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Fusion frame, relay fusion frame and signal reconstruction

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Abstract. In this paper, we demonstrate that any signal exhibiting a certain sparse pattern can be recovered in a stable and resilient manner through the utilization of the fusion frame approach. The theoretical analysis highlights that the deviation of the approximate solution is effectively controlled. Furthermore, the adoption of different norms contributes to further reinforcing the guarantees of robustness and stability. Driven by the ideas of compressed sensing and fusion frames, we extend the setting to relay fusion frames. With the help of operator theory, we provide several recovery guarantee conditions based on the relay fusion frames. Finally, the relationship between relay fusion frames and compressed sensing is elucidated.

1. Introduction

Compressed sensing is an emerging field in signal processing that enables signal acquisition using very few measurements compared to the signal dimension, as long as the signal is sparse in some basis. This is based on the structural assumption such signals are satisfying—having a sparse and redundant representation over a specific dictionary. The field of compressed sensing was initiated with the papers [5] by Candès, Romberg and Tao and [14] by Donoho who coined the term compressed sensing.

The compressed sensing problem [15] consists in reconstructing an s-sparse vector $\mathbf{x} \in \mathbb{K}^N$ from

$$\mathbf{y} = \mathbf{A}\mathbf{x},$$

where $\mathbf{A} \in \mathbb{K}^{m\times N}$ is the so-called measurement matrix. Here $\mathbb{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$. With $m < N$, this system of linear equations is underdetermined, but the sparsity assumption hopefully helps in identifying the original vector $\mathbf{x}$. In a traditional sampling system, reconstructing a vector $\mathbf{x} \in \mathbb{K}^N$ from its measurement vector $\mathbf{y} \in \mathbb{K}^m$ amounts to solving the ℓ^0-minimization problem

$$\underset{\mathbf{z}\in\mathbb{K}^N}{\text{minimize}}\ \|\mathbf{z}\|_0 \quad \text{subject to} \quad \|\mathbf{A}\mathbf{z}-\mathbf{y}\|_2 \le \eta,\ \eta \ge 0.$$

Unfortunately, ℓ^0-minimization, the ideal recovery scheme, is NP-hard in general [22], hence is infeasible. A very popular and by now well-understood method is basis pursuit denoising or ℓ^1-minimization, which consists in finding the minimizer of the problem

$$\underset{\mathbf{z}\in\mathbb{K}^N}{\text{minimize}}\ \|\mathbf{z}\|_1 \quad \text{subject to} \quad \|\mathbf{A}\mathbf{z}-\mathbf{y}\|_2 \le \eta,\ \eta \ge 0. \tag{BPDN}$$

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It is well known that for a given sparsity s of the signal $\mathbf{x}$, the number of random subgaussian linear measurements needs to grow as $m \gtrsim s\log(N/s)$ for the solution of (BPDN) to be a good enough approximation to $\mathbf{x}$. However, when the number of measurements m cannot be chosen based on the sparsity of the signals to recover, there is a limit on the sparsity of the vectors that can be recovered by $s \lesssim m\log(N/m)$. In other words, the problems arise when the signals being sampled are too dense for the usual mathematical theories while the quality of the sensors is constrained. The idea of splitting dense information into subchannels and utilizing fusion frames to fuse local estimates overcomes this limitation. The local pieces of information are computed as solutions to the problems

$$\underset{\mathbf{z}\in\mathbb{K}^N}{\text{minimize}}\ \|\mathbf{z}\|_1 \quad \text{subject to}\ \|\mathbf{A}\pi_{W_i}\mathbf{z}-\mathbf{y}^{(i)}\|_2 \le \eta_i,\ \eta_i \ge 0. \tag{$\mathcal{P}_{1,\eta_i}$}$$

where $\mathbf{A} \in \mathbb{K}^{m\times N}$ is the measurement matrix and $\pi_{W_i} \in \mathbb{K}^{N\times N}$ is the orthogonal projection onto subspace W_i. In the perfect measurements (noiseless) case, the problem is solved by the basis pursuit

$$\underset{\mathbf{z}\in\mathbb{K}^N}{\text{minimize}}\ \|\mathbf{z}\|_1 \quad \text{subject to}\ \mathbf{A}\pi_{W_i}\mathbf{z}=\mathbf{y}^{(i)}. \tag{$\mathcal{P}_{1,0}$}$$

Frames, fusion frames and compressed sensing are hot topics today because of their broad applications to problems in signal processing and much more. We refer the reader to some recent tutorials on the subjects and their references [1, 3, 4, 8, 11, 12, 16]. In the remainder of this introduction we state the main definitions and notations. Some of what we describe in the following is known and is standard in the literature.

1.1. *Frames and fusion frames*

Letting $\mathbb{I}$ be a countable index set, a sequence of vectors $\{f_i\}_{i\in\mathbb{I}}$ lying in some Hilbert space $\mathcal{H}$ is said to be a frame [18] for $\mathcal{H}$ if there exist frame bounds $\alpha, \beta > 0$ such that

$$\alpha\|f\|^2 \le \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \le \beta\|f\|^2,\ \forall f\in\mathcal{H}. \tag{1}$$

More generally, $\{f_i\}_{i\in\mathbb{I}}$ is called a Bessel sequence if at least the upper bound in (1) is satisfied. In particular, the Bessel sequence $\{g_i\}_{i\in\mathbb{I}}$ is called a dual of the frame $\{f_i\}_{i\in\mathbb{I}}$ if the following formula holds, for all $f\in\mathcal{H}$:

$$f=\sum_{i\in\mathbb{I}}\langle f, g_i\rangle f_i.$$

Fusion frames are generalizations of frames that provide a richer description of signal spaces, which were introduced in [7] (under the name frames of subspaces) and further developed in [9], and have quickly become a major tool in the implementation of distributed systems [10, 20, 21]. It can be regarded as a frame-like collection of subspaces in a Hilbert space, which clearly generalizes classical vector frames.

Definition 1.1. *Let $\mathbb{I}$ be a countable (or finite) index set and $\{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of a Hilbert space $\mathcal{H}$. Let $\{w_i\}_{i\in\mathbb{I}} \in \ell^\infty(\mathbb{I})$ such that $w_i > 0$ for every $i\in\mathbb{I}$. The family $\{(W_i, w_i)\}_{i\in\mathbb{I}}$ is said to be a fusion frame for $\mathcal{H}$ if there exist numbers $0<\alpha\le\beta<\infty$ which satisfy that*

$$\alpha\|f\|^2 \le \sum_{i\in\mathbb{I}} w_i^2\|\pi_{W_i}(f)\|^2 \le \beta\|f\|^2,\ \forall f\in\mathcal{H}, \tag{2}$$

where π_{W_i} is the orthogonal projection onto W_i. The constants α, β are called fusion frame bounds.

For the sake of brevity, we sometimes write $\mathcal{W}_w$ instead of $\{(W_i, w_i)\}_{i\in\mathbb{I}}$. Let $I_{\mathcal{H}}$ be the identity operator on Hilbert space $\mathcal{H}$. The decomposition of any signal $f\in\mathcal{H}$ according to a fusion frame $\mathcal{W}_w$ is given by the fusion frame measurements $\left\{w_i\pi_{W_i}f\right\}_{i\in\mathbb{I}}$. These completely characterize the signal f, which can be reconstructed from those by performing

$$f=\sum_{i\in\mathbb{I}} w_i S_{\mathcal{W}_w}^{-1}(w_i\pi_{W_i}f),$$

where $S_{\mathcal{W}_w}(f) = \sum_{i\in\mathbb{I}} w_i^2 \pi_{W_i}(f)$ is the fusion frame operator known to be self-adjoint and positive with $\alpha I_{\mathcal{H}} \leq S_{\mathcal{R}} \leq \beta I_{\mathcal{H}}$. Therefore, a distributed fusion processing is feasible in an elegant way. As presented above, given some local information $\mathbf{x}^{(i)} := \pi_{W_i}(\mathbf{x})$, for $i \in \mathbb{I}$, a vector can easily be reconstructed by applying the inverse fusion frame operator

$$\mathbf{x} := S_{\mathcal{W}_w}^{-1}\left(\sum_{i\in\mathbb{I}} w_i^2 \mathbf{x}^{(i)}\right).$$

For simplicity, fusion frame $\{(W_i, 1)\}_{i=1}^n$ will be abbreviated as $\mathcal{W}$ throughout the paper.

1.2. Distributed sparse and partial properties

We first introduce the notations $[N]$ for the set $\{1, ..., N\}$ and $| S |$ for the cardinality of a set S. Furthermore, we write $\overline{S}$ for the complement $[N]\backslash S$ of a set S in $[N]$. For a matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ and a subset $S \subset [N]$, we use the notation $\mathbf{A}_S$ to indicate the column submatrix of $\mathbf{A}$ consisting of the columns indexed by S. Similarly, for a vector $\mathbf{x} \in \mathbb{K}^N$ and a subset $S \subset [N]$, we denote by $\mathbf{x}_S$ the vector in $\mathbb{K}^N$ which coincides with $\mathbf{x}$ on the entries in S and is zero on the entries outside S.

The sparsity is used in the compressed sensing literature as a underlying hypothesis for solving the resulting underdetermined systems of linear equations. A vector $\mathbf{x} \in \mathbb{K}^N$ is called s-sparse if it has at most s nonzero entries, in other words, if

$$\|\mathbf{x}\|_0 :=| \{j : x_j \neq 0\} |$$

is smaller than or equal to s.

