \right|} \right. \\ &\quad \left. + \frac{\overline{\Psi}\zeta_3 \widehat{\varkappa_3} (\varpi(b) - \varpi(0))^{\frac{\varkappa_1}{\kappa}}}{\Gamma_{\kappa}(\varkappa_1 + \kappa\varkappa_4)} \right] |x(t) - y(t)|, \end{aligned}$$

where for $\tilde{\zeta}_3 \in \mathbb{R}^+$, let

$$\|x^t - y^t\|_{[-\lambda, \tilde{\lambda}]} \le \tilde{\zeta}_3 |x(t) - y(t)|, \text{ for all } t \in (0, b].$$

Then,

$$\begin{aligned} |x(t) - y(t)| &\le \epsilon \frac{\overline{\Theta}}{1 - \mathcal{L}_1}\, {}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0+} \mathbb{E}^{\varkappa_1}_{\kappa} \left((\varpi(t) - \varpi(0))^{\frac{\varkappa_1}{\kappa}} \right) \\ &\quad + \frac{(1 + \tilde{\zeta}_3)\tilde{\zeta}_1 \overline{\Theta}}{(1 - \mathcal{L}_1)(1 - \tilde{\zeta}_2)} \left({}^{T}_{\varkappa_3}\mathcal{J}^{\varkappa_1,\kappa;\varpi}_{0^+} |x(\gamma) - y(\gamma)| \right)(t), \end{aligned}$$

where

$$\mathcal{L}_1 = \left[\frac{\overline{\Psi}\zeta_3|\wp_2|\widehat{\varkappa_3}^2\,(\varpi(b)-\varpi(0))^{1-\varkappa_4+\frac{\varkappa_1}{\kappa}}}{\Gamma_\kappa(\kappa\varkappa_4)\Gamma_\kappa(2\kappa-\kappa\varkappa_4+\varkappa_1)\left|\wp_1+\wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|}+\frac{\overline{\Psi}\zeta_3\widehat{\varkappa_3}\,(\varpi(b)-\varpi(0))^{1-\varkappa_4+\frac{\varkappa_1}{\kappa}}}{\Gamma_\kappa(\varkappa_1+\kappa\varkappa_4)}\right].$$

Using Lemma 2.8 and Lemma 2.13, we get

$$\begin{aligned}|x(t)-y(t)| &\le \epsilon\frac{\overline{\Theta}}{(1-\mathcal{L}_1)}\widehat{\varkappa_3}\left[\mathbb{E}_\kappa^{\varkappa_1}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)-1\right]\\ &\quad+\frac{(1+\tilde{\zeta}_3)\tilde{\zeta}_1\overline{\Theta}}{(1-\mathcal{L}_1)(1-\tilde{\zeta}_2)}\int_0^t \varpi'(\gamma)\,e^{-\varkappa_3(\varpi(t)-\varpi(\gamma))}\bar{\Pi}_{\varkappa_1}^{\kappa,\varpi}(t,\gamma)|x(\gamma)-y(\gamma)|d\gamma.\end{aligned}$$

By applying Theorem 2.14, we obtain

$$\begin{aligned}&|x(t)-y(t)|\\ &\le \frac{\overline{\Theta}}{(1-\mathcal{L}_1)}\epsilon\widehat{\varkappa_3}\mathbb{E}_\kappa^{\varkappa_1}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)\\ &\quad+\int_0^t\sum_{i=1}^{\infty}\left(\frac{\overline{\Theta}(1+\tilde{\zeta}_3)\tilde{\zeta}_1\widehat{\varkappa_3}}{(1-\mathcal{L}_1)(1-\tilde{\zeta}_2)}\right)^i\varpi'(\gamma)\,\bar{\Pi}_{i\varkappa_1}^{\kappa,\varpi}(t,\gamma)\frac{\overline{\Theta}}{(1-\mathcal{L}_1)}\epsilon\widehat{\varkappa_3}\mathbb{E}_\kappa^{\varkappa_1}\left((\varpi(\gamma)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)d\gamma.\end{aligned}$$

Thus,

$$|x(t)-y(t)| \le \epsilon\frac{\overline{\Theta}}{(1-\mathcal{L}_1)}\widehat{\varkappa_3}\mathbb{E}_\kappa^{\varkappa_1}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)\mathbb{E}_\kappa^{\varkappa_1}\left[\frac{\overline{\Theta}(1+\tilde{\zeta}_3)\tilde{\zeta}_1\widehat{\varkappa_3}}{(1-\mathcal{L}_1)(1-\tilde{\zeta}_2)}\,(\varpi\,(b)-\varpi\,(0))^{\frac{\varkappa_1}{\kappa}}\right].$$

