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On subgradient extragradient rule with inertial correction for bilevel pseudomonotone VIP with constraints of equilibria system and split CFPP

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Abstract. In this paper, let the BPVIP, GEPS, and SCFPP represent a bilevel pseudomonotone variational inequality problem, a generalized equilibrium problems system, and a split common fixed point problem involving demimetric mappings in real Hilbert spaces, respectively. We devise a composite subgradient extragradient rule with an inertial correction term for solving the BPVIP with constraints of GEPS and SCFPP, where the rule exploits the inertial technique with a correction term and a self-adaptive stepsize strategy. The BPVIP consists of the upper-level VIP for one strongly monotone operator and the lower-level VIP for another pseudomonotone operator. The strong convergence result for the designed algorithm is established under certain suitable conditions. In addition, the main result is employed to handle a bilevel split pseudomonotone variational inequality problem (BSPVIP). Lastly, an illustrated instance is utilized to back up the applicability and performability of the suggested rule.

1. Introduction

Let $\emptyset \neq C \subset \mathcal{H}$ with C being convex and closed in real Hilbert space $\mathcal{H}$. We use the $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to indicate the inner product and induced norm of $\mathcal{H}$, respectively. Let P_C be the metric projection from $\mathcal{H}$ onto C. Let $T : C \to \mathcal{H}$ be a nonlinear operator. Denote by $\mathrm{Fix}(T)$ and $\mathbf{R}$ the fixed-point set of T and the real-number set, respectively. We use the $\rightharpoonup$ and $\to$ to represent the weak and strong convergence in $\mathcal{H}$, respectively. Presume that $A, F : \mathcal{H} \to \mathcal{H}$ both are nonlinear mappings. Then the bilevel variational inequality problem (BVIP) is defined as follows:

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$$\text{Find } u^* \in \Omega \text{ such that } \langle Fu^*, v - u^* \rangle \geq 0 \ \ \forall v \in \Omega,$$

where $\Omega := \mathrm{VI}(C, A)$ is the solution set of the variational inequality problem (VIP) below:

$$\text{Find } v^* \in C \text{ such that } \langle Av^*, w - v^* \rangle \geq 0 \ \ \forall w \in C.$$

It is evident that the above VIP is equivalent to the fixed point problem (FPP) below:

$$\text{Find } v^* \in C \text{ such that } v^* = P_C(v^* - \ell A v^*),$$

where ℓ is any positive number. In 1976, the Korpelevich extragradient approach was first put forth in [29] for approximating a point of $\mathrm{VI}(C, A)$. It is now one of the most effective methods. Till now, the literature on the VIP is vast and the Korpelevich extragradient method has caught the broad attention of many authors, who facilitated it in different matters; see e.g., [1, 2, 6, 7, 11–17, 20–22, 27, 28, 30–34, 36, 41] and references therein, to name but a few.

For $i = 1, 2, ..., N$, let $\mathcal{H}_i$ be a real Hilbert space and $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i$ be a bounded linear operator, and suppose that $T_i : \mathcal{H} \to \mathcal{H}$ and $S_i : \mathcal{H}_i \to \mathcal{H}_i$ are nonlinear operators. The split common fixed point problem (SCFPP) is formulated below:

$$\text{Find } x^* \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \text{ such that } \mathcal{T}_i x^* \in \mathrm{Fix}(S_i), \quad \forall i \in \{1, 2, ..., N\}. \tag{1.1}$$

To our knowledge, the SCFPP is a generalization of the split feasibility problem. It has attracted extensive research from numerous scholars due to its applications in different disciplines such as image reconstruction, computer tomography, and radiation therapy treatment planning [18]. In recent years, the SCFPP has been investigated for various classes of operators, and iterative schemes for solving it have also attracted wide attention from numerous researchers, see, e.g. [23, 24].

Very recently, invoking the inertial technique with a correction term and a self-adaptive stepsize strategy, Eslamian and Kamandi [25] introduced a new iterative scheme for resolving the strongly monotone VIP over the solution set of the SCFPP with demimetric mappings in real Hilbert spaces. Presume that the conditions hold below: (i) for $i = 1, 2, ..., N$, $S_i : \mathcal{H}_i \to \mathcal{H}_i$ is ξ_i-demimetric mapping with $\xi_i \in (-\infty, 1)$ such that $I - S_i$ is demiclosed at zero, (ii) for $i = 1, 2, ..., N$, $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i$ is a nonzero bounded linear operator with the adjoint operator $\mathcal{T}_i^* : \mathcal{H}_i \to \mathcal{H}$ such that $\Omega = \bigcap_{i=1}^{N} \mathcal{T}_i^{-1}\mathrm{Fix}(S_i) \neq \emptyset$, (iii) $F : \mathcal{H} \to \mathcal{H}$ is κ-Lipschitzian and η-strongly monotone, and (iv) the sequences $\{\gamma_n\} \subset [0, 1)$, $\{\sigma_n\} \subset (0, 1)$ and $\{\varsigma_n\} \subset (0, \infty)$, satisfy $\limsup_{n\to\infty} \gamma_n < 1$, $\sum_{n=0}^{\infty} \sigma_n = \infty$, $\lim_{n\to\infty} \sigma_n = 0$ and $\lim_{n\to\infty} \frac{\varsigma_n}{\sigma_n} = 0$.

Algorithm 1.1 (see [25, Algorithm 1]).
Initialization: Let $\alpha > 0$, $\beta > 0$ and $x_1, x_0, p_0 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Given the iterates x_{n-1}, x_n $(n \geq 1)$, calculate x_{n+1} as follows:

Step 1. Compute $p_n = x_n + \alpha_n(x_{n-1} - x_n) + \beta_n(p_{n-1} - x_{n-1})$ for $\alpha_n \in [0, \overline{\alpha}_n]$ and $\beta_n \in [0, \overline{\beta}_n]$ where

$$\overline{\alpha}_n = \begin{cases} \min\{\alpha, \frac{\varsigma_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise}, \end{cases} \quad \text{and}$$

$$\overline{\beta}_n = \begin{cases} \min\{\beta, \frac{\varsigma_n}{\|p_{n-1} - x_{n-1}\|}\} & \text{if } p_{n-1} \neq x_{n-1}, \\ \beta & \text{otherwise}. \end{cases}$$

Step 2. Select two indices $i_n, \imath_n \in \{1, 2, ..., N\}$ such that $\|(I - S_{i_n})\mathcal{T}_{i_n} p_n\| = \max_{i\in\{1,2,...,N\}} \|(I - S_i)\mathcal{T}_i p_n\|$ and $\|(I - S_{\imath_n})\mathcal{T}_{\imath_n} p_n\| = \min_{i\in\{1,2,...,N\}} \|(I - S_i)\mathcal{T}_i p_n\|$.

Step 3. Compute $u_n = p_n - \tau_{n,i_n}\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n} p_n$ and $v_n = p_n - \tau_{n,\imath_n}\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n} p_n$, where the stepsizes are picked in such a way that for small enough $\epsilon > 0$,

$$\tau_{n,i} \in \left(\epsilon, \frac{(1-\xi_{i_n})\|(I-S_i)\mathcal{T}_i p_n\|^2}{\|\mathcal{T}_i^*(I-S_i)\mathcal{T}_i p_n\|^2} - \epsilon\right) \quad \text{if } (I - S_i)\mathcal{T}_i p_n \neq 0;$$

otherwise $\tau_{n,i} = \tau_i$ is arbitrary nonnegative number.

Step 4. Compute $q_n = (1 - \gamma_n)u_n + \gamma_n v_n$ and $x_{n+1} = (I - \sigma_n F)q_n$. Again put $n := n + 1$ and go to Step 1.

It was shown in [25] that $x_n \to x^\dagger \in \Omega$, which is only a solution to the VIP: $\langle Fx^\dagger, x - x^\dagger \rangle \geq 0 \ \forall x \in \Omega$.

On the other hand, suppose that $\mathcal{H}$ and $\mathcal{H}_1$ are two real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of $\mathcal{H}$ and $\mathcal{H}_1$, respectively. Let $\mathcal{T}:\mathcal{H}\to\mathcal{H}_1$ denote a bounded linear operator and $A,F:\mathcal{H}\to\mathcal{H}$ and $B:\mathcal{H}_1\to\mathcal{H}_1$ be nonlinear mappings. Then, the bilevel split VIP (BSVIP) (see [3]) is specified below:

$$\text{Seek } y^\dagger\in\Omega \text{ such that } \langle Fy^\dagger, y-y^\dagger\rangle\geq 0\ \ \forall y\in\Omega, \tag{1.2}$$

where $\Omega:=\{y\in \mathrm{VI}(C,A):\mathcal{T}y\in\mathrm{VI}(Q,B)\}$ is the solution set of the split VIP (SVIP), which was introduced by Censor et al. [19]. From the aforementioned, the SVIP is to find $y^\dagger\in\mathrm{VI}(C,A)$ such that $\mathcal{T}y^\dagger\in\mathrm{VI}(Q,B)$. They proposed and analyzed the following iterative method for approximating a solution of the SVIP, i.e., for any initial $p_1\in\mathcal{H}$, the sequence $\{p_n\}$ is generated by

$$p_{n+1}=P_C(I-\lambda A)(p_n+\gamma\mathcal{T}^*(P_Q(I-\lambda B)-I)\mathcal{T}p_n)\quad\forall n\geq 1, \tag{1.3}$$

where A and B both are inverse-strongly monotone and $\mathcal{T}$ is a non-zero bounded linear operator. Under certain mild restrictions, it was shown in [19] that $p_n\rightharpoonup p^\dagger\in\Omega$.