Definition 1.2. *A signal $\mathbf{x} \in \mathbb{K}^N$ is said to be $\mathbf{s} = (s_1, \cdots, s_n)$-distributed sparse with respect to a fusion frame $\mathcal{W}$, if $\|\pi_{W_i}(\mathbf{x})\|_0 \leq s_i$, for every $1 \leq i \leq n$. $\mathbf{s}$ is called the sparsity pattern of $\mathbf{x}$ with respect to $\mathcal{W}$.*

In practice, one encounters vectors that are not exactly s-sparse but compressible in the sense that they are well approximated by sparse ones. This is quantified by the ℓ^p-error of best s-term approximation to $\mathbf{x}$ given by

$$\sigma_s(\mathbf{x})_p := \inf_{\|\mathbf{z}\|_0\leq s} \|\mathbf{x} - \mathbf{z}\|_p, \ p > 0.$$

It is well known that for $q > p > 0$ and any $\mathbf{x} \in \mathbb{K}^N$,

$$\sigma_s(\mathbf{x})_p \leq \frac{1}{s^{1/p-1/q}} \|\mathbf{x}\|_p. \tag{3}$$

As it will be useful later, we also need to introduce the local best approximations.

Definition 1.3. *Let $\mathcal{W}$ be a fusion frame and let $\mathbf{x} \in \mathbb{K}^N$. For $p > 0$ and a sparsity pattern $\mathbf{s} = (s_1, \cdots, s_n)$ with $s_i \in \mathbb{N}$ for all $1 \leq i \leq n$, the ℓ^p-error of best $\mathbf{s}$-term approximations is defined as the vector*

$$\sigma_{\mathbf{s}}^{\mathcal{W}}(\mathbf{x})_p := \left(\sigma_{s_1}(P_1\mathbf{x}), \sigma_{s_2}(P_2\mathbf{x}), \ldots, \sigma_{s_n}(P_n\mathbf{x})\right)^{\mathrm{T}}.$$

In the compressed sensing literature, the null space property (NSP) has been used as a necessary and sufficient condition for the sparse recovery problem through ℓ^0-minimization. The definition below extends this concept to the context of distributed sparsity of fusion frames. It requires the NSP property to be valid for all local subspaces up to a certain local sparsity level with respect to sparsity pattern $\mathbf{s}$. To this end, we equip each subspace W_i of $\mathbb{K}^N$ with a sub-index set Δ_i with $\Delta_i \subseteq [N]$ such that $\pi_{W_i}(\mathbf{x}) = \mathbf{x}_{\Delta_i}$. Clearly, if $\{W_i\}_i$ forms a fusion frame for $\mathbb{K}^N$, then $\bigcup_{i=1}^n \Delta_i = [N]$ and $| \Delta_i |:= \mathrm{rank}(\pi_{W_i}) = \dim W_i$. In this case, we have that the sparsity of each local signal is at most $N_i :=| \Delta_i |$.

Definition 1.4. *(Distributed partial null space property (DP-NSP)). Let $\mathcal{W}$ be a fusion frame for $\mathbb{K}^N$ and let $\mathbf{s}$ be a sparsity pattern (with entries s_i). A sensing matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ is said to fulfill the DP-NSP with sparsity pattern $\mathbf{s}$ with respect to $\mathcal{W}$ and constants $\rho_1, \cdots, \rho_n \in (0,1)$ and $\tau_1, \cdots, \tau_n > 0$ if*

$$\|(\pi_{W_i}\mathbf{v})_{S_i}\|_1 \le \rho_i \|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1 + \tau_i \|\mathbf{A}\mathbf{v}\|_2$$

for all $\mathbf{v} \in \mathbb{K}^N, 1 \le i \le n, S_i \subset \Delta_i$ with $| S_i | \le s_i$.

In compressed sensing, the analysis of recovery algorithms usually involves a quantity that measures the suitability of the measurement matrix. The restricted isometry property (RIP) [2, 6] is a very simple such measure of quality. The restricted isometry constant δ_s of a matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ is defined as the smallest $\delta \ge 0$ such that for all s-sparse $\mathbf{x}$,

$$(1-\delta)\|\mathbf{x}\|_2^2 \le \|\mathbf{A}\mathbf{x}\|_2^2 \le (1+\delta)\|\mathbf{x}\|_2^2. \tag{4}$$

In this case, we call $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfies RIP(s, δ). The following definition generalizes the concept of RIP to the distributed sparse signal model.

Definition 1.5. *(Partial-RIP (P-RIP)). Let $\mathcal{W}$ be a fusion frame for $\mathbb{K}^N$ and let $\mathbf{A} \in \mathbb{K}^{m\times N}$. Assume that $\mathbf{A}$ satisfies the RIP(s_i, δ_i) on W_i, with $\delta_i \in (0,1), i \in \{1, \dots, n\}$. Then, we say that $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfies the partial RIP with respect to $\mathcal{W}$, with bounds $\delta_1, \dots, \delta_n$ and sparsity pattern $\mathbf{s} = (s_1, \cdots, s_n)$.*

1.3. Outline

The subsequent sections of this paper are structured as follows. In Section 2, we establish that leveraging the fusion frame methodology, any signal exhibiting a sparsity pattern $\mathbf{s}$ can be retrieved in a stable and resilient fashion through the utilization of the aforementioned tools. Our theoretical analysis reveals that the deviation in the approximation of the solution is effectively managed. Moreover, the robustness and stability guarantees are further fortified by substituting the ℓ^1-error bound with an ℓ^p-error bound, where $p \ge 1$. Motivated by the principles of compressed sensing and fusion frames, Section 3 reintroduces the concept of relay fusion frames, which expands their applicability to encompass the recovery of arbitrary signals within the ambient space, without the necessity of a sparsity assumption. By harnessing the principles of operator theory, we formulate several recovery criteria that are firmly rooted within the relay fusion frame setting. Finally, we elucidate the relationship between relay fusion frames and compressed sensing.

2. Recovery based on fusion frames

Employing the aforementioned tools, our focus lies in the stable and robust recovery of any signal exhibiting a sparsity pattern $\mathbf{s}$, leveraging the fusion frame approach that was detailed in the preceding section.

2.1. DP-NSP based recovery

Theorem 2.1. *Let $\mathbf{A} \in \mathbb{K}^{m\times N}$ and $\mathcal{W}$ a fusion frame for $\mathbb{K}^N$ with lower fusion frame bound $\alpha > 0$ and fusion frame operator $S_{\mathcal{W}}$. Let $(\mathbf{y}^{(i)})_{i=1}^n$ be the linear measurements $\mathbf{y}^{(i)} = \mathbf{A}\pi_{W_i}\mathbf{x} + \mathbf{e}^{(i)}$ for some bounded noise vectors $\mathbf{e}^{(i)}$ such that $\|\mathbf{e}^{(i)}\|_2 \le \eta_i$, $i = 1, 2, \cdots, n$. Denote by $\widehat{\mathbf{x}^{(i)}}$ the solution to the local basis pursuit problem $(\mathcal{P}_{1,\eta_i})$. If the matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfies the DP-NSP with sparsity pattern $\mathbf{s}$ with constants $\rho_1, \cdots, \rho_n \in (0,1)$ and $\tau_1, \cdots, \tau_n > 0$ with respect to $\mathcal{W}$, then $\widehat{\mathbf{x}} = S_{\mathcal{W}}^{-1} \sum_i \widehat{\mathbf{x}^{(i)}}$ approximates $\mathbf{x}$ in the following sense:*

$$\|\widehat{\mathbf{x}} - \mathbf{x}\|_2 \le \frac{2}{\alpha}\left(\sum_{i=1}^n \frac{1+\rho_i}{1-\rho_i}\sigma_{s_i}(\pi_{W_i}\mathbf{x})_1 + \sum_{i=1}^n \frac{2\tau_i\eta_i}{1-\rho_i} \right). \tag{5}$$

Before turning to the proof of Theorem 2.1, we isolate the following observation, as it will also be needed later, which gives a characterization of a matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfying the DP-NSP with sparsity pattern $\mathbf{s}$.

Theorem 2.2. *The matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfies the DP-NSP with sparsity pattern $\mathbf{s}$ with constants $\rho_1, \cdots, \rho_n \in (0,1)$ and $\tau_1, \cdots, \tau_n > 0$ with respect to $\mathcal{W}$ if and only if*

$$\|\pi_{W_i}(\mathbf{z}-\mathbf{x})\|_1 \le \frac{1+\rho_i}{1-\rho_i}\Big(\|\pi_{W_i}\mathbf{z}\|_1 - \|\pi_{W_i}\mathbf{x}\|_1 + 2\|(\pi_{W_i}\mathbf{x})_{\overline{S_i}}\|_1\Big) + \frac{2\tau_i}{1-\rho_i}\|\mathbf{A}(\mathbf{z}-\mathbf{x})\|_2 \tag{6}$$

for all $\mathbf{x}, \mathbf{z} \in \mathbb{K}^N$, $1 \le i \le n$, $S_i \subset \Delta_i$ with $|S_i| \le s_i$.