For each $t\in[-\lambda,0]$, we have

$$\begin{aligned}|x(t)-\vartheta(t)| &\le |\nu_1|\\ &\le \epsilon\Delta_1.\end{aligned}$$

For each $t\in[b,b+\tilde{\lambda}]$, we have

$$\begin{aligned}|x(t)-\tilde{\vartheta}(t)| &\le |\nu_2|\\ &\le \epsilon\Delta_2.\end{aligned}$$

Then, for each $t\in[-\lambda,b+\tilde{\lambda}]$, we have

$$|x(t)-y(t)| \le a_{\mathbb{E}_\kappa^{\varkappa_1}}\epsilon\left[\mathbb{E}_\kappa^{\varkappa_1}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)+\Delta_1+\Delta_2\right],$$

where

$$a_{\mathbb{E}_\kappa^{\varkappa_1}} = 1+\frac{\overline{\Theta}}{(1-\mathcal{L}_1)}\widehat{\varkappa_3}\mathbb{E}_\kappa^{\varkappa_1}\left[\frac{\overline{\Theta}(1+\tilde{\zeta}_3)\tilde{\zeta}_1\widehat{\varkappa_3}}{(1-\mathcal{L}_1)(1-\tilde{\zeta}_2)}\,(\varpi\,(b)-\varpi\,(0))^{\frac{\varkappa_1}{\kappa}}\right].$$

Hence, the problem (1)-(4) is κ-M-L-U-H stable with respect to $\mathbb{E}_\kappa^{\varkappa_1}\left((\varpi(t)-\varpi(0))^{\frac{\varkappa_1}{\kappa}}\right)$. If we set $v(\epsilon)=a_{\mathbb{E}_\kappa^{\varkappa_1}}\epsilon$, then the problem (1)-(4) is also generalized κ-M-L-U-H stable. □

5. Examples

Example 5.1. *By taking $\varkappa_2=\varkappa_1=\mu=\frac{1}{2}$, $\varkappa_3=1$, $\kappa=\frac{1}{2}$, $\varpi(t)=t^2$, $b=1$, $\wp_1=\wp_2=1$, $\lambda=\tilde{\lambda}=2$ and $\wp_3=e$, from the problem* (1)-(4)*, we obtain the following:*

$$\left({}^{TH}_{\frac{1}{2}}\mathcal{D}_{1^+}^{\frac{1}{2},\frac{1}{2},1;t^2}\frac{x(s)}{\Theta(s,x(s))}\right)(t)=\Psi\left(t,x^t(\cdot),x(\tfrac{t}{2}),\left({}^{TH}_{\frac{1}{2}}\mathcal{D}_{1^+}^{\frac{1}{2},\frac{1}{2},1;t^2}\frac{x(s)}{\Theta(t,x(s))}\right)(t)\right),\quad t\in(0,1], \tag{23}$$

$$\left({}_{2}^{T}\mathcal{J}_{1^+}^{\frac{2}{9},\frac{2}{3};\varpi}x\right)(0^+) + \left({}_{2}^{T}\mathcal{J}_{1^+}^{\frac{2}{9},\frac{2}{3};\varpi}x\right)(1) = e, \tag{24}$$

$$x(t) = \vartheta(t), \;\; t \in [-2, 0], \; \lambda > 0, \tag{25}$$

$$x(t) = \tilde{\vartheta}(t), \;\; t \in [1, 3], \; \tilde{\lambda} > 0, \tag{26}$$

where $F = [0, 1]$, $\varkappa_4 = \frac{1}{\kappa}(\varkappa_2(\kappa - \varkappa_1) + \varkappa_1) = 1$ *and*

$$\Psi(t, x, y) = \frac{|\cos(t)| + \|x\|_{[-\lambda,\tilde{\lambda}]} + y + z}{315e^{-t+3}(1 + |\cos(t)| + \|x\|_{[-\lambda,\tilde{\lambda}]} + y + z)}, \; t \in F, \; x \in C_{1;t^2}([-2, 2], \mathbb{R}), \; y, z \in \mathbb{R},$$

and

$$\Theta(t, x) = \frac{\ln(t + e)}{512e^{-t+3}(1 + |x|)} + \frac{1}{33}, \; t \in F, \; x \in \mathbb{R}.$$