It is noteworthy that the VIP can be rewritten as the FPP: $Sy=P_Q(y-\mu By),\ \mu>0$, with $\mathrm{VI}(Q,B)=\mathrm{Fix}(S)$, where $\mathrm{Fix}(S)$ is the fixed point set of the mapping S. Then BSVIP (1.2) is rewritten as the problem below:

$$\text{Seek } y^\dagger\in\Omega \text{ such that } \langle Fy^\dagger, y-y^\dagger\rangle\geq 0\ \ \forall y\in\Omega, \tag{1.4}$$

where $\Omega:=\{y\in\mathrm{VI}(C,A):\mathcal{T}y\in\mathrm{Fix}(S)\}$. Inspired by the problems (1.1) and (1.4), we are devoted to studying the bilevel pseudomonotone VIP (BPVIP) with the GEPS and SCFPP constraints in real Hilbert spaces. Here the GEPS is a generalized equilibrium problems system, which is the problem of finding $(u^\dagger,v^\dagger)\in C\times C$ fulfilling

$$\begin{cases}\Theta_1(u^\dagger,x)+\langle B_1v^\dagger,x-u^\dagger\rangle+\frac{1}{\eta_1}\langle u^\dagger-v^\dagger,x-u^\dagger\rangle\geq 0\quad\forall x\in C,\\ \Theta_2(v^\dagger,y)+\langle B_2u^\dagger,y-v^\dagger\rangle+\frac{1}{\eta_2}\langle v^\dagger-u^\dagger,y-v^\dagger\rangle\geq 0\quad\forall y\in C,\end{cases} \tag{1.5}$$

where $B_1,B_2:\mathcal{H}\to\mathcal{H}$ are both nonlinear mappings, $\Theta_1,\Theta_2:C\times C\to\mathbf{R}$ are two bifunctions, and $\eta_1,\eta_2>0$ are two coefficients.

To solve the BPVIP with the GEPS and SCFPP constraints, we devise a composite subgradient extragradient rule with an inertial correction term. This rule incorporates the modified inertial subgradient extragradient algorithm (see [8]) for solving the GEPS with the VIP and CFPP constraints.

Let the mapping S_r be nonexpansive on $\mathcal{H}$ for $r=1,...,N$ and $S:\mathcal{H}\to\mathcal{H}$ be a ϑ_n-asymptotically nonexpansive mapping. Let $A:\mathcal{H}\to\mathcal{H}$ be an L-Lipschitzian pseudomonotone mapping satisfying $\|Ah\|\leq\liminf_{n\to\infty}\|Ah_n\|$ provided $h_n\rightharpoonup h$. Let $\Theta_1,\Theta_2:C\times C\to\mathbf{R}$ be two bifunctions. Let $B_1,B_2:\mathcal{H}\to\mathcal{H}$ be ρ-ism and σ-ism, respectively. Let $f:\mathcal{H}\to\mathcal{H}$ be δ-contractive and $F:\mathcal{H}\to\mathcal{H}$ be κ-Lipschitzian η-strongly monotone with $\delta<\tau:=1-\sqrt{1-\alpha(2\eta-\alpha\kappa^2)}$ for $\alpha\in(0,\frac{2\eta}{\kappa^2})$. Let $\{\epsilon_n\}\subset[0,1]$, $\{\beta_n\}\subset(0,1]$ and $\{\alpha_n\},\{\sigma_n\}\subset(0,1)$ such that (i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^\infty\alpha_n=\infty$; (ii) $\lim_{n\to\infty}\frac{\vartheta_n}{\alpha_n}=0$ and $\sup_{n\geq1}\frac{\epsilon_n}{\alpha_n}<\infty$; and (iii) $0<\liminf_{n\to\infty}\sigma_n\leq\limsup_{n\to\infty}\sigma_n<1$ and $\limsup_{n\to\infty}\beta_n<1$.

Let $\gamma>0,\ \mu\in(0,1),\ \ell\in(0,1),\ \eta_1\in(0,2\rho)$, and $\eta_2\in(0,2\sigma)$ be five constants. Set $S_0:=S,\ S_n:=S_{n\mathrm{mod}N}$ and $G:=T_{\eta_1}^{\Theta_1}(I-\eta_1B_1)T_{\eta_2}^{\Theta_2}(I-\eta_2B_2)$. Suppose that $\Omega:=\bigcap_{r=0}^N\mathrm{Fix}(S_r)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)\neq\emptyset$.

Algorithm 1.2 (see [8, Algorithm 3.1]).

Initialization: Let $x_1,x_0\in\mathcal{H}$ be arbitrary. Given x_n,x_{n-1} $(n\geq1)$, calculate x_{n+1} below:

Step 1. Put $q_n=S^nx_n+\epsilon_n(S^nx_n-S^nx_{n-1})$, and compute $p_n=\beta_nq_n+(1-\beta_n)u_n$, where $u_n=T_{\eta_1}^{\Theta_1}(I-\eta_1B_1)v_n$ and $v_n=T_{\eta_2}^{\Theta_2}(I-\eta_2B_2)p_n$.

Step 2. Calculate $y_n=P_C(p_n-\zeta_nAp_n)$, where ζ_n is the largest $\zeta\in\{\gamma,\gamma\ell,\gamma\ell^2,...\}$ such that $\zeta\|Ap_n-Ay_n\|\leq\mu\|p_n-y_n\|$.

Step 3. Calculate $t_n = \sigma_n x_n + (1 - \sigma_n) z_n$, where $z_n = P_{C_n}(p_n - \zeta_n A y_n)$ and $C_n := \{y \in \mathcal{H} : \langle p_n - \zeta_n A p_n - y_n, y - y_n \rangle \le 0\}$.

Step 4. Compute $x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n \alpha F) S_n t_n$.

Again put $n := n + 1$ and go to Step 1.

It was shown in [8] that $x_n \to x^\dagger \in \Omega$, which is only a solution to the VIP: $\langle (\alpha F - f)x^\dagger, y - x^\dagger \rangle \ge 0 \ \forall y \in \Omega$. In this paper, inspired by [8, 25], we prove the strong convergence of our proposed algorithm to the unique solution of the BPVIP with the GEPS and SCFPP constraints under certain mild conditions. Moreover, the main result is invoked to treat a BSPVIP with the GEPS constraint. Lastly, some examples are given to show the implementations and performance of the proposed rule.

We organize the paper as follows: In Sect. 2, we recall certain definitions and basic tools for later use. Sect. 3 explores the convergence criteria of the proposed algorithm. Lastly, Sect. 4 invokes our main results to deal with the BSPVIP with the GEPS constraint in an illustrated instance. Our results are improvements and extensions of the corresponding ones in [8, 25].

2. Preliminaries

Suppose that C is a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$. A mapping $S : C \to \mathcal{H}$ is said to be nonexpansive if $\|Sx - Sy\| \le \|x - y\| \ \forall x, y \in C$. Given a sequence $\{x_n\} \subset \mathcal{H}$, we denote by $x_n \to x$ (resp., $x_n \rightharpoonup x$) the strong (resp., weak) convergence of $\{x_n\}$ to x. For each $x \in \mathcal{H}$, we know that there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\| \ \forall y \in C$. The operator P_C is called the metric projection of $\mathcal{H}$ onto C. According to [26], we know that the following properties hold:

(a) $\langle x - y, P_C x - P_C y \rangle \ge \|P_C x - P_C y\|^2 \ \forall x, y \in \mathcal{H}$;

(b) $z = P_C x \ \Leftrightarrow \ \langle x - z, y - z \rangle \le 0 \ \forall x \in \mathcal{H}, y \in C$;

(c) $\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2 \ \forall x \in \mathcal{H}, y \in C$;

(d) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle \ \forall x, y \in \mathcal{H}$;

(e) $\|sx + (1 - s)y\|^2 = s\|x\|^2 + (1 - s)\|y\|^2 - s(1 - s)\|x - y\|^2 \ \forall x, y \in \mathcal{H}, \ s \in [0, 1]$.