Proof. First, we assume that the matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfies inequality (6) for all vectors $\mathbf{x}, \mathbf{z} \in \mathbb{K}^N$. Fix arbitrary $i \in \{1, \cdots, n\}$. Then for any $\mathbf{v} \in \mathbb{K}^N$, taking $\mathbf{x} = -\mathbf{v}_{S_i}$ and $\mathbf{z} = \mathbf{v}_{\overline{S_i}}$ yields

$$\begin{aligned}\|\pi_{W_i}\mathbf{v}\|_1 &= \|\pi_{W_i}\mathbf{v}_{\Delta_i}\|_1\\ &\le \tfrac{1+\rho_i}{1-\rho_i}\Big(\|\pi_{W_i}\mathbf{v}_{\overline{S_i}}\|_1 - \|\pi_{W_i}\mathbf{v}_{S_i}\|_1 + 2\|(\pi_{W_i}\mathbf{v}_{S_i})_{\overline{S_i}}\|_1\Big) + \tfrac{2\tau_i}{1-\rho_i}\|\mathbf{A}\mathbf{v}_{\Delta_i}\|_2\\ &\le \tfrac{1+\rho_i}{1-\rho_i}\Big(\|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1 - \|(\pi_{W_i}\mathbf{v})_{S_i}\|_1\Big) + \tfrac{2\tau_i}{1-\rho_i}\|\mathbf{A}\mathbf{v}\|_2.\end{aligned}$$

Rearranging the terms gives

$$(1-\rho_i)\Big(\|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1 + \|(\pi_{W_i}\mathbf{v})_{S_i}\|_1\Big) \le (1+\rho_i)\Big(\|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1 - \|(\pi_{W_i}\mathbf{v})_{S_i}\|_1\Big) + 2\tau_i\|\mathbf{A}\mathbf{v}\|_2,$$

that is to say

$$\|(\pi_{W_i}\mathbf{v})_{S_i}\|_1 \le \rho_i\|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1 + \tau_i\|\mathbf{A}\mathbf{v}\|_2.$$

This is the DP-NSP with sparsity pattern $\mathbf{s}$ with constants $\rho_1, \cdots, \rho_n \in (0,1)$ and $\tau_1, \cdots, \tau_n > 0$ with respect to $\mathcal{W}$.

Conversely, we assume that the matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfies the DP-NSP with sparsity pattern $\mathbf{s}$ with constants $\rho_1, \cdots, \rho_n \in (0,1)$ and $\tau_1, \cdots, \tau_n > 0$ with respect to $\mathcal{W}$. For $\mathbf{x}, \mathbf{z} \in \mathbb{K}^N$, setting $\mathbf{v} := \mathbf{z} - \mathbf{x}$, the DP-NSP yields for any $i \in \{1, \cdots, n\}$

$$\|(\pi_{W_i}\mathbf{v})_{S_i}\|_1 \le \rho_i\|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1 + \tau_i\|\mathbf{A}\mathbf{v}\|_2. \tag{7}$$

Observe that

$$\|\pi_{W_i}\mathbf{x}\|_1 = \|(\pi_{W_i}\mathbf{x})_{\overline{S_i}}\|_1 + \|(\pi_{W_i}\mathbf{x})_{S_i}\|_1 \le \|(\pi_{W_i}\mathbf{x})_{\overline{S_i}}\|_1 + \|(\pi_{W_i}\mathbf{x} - \pi_{W_i}\mathbf{z})_{S_i}\|_1 + \|(\pi_{W_i}\mathbf{z})_{S_i}\|_1,$$

$$\|(\pi_{W_i}\mathbf{x} - \pi_{W_i}\mathbf{z})_{\overline{S_i}}\|_1 \le \|(\pi_{W_i}\mathbf{x})_{\overline{S_i}}\|_1 + \|(\pi_{W_i}\mathbf{z})_{\overline{S_i}}\|_1.$$

Adding these two inequalities together gives

$$\|(\pi_{W_i}\mathbf{x} - \pi_{W_i}\mathbf{z})_{\overline{S_i}}\|_1 \le \|\pi_{W_i}\mathbf{z}\|_1 - \|\pi_{W_i}\mathbf{x}\|_1 + \|(\pi_{W_i}\mathbf{x} - \pi_{W_i}\mathbf{z})_{S_i}\|_1 + 2\|(\pi_{W_i}\mathbf{x})_{\overline{S_i}}\|_1. \tag{8}$$

Combining inequalities (7) and (8) gives

$$(1-\rho_i)\|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1 \le \|\pi_{W_i}\mathbf{z}\|_1 - \|\pi_{W_i}\mathbf{x}\|_1 + 2\|(\pi_{W_i}\mathbf{x})_{\overline{S_i}}\|_1 + \tau_i\|\mathbf{A}\mathbf{v}\|_2.$$

Now using the DP-NSP once again, we derive

$$\begin{aligned}\|\pi_{W_i}\mathbf{v}\|_1 &= \|(\pi_{W_i}\mathbf{v})_{S_i}\|_1 + \|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1\\ &\le (1+\rho_i)\|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1 + \tau_i\|\mathbf{A}\mathbf{v}\|_2\\ &\le \tfrac{1+\rho_i}{1-\rho_i}\Big(\|\pi_{W_i}\mathbf{z}\|_1 - \|\pi_{W_i}\mathbf{x}\|_1 + 2\|(\pi_{W_i}\mathbf{x})_{\overline{S_i}}\|_1\Big) + \tfrac{2\tau_i}{1-\rho_i}\|\mathbf{A}(\mathbf{z}-\mathbf{x})\|_2,\end{aligned}$$

which is the desired inequality. □

Proof of Theorem 2.1. Let $\widehat{\mathbf{x}^{(i)}}, i \in \{1, \cdots, n\}$ be solutions to the noisy $(\mathcal{P}_{1,\eta_i})$ basis pursuit problems. By applying the inverse fusion frame operator and fusion processing, it follows that $\widehat{\mathbf{x}} = S_{\mathcal{W}}^{-1} \sum_i \widehat{\mathbf{x}^{(i)}}$. This yields

$$\|\widehat{\mathbf{x}} - \mathbf{x}\|_2 \leq \frac{1}{\alpha} \sum_{i=1}^{n} \left\| \pi_{W_i}\mathbf{x} - \widehat{\mathbf{x}^{(i)}} \right\|_2 \leq \frac{1}{\alpha} \sum_{i=1}^{n} \left\| \pi_{W_i}\mathbf{x} - \widehat{\mathbf{x}^{(i)}} \right\|_1. \tag{9}$$

For each $i \in \{1, \cdots, n\}$, we estimate the error on subspace W_i in the ℓ^1 sense. It is easy to see that $\pi_{W_i}\widehat{\mathbf{x}^{(i)}} = \widehat{\mathbf{x}^{(i)}}$. Put $\mathbf{v}_i := \pi_{W_i}(\mathbf{x} - \widehat{\mathbf{x}^{(i)}}) = \pi_{W_i}\mathbf{x} - \widehat{\mathbf{x}^{(i)}} = \pi_{W_i}(\pi_{W_i}\mathbf{x}) - \widehat{\mathbf{x}^{(i)}}$. Now employing the inequality (6), we have

$$\|\mathbf{v}_i\|_1 \leq \frac{1+\rho_i}{1-\rho_i}\Big(\|\widehat{\mathbf{x}^{(i)}}\|_1 - \|\pi_{W_i}\mathbf{x}\|_1 + 2\|(\pi_{W_i}\mathbf{x})_{\overline{S_i}}\|_1\Big) + \frac{2\tau_i}{1-\rho_i}\|\mathbf{A}\mathbf{v}_i\|_2. \tag{10}$$

Take S_i to be the set of best s_i components of $\mathbf{x}$ supported on W_i so that $\|(\pi_{W_i}\mathbf{x})_{\overline{S_i}}\|_1 = \sigma_{s_i}(\pi_{W_i}\mathbf{x})_1$. Since $\widehat{\mathbf{x}^{(i)}}$ is a minimizer solution of $(\mathcal{P}_{1,\eta_i})$, $\|\widehat{\mathbf{x}^{(i)}}\|_1 \leq \|\pi_{W_i}\mathbf{x}\|_1$. This implies that

$$\|\mathbf{v}_i\|_1 \leq \frac{2(1+\rho_i)}{1-\rho_i}\sigma_{s_i}(\pi_{W_i}\mathbf{x})_1 + \frac{2\tau_i}{1-\rho_i}\|\mathbf{A}\mathbf{v}_i\|_2. \tag{11}$$

Summing up the contributions for all $i \in \{1, \cdots, n\}$ we obtain

$$\sum_{i=1}^{n} \|\mathbf{v}_i\|_1 = \sum_{i=1}^{n} \left\| \pi_{W_i}\mathbf{x} - \widehat{\mathbf{x}^{(i)}} \right\|_1 \leq \sum_{i=1}^{n} \frac{2(1+\rho_i)}{1-\rho_i}\sigma_{\mathbf{s}}(\pi_{W_i}\mathbf{x})_1 + \sum_{i=1}^{n} \frac{4\tau_i\eta_i}{1-\rho_i}. \tag{12}$$

The last step involved the inequality $\|\mathbf{A}\mathbf{v}_i\|_2 \leq 2\eta_i$, which follows from the optimization constraint as

$$\|\mathbf{A}\mathbf{v}_i\|_2 \leq \|\mathbf{A}\pi_{W_i}\widehat{\mathbf{x}^{(i)}} - \mathbf{y}^{(i)}\|_2 + \|\mathbf{y}^{(i)} - \mathbf{A}\pi_{W_i}\mathbf{x}\|_2 \leq 2\eta_i.$$

Finally, combining inequalities (9) and (12), we arrive at the desired result. □

Remark 2.3. *Note that if we let* $\vec{\rho} = \left(\frac{1+\rho_i}{1-\rho_i}\right)_{i=1}^{n}$, $\vec{\tau} = \left(\frac{2\tau_i}{1-\rho_i}\right)_{i=1}^{n}$, $\vec{\eta} = (\eta_i)_{i=1}^{n}$, *then error estimate* (5) *can be simply represented as follows*

$$\|\widehat{\mathbf{x}} - \mathbf{x}\|_2 \leq \frac{2}{\alpha}\Big(\langle \vec{\rho}, \sigma_{\mathbf{s}}^{\mathcal{W}}(\mathbf{x})_1 \rangle + \langle \vec{\tau}, \vec{\eta} \rangle\Big). \tag{13}$$

Assuming perfect measurements (that is, $\eta_i = 0$*), the error bound* (13) *yields*

$$\|\widehat{\mathbf{x}} - \mathbf{x}\|_2 \leq \frac{2}{\alpha}\langle \vec{\rho}, \sigma_{\mathbf{s}}^{\mathcal{W}}(\mathbf{x})_1 \rangle. \tag{14}$$

2.2. RP-NSP based recovery

We now turn to another main result of this section. It enhances the previous stability and robustness result by replacing the ℓ^1-error estimate by an ℓ^p-error estimate for $p \geq 1$. A strengthening of the DP-NSP is required.