We have

$$C_{1;t^2}(F) = \{x : (0, 1] \to \mathbb{R} : x \in C(F, \mathbb{R})\},$$

and

$$\mathbb{F} = \left\{x : [-2, 3] \to \mathbb{R} : \; x|_{[-2,0]} \in C, \, x|_{[2,3]} \in \tilde{C} \text{ and } \; x|_{(0,1]} \in C_{1;t^2}(F, \mathbb{R})\right\}$$

It is clear that the functions Ψ *and* Θ *are continuous on* F*. Then, the condition (*Ax1*) is satisfied.*
For each $x, y \in C_{1;t^2}([-2, 2], \mathbb{R})$ *and* $\bar{x}, \bar{y}, z, \bar{z} \in \mathbb{R}$ *and* $t \in F$*, we have*

$$|\Psi(t, x, \bar{x}, z) - \Psi(t, y, \bar{y}, \bar{z})| \leq \frac{1}{315e^{-t+3}}\left(\|x - y\|_{[-\lambda,\tilde{\lambda}]} + |\bar{x} - \bar{y}| + |z - \bar{z}|\right), \; t \in F,$$

and

$$|\Theta(t, x) - \Theta(t, y)| \leq \frac{\ln(t + e)}{512e^{-t+3}}|x - y|, \; t \in F,$$

*and so the condition (*Ax2*) is satisfied with* $\zeta_1 = \zeta_2 = \dfrac{1}{315e^3}$ *and* $\zeta_3 = \dfrac{\ln(1 + e)}{512e^3}$.
For each $x \in C_{1;t^2}([-2, 2], \mathbb{R})$ *and* $\bar{x}, z \in \mathbb{R}$ *and* $t \in F$*, we have*

$$|\Psi(t, x, \bar{x}, z)| \leq \frac{1}{315e^{-t+3}}\left(\|x\|_{[-\lambda,\tilde{\lambda}]} + |\bar{x}| + |z|\right), \; t \in F,$$

and

$$|\Theta(t, x)| \leq \frac{\ln(t + e)}{512e^{-t+3}}|x| + \frac{1}{33}, \; t \in F,$$

*and so the condition (*Ax3*) is satisfied with* $\overline{\Psi} = \dfrac{1}{315e^3}$ *and* $\overline{\Theta} = \dfrac{\ln(1 + e)}{512e^3} + \dfrac{1}{33}$.
Also, the condition (18) of Theorem 3.3 is satisfied. Indeed, we have

$$\begin{aligned}
\mathcal{L}_1 &= \left[\frac{|\wp_2|\widehat{\varkappa_3}^2\,(\varpi(b) - \varpi(0))^{1-\varkappa_4+\frac{\varkappa_1}{\kappa}}}{\Gamma_\kappa(\kappa\varkappa_4)\Gamma_\kappa(2\kappa - \kappa\varkappa_4 + \varkappa_1)\left|\wp_1 + \wp_2 e^{-\varkappa_3(\varpi(b)-\varpi(0))}\right|} + \frac{\widehat{\varkappa_3}\,(\varpi(b) - \varpi(0))^{1-\varkappa_4+\frac{\varkappa_1}{\kappa}}}{\Gamma_\kappa(\varkappa_1 + \kappa\varkappa_4)}\right] \\
&\quad \times \left(\frac{2\overline{\Theta}\zeta_1}{1 - \zeta_2} + \overline{\Psi}\zeta_3\right) \\
&= \left[\frac{2\ln(1 + e)}{512e^3(315e^3 - 1)} + \frac{2}{10395e^3 - 33} + \frac{\ln(1 + e)}{315e^3 \times 512e^3}\right] \\
&\quad \times \left[\frac{1}{\Gamma_{\frac{1}{2}}(\frac{1}{2})\Gamma_{\frac{1}{2}}(1)\,(1 + e^{-1})} + \frac{1}{\Gamma_{\frac{1}{2}}(1)}\right] \\
&= \left[\frac{2\ln(1 + e)}{512e^3(315e^3 - 1)} + \frac{2}{10395e^3 - 33} + \frac{\ln(1 + e)}{315e^3 \times 512e^3}\right] \times \left[\frac{2}{1 + e^{-1}} + 2\right] \\
&\approx 3.33786651264425 \cdot 10^{-5}.
\end{aligned}$$

Then the problem (23)-(26) *has a unique solution in* $C_{1;t^2}([0, 1], \mathbb{R})$ *and is* κ*-M-L-U-H stable with respect to* $\mathbb{E}_{\frac{1}{2}}^{\frac{1}{2}}\left(t^2\right)$.

Declarations

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