Recall also that $S : C \to \mathcal{H}$ is called (see [5])

(i) L-Lipschitz continuous or L-Lipschitzian if $\exists L > 0$ s.t. $\|Sx - Sy\| \le L\|x - y\| \ \forall x, y \in C$;

(ii) α-strongly monotone if $\exists \alpha > 0$ such that $\langle Sx - Sy, x - y \rangle \ge \alpha \|x - y\|^2 \ \forall x, y \in C$;

(iii) β-inverse-strongly monotone (β-ism) if $\langle Sx - Sy, x - y \rangle \ge \beta \|Sx - Sy\|^2 \ \forall x, y \in C$;

(iv) pseudomonotone if $\langle Sx, y - x \rangle \ge 0 \Rightarrow \langle Sy, y - x \rangle \ge 0 \ \forall x, y \in C$;

(v) quasimonotone if $\langle Sx, y - x \rangle > 0 \Rightarrow \langle Sy, y - x \rangle \ge 0 \ \forall x, y \in C$;

(vi) ξ-demicontractive if $\exists \xi \in (0, 1)$ such that $\|Sx - p\|^2 \le \|x - p\|^2 + \xi \|x - Sx\|^2 \ \forall x \in C, \ p \in \mathrm{Fix}(S) \ne \emptyset$;

(vii) ξ-demimetric if $\exists \xi \in (-\infty, 1)$ such that $\langle x - Sx, x - p \rangle \ge \frac{1-\xi}{2} \|x - Sx\|^2 \ \forall x \in C, \ p \in \mathrm{Fix}(S) \ne \emptyset$;

(viii) sequentially weakly continuous on C if $\forall \{x_n\} \subset C$, the relation holds: $x_n \rightharpoonup x \Rightarrow Sx_n \rightharpoonup Sx$.

It is easy to see that (ii)⇒(iii)⇒(iv)⇒(v). However, the converse is not generally true.

Definition 2.1. A mapping $S : C \to \mathcal{H}$ is said to satisfy the demiclosedness principle if $I - S$ is demiclosed at zero, that is, for any sequence $\{x_n\} \subset C$ satisfying both $x_n \rightharpoonup x$ and $(I - S)x_n \to 0$, one has $x \in \mathrm{Fix}(S)$.

From [26], it is known that if S is a nonexpansive self-mapping on C, then $I - S$ is demiclosed at zero, that is, for any $\{x_n\} \subset C$ satisfying both $x_n \rightharpoonup x$ and $(I - S)x_n \to 0$, one has $x \in \mathrm{Fix}(S)$.

Lemma 2.1 [35]. If $S : C \to \mathcal{H}$ is a ξ-demimetric mapping, then $\mathrm{Fix}(S)$ is closed and convex.

Lemma 2.2 [38]. Let $\lambda \in (0, 1]$, $S : C \to \mathcal{H}$ be a nonexpansive mapping, and the mapping $S^\lambda : C \to \mathcal{H}$ be defined by $S^\lambda x := Sx - \lambda \alpha F(Sx) \ \forall x \in C$, where $F : \mathcal{H} \to \mathcal{H}$ is κ-Lipschitzian and η-strongly monotone. Then S^λ is a contraction provided $0 < \alpha < \frac{2\eta}{\kappa^2}$, i.e., $\|S^\lambda x - S^\lambda y\| \le (1 - \lambda \zeta)\|x - y\| \ \forall x, y \in C$, where

$\zeta = 1 - \sqrt{1 - \alpha(2\eta - \alpha\kappa^2)} \in (0, 1]$.

Lemma 2.3. Assume that $A : C \to \mathcal{H}$ is pseudomonotone and continuous. Then $u \in C$ is a solution to the VIP $\langle Au, v - u\rangle \geq 0\ \forall v \in C$, if and only if $\langle Av, v - u\rangle \geq 0\ \forall v \in C$.

Proof. It is easy to check that the conclusion is valid.

Lemma 2.4 [38]. Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the conditions: $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n\gamma_n\ \forall n \geq 1$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that (i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$, and (ii) $\limsup_{n\to\infty} \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\lambda_n\gamma_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 [30]. Let $\{\mathbf{\Lambda}_m\}$ be a sequence of real numbers that do not decrease at infinity in the sense that $\exists\{\mathbf{\Lambda}_{m_k}\} \subset \{\mathbf{\Lambda}_m\}$ such that $\mathbf{\Lambda}_{m_k} < \mathbf{\Lambda}_{m_k+1}\ \forall k \geq 1$. Let the sequence $\{\phi(m)\}_{m\geq m_0}$ of integers be formulated below:

$$\phi(m) = \max\{k \leq m : \mathbf{\Lambda}_k < \mathbf{\Lambda}_{k+1}\},$$

with integer $m_0 \geq 1$ satisfying $\{k \leq m_0 : \mathbf{\Lambda}_k < \mathbf{\Lambda}_{k+1}\} \neq \emptyset$. Then

(i) $\phi(m_0) \leq \phi(m_0 + 1) \leq \cdots$ and $\phi(m) \to \infty$;

(ii) $\mathbf{\Lambda}_{\phi(m)} \leq \mathbf{\Lambda}_{\phi(m)+1}$ and $\mathbf{\Lambda}_m \leq \mathbf{\Lambda}_{\phi(m)+1}\ \forall m \geq m_0$.

For a given bifunction $\Theta : C \times C \to \mathbf{R}$, one needs to make the hypotheses below:

(H1) $\Theta(y, y) = 0\ \ \forall y \in C$;

(H2) $\Theta(u, y) + \Theta(y, u) \leq 0\ \ \forall y, u \in C$;

(H3) $\lim_{\lambda\to0^+} \Theta((1 - \lambda)y + \lambda v, u) \leq \Theta(y, u)\ \ \forall y, u, v \in C$;

(H4) for each $y \in C$, $\Theta(y, \cdot)$ is convex and lower semicontinuous (l.s.c.).

In 1994, Blum and Oettli [4] derived this lemma.

Proposition 2.1 [4]. Assume that $\Theta : C \times C \to \mathbf{R}$ fulfills the hypotheses (H1)-(H4). For each $x \in \mathcal{H}$ and $\ell > 0$, let $T_\ell^\Theta : \mathcal{H} \to C$ be a mapping defined below:

$$T_\ell^\Theta(x) := \{y \in C : \Theta(y, z) + \frac{1}{\ell}\langle z - y, y - x\rangle \geq 0\ \ \forall z \in C\}.$$

Then, (i) T_ℓ^Θ is single-valued and satisfies $\|T_\ell^\Theta y - T_\ell^\Theta u\|^2 \leq \langle T_\ell^\Theta y - T_\ell^\Theta u, y - u\rangle\ \ \forall y, u \in \mathcal{H}$; and (ii) $\mathrm{Fix}(T_\ell^\Theta) = \mathrm{EP}(\Theta)$ (i.e., the solution set of the equilibrium problem), and $\mathrm{EP}(\Theta)$ is closed and convex.

To solve the problem (1.5), the authors in [9] used a fixed-point technique. Indeed, the GEPS (1.5) can be transformed into a fixed-point problem.

Proposition 2.2 [9]. Suppose that the bifunctions $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ satisfy the hypotheses (H1)-(H4) and $B_1, B_2 : \mathcal{H} \to \mathcal{H}$ are ρ-ism and σ-ism, respectively. Then, $(u^\dagger, v^\dagger) \in C \times C$ is a solution of GEPS (1.5) if and only if $u^\dagger \in \mathrm{Fix}(G)$, where $G := T_{\eta_1}^{\Theta_1}(I - \eta_1 B_1)T_{\eta_2}^{\Theta_2}(I - \eta_2 B_2)$ and $v^\dagger = T_{\eta_2}^{\Theta_2}(I - \eta_2 B_2)u^\dagger$ for $\eta_1 \in (0, 2\rho)$ and $\eta_2 \in (0, 2\sigma)$.

If $\Theta_1 = \Theta_2 = 0$, then GEPS (1.5) reduces to the generalized variational inequalities system (GVIS) [10]: Find $(u^\dagger, v^\dagger) \in C \times C$ satisfying

$$\begin{cases} \langle \eta_1 B_1 v^\dagger + u^\dagger - v^\dagger, x - u^\dagger\rangle \geq 0 & \forall x \in C, \\ \langle \eta_2 B_2 u^\dagger + v^\dagger - u^\dagger, y - v^\dagger\rangle \geq 0 & \forall y \in C, \end{cases} \tag{2.1}$$

for coefficients $\eta_1, \eta_2 > 0$. From Proposition 2.2 it is easily known that $(u^\dagger, v^\dagger) \in C \times C$ is a solution of GVIS (2.1) if and only if $u^\dagger \in \mathrm{Fix}(G)$, where $G := P_C(I - \eta_1 B_1)P_C(I - \eta_2 B_2)$ and $v^\dagger = P_C(I - \eta_2 B_2)u^\dagger$ for $\eta_1 \in (0, 2\rho)$

and $\eta_2 \in (0, 2\sigma)$.

Proposition 2.3 [9]. Suppose that$B : \mathcal{H} \to \mathcal{H}$ is a β-ism. Then,

$$\|(I - \eta B)v - (I - \beta B)y\|^2 \le \|v - y\|^2 - \eta(2\beta - \eta)\|Bv - By\|^2 \quad \forall v, y \in \mathcal{H}, \forall \beta \ge 0.$$

In particular, if $0 \le \eta \le 2\beta$, then $I - \eta B$ is nonexpansive.