Definition 2.4. *(Robust and stable partial null space property (RP-NSP)). Let* n *be an integer and* $\mathcal{W}$ *a fusion frame for* $\mathbb{K}^N$*. Let* $\mathbf{s} = (s_1, \cdots, s_n)$ *be a sequence of non negative numbers representing the sparsity pattern with respect to* $\mathcal{W}$*. For a number* $q \geq 1$*, a sensing matrix* $\mathbf{A} \in \mathbb{K}^{m\times N}$ *is said to satisfy the* ℓ^q*-RP-NSP with pattern* $\mathbf{s}$ *with respect to* $\mathcal{W}$ *and with constants* $\rho_1, \cdots, \rho_n \in (0,1)$ *and* $\tau_1, \cdots, \tau_n > 0$ *if*

$$\|(\pi_{W_i}\mathbf{v})_{S_i}\|_q \leq \frac{\rho_i}{s_i^{1-1/q}}\|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1 + \tau_i\|\mathbf{A}\mathbf{v}\|_2.$$

for all $\mathbf{v} \in \mathbb{K}^N, 1 \leq i \leq n, S_i \subset \Delta_i$ *with* $|S_i| \leq s_i$*.*

Armed with this notion, we can now give the following recovery guarantee result.

Theorem 2.5. *Let $\mathbf{A} \in \mathbb{K}^{m\times N}$ and $\mathcal{W}$ a fusion frame for $\mathbb{K}^N$ with lower fusion frame bound $\alpha > 0$ and fusion frame operator $S_{\mathcal{W}}$. Let $(\mathbf{y}^{(i)})_{i=1}^n$ be the linear measurements $\mathbf{y}^{(i)} = \mathbf{A}\pi_{W_i}\mathbf{x} + \mathbf{e}^{(i)}$ for some bounded noise vectors $\mathbf{e}^{(i)}$ such that $\|\mathbf{e}^{(i)}\|_2 \le \eta_i$, $i = 1, 2, \cdots, n$. Denote by $\widehat{\mathbf{x}^{(i)}}$ the solution to the local basis pursuit problems $(\mathcal{P}_{1,\eta_i})$. If the matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfies the ℓ^q-RP-NSP with sparsity pattern $\mathbf{s}$ with constants $\rho_1, \cdots, \rho_n \in (0, 1)$ and $\tau_1, \cdots, \tau_n > 0$ with respect to $\mathcal{W}$, then $\widehat{\mathbf{x}} = S_{\mathcal{W}}^{-1}\sum_i \widehat{\mathbf{x}^{(i)}}$ approximates $\mathbf{x}$ in the following sense:*

$$\|\widehat{\mathbf{x}} - \mathbf{x}\|_p \le \frac{2}{\alpha}\Big(\langle \vec{\rho}_{\mathbf{s}}, \sigma_{\mathbf{s}}^{\mathcal{W}}(\mathbf{x})_1\rangle + \langle \vec{\tau}_{\mathbf{s}}, \vec{\eta}\rangle\Big), \quad 1 \le p \le q, \tag{15}$$

where $\vec{\rho}_{\mathbf{s}} = \Big(\frac{(1+\rho_i)^2}{s_i^{1-1/p}(1-\rho_i)}\Big)_{i=1}^n$, $\vec{\tau}_{\mathbf{s}} = \Big(\frac{s_i^{1/p-1/q}\tau_i(3+\rho_i)}{1-\rho_i}\Big)_{i=1}^n$, $\vec{\eta} = (\eta_i)_{i=1}^n$.

For the proof of Theorem 2.5, we establish the auxiliary result Lemma 2.6 below.

Lemma 2.6. *Given $1 \le p \le q$, suppose that the sensing matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfies the ℓ^q-RP-NSP with pattern $\mathbf{s}$ with respect to $\mathcal{W}$ and with constants $\rho_1, \cdots, \rho_n \in (0, 1)$ and $\tau_1, \cdots, \tau_n > 0$. Then*

$$\begin{aligned}\|\pi_{W_i}(\mathbf{z} - \mathbf{x})\|_p \quad &\le \tfrac{(1+\rho_i)^2}{s_i^{1-1/p}(1-\rho_i)}\Big(\|\pi_{W_i}\mathbf{z}\|_1 - \|\pi_{W_i}\mathbf{x}\|_1 + 2\sigma_{s_i}(\pi_{W_i}\mathbf{x})_1\Big) \\ &\quad + \tfrac{s_i^{1/p-1/q}\tau_i(3+\rho_i)}{1-\rho_i}\|\mathbf{A}(\mathbf{z} - \mathbf{x})\|_2\end{aligned} \tag{16}$$

for all $\mathbf{x}, \mathbf{z} \in \mathbb{K}^N$, $1 \le i \le n$, $S_i \subset \Delta_i$ with $|S_i| \le s_i$.

Proof. In view of inequality $\|\mathbf{v}_{S_i}\|_p \le s_i^{1/p-1/q}\|\mathbf{v}_{S_i}\|_q$ for all $\mathbf{v} \in \mathbb{K}^N$, $1 \le p \le q$, we observe that the ℓ^q-RP-NSP implies that, for any $1 \le i \le n$, $S_i \subset \Delta_i$ with $|S_i| \le s_i$,

$$\|(\pi_{W_i}\mathbf{v})_{S_i}\|_p \le \frac{\rho_i}{s_i^{1-1/p}}\|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1 + \tau_i s_i^{1/p-1/q}\|\mathbf{A}\mathbf{v}\|_2 \ \text{ for all } \ \mathbf{v} \in \mathbb{K}^N. \tag{17}$$

In particular, it holds

$$\|(\pi_{W_i}\mathbf{v})_{S_i}\|_1 \le \rho_i\|(\pi_{W_i}\mathbf{v})_{\overline{S_i}}\|_1 + \tau_i s_i^{1-1/q}\|\mathbf{A}\mathbf{v}\|_2$$

for all $\mathbf{v} \in \mathbb{K}^N$, $1 \le i \le n$, $S_i \subset \Delta_i$ with $|S_i| \le s_i$. Thus, for all $\mathbf{x}, \mathbf{z} \in \mathbb{K}^N$ and each $i \in \{1, \cdots, n\}$, applying Theorem 2.2 leads to

$$\|\pi_{W_i}(\mathbf{z} - \mathbf{x})\|_1 \le \frac{1 + \rho_i}{1 - \rho_i}\Big(\|\pi_{W_i}\mathbf{z}\|_1 - \|\pi_{W_i}\mathbf{x}\|_1 + 2\sigma_{s_i}(\pi_{W_i}\mathbf{x})_1\Big) + \frac{2\tau_i s_i^{1-1/q}}{1 - \rho_i}\|\mathbf{A}(\mathbf{z} - \mathbf{x})\|_2. \tag{18}$$

Then, choosing S_i as an index set of s_i largest entries of $\mathbf{z} - \mathbf{x}$, we use inequality (3) to notice that

$$\|\pi_{W_i}(\mathbf{z} - \mathbf{x})\|_p \le \|\Big(\pi_{W_i}(\mathbf{z} - \mathbf{x})\Big)_{\overline{S_i}}\|_p + \frac{1}{s_i^{1-1/p}}\|\pi_{W_i}(\mathbf{z} - \mathbf{x})\|_1.$$

In terms of (17), we derive

$$\begin{aligned}&\|\pi_{W_i}(\mathbf{z} - \mathbf{x})\|_p \\ &\le \tfrac{1}{s_i^{1-1/p}}\|\pi_{W_i}(\mathbf{z} - \mathbf{x})\|_1 + \tfrac{\rho_i}{s_i^{1-1/p}}\|\Big(\pi_{W_i}(\mathbf{z} - \mathbf{x})\Big)_{S_i}\|_1 + \tau_i s_i^{1/p-1/q}\|\mathbf{A}(\mathbf{z} - \mathbf{x})\|_2 \\ &\le \tfrac{1+\rho_i}{s_i^{1-1/p}}\|\pi_{W_i}(\mathbf{z} - \mathbf{x})\|_1 + \tau_i s_i^{1/p-1/q}\|\mathbf{A}(\mathbf{z} - \mathbf{x})\|_2 \\ &\le \tfrac{(1+\rho_i)^2}{s_i^{1-1/p}(1-\rho_i)}\Big(\|\pi_{W_i}\mathbf{z}\|_1 - \|\pi_{W_i}\mathbf{x}\|_1 + 2\sigma_{s_i}(\pi_{W_i}\mathbf{x})_1\Big) + \tfrac{s_i^{1/p-1/q}\tau_i(3+\rho_i)}{1-\rho_i}\|\mathbf{A}(\mathbf{z} - \mathbf{x})\|_2,\end{aligned}$$

which proves our claim. □

We now give a sketch of the proof of Theorem 2.5.