Proposition 2.4 [9]. Let $B_1, B_2 : \mathcal{H} \to \mathcal{H}$ be ρ-ism and σ-ism, respectively. Suppose that the bifunctions $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ satisfy the hypotheses (H1)-(H4). Then, $G := T^{\Theta_1}_{\eta_1}(I - \eta_1 B_1)T^{\Theta_2}_{\eta_2}(I - \eta_2 B_2)$ is nonexpansive for $0 < \eta_1 \le 2\rho$ and $0 < \eta_2 \le 2\sigma$.

Corollary 2.1 [10]. Let $B_1 : \mathcal{H} \to \mathcal{H}$ be ρ-ism and $B_2 : \mathcal{H} \to \mathcal{H}$ be σ-ism. Define an operator $G : \mathcal{H} \to C$ by $G := P_C(I - \eta_1 B_1)P_C(I - \eta_2 B_2)$. Then G is nonexpansive for $\eta_1 \in (0, 2\rho)$ and $\eta_2 \in (0, 2\sigma)$.

3. Main Results

In this section, suppose that $\mathcal{H}$ and $\mathcal{H}_i$ both are real Hilbert spaces for $i = 1, 2, ..., N$ and the feasible set C is nonempty, closed and convex in $\mathcal{H}$. To analyze the convergence of our proposed approach for settling the BPVIP with the SCFPP constraint, we assume that the following hold:

(C1) $f : \mathcal{H} \to \mathcal{H}$ is δ-contractive map and $F : \mathcal{H} \to \mathcal{H}$ is η-strongly monotone κ-Lipschitzian mapping with $\delta < \zeta := 1 - \sqrt{1 - \alpha(2\eta - \alpha\kappa^2)}$ for $\alpha \in (0, \frac{2\eta}{\kappa^2})$.
(C2) $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ are both bifunctions fulfilling the hypotheses (H1)-(H4), and $B_1, B_2 : \mathcal{H} \to \mathcal{H}$ are ρ-ism and σ-ism, respectively.
(C3) $A : \mathcal{H} \to \mathcal{H}$ is pseudomonotone and L-Lipschitzian mapping such that $\|Au\| \le \liminf_{n\to\infty} \|Au_n\|$ for each $\{u_n\} \subset C$ with $u_n \rightharpoonup u$.
(C4) for $i = 1, 2, ..., N$, $S_i : \mathcal{H}_i \to \mathcal{H}_i$ is ξ_i-demimetric mapping with $\xi_i \in (-\infty, 1)$ such that $I - S_i$ is demiclosed at zero.
(C5) for $i = 1, 2, ..., N$, $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i$ is a nonzero bounded linear operator with the adjoint $\mathcal{T}_i^* : \mathcal{H}_i \to \mathcal{H}$.
(C6) $\Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(G) \cap (\bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)) \ne \emptyset$ where $G := T^{\Theta_1}_{\eta_1}(I - \eta_1 B_1)T^{\Theta_2}_{\eta_2}(I - \eta_2 B_2)$ for $\eta_1 \in (0, 2\rho)$ and $\eta_2 \in (0, 2\sigma)$.
(C7) the sequences $\{\lambda_n\} \subset (0, 1]$, $\{\gamma_n\} \subset [0, 1)$, $\{\sigma_n\} \subset (0, 1)$ and $\{\varepsilon_n\} \subset (0, \infty)$, satisfy $\limsup_{n\to\infty} \lambda_n < 1$, $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$, $\sum_{n=0}^\infty \sigma_n = \infty$, $\lim_{n\to\infty} \sigma_n = 0$ and $\lim_{n\to\infty} \frac{\varepsilon_n}{\sigma_n} = 0$.

Under the above conditions, we introduce and consider the BPVIP with the GEPS and SCFPP constraints specified below:

$$\text{Seek } x^* \in \Omega \text{ such that } \langle(\alpha F - f)x^*, y - x^*\rangle \ge 0 \ \ \forall y \in \Omega.$$

Algorithm 3.1.
Initialization: Let $\gamma > 0$, $\nu \in (0, 1)$, $\ell \in (0, 1)$, $\{\tau_i\}_{i=1}^N \subset [0, \infty)$, $\bar{\alpha}, \bar{\beta} \in [0, 1]$ and $x_1, x_0, w_0 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Given the iterates x_{n-1}, x_n $(n \ge 1)$, compute x_{n+1} as follows:
Step 1. Put $w_n = x_n + \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1})$ and calculate

$$\begin{cases} p_n = \lambda_n w_n + (1 - \lambda_n)g_n, \\ h_n = T^{\Theta_2}_{\eta_2}(I - \eta_2 B_2)p_n, \\ g_n = T^{\Theta_1}_{\eta_1}(I - \eta_1 B_1)h_n, \end{cases}$$

where $\alpha_n \in [0, \overline{\alpha}_n]$ and $\beta_n \in [0, \overline{\beta}_n]$ such that $\overline{\alpha}_n = \begin{cases} \min\{\bar{\alpha}, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \ne x_{n-1}, \\ \bar{\alpha} & \text{otherwise,} \end{cases}$ and

$$\overline{\beta}_n = \begin{cases} \min\{\bar{\beta}, \frac{\varepsilon_n}{\|w_{n-1} - x_{n-1}\|}\} & \text{if } w_{n-1} \ne x_{n-1}, \\ \bar{\beta} & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Calculate $y_n = P_C(p_n - \zeta_n A p_n)$ and $q_n = P_{C_n}(p_n - \zeta_n A y_n)$ where ζ_n is the largest $\zeta \in \{\gamma, \gamma\ell, \gamma\ell^2, ...\}$ satisfying

$$\zeta\|Ap_n - Ay_n\| \le \nu\|p_n - y_n\|, \tag{3.2}$$

and $C_n = \{y \in \mathcal{H} : \langle p_n - \zeta_n A p_n - y_n, y - y_n\rangle \le 0\}$.

Step 3. Choose two indices $i_n, \imath_n \in \{1, 2, ..., N\}$ such that $\|(I - S_{i_n})\mathcal{T}_{i_n} q_n\| = \max_{i\in\{1,2,...,N\}} \|(I - S_i)\mathcal{T}_i q_n\|$ and $\|(I - S_{\imath_n})\mathcal{T}_{\imath_n} q_n\| = \min_{i\in\{1,2,...,N\}} \|(I - S_i)\mathcal{T}_i q_n\|$.

Step 4. Calculate $u_n = q_n - \tau_{n,i_n}\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n} q_n$ and $v_n = q_n - \tau_{n,\imath_n}\mathcal{T}^*_{\imath_n}(I - S_{\imath_n})\mathcal{T}_{\imath_n} q_n$, where $\tau_{n,i}$ is chosen to be the bounded sequence satisfying

$$0 < \epsilon \le \tau_{n,i} \le \frac{(1 - \xi_{i_n})\|(I - S_i)\mathcal{T}_i q_n\|^2}{\|\mathcal{T}^*_i(I - S_i)\mathcal{T}_i q_n\|^2} - \epsilon \quad \text{if } (I - S_i)\mathcal{T}_i q_n \ne 0, \tag{3.3}$$

otherwise set $\tau_{n,i} = \tau_i$.

Step 5. Calculate $z_n = (1 - \gamma_n)u_n + \gamma_n v_n$ and $x_{n+1} = \sigma_n f(x_n) + (I - \sigma_n \alpha F)z_n$.

Again set $n := n + 1$ and return to Step 1.

First of all, it is easy to see that $\mathrm{VI}(C, A)$ is nonempty, closed and convex in $\mathcal{H}$. In terms of Proposition 2.4, we know that $\mathrm{Fix}(G)$ is also nonempty, closed and convex in $\mathcal{H}$.

We claim that Ω is nonempty, closed, and convex in $\mathcal{H}$. Indeed, the conditions (C4)-(C5) ensure that $\bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)$ is a closed and convex set. To show this, let $\{x_n\} \subset \bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)$ s.t. $x_n \to x^*$. It is clear that $\mathcal{T}_i x_n \to \mathcal{T}_i x^*$ for $i = 1, 2, ..., N$. By the demiclosedness of $I - S_i$ at zero, we obtain $x^* \in \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)$. Hence, $\bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)$ is closed in $\mathcal{H}$. In addition, let us show the convexity of $\bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)$. Consider $x, y \in \bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)$ and $\alpha \in [0, 1]$. This ensures that $\mathcal{T}_i x, \mathcal{T}_i y \in \mathrm{Fix}(S_i)$ for each i. Because $\mathrm{Fix}(S_i)$ is convex (due to Lemma 2.1), $\alpha\mathcal{T}_i x + (1-\alpha)\mathcal{T}_i y \in \mathrm{Fix}(S_i)$, which implies that $\mathcal{T}_i(\alpha x + (1-\alpha)y) = \alpha\mathcal{T}_i x + (1-\alpha)\mathcal{T}_i y \in \mathrm{Fix}(S_i)$. As a result, $\alpha x + (1-\alpha)y \in \bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)$. Therefore, $\Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(G) \cap (\bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i))$ is nonempty, closed, and convex in $\mathcal{H}$. Since Ω is a nonempty closed convex set, then there exists a unique solution $x^* \in \Omega$ to the following VIP by condition (C1).

$$\langle(\alpha F - f)x^*, y - x^*\rangle \ge 0 \quad \forall y \in \Omega. \tag{3.4}$$

Clearly, the VIP (3.4) is equivalent to the fixed-point equation $x^* = P_\Omega(f + I - \alpha F)x^*$. Now, we show that $P_\Omega(f + I - \alpha F)$ is a contraction mapping. As a matter of fact, for all $x, y \in C$, by Lemma 2.2 we have

$$\|P_\Omega(f + I - \alpha F)x - P_\Omega(f + I - \alpha F)y\| \le [1 - (\zeta - \delta)]\|x - y\|,$$

which implies that $P_\Omega(f + I - \alpha F)$ is a contraction mapping. Banach's contraction mapping principle guarantees that $P_\Omega(f + I - \alpha F)$ has a unique fixed point. Say $x^* \in C$, i.e., $x^* = P_\Omega(f + I - \alpha F)x^*$.

Remark 3.1. From the definitions of $\overline{\alpha}_n$ and $\overline{\beta}_n$ we claim that $\lim_{n\to\infty} \frac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| = 0$ and $\lim_{n\to\infty} \frac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\| = 0$. Indeed, we have $\alpha_n\|x_n - x_{n-1}\| \le \varepsilon_n\ \forall n \ge 1$, which together with $\lim_{n\to\infty} \frac{\varepsilon_n}{\sigma_n} = 0$, implies that $\frac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| \le \frac{\varepsilon_n}{\sigma_n} \to 0$ as $n \to \infty$. In a similar way we find that $\lim_{n\to\infty} \frac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\| = 0$.

Lemma 3.1 [8]. $\min\{\gamma, \frac{\nu\ell}{L}\} \le \zeta_n \le \gamma$.

The following lemmas are quite helpful for the convergence analysis of our algorithm.

Lemma 3.2. Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then, the stepsize $\tau_{n,i}$ formulated in (3.3) is well-defined.

Proof. It is enough to only show that $\|\mathcal{T}_i^*(I-S_i)\mathcal{T}_i q_n\|^2 \neq 0$. Choose any $p \in \Omega$. Because S_i is ξ_i-demimetric mapping, we obtain

$$\begin{aligned} \|q_n - p\|\|\mathcal{T}_i^*(I - S_i)\mathcal{T}_i q_n\| &\geq \langle q_n - p, \mathcal{T}_i^*(I - S_i)\mathcal{T}_i q_n\rangle \\ &= \langle \mathcal{T}_i q_n - \mathcal{T}_i p, (I - S_i)\mathcal{T}_i q_n\rangle \\ &\geq \tfrac{1-\xi_i}{2}\|(I - S_i)\mathcal{T}_i q_n\|^2. \end{aligned} \tag{3.5}$$

In the case of $(I - S_i)\mathcal{T}_i q_n \neq 0$, we know that $\|(I - S_i)\mathcal{T}_i q_n\|^2 > 0$ and hence $\|\mathcal{T}_i^*(I - S_i)\mathcal{T}_i q_n\|^2 > 0$.

Lemma 3.3. Let $p \in \Omega$ and $q = T_{\eta_2}^{\Theta_2}(I - \eta_2 B_2)p$. Then,

$$\begin{aligned} \|q_n - p\|^2 &\leq \|w_n - p\|^2 - (1 - \lambda_n)[\|w_n - p_n\|^2 + \eta_1(2\rho - \eta_1)\|B_1 h_n - B_1 q\|^2 \\ &\quad + \eta_2(2\sigma - \eta_2)\|B_2 p_n - B_2 p\|^2] - (1 - \nu)[\|y_n - q_n\|^2 + \|y_n - p_n\|^2], \end{aligned} \tag{3.6}$$

where $h_n = T_{\eta_2}^{\Theta_2}(I - \eta_2 B_2)p_n$.

Proof. According to Proposition 2.4, there exists a unique point $p_n \in \mathcal{H}$ such that $p_n = \lambda_n w_n + (1-\lambda_n)Gp_n$. Since $p \in C_n$, we have

$$\begin{aligned} \|q_n - p\|^2 &\leq \langle p_n - \zeta_n A y_n - p, q_n - p\rangle \\ &= \tfrac{1}{2}(\|p_n - p\|^2 - \|q_n - p_n\|^2 + \|q_n - p\|^2) - \zeta_n\langle A y_n, q_n - p\rangle, \end{aligned} \tag{3.7}$$

which implies that

$$\|q_n - p\|^2 \leq \|p_n - p\|^2 - \|q_n - p_n\|^2 - 2\zeta_n\langle A y_n, q_n - p\rangle. \tag{3.8}$$

Noting that $q_n = P_{C_n}(p_n - \zeta_n A y_n)$, we have $\langle p_n - \zeta_n A p_n - y_n, q_n - y_n\rangle \leq 0$. Owing to the pseudomonotonicity of A, by (3.2), we get

$$\begin{aligned} \|q_n - p\|^2 &\leq \|p_n - p\|^2 - \|q_n - p_n\|^2 - 2\zeta_n\langle A y_n, y_n - p + q_n - y_n\rangle \\ &\leq \|p_n - p\|^2 - \|q_n - p_n\|^2 - 2\zeta_n\langle A y_n, q_n - y_n\rangle \\ &= \|p_n - p\|^2 + 2\langle p_n - \zeta_n A y_n - y_n, q_n - y_n\rangle - \|y_n - p_n\|^2 - \|q_n - y_n\|^2 \\ &= \|p_n - p\|^2 - \|q_n - y_n\|^2 + 2\langle p_n - \zeta_n A p_n - y_n, q_n - y_n\rangle - \|y_n - p_n\|^2 \\ &\quad + 2\zeta_n\langle A p_n - A y_n, q_n - y_n\rangle \\ &\leq \|p_n - p\|^2 + 2\nu\|p_n - y_n\|\|q_n - y_n\| - \|q_n - y_n\|^2 - \|y_n - p_n\|^2 \\ &\leq \|p_n - p\|^2 - \|y_n - p_n\|^2 + \nu(\|p_n - y_n\|^2 + \|q_n - y_n\|^2) - \|q_n - y_n\|^2 \\ &= \|p_n - p\|^2 - (1 - \nu)[\|y_n - p_n\|^2 + \|y_n - q_n\|^2]. \end{aligned} \tag{3.9}$$

Observe that $g_n = T_{\eta_1}^{\Theta_1}(I - \eta_1 B_1)h_n$, $h_n = T_{\eta_2}^{\Theta_2}(I - \eta_2 B_2)p_n$, and $q = T_{\eta_2}^{\Theta_2}(I - \eta_2 B_2)p$. Then $g_n = Gp_n$. Applying Proposition 2.3 to get $\|g_n - p\|^2 \leq \|h_n - q\|^2 - \eta_1(2\rho - \eta_1)\|B_1 h_n - B_1 q\|^2$ and $\|h_n - q\|^2 \leq \|p_n - p\|^2 - \eta_2(2\sigma - \eta_2)\|B_2 p_n - B_2 p\|^2$. Thus,

$$\|g_n - p\|^2 \leq \|p_n - p\|^2 - \eta_1(2\rho - \eta_1)\|B_1 h_n - B_1 q\|^2 - \eta_2(2\sigma - \eta_2)\|B_2 p_n - B_2 p\|^2. \tag{3.10}$$

Besides, thanks to $p_n = \lambda_n w_n + (1 - \lambda_n)g_n$, we get $\|p_n - p\|^2 \leq \lambda_n\langle w_n - p, p_n - p\rangle + (1 - \lambda_n)\|p_n - p\|^2$, which results in $\|p_n - p\|^2 \leq \langle w_n - p, p_n - p\rangle = \frac{1}{2}[\|w_n - p\|^2 + \|p_n - p\|^2 - \|w_n - p_n\|^2]$. So,

$$\|p_n - p\|^2 \leq \|w_n - p\|^2 - \|w_n - p_n\|^2. \tag{3.11}$$

Thus,

$$\begin{aligned} \|p_n - p\|^2 &\leq (1 - \lambda_n)\|g_n - p\|^2 + \lambda_n\|w_n - p\|^2 \\ &\leq \lambda_n\|w_n - p\|^2 + (1 - \lambda_n)[\|p_n - p\|^2 - \eta_1(2\rho - \eta)\|B_1 h_n - B_1 q\|^2 \\ &\quad - \eta_2(2\sigma - \eta_2)\|B_2 p_n - B_2 p\|^2] \\ &\leq \lambda_n\|w_n - p\|^2 + (1 - \lambda_n)[\|w_n - p\|^2 - \|w_n - p_n\|^2 - \eta_2(2\sigma - \eta_2) \\ &\quad \times\|B_2 p_n - B_2 p\|^2 - \eta_1(2\rho - \eta_1)\|B_1 h_n - B_1 q\|^2] \\ &= \|w_n - p\|^2 - (1 - \lambda_n)[\|w_n - p_n\|^2 + \eta_1(2\rho - \eta_1)\|B_1 h_n - B_1 q\|^2 \\ &\quad + \eta_2(2\sigma - \eta_2)\|B_2 p_n - B_2 p\|^2], \end{aligned} \tag{3.12}$$

which, together with (3.9), yields

$$\begin{aligned}\|q_n - p\|^2 &\le \|p_n - p\|^2 - (1-\nu)[\|y_n - p_n\|^2 + \|y_n - q_n\|^2] \\ &\le \|w_n - p\|^2 - (1-\lambda_n)[\|w_n - p_n\|^2 + \eta_1(2\rho - \eta_1)\|B_1 h_n - B_1 q\|^2 \\ &\quad + \eta_2(2\sigma - \eta_2)\|B_2 p_n - B_2 p\|^2] - (1-\nu)[\|y_n - p_n\|^2 + \|y_n - q_n\|^2].\end{aligned} \tag{3.13}$$

This ensures that the conclusion holds.