Proof of Theorem 2.5. Let us follow the strategy used in the proof of Theorem 2.1. It is readily to observe that

$$\|\widehat{\mathbf{x}}-\mathbf{x}\|_p \leq \frac{1}{\alpha}\sum_{i=1}^{n}\left\|\pi_{W_i}\mathbf{x}-\widehat{\mathbf{x}^{(i)}}\right\|_p. \tag{19}$$

Put $\mathbf{v}_i := \pi_{W_i}(\mathbf{x}-\widehat{\mathbf{x}^{(i)}})$ and employing the inequality (16), we obtain

$$\|\mathbf{v}_i\|_p \leq \frac{(1+\rho_i)^2}{s_i^{1-1/p}(1-\rho_i)}\Big(\|\pi_{W_i}\mathbf{z}\|_1 - \|\pi_{W_i}\mathbf{x}\|_1 + 2\sigma_{s_i}(\pi_{W_i}\mathbf{x})_1\Big) + \frac{s_i^{1/p-1/q}\tau_i(3+\rho_i)}{1-\rho_i}\|\mathbf{A}\mathbf{v}_i\|_2 \tag{20}$$

Summing up the contributions for all $i \in \{1,\cdots,n\}$ and applying inequality (19) finishes the proof. □

2.3. P-RIP based recovery

Before proceeding we recall an important fact from the standard compressed sensing literature. That is, if the restricted isometry constant of $\mathbf{A} \in \mathbb{K}^{m\times N}$ obeys $\delta_{2s} \leq 4/\sqrt{41}$, then the matrix $\mathbf{A}$ satisfies the ℓ^2-robust null space property with constants $0 < \rho < 1$ and $\tau > 0$ depending only on δ_{2s}, where ρ and τ can be respectively taken as

$$\rho := \frac{\delta_{2s}}{\sqrt{1-\delta_{2s}^2}-\delta_{2s}/4} < 1, \;\; \tau := \frac{\sqrt{1+\delta_{2s}}}{\sqrt{1-\delta_{2s}^2}-\delta_{2s}/4}.$$

Note that the matrix $\mathbf{A}$ satisfies P-RIP conditions, if it satisfies RIP-like conditions on every subset of vectors in range of π_{W_i}. Therefore, if $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfies the P-RIP($2\mathbf{s}$, δ) with $\delta = (\delta_1,\ldots,\delta_n)$ with sparsity pattern $\mathbf{s} = (s_1,\cdots,s_n)$ and $\delta = (\delta_1,\ldots,\delta_n)$ with $\delta_i \leq 4/\sqrt{41}$, for all $1 \leq i \leq n$, then, the matrix $\mathbf{A}$ satisfies the ℓ^2-RP-NSP with constants $(\rho_i,\tau_i)_{i=1}^n$, where

$$\rho_i := \frac{\delta_i}{\sqrt{1-\delta_i^2}-\delta_i/4} < 1, \;\; \tau_i := \frac{\sqrt{1+\delta_i}}{\sqrt{1-\delta_i^2}-\delta_i/4}.$$

This obsevation shows the existence of random matrices satisfying the RP-NSP. Hence, by combining Theorem 2.5, we can show that the P-RIP is sufficient for stable and robust recovery.

Theorem 2.7. *Let $\mathbf{A} \in \mathbb{K}^{m\times N}$ and $\mathcal{W}$ a fusion frame for $\mathbb{K}^N$ with lower fusion frame bound $\alpha > 0$ and fusion frame operator $S_{\mathcal{W}}$. Let $(\mathbf{y}^{(i)})_{i=1}^n$ be the linear measurements $\mathbf{y}^{(i)} = \mathbf{A}\pi_{W_i}\mathbf{x} + \mathbf{e}^{(i)}$ for some bounded noise vectors $\mathbf{e}^{(i)}$ such that $\|\mathbf{e}^{(i)}\|_2 \leq \eta_i$, $i = 1,2,\cdots,n$. Denote by $\widehat{\mathbf{x}^{(i)}}$ the solution to the local basis pursuit problems $(\mathcal{P}_{1,\eta_i})$. If the matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfies the P-RIP($2\mathbf{s}$, δ) with sparsity pattern $\mathbf{s} = (s_1,\cdots,s_n)$ and $\delta = (\delta_1,\ldots,\delta_n)$ with $\delta_i \leq 4/\sqrt{41}$, for all $1 \leq i \leq n$. Then $\widehat{\mathbf{x}} = S_{\mathcal{W}}^{-1}\sum_i \widehat{\mathbf{x}^{(i)}}$ approximates $\mathbf{x}$ in the following sense:*

$$\|\widehat{\mathbf{x}}-\mathbf{x}\|_2 \leq \frac{1}{\alpha}\sum_{i=1}^{n}\Big(\frac{\xi_i\sigma_{\mathbf{s}}^{\mathcal{W}}(\mathbf{x})_{1,i}}{\sqrt{s_i}} + \zeta_i\eta_i\Big), \tag{21}$$

where ξ_i and ζ_i depend only on the RIP constants δ_i.

Proof. Since the matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfies the P-RIP($2\mathbf{s}$, δ) with sparsity pattern $\mathbf{s} = (s_1,\cdots,s_n)$ and $\delta = (\delta_1,\ldots,\delta_n)$ with $\delta_i \leq 4/\sqrt{41}$, for all $1 \leq i \leq n$, the matrix $\mathbf{A}$ satisfies the ℓ^2-RP-NSP with constants

$$\rho_i := \frac{\delta_i}{\sqrt{1-\delta_i^2}-\delta_i/4} < 1, \;\; \tau_i := \frac{\sqrt{1+\delta_i}}{\sqrt{1-\delta_i^2}-\delta_i/4}, \;\; 1 \leq i \leq n.$$

Now the conclusions follow from Theorem 2.5. □

2.4. Tangent cone based recovery

We end up this section with the following characterization of exact distributed sparse recovery via ℓ^1-minimization, which involves tangent cones to the ℓ^1-ball. Once again, we need to define the concept of local tangent cone with respect to fusion frame for the distributed sparse signal model. In order to properly state our result, we recall some basic concepts from convex analysis. A convex set $C \subset \mathbb{K}^N$ is called a *cone* if it is closed under positive linear combinations. In addition, if C is convex, then C is called a *convex cone*. The *conic hull* cone(T) of a set $T \subset \mathbb{K}^N$ is the smallest convex cone containing T [13]. Given some nonzero $\mathbf{x} \in \mathbb{K}^N$, we define the *local tangent cone* at $\mathbf{x}$ with respect to fusion frame $\mathcal{W}$ as

$$T_{\pi_{W_i}}(\mathbf{x}) = \mathrm{cone}\{\pi_{W_i}(\mathbf{z}-\mathbf{x}) : \mathbf{z} \in \mathbb{K}^N, \|\pi_{W_i}\mathbf{z}\|_1 \le \|\pi_{W_i}\mathbf{x}\|_1\}, 1 \le i \le n,$$

where the notation cone represents the conic hull.

The following theorem characterizes when $\mathbf{x}$ can be well approximated using the convex programs $\mathcal{P}_{1,\eta_i}, 1 \le i \le n$.

Theorem 2.8. *Let $\mathbf{A} \in \mathbb{K}^{m\times N}$ and $\mathcal{W}$ a fusion frame for $\mathbb{K}^N$ with lower fusion frame bound $\alpha > 0$ and fusion frame operator $S_{\mathcal{W}}$. Let $(\mathbf{y}^{(i)})_{i=1}^n$ be the linear measurements $\mathbf{y}^{(i)} = \mathbf{A}\pi_{W_i}\mathbf{x}+\mathbf{e}^{(i)}$ for some bounded noise vectors $\mathbf{e}^{(i)}$ such that $\|\mathbf{e}^{(i)}\|_2 \le \eta_i$, $i = 1, 2, \cdots, n$. Denote by $\widehat{\mathbf{x}^{(i)}}$ the solution to the local basis pursuit problems ($\mathcal{P}_{1,\eta_i}$). If*

$$\inf_{\mathbf{v}_i \in T_{\pi_{W_i}}(\mathbf{x}), \|\mathbf{v}_i\|_2=1} \|\mathbf{A}\mathbf{v}_i\|_2 \ge \tau_i,$$

for some constants $\tau_1, \cdots, \tau_n > 0$ with respect to $\mathcal{W}$, for all $1 \le i \le n$, then $\widehat{\mathbf{x}} = S_{\mathcal{W}}^{-1}\sum_i \widehat{\mathbf{x}^{(i)}}$ approximates $\mathbf{x}$ in the following sense:

$$\|\widehat{\mathbf{x}}-\mathbf{x}\|_2 \le \frac{2}{\alpha}\sum_{i=1}^n \frac{\eta_i}{\tau_i}. \tag{22}$$