Next, we show that the sequence $\{x_n\}$ in Algorithm 3.1 is bounded.

Lemma 3.4. Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then, $\{x_n\}$ is bounded.

Proof. We consider $x^* \in \Omega$ the unique solution of the VIP (3.4). This also means that there exists the unique solution $x^* \in \Omega$ to the BPVIP with the GEPS and SCFPP constraint. By the definition of w_n and the triangle inequality we obtain that

$$\begin{aligned}\|w_n - x^*\| &= \|x_n + \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1}) - x^*\| \\ &\le \|x_n - x^*\| + \alpha_n\|x_n - x_{n-1}\| + \beta_n\|w_{n-1} - x_{n-1}\| \\ &= \|x_n - x^*\| + \sigma_n \cdot \tfrac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| + \sigma_n \cdot \tfrac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\| \\ &\le \|x_n - x^*\| + 2\sigma_n M_1,\end{aligned} \tag{3.14}$$

where $\sup_{n\ge1}\{\frac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\|, \frac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\|\} \le M_1$ for some $M_1 > 0$.

Using the fact that S_{i_n} is a ξ_{i_n}-demimetric mapping, we get

$$\begin{aligned}\|u_n - x^*\|^2 &= \|q_n - \tau_{n,i_n}\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n - x^*\|^2 \\ &= \|q_n - x^*\|^2 - 2\langle q_n - x^*, \tau_{n,i_n}\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\rangle + \|\tau_{n,i_n}\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &= \|q_n - x^*\|^2 - 2\tau_{n,i_n}\langle \mathcal{T}_{i_n}q_n - \mathcal{T}_{i_n}x^*, (I - S_{i_n})\mathcal{T}_{i_n}q_n\rangle + \tau^2_{n,i_n}\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &\le \|q_n - x^*\|^2 - 2\tau_{n,i_n}\tfrac{1-\xi_{i_n}}{2}\|(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 + \tau^2_{n,i_n}\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &= \|q_n - x^*\|^2 + \tau_{n,i_n}(\tau_{n,i_n}\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 - (1 - \xi_{i_n})\|(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2).\end{aligned} \tag{3.15}$$

For each $n \ge 1$, from the definition of τ_{n,i_n} in (3.3) it follows that

$$\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2(\epsilon + \tau_{n,i_n}) \le (1 - \xi_{i_n})\|(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2,$$

and hence $\epsilon\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \le (1 - \xi_{i_n})\|(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 - \tau_{n,i_n}\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2$. This immediately arrives at

$$\begin{aligned}\tau_{n,i_n}\epsilon\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 &\le \tau_{n,i_n}[(1 - \xi_{i_n})\|(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &\quad - \tau_{n,i_n}\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2].\end{aligned} \tag{3.16}$$

So it follows from (3.15) and (3.16) that

$$\|u_n - x^*\|^2 \le \|q_n - x^*\|^2 - \tau_{n,i_n}\epsilon\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2. \tag{3.17}$$

Similarly, we obtain

$$\|v_n - x^*\|^2 \le \|q_n - x^*\|^2 - \tau_{n,\iota_n}\epsilon\|\mathcal{T}^*_{\iota_n}(I - S_{\iota_n})\mathcal{T}_{\iota_n}q_n\|^2. \tag{3.18}$$

From the convexity of the function $\|\cdot\|^2$ and inequalities (3.17)-(3.18) we have

$$\begin{aligned}\|z_n - x^*\|^2 &\le (1-\gamma_n)\|u_n - x^*\|^2 + \gamma_n\|v_n - x^*\|^2 \\ &\le \|q_n - x^*\|^2 - (1-\gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &\quad - \gamma_n\tau_{n,\iota_n}\epsilon\|\mathcal{T}^*_{\iota_n}(I - S_{\iota_n})\mathcal{T}_{\iota_n}q_n\|^2.\end{aligned} \tag{3.19}$$

In addition, by Lemma 3.3, we get

$$\begin{aligned}\|q_n - x^*\|^2 &\le \|w_n - x^*\|^2 - (1-\lambda_n)[\|w_n - p_n\|^2 + \eta_1(2\rho - \eta_1)\|B_1 h_n - B_1 q\|^2 \\ &\quad + \eta_2(2\sigma - \eta_2)\|B_2 p_n - B_2 p\|^2] - (1-\nu)[\|y_n - q_n\|^2 + \|y_n - p_n\|^2] \\ &\le \|w_n - x^*\|^2.\end{aligned} \tag{3.20}$$

Combining (3.14), (3.19), and (3.20), we obtain

$$\|z_n - x^*\| \le \|q_n - x^*\| \le \|w_n - x^*\| \le \|x_n - x^*\| + 2\sigma_n M_1 \quad \forall n \ge 1. \tag{3.21}$$

For $\alpha \in (0, \frac{2\eta}{\kappa^2})$ and $\{\sigma_n\} \subset (0,1)$, we reduce from Lemma 2.2 that

$$\|(I - \sigma_n \alpha F)z_n - (I - \sigma_n \alpha F)x^*\| \le (1 - \sigma_n \zeta)\|z_n - x^*\|, \tag{3.22}$$

where $\zeta = 1 - \sqrt{1 - \alpha(2\eta - \alpha\kappa^2)} \in (0, 1]$. Using inequalities (3.21) and (3.22) we obtain that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\sigma_n(f(x_n) - f(x^*)) + (I - \sigma_n\alpha F)z_n - (I - \sigma_n\alpha F)x^* + \sigma_n(f - \alpha F)x^*\| \\ &\le \sigma_n\|f(x_n) - f(x^*)\| + \|(I - \sigma_n\alpha F)z_n - (I - \sigma_n\alpha F)x^*\| + \sigma_n\|(f - \alpha F)x^*\| \\ &\le \sigma_n\delta\|x_n - x^*\| + (1 - \sigma_n\zeta)\|z_n - x^*\| + \sigma_n\|(f - \alpha F)x^*\| \\ &\le \sigma_n\delta\|x_n - x^*\| + (1 - \sigma_n\zeta)[\|x_n - x^*\| + 2\sigma_n M_1] + \sigma_n\|(f - \alpha F)x^*\| \\ &\le [1 - \sigma_n(\zeta - \delta)]\|x_n - x^*\| + \sigma_n[2M_1 + \|(f - \alpha F)x^*\|] \\ &= [1 - \sigma_n(\zeta - \delta)]\|x_n - x^*\| + \sigma_n(\zeta - \delta) \cdot \tfrac{2M_1 + \|(f - \alpha F)x^*\|}{\zeta - \delta} \\ &\le \max\{\|x_n - x^*\|, \tfrac{2M_1 + \|(f - \alpha F)x^*\|}{\zeta - \delta}\}, \end{aligned}$$

By induction, we obtain $\|x_n - x^*\| \le \max\{\|x_1 - x^*\|, \frac{2M_1 + \|(f - \alpha F)x^*\|}{\zeta - \delta}\}$ $\forall n \ge 1$. Thus, $\{x_n\}$ is bounded, and so are the sequences $\{q_n\}, \{u_n\}, \{v_n\}, \{w_n\}, \{y_n\}, \{z_n\}, \{Fz_n\}$ and $\{f(x_n)\}$.

Lemma 3.5. Let $\{q_n\}, \{w_n\}, \{x_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by Algorithm 3.1. Suppose that $p_n - y_n \to 0$, $q_n - y_n \to 0$, $w_n - p_n \to 0$ and $q_n - u_n \to 0$. Then $\omega_w(\{x_n\}) \subset \Omega$, with $\omega_w(\{x_n\}) = \{z \in \mathcal{H} : x_{n_k} \rightharpoonup z \text{ for some } \{x_{n_k}\} \subset \{x_n\}\}$.

Proof. Take a fixed $z \in \omega_w(\{x_n\})$ arbitrarily. Then, $\exists \{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup z \in \mathcal{H}$. From Algorithm 3.1, we get $w_n - x_n = \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1})$ $\forall n \ge 1$, and hence

$$\begin{aligned} \|w_n - x_n\| &= \|\alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1})\| \\ &\le \alpha_n\|x_n - x_{n-1}\| + \beta_n\|w_{n-1} - x_{n-1}\| \\ &= \sigma_n \tfrac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| + \sigma_n \tfrac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\|. \end{aligned}$$

Using Remark 3.1, we have

$$\lim_{n\to\infty} \|w_n - x_n\| = 0, \tag{3.23}$$

Due to $w_n - x_n \to 0$, we know that $\exists \{w_{n_k}\} \subset \{w_n\}$ s.t. $w_{n_k} \rightharpoonup z \in \mathcal{H}$. In what follows, let us show that $z \in \Omega$.