Proof. For each $i \in \{1, \cdots, n\}$, since $\widehat{\mathbf{x}^{(i)}}$ is a minimizer solution of ($\mathcal{P}_{1,\eta_i}$), $\|\widehat{\mathbf{x}^{(i)}}\|_1 \le \|\pi_{W_i}\mathbf{x}\|_1$. This yields $\mathbf{v}_i := \frac{\widehat{\mathbf{x}^{(i)}}-\pi_{W_i}\mathbf{x}}{\|\widehat{\mathbf{x}^{(i)}}-\pi_{W_i}\mathbf{x}\|_2} \in T_{\pi_{W_i}}(\mathbf{x})$. (Note that $\widehat{\mathbf{x}^{(i)}} - \pi_{W_i}\mathbf{x} \ne 0$ can be safely assumed.) Since $\|\mathbf{v}_i\|_2 = 1$, the assumption implies $\|\mathbf{A}\mathbf{v}_i\|_2 \ge \tau_i$, that is,

$$\|\mathbf{A}\pi_{W_i}(\widehat{\mathbf{x}^{(i)}}-\mathbf{x})\|_2 \ge \tau_i\|\widehat{\mathbf{x}^{(i)}} - \pi_{W_i}\mathbf{x}\|_2.$$

It follows by the triangle inequality that

$$\|\mathbf{A}\pi_{W_i}(\widehat{\mathbf{x}^{(i)}}-\mathbf{x})\|_2 \le \|\mathbf{A}\pi_{W_i}\widehat{\mathbf{x}^{(i)}}-\mathbf{y}^{(i)}\|_2 + \|\mathbf{y}^{(i)} - \mathbf{A}\pi_{W_i}\mathbf{x}\|_2 \le 2\eta_i,$$

which allows us to conclude that

$$\|\widehat{\mathbf{x}}-\mathbf{x}\|_2 \le \frac{1}{\alpha}\sum_{i=1}^n \left\|\pi_{W_i}\mathbf{x}-\widehat{\mathbf{x}^{(i)}}\right\|_2 \le \frac{2}{\alpha}\sum_{i=1}^n \frac{\eta_i}{\tau_i}. \tag{23}$$

□

3. Recovery based on relay fusion frames

Throughout this section, $\mathbb{I}$ will denote a generic countable (or finite) index set. Let $\mathcal{H}$ and $\mathcal{K}$ (resp. $\mathcal{K}_i, i \in \mathbb{I}$) be separable complex Hilbert spaces and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ (resp. $\mathcal{B}(\mathcal{H},\mathcal{K}_i), i \in \mathbb{I}$) be the space of all the bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ (resp. $\mathcal{K}_i, i \in \mathbb{I}$). If $\mathcal{H} = \mathcal{K}$ we write $\mathcal{B}(\mathcal{H})$. Usually, it will be clear from the context which norm we use. If $W \subseteq \mathcal{H}$ is a closed subspace, we let $\pi_W \in \mathcal{B}(\mathcal{H})$ denote the orthogonal projection onto the subspace W. In particular, we use the notation $\{W_i\}_{i\in\mathbb{I}} \sqsubset \mathcal{H}$ to represent a family of closed subspaces $\{W_i\}_{i\in\mathbb{I}}$ of a Hilbert space $\mathcal{H}$, for the sake of brevity.

The purpose of this section is to introduce a generalization of the distributed sparse recovery to the case where the measurement model is replaced by a so-called relay fusion frame. This model takes the form

$$\mathbf{y}_i = \mathbf{A}_i \pi_{W_i}(f), \ \ f \in \mathcal{H}$$

where $\mathbf{A}_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i), i \in \mathbb{I}$. We examine the question of reconstructing vectors from these linear measurements.

3.1. Relay fusion frames and their operators

Definition 3.1. *Let $\{W_i\}_{i\in\mathbb{I}} \sqsubset \mathcal{H}$, $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights, i.e. $v_i > 0$ for every $i \in \mathbb{I}$, and $\mathbf{A}_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$ for all $i \in \mathbb{I}$. Then $\{(W_i, \mathbf{A}_i, v_i)\}_{i\in\mathbb{I}}$ is said to be a relay fusion frame, or simply r-fusion frame, if there exist constants $0 < \alpha \le \beta < \infty$ such that*

$$\alpha\|f\|^2 \le \sum_{i\in\mathbb{I}} v_i^2 \|\mathbf{A}_i \pi_{W_i}(f)\|^2 \le \beta\|f\|^2, \ \forall f \in \mathcal{H}. \tag{24}$$

Remark 3.2. *This definition is consistent with the definition of the classical g-frame [23] in the sense that the case where $v_i = 1$ and $W_i = \mathcal{H}$ (that is, $\pi_{W_i} = I_{\mathcal{H}}$) recovers the usual g-frame.*

We point out in passing that this structure can be used as a special frame for research. In general, let

$$\mathcal{R}_{\ell^2} = \Big\{ \{f_i\}_{i\in\mathbb{I}} \ \mid \ f_i \in \mathcal{K}_i \text{ and } \sum_{i\in\mathbb{I}} \|f_i\|^2 < \infty \Big\}.$$

Define the *analysis operator* $T_{\mathcal{R}} : \mathcal{H} \mapsto \mathcal{R}_{\ell^2}$ by

$$T_{\mathcal{R}}(f) = \Big\{ v_i \mathbf{A}_i \pi_{W_i}(f) \Big\}_{i\in\mathbb{I}}, \ \forall f \in \mathcal{H}.$$

Then

$$T_{\mathcal{R}}^*(f) = \sum_{i\in\mathbb{I}} v_i \pi_{W_i} \mathbf{A}_i^* f_i, \ \forall f = \{f_i\}_{i\in\mathbb{I}} \in \mathcal{R}_{\ell^2}.$$

The new *r-fusion frame operator* becomes

$$S_{\mathcal{R}}(f) = T_{\mathcal{R}}^* T_{\mathcal{R}}(f) = \sum_{i\in\mathbb{I}} v_i^2 \pi_{W_i} \mathbf{A}_i^* \mathbf{A}_i \pi_{W_i}(f), \ \forall f \in \mathcal{H}. \tag{25}$$

It is also true that the r-fusion frame condition (24) is equivalent to that $\alpha I_{\mathcal{H}} \le S_{\mathcal{R}} \le \beta I_{\mathcal{H}}$. It shows that the r-fusion frame operator $S_{\mathcal{R}}$ is a positive, self-adjoint and invertible operator. This means that recovery of any $f \in \mathcal{H}$ is possible, if $S_{\mathcal{R}}(f)$ is known.

3.2. Relay fusion frame systems based recovery

Another way to recover elements in $\mathcal{H}$ is through relay fusion frame systems. Let $\{W_i\}_{i\in\mathbb{I}} \sqsubset \mathcal{H}$, $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights, and $\mathbf{A}_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$ for all $i \in \mathbb{I}$. Assume further that $\{f_{ij}\}_{j\in\mathbb{J}_i}$ is a frame for $\mathcal{K}_i$ with dual frame $\{g_{ij}\}_{j\in\mathbb{J}_i}$, for all $i \in \mathbb{I}$. Standard arguments show that $\{(W_i, \mathbf{A}_i, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$ if and only if $\{v_i \pi_{W_i} \mathbf{A}_i^* f_{ij}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ is a frame for $\mathcal{H}$. In this case, we call $\{(W_i, \mathbf{A}_i, v_i, \{f_{ij}\}_{j\in\mathbb{J}_i})\}_{i\in\mathbb{I}}$ is an *relay fusion frame system* for $\mathcal{H}$. In the following we will show that the sequence $\{v_i S_{\mathcal{R}}^{-1} \pi_{W_i} \mathbf{A}_i^* g_{ij}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ is a dual frame for the frame $\{v_i \pi_{W_i} \mathbf{A}_i^* f_{ij}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$. Before that, we are actually going to prove a stronger "if and only if" theorem below.

Theorem 3.3. *For each $i \in \mathbb{I}$, let $T_i, S_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$ and $\{f_{ij}\}_{j\in\mathbb{J}_i}$ be a frame for each $\mathcal{K}_i$ with dual frame $\{g_{ij}\}_{j\in\mathbb{J}_i}$. If $\{T_i^* f_{ij}\}_{j\in\mathbb{J}_i}$ and $\{S_i^* g_{ij}\}_{j\in\mathbb{J}_i}$ are Bessel sequences in $\mathcal{H}$, then they are dual frames in $\mathcal{H}$ if and only if $\sum_{i\in\mathbb{I}} S_i^* T_i = I_{\mathcal{H}}$.*

Proof. It is easy to see that for any $f \in \mathcal{H}$,

$$f = \sum_{i\in\mathbb{I}} \sum_{j\in\mathbb{J}_i} \langle f, T_i^* f_{ij}\rangle S_i^* g_{ij} = \sum_{i\in\mathbb{I}} S_i^* T_i(f),$$

which finishes the proof. □

Bearing in mind that for each $f \in \mathcal{H}$, we have the *reconstruction formula*

$$f = \sum_{i\in\mathbb{I}} v_i^2 S_{\mathcal{R}}^{-1} \pi_{W_i} \mathbf{A}_i^* \mathbf{A}_i \pi_{W_i}(f) = \sum_{i\in\mathbb{I}} v_i^2 \pi_{W_i} \mathbf{A}_i^* \mathbf{A}_i \pi_{W_i} S_{\mathcal{R}}^{-1}(f).$$

Therefore, a direct consequence of Theorem 3.3 is that $\{v_i S_{\mathcal{R}}^{-1} \pi_{W_i} \mathbf{A}_i^* g_{ij}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ is a dual frame for the frame $\{v_i \pi_{W_i} \mathbf{A}_i^* f_{ij}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$.