Indeed, from $y_n = P_C(p_n - \zeta_n A p_n)$, we have $\langle p_n - \zeta_n A p_n - y_n, y_n - y\rangle \ge 0$ $\forall y \in C$, and hence

$$\frac{1}{\zeta_n}\langle p_n - y_n, y - y_n\rangle + \langle Ap_n, y_n - p_n\rangle \le \langle Ap_n, y - p_n\rangle \quad \forall y \in C. \tag{3.24}$$

According to the Lipschitz continuity of A, $\{Ap_{n_k}\}$ is bounded. Note that $\zeta_n \ge \min\{\gamma, \frac{\nu\ell}{L}\}$. So, from (3.24) we get $\liminf_{k\to\infty}\langle Ap_{n_k}, y - p_{n_k}\rangle \ge 0$ $\forall y \in C$. Observe that $\langle Ay_n, y - y_n\rangle = \langle Ay_n - Ap_n, y - p_n\rangle + \langle Ap_n, y - p_n\rangle + \langle Ay_n, p_n - y_n\rangle$. Since $p_n - y_n \to 0$, we obtain $Ap_n - Ay_n \to 0$ from L-Lipschitz continuity of A, which together with (3.24) gives

$$\liminf_{k\to\infty}\langle Ay_{n_k}, y - y_{n_k}\rangle \ge 0 \quad \forall y \in C. \tag{3.25}$$

We now take a sequence $\{\delta_k\} \subset (0,1)$ such that $\delta_k \downarrow 0$ as $k \to \infty$. For all $k \ge 1$, we denote by m_k the smallest positive integer such that

$$\langle Ay_{n_j}, y - y_{n_j}\rangle + \delta_k \ge 0 \quad \forall j \ge m_k. \tag{3.26}$$

Since $\{\delta_k\}$ is decreasing, it is clear that $\{m_k\}$ is increasing.

Again from the assumption on A, we know that $\liminf_{k\to\infty}\|Ay_{n_k}\| \geq \|Az\|$. If $Az = 0$, then z is a solution, i.e., $z \in \mathrm{VI}(C, A)$. Let $Az \neq 0$. Then we have $0 < \|Az\| \leq \liminf_{k\to\infty}\|Ay_{n_k}\|$. Without loss of generality, we may assume that $Ay_{n_k} \neq 0\ \forall k \geq 1$. Noticing that $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $Ay_{n_k} \neq 0\ \forall k \geq 1$, we set $v_{m_k} = \frac{Ay_{m_k}}{\|Ay_{m_k}\|^2}$, we get $\langle Ay_{m_k}, v_{m_k}\rangle = 1\ \forall k \geq 1$. So, from (3.26) we get $\langle Ay_{m_k}, y + \delta_k v_{m_k} - y_{m_k}\rangle \geq 0\ \forall k \geq 1$. Again, by the pseudomonotonicity of A, we have $\langle A(y + \delta_k v_{m_k}), y + \delta_k v_{m_k} - y_{m_k}\rangle \geq 0\ \forall k \geq 1$. This immediately yields

$$\langle Ay, y - y_{m_k}\rangle \geq \langle Ay - A(y + \delta_k v_{m_k}), y + \delta_k v_{m_k} - y_{m_k}\rangle - \delta_k\langle Ay, v_{m_k}\rangle \quad \forall k \geq 1. \tag{3.27}$$

We claim that $\lim_{k\to\infty}\delta_k v_{m_k} = 0$. Indeed, from $x_{n_k} \rightharpoonup z$ and $x_n - y_n \to 0$ (due to $w_n - x_n \to 0$, $p_n - y_n \to 0$ and $w_n - p_n \to 0$), we obtain $y_{n_k} \rightharpoonup z$. So, $\{y_n\} \subset C$ guarantees $z \in C$. Note that $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $\delta_k \downarrow 0$ as $k \to \infty$. So it follows that

$$0 \leq \limsup_{k\to\infty}\|\delta_k v_{m_k}\| = \limsup_{k\to\infty}\frac{\delta_k}{\|Ay_{m_k}\|} \leq \frac{\limsup_{k\to\infty}\delta_k}{\liminf_{k\to\infty}\|Ay_{n_k}\|} = 0.$$

Hence we get $\delta_k v_{m_k} \to 0$.

Next, we claim that $z \in \Omega$. Indeed, letting $k \to \infty$, we deduce that the right-hand side of (3.27) tends to zero by the uniform continuity of A, the boundedness of $\{y_{m_k}\}, \{v_{m_k}\}$ and the limit $\lim_{k\to\infty}\delta_k v_{m_k} = 0$. Thus, we get $\langle Ay, y - z\rangle = \liminf_{k\to\infty}\langle Ay, y - y_{m_k}\rangle \geq 0\ \forall y \in C$. By Lemma 2.3 we have $z \in \mathrm{VI}(C, A)$. Furthermore, we claim that $\mathcal{T}_i z \in \mathrm{Fix}(S_i)$ for $i = 1, 2, ..., N$. In fact, noticing $u_n = q_n - \tau_{n,i_n}\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n$, from $0 < \epsilon \leq \tau_{n,i_n}$ and $q_n - u_n \to 0$, we get

$$\epsilon\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\| \leq \tau_{n,i_n}\|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\| = \|q_n - u_n\| \to 0 \quad (n \to \infty),$$

which together with the ξ_{i_n}-demimetric property of S_{i_n}, leads to

$$\begin{aligned}\tfrac{1-\xi_{i_n}}{2}\max_{i\in\{1,2,...,N\}}\|(I - S_i)\mathcal{T}_i q_n\|^2 &= \tfrac{1-\xi_{i_n}}{2}\|(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &\leq \langle (I - S_{i_n})\mathcal{T}_{i_n}q_n, \mathcal{T}_{i_n}(q_n - x^*)\rangle \\ &\leq \|\mathcal{T}^*_{i_n}(I - S_{i_n})\mathcal{T}_{i_n}q_n\|\|q_n - x^*\| \to 0 \quad (n \to \infty).\end{aligned} \tag{3.28}$$

This ensures that $(I - S_i)\mathcal{T}_i q_n \to 0$ for $i = 1, 2, ..., N$. By $p_n - y_n \to 0$, $q_n - y_n \to 0$, $w_n - p_n \to 0$ (due to the assumptions) and $w_n - x_n \to 0$, it follows that

$$\|q_n - x_n\| \leq \|q_n - y_n\| + \|y_n - p_n\| + \|p_n - w_n\| + \|w_n - x_n\| \to 0 \quad (n \to \infty),$$

which together with $x_{n_k} \rightharpoonup z$, leads to $q_{n_k} \rightharpoonup z$. Since each $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i$ is a bounded linear operator, we know that $\mathcal{T}_i$ is weakly continuous from $\mathcal{H}$ to $\mathcal{H}_i$ for $i = 1, 2, ..., N$. Hence, we obtain that $\mathcal{T}_i q_{n_k} \rightharpoonup \mathcal{T}_i z$ for $i = 1, 2, ..., N$. Using the demiclosedness assumption of each $(I - S_i)$ at zero, we infer from $(I - S_i)\mathcal{T}_i q_{n_k} \to 0$ that $\mathcal{T}_i z \in \mathrm{Fix}(S_i)$ for $i = 1, 2, ..., N$. As a result, $z \in \bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)$. Finally, using the definition of p_n, we have

$$(1 - \lambda_n)\|Gp_n - p_n\| = (1 - \lambda_n)\|g_n - p_n\| = \lambda_n\|w_n - p_n\| \leq \|w_n - p_n\| \to 0 \quad (n \to \infty).$$

Since $0 < \liminf_{n\to\infty}(1 - \lambda_n)$, we get $\lim_{n\to\infty}\|Gp_n - p_n\| = 0$. Note that G is nonexpansive (due to Proposition 2.4) and $p_{n_k} \rightharpoonup z$ (due to $w_n - p_n \to 0$). Thus, we deduce from the demiclosedness of $I - G$ at zero that $z \in \mathrm{Fix}(G)$. Therefore, $z \in \mathrm{VI}(C, A) \cap \mathrm{Fix}(G) \cap (\bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)) = \Omega$. This completes the proof.

Theorem 3.1. Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then $\{x_n\}$ converges strongly to the unique solution $z^* \in \Omega$ to the BPVIP with the GEPS and SCFPP constraints.