Corollary 3.4. *Let $\{(W_i, \mathbf{A}_i, v_i, \{f_{ij}\}_{j\in\mathbb{J}_i})\}_{i\in\mathbb{I}}$ be an r-fusion frame system for $\mathcal{H}$ with associated r-fusion frame operator $S_{\mathcal{R}}$ and let $\{g_{ij}\}_{j\in\mathbb{J}_i}$ be local dual frames with respect to $\{f_{ij}\}_{j\in\mathbb{J}_i}$, $i \in \mathbb{I}$. Then $\{v_i S_{\mathcal{R}}^{-1} \pi_{W_i} \mathbf{A}_i^* g_{ij}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ is a dual frame for the frame $\{v_i \pi_{W_i} \mathbf{A}_i^* f_{ij}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$.*

It is interesting to observe that a "dual" relation also holds.

Corollary 3.5. *Let $\{(W_i, \mathbf{A}_i, v_i, \{f_{ij}\}_{j\in\mathbb{J}_i})\}_{i\in\mathbb{I}}$ be an r-fusion frame system for $\mathcal{H}$ with associated r-fusion frame operator $S_{\mathcal{R}}$ and let $\{g_{ij}\}_{j\in\mathbb{J}_i}$ be local dual frames with respect to $\{f_{ij}\}_{j\in\mathbb{J}_i}$, $i \in \mathbb{I}$. Then $\{v_i S_{\mathcal{R}}^{-1} \pi_{W_i} \mathbf{A}_i^* f_{ij}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$ is a dual frame for the frame $\{v_i \pi_{W_i} \mathbf{A}_i^* g_{ij}\}_{i\in\mathbb{I},j\in\mathbb{J}_i}$.*

The above findings justify that r-fusion frame is convenient in that it allows us to recover any signal, whether it is sparse or not.

3.3. Relay fusion frame algorithm based recovery

In order for reconstruction formula to be useful, we need to invert the r-fusion frame operator, which is often a challenging task. Another option is to use an algorithm to obtain approximations of f. As a reminder, we recall from the literature that the algorithm starts with an initial vector $f^{(0)} \in \mathcal{H}$, typically $f^{(0)} = 0$, and produces a sequence $(f^{(k)})$ defined inductively by

$$f^{(k)} = f^{(k-1)} + \frac{2}{\alpha+\beta} S_{\mathcal{R}}\big(f - f^{(k-1)}\big),\ k \geq 1. \tag{26}$$

Theorem 3.6. *Let $\{(W_i, \mathbf{A}_i, v_i)\}_{i\in\mathbb{I}}$ be an r-fusion frame for $\mathcal{H}$ with r-fusion frame operator $S_{\mathcal{R}}$ and r-fusion frame bounds α, β. Then for every $f \in \mathcal{H}$, the sequence $(f^{(k)})$ defined by* (26) *converges to f with the error estimate*

$$\|f - f^{(k)}\| \leq \Big(\frac{\beta-\alpha}{\beta+\alpha}\Big)^k \|f\|.$$

Proof. According to equation (25), we have

$$\langle S_{\mathcal{R}} f, f\rangle = \Big\langle \sum_{i\in\mathbb{I}} v_i^2 \pi_{W_i} \mathbf{A}_i^* \mathbf{A}_i \pi_{W_i} f, f \Big\rangle = \sum_{i\in\mathbb{I}} v_i^2 \|\mathbf{A}_i \pi_{W_i} f\|^2.$$

Thus

$$\langle (I_{\mathcal{H}} - \frac{2}{\alpha+\beta} S_{\mathcal{R}}) f, f\rangle = \|f\|^2 - \frac{2}{\alpha+\beta} \sum_{i\in\mathbb{I}} v_i^2 \|\mathbf{A}_i \pi_{W_i} f\|^2, \forall f \in \mathcal{H}.$$

By the r-fusion frame condition, we obtain

$$\langle (I_{\mathcal{H}} - \frac{2}{\alpha+\beta} S_{\mathcal{R}}) f, f\rangle \leq \|f\|^2 - \frac{2\alpha}{\alpha+\beta} \|f\|^2 = \frac{\beta-\alpha}{\alpha+\beta} \|f\|^2, \forall f \in \mathcal{H}.$$

Similarly,

$$-\frac{\beta-\alpha}{\alpha+\beta}\|f\|^2 \leq \langle (I_{\mathcal{H}} - \frac{2}{\alpha+\beta}S_{\mathcal{R}})f, f\rangle, \forall f \in \mathcal{H}.$$

From the two inequalities obtained above, it can be seen that

$$\left\| I_{\mathcal{H}} - \frac{2}{\alpha+\beta}S_{\mathcal{R}} \right\| \leq \frac{\beta-\alpha}{\alpha+\beta}.$$

In the light of the definition of $(f^{(k)})$, we derive

$$f - f^{(k)} = f - f^{(k-1)} - \frac{2}{\alpha+\beta}S_{\mathcal{R}}\Big(f - f^{(k-1)}\Big) = \Big(I_{\mathcal{H}} - \frac{2}{\alpha+\beta}S_{\mathcal{R}}\Big)(f - f^{(k-1)}).$$

By repeating this process yields that

$$f - f^{(k)} = \Big(I_{\mathcal{H}} - \frac{2}{\alpha+\beta}S_{\mathcal{R}}\Big)^k (f - f^{(0)}).$$

Therefore,

$$\|f - f^{(k)}\| = \left\| \Big(I_{\mathcal{H}} - \frac{2}{\alpha+\beta}S_{\mathcal{R}}\Big)^k (f - f^{(0)}) \right\| \leq \Big(\frac{\beta-\alpha}{\alpha+\beta}\Big)^k \|f\|.$$

□

The vectors $f^{(k)}$ in (26) converge to f as $k \to \infty$. In particular, every $f \in \mathcal{H}$ can be reconstructed from the r-fusion frame coefficients $T_{\mathcal{R}}(f) = \Big\{ v_i \mathbf{A}_i \pi_{W_i}(f) \Big\}_{i \in \mathbb{I}}$, since $S_{\mathcal{R}}(f)$ only requires the knowledge of those coefficients. However, the rate of convergence of the r-fusion frame algorithm depends crucially on good estimates for the r-fusion frame bounds. For r-fusion frames with a large ratio $\beta/\alpha \gg 1$, the algorithm might require too many iterations to be of good use. To save the algorithm, Gröchenig discussed in [17] two representative acceleration methods: Chebyshev acceleration and Conjugate Gradient acceleration, and showed that they lead to faster convergence. We refer to the original paper for the details.

3.4. Relay fusion frames and compressed sensing

We first indicate the link among r-fusion frame and fusion frame and g-frame.

Theorem 3.7. *Let $\mathbb{I}$ be a finite index set and $\{(W_i, \mathbf{A}_i, v_i)\}_{i\in\mathbb{I}}$ an r-fusion frame for $\mathcal{H}$. Then $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathcal{H}$ and $\{\mathbf{A}_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)\}_{i\in\mathbb{I}}$ is a g-frame for $\mathcal{H}$.*

Proof. We only need to illustrate the lower fusion frame bound and lower g-frame bound for $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ and $\{\mathbf{A}_i \in \mathcal{B}(\mathcal{H}, \mathcal{K}_i)\}_{i\in\mathbb{I}}$, respectively. Assume that $\{(W_i, \mathbf{A}_i, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$ with r-fusion frame bounds α, β. Then for arbitrary element f of $\mathcal{H}$, we have

$$\begin{aligned} \alpha\|f\|^2 &\leq \sum_{i\in\mathbb{I}} v_i^2 \|\mathbf{A}_i \pi_{W_i}(f)\|^2 \\ &\leq \sum_{i\in\mathbb{I}} v_i^2 \|\mathbf{A}_i\|^2 \|\pi_{W_i}(f)\|^2 \\ &\leq \max_{i\in\mathbb{I}}\{\|\mathbf{A}_i\|^2\} \sum_{i\in\mathbb{I}} v_i^2 \|\pi_{W_i}(f)\|^2. \end{aligned}$$

We conclude that $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathcal{H}$. Similarly, it is easy to observe that for each $f \in \mathcal{H}$

$$\begin{aligned} \alpha\|f\|^2 &\leq \sum_{i\in\mathbb{I}} v_i^2 \|\mathbf{A}_i \pi_{W_i}(f)\|^2 \\ &\leq \sum_{i\in\mathbb{I}} v_i^2 \|\mathbf{A}_i(f)\|^2 \\ &\leq \max_{i\in\mathbb{I}}\{v_i^2\} \sum_{i\in\mathbb{I}} \|\mathbf{A}_i(f)\|^2. \end{aligned}$$

This proves the claim. □

It is readily to see that the assertions of Theorem 3.7 are not true in general if we assume that $\mathbb{I}$ is a countable infinite index set instead of assuming that $\mathbb{I}$ is a finite index set. In addition, one may wonder whether the Theorem 3.7 holds in reverse? In fact, the answer is negative.

Example 3.8. *Let W_1, W_2 be closed non-trivial subspaces of $\mathcal{H}$ and let $\mathcal{H} = W_1 \oplus W_2$ and $v_1 = v_2 = 1$. Assume that $\mathbf{A}_i \in \mathcal{B}(\mathcal{H}), i = 1,2$ and $\mathbf{A}_1 = \mathbf{0}$ and $\mathbf{A}_2$ is any bounded invertible linear operator on $\mathcal{H}$. Obviously, $\{(W_i, v_i)\}_{i=1,2}$ is a Parseval fusion frame for $\mathcal{H}$ and $\{\mathbf{A}_i \in \mathcal{B}(\mathcal{H})\}_{i=1,2}$ is a g-frame for $\mathcal{H}$, respectively. However, $\{(W_i, \mathbf{A}_i, v_i)\}_{i\in\mathbb{I}}$ can never be an r-fusion frame for $\mathcal{H}$.*

Fortunately, if all relay operators $\mathbf{A}_i$ are restricted to be the same, a stronger "if and only if" theorem can be obtained.

Theorem 3.9. *Let $\mathbb{I}$ be a finite index set. Then $\{(W_i, \mathbf{A}, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$ if and only if $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathcal{H}$ and $\mathbf{A}$ is bounded below from $\mathcal{H}$ into $\mathcal{K}$.*

Proof. Suppose that $\{(W_i, v_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds α, β and operator $\mathbf{A}$ is bounded below from $\mathcal{H}$ into $\mathcal{K}$ so that $\|\mathbf{A}f\| \geqslant C\|f\|$ with $C > 0$ for every $f \in \mathcal{H}$. Then for all $f \in \mathcal{H}$,

$$\begin{aligned}
\alpha C\|f\|^2 &\le C\sum_{i\in\mathbb{I}} v_i^2\|\pi_{W_i}(f)\|^2 \\
&\le \sum_{i\in\mathbb{I}} v_i^2\|\mathbf{A}\pi_{W_i}(f)\|^2 \\
&\le \|\mathbf{A}\|^2\sum_{i\in\mathbb{I}} v_i^2\|\pi_{W_i}(f)\|^2 \\
&\le \beta\|\mathbf{A}\|^2\|f\|^2.
\end{aligned}$$

This implies that $\{(W_i, \mathbf{A}, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathcal{H}$.

The converse proof strategy is similar to the proof of Theorem 3.7. □

Armed with this fact, let us consider some implications of the result in Theorem 3.9 for a couple of specific measurement matrix $\mathbf{A} \in \mathbb{K}^{m\times N}$. The special case of an s-sparse vector $\mathbf{x} \in \mathbb{K}^N$ is also worth a separate look. By Ω_s we denote the set of all s-sparse vectors $\mathbf{x} \in \mathbb{K}^N$.

Theorem 3.10. *Let $\mathcal{W}$ be a fusion frame for $\mathbb{K}^N$. Then there exists a measurement matrix $\mathbf{A} \in \mathbb{K}^{N\times N}$ such that every vector $\mathbf{x} \in \mathbb{K}^N$ can be recovered via an r-fusion frame procedure. In particular, if $\mathbf{x} \in \Omega_s$, then $\mathbf{A} \in \mathbb{K}^{N\times N}$ contains a submatrix $\mathbf{A}_S$ as a map from $\mathbb{K}^S$ to $\mathbb{K}^N$ such that $\mathbf{x}$ can be recovered via an r-fusion frame procedure (generated by $\mathbf{A}_S$).*

Proof. Let us fix $t_N > \cdots > t_2 > t_1 > 0$ and consider the matrix $\mathbf{A} \in \mathbb{K}^{N\times N}$ defined by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_N \\ \vdots & \vdots & \cdots & \vdots \\ t_1^{N-1} & t_2^{N-1} & \cdots & t_N^{N-1} \end{bmatrix}.$$

The square matrix $\mathbf{A}$ is a Vandermonde matrix. The determinant of $\mathbf{A}$ equals

$$\det \mathbf{A} = \prod_{1\le k<l\le N} (t_l - t_k) > 0.$$

This shows that $\mathbf{A}$ is invertible, in particular injective. Therefore, $\{(W_i, \mathbf{A}, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\mathbb{K}^N$. In particular, for an s-sparse vector $\mathbf{x} \in \mathbb{K}^N$ with

$$S = \operatorname{supp}(\mathbf{x}) := \{j \in [N] : x_j \neq 0\},$$

we have $\mathbf{A}\pi_{W_i}\mathbf{x} = \mathbf{A}_S\pi_{W_i}\mathbf{x}_S$ for every $i \in \mathbb{I}$. Note that $S = \operatorname{supp}(\mathbf{x})$ ranges through all possible subsets of $[N]$ of cardinality $|S| = s$ when $\mathbf{x}$ ranges through all possible s-sparse vectors. Thus, for each $\mathbf{x} \in \Omega_s$,

$$\|\mathbf{A}^{-1}\|^{-2}\|\mathbf{x}_S\|^2 \le \sum_{i\in\mathbb{I}} v_i^2\|\mathbf{A}\pi_{W_i}\mathbf{x}\|^2 = \sum_{i\in\mathbb{I}} v_i^2\|\mathbf{A}_S\pi_{W_i}\mathbf{x}_S\|^2 \le \|\mathbf{A}\|^2\|\mathbf{x}_S\|^2.$$

□

In view of the proof of the Theorem 3.10, many other matrices meet the requirement of the matrix $\mathbf{A}$. Instead of the $N \times N$ Vandermonde matrix associated with $t_N > \cdots > t_2 > t_1 > 0$, we can choose any matrix that is totally positive, i.e., that satisfies $\det \mathbf{A}_{I,J} > 0$ for any sets $I, J \subset [N]$ of the same cardinality, where $\mathbf{A}_{I,J}$ represents the submatrix of $\mathbf{A}$ with rows indexed by I and columns indexed by J. In addition, the partial Fourier matrices can also be used as candidates. More information can be extracted from the Theorem 3.10, that is, if $\mathbf{x} \in \mathbb{K}^N$ is an s-sparse vector, then we only need to construct an r-fusion frame for Ω_s, rather than for the whole ambient space. This simple observation might be useful in practical applications to reduce the computational complexity and thus improve computational efficiency. Moreover, given any $\mathbf{s}$-distributed sparse vector $\mathbf{x} \in \mathbb{K}^N$, a set S' (at most $[N]$) can always be found such that $\mathbf{x} = \mathbf{x}_{S'}$. Therefore, according to Theorem 3.10, there exists a measurement submatrix $\mathbf{A}_{S'} \in \mathbb{K}^{S' \times N}$ so that every $\mathbf{s}$-distributed sparse vector $\mathbf{x}$ can be recovered via associated r-fusion frame procedure.

Let $\Omega_{\mathbf{s}}^{\mathcal{W}}$ be the set of all $\mathbf{s}$-distributed sparse vectors with respect to the family of subspaces $(W_i)_i$. The following theorem states that the P-RIP is a sufficient condition for a setting to be an r-fusion frame. Thus, any $\mathbf{s}$-distributed sparse vector can be recovered by an r-fusion frame system under the assumption of the P-RIP.

Theorem 3.11. *Let $\mathcal{W}$ be a fusion frame for $\mathbb{K}^N$ with frame bounds α, β. Let $\mathbf{A} \in \mathbb{K}^{m\times N}$ satisfy the P-RIP with respect to $\mathcal{W}$, with bounds $\delta_1, \ldots, \delta_n$ and sparsity pattern $\mathbf{s} = (s_1, \cdots, s_n)$, and let*

$$\alpha_0 = \alpha \min_i\{1 - \delta_i\}, \beta_0 = \beta \max_i\{1 + \delta_i\}.$$

Then $\{(W_i, \mathbf{A}, v_i)\}_{i\in\mathbb{I}}$ is an r-fusion frame for $\Omega_{\mathbf{s}}^{\mathcal{W}}$ with r-fusion frame bounds α_0, β_0.

Proof. For any $\mathbf{x} \in \Omega_{\mathbf{s}}^{\mathcal{W}}$, according to the inequality (4), we know

$$\sum_{i\in\mathbb{I}}(1 - \delta_i)v_i^2\|\pi_{W_i}\mathbf{x}\|^2 \le \sum_{i\in\mathbb{I}} v_i^2\|\mathbf{A}\pi_{W_i}\mathbf{x}\|^2 \le \sum_{i\in\mathbb{I}}(1 + \delta_i)v_i^2\|\pi_{W_i}\mathbf{x}\|^2.$$

Further using the fusion frame inequality (2), we get

$$\begin{aligned}
\alpha \min_i\{1 - \delta_i\}\|\mathbf{x}\|^2 &\le \min_i\{1 - \delta_i\} \sum_{i\in\mathbb{I}} v_i^2\|\pi_{W_i}\mathbf{x}\|^2 \\
&\le \sum_{i\in\mathbb{I}}(1 - \delta_i)v_i^2\|\pi_{W_i}\mathbf{x}\|^2 \\
&\le \sum_{i\in\mathbb{I}} v_i^2\|\mathbf{A}\pi_{W_i}\mathbf{x}\|^2 \\
&\le \sum_{i\in\mathbb{I}}(1 + \delta_i)v_i^2\|\pi_{W_i}\mathbf{x}\|^2 \\
&\le \max_i\{1 + \delta_i\} \sum_{i\in\mathbb{I}} v_i^2\|\pi_{W_i}\mathbf{x}\|^2 \\
&\le \beta \max_i\{1 + \delta_i\}\|\mathbf{x}\|^2,
\end{aligned}$$

which concludes the proof. □

Remark 3.12. *We would like to add a few comments about r-fusion frames and compressed sensing. Note that Theorem 3.11 can be used as special bridging results for fusion frame. Recall that the bridging problem investigate conditions on relay operator to ensure that a given fusion frame can form an r-fusion frame, cf.[19]. In compressed sensing terms, this is equivalent to investigate conditions on measurement matrix $\mathbf{A}$ which ensure exact or approximate reconstruction of the original sparse or compressible vector $\mathbf{x}$. In other words, the recovery guarantee conditions on $\mathbf{A}$ provide solutions to the bridging problem for fusion frames.*

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