Proof. First of all, in terms of Lemma 3.4 we obtain that $\{x_n\}$ is bounded. It is known that there exists the unique solution $x^* \in \Omega$ to the BPVIP with the GEPS and SCFPP constraints, that is, the VIP (3.4) has the unique solution $x^* \in \Omega$. By the definition of w_n and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|w_n - x^*\|^2 &= \|x_n - x^*\|^2 + \|\alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1})\|^2 \\ &\quad +2\langle x_n - x^*, \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1})\rangle \\ &= \|x_n - x^*\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + \beta_n^2\|w_{n-1} - x_{n-1}\|^2 \\ &\quad +2\alpha_n\beta_n\langle x_n - x_{n-1}, w_{n-1} - x_{n-1}\rangle + 2\alpha_n\langle x_n - x^*, x_n - x_{n-1}\rangle \\ &\quad +2\beta_n\langle x_n - x^*, w_{n-1} - x_{n-1}\rangle \\ &\le \|x_n - x^*\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + \beta_n^2\|w_{n-1} - x_{n-1}\|^2 \\ &\quad +2\alpha_n\beta_n\|x_n - x_{n-1}\|\|w_{n-1} - x_{n-1}\| + 2\alpha_n\|x_n - x^*\|\|x_n - x_{n-1}\| \\ &\quad +2\beta_n\|x_n - x^*\|\|w_{n-1} - x_{n-1}\| \\ &= \|x_n - x^*\|^2 + \beta_n\|w_{n-1} - x_{n-1}\|(2\|x_n - x^*\| + \beta_n\|w_{n-1} - x_{n-1}\|) \\ &\quad +\alpha_n\|x_n - x_{n-1}\|(\alpha_n\|x_n - x_{n-1}\| + 2\beta_n\|w_{n-1} - x_{n-1}\| + 2\|x_n - x^*\|).\end{aligned}$$

Since $\{x_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{w_n\}$ are bounded, it can be readily seen that

$$\|w_n - x^*\|^2 \le \|x_n - x^*\|^2 + M_2\alpha_n\|x_n - x_{n-1}\| + M_3\beta_n\|w_{n-1} - x_{n-1}\|, \tag{3.29}$$

where $\sup_{n\ge1}\{\alpha_n\|x_n - x_{n-1}\| + 2\beta_n\|w_{n-1} - x_{n-1}\| + 2\|x_n - x^*\|\} \le M_2$ and $\sup_{n\ge1}\{2\|x_n - x^*\| + \beta_n\|w_{n-1} - x_{n-1}\|\} \le M_3$ for some $M_2 > 0$ and $M_3 > 0$.

We divide the rest of the proof into several steps to show the theorem's conclusion.

Step 1. We claim that

$$\begin{aligned}&(1 - \sigma_n\zeta)\{(1 - \lambda_n)\|w_n - p_n\|^2 + (1 - \nu)[\|y_n - q_n\|^2 + \|y_n - p_n\|^2] \\ &\quad +(1 - \gamma_n)\epsilon^2\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 + \gamma_n\epsilon^2\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\} \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \sigma_nM_5 \quad \forall n \ge 1,\end{aligned}$$

for some $M_5 > 0$. In fact, noticing the inequality $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle\ \forall x, y \in \mathcal{H}$, from (3.22) we obtain

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 = \|\sigma_nf(x_n) + (I - \sigma_n\alpha F)z_n - x^*\|^2 \\ &= \|\sigma_n(f(x_n) - f(x^*)) + (I - \sigma_n\alpha F)z_n - (I - \sigma_n\alpha F)x^* + \sigma_n(f - \alpha F)x^*\|^2 \\ &\le \|\sigma_n(f(x_n) - f(x^*)) + (I - \sigma_n\alpha F)z_n - (I - \sigma_n\alpha F)x^*\|^2 \\ &\quad -2\sigma_n\langle(\alpha F - f)x^*, x_{n+1} - x^*\rangle \\ &\le [\sigma_n\delta\|x_n - x^*\| + (1 - \sigma_n\zeta)\|z_n - x^*\|]^2 - 2\sigma_n\langle(\alpha F - f)x^*, x_{n+1} - x^*\rangle \\ &\le \sigma_n\delta\|x_n - x^*\|^2 + (1 - \sigma_n\zeta)\|z_n - x^*\|^2 + \sigma_nM_4 \quad \forall n \ge 1,\end{aligned} \tag{3.30}$$

(due to $\sigma_n\delta + (1 - \sigma_n\zeta) = 1 - \sigma_n(\zeta - \delta) < 1$) where $\sup_{n\ge1}\{2\|(\alpha F - f)x^*\|\|x_n - x^*\|\} \le M_4$ for some $M_4 > 0$.

Using Lemma 3.3 we deduce from (3.19), (3.29), and (3.30) that for all $n \ge n_0$,

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \le \sigma_n\delta\|x_n - x^*\|^2 + (1 - \sigma_n\zeta)\|z_n - x^*\|^2 + \sigma_nM_4 \\ &\le \sigma_n\delta\|x_n - x^*\|^2 + (1 - \sigma_n\zeta)[\|q_n - x^*\|^2 - (1 - \gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &\quad -\gamma_n\tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2] + \sigma_nM_4 \\ &\le \sigma_n\delta\|x_n - x^*\|^2 + (1 - \sigma_n\zeta)\{\|w_n - x^*\|^2 - (1 - \lambda_n)\|w_n - p_n\|^2 \\ &\quad -(1 - \nu)[\|y_n - q_n\|^2 + \|y_n - p_n\|^2] - (1 - \gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &\quad -\gamma_n\tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\} + \sigma_nM_4 \\ &\le \sigma_n\delta\|x_n - x^*\|^2 + (1 - \sigma_n\zeta)\{\|x_n - x^*\|^2 + M_2\alpha_n\|x_n - x_{n-1}\| \\ &\quad +M_3\beta_n\|w_{n-1} - x_{n-1}\| - (1 - \lambda_n)\|w_n - p_n\|^2 - (1 - \nu)[\|y_n - q_n\|^2 \\ &\quad +\|y_n - p_n\|^2] - (1 - \gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &\quad -\gamma_n\tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\} + \sigma_nM_4 \\ &\le [1 - \sigma_n(\zeta - \delta)]\|x_n - x^*\|^2 + M_2\alpha_n\|x_n - x_{n-1}\| + M_3\beta_n\|w_{n-1} - x_{n-1}\| \\ &\quad -(1 - \sigma_n\zeta)\{(1 - \lambda_n)\|w_n - p_n\|^2 + (1 - \nu)[\|y_n - q_n\|^2 \\ &\quad +\|y_n - p_n\|^2] + (1 - \gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &\quad +\gamma_n\tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\} + \sigma_nM_4\end{aligned}$$

$$\begin{aligned}
&\le \|x_n - x^*\|^2 + M_2\alpha_n\|x_n - x_{n-1}\| + M_3\beta_n\|w_{n-1} - x_{n-1}\| \\
&\quad -(1-\sigma_n\zeta)\{(1-\lambda_n)\|w_n - p_n\|^2 + (1-\nu)[\|y_n - q_n\|^2 \\
&\quad +\|y_n - p_n\|^2] + (1-\gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\
&\quad +\gamma_n\tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\} + \sigma_n M_4 \\
&= \|x_n - x^*\|^2 - (1-\sigma_n\zeta)\{(1-\lambda_n)\|w_n - p_n\|^2 + (1-\nu)[\|y_n - q_n\|^2 \\
&\quad +\|y_n - p_n\|^2] + (1-\gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\
&\quad +\gamma_n\tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\} + \sigma_n[M_2\tfrac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| \\
&\quad +M_3\tfrac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\| + M_4] \\
&\le \|x_n - x^*\|^2 - (1-\sigma_n\zeta)\{(1-\lambda_n)\|w_n - p_n\|^2 + (1-\nu)[\|y_n - q_n\|^2 \\
&\quad +\|y_n - p_n\|^2] + (1-\gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\
&\quad +\gamma_n\tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\} + \sigma_n M_5,
\end{aligned} \tag{3.31}$$

where $\sup_{n\ge1}\{M_2\frac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| + M_3\frac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\| + M_4\} \le M_5$ for some $M_5 > 0$.

Step 2. We claim that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le [1 - \sigma_n(\zeta-\delta)]\|x_n - x^*\|^2 + \sigma_n(\zeta-\delta)\{\tfrac{M_2}{\zeta-\delta}\cdot\tfrac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| \\
&\quad +\tfrac{M_3}{\zeta-\delta}\cdot\tfrac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\| + \tfrac{2}{\zeta-\delta}\langle(f-\alpha F)x^*, x_{n+1} - x^*\rangle\} \quad \forall n \ge 1.
\end{aligned}$$

Indeed, from (3.21), (3.29), and (3.30) it follows that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \sigma_n\delta\|x_n - x^*\|^2 + (1-\sigma_n\zeta)\|z_n - x^*\|^2 + 2\sigma_n\langle(f-\alpha F)x^*, x_{n+1} - x^*\rangle \\
&\le \sigma_n\delta\|x_n - x^*\|^2 + (1-\sigma_n\zeta)\|w_n - x^*\|^2 + 2\sigma_n\langle(f-\alpha F)x^*, x_{n+1} - x^*\rangle \\
&\le \sigma_n\delta\|x_n - x^*\|^2 + (1-\sigma_n\zeta)[\|x_n - x^*\|^2 + M_2\alpha_n\|x_n - x_{n-1}\| \\
&\quad +M_3\beta_n\|w_{n-1} - x_{n-1}\|] + 2\sigma_n\langle(f-\alpha F)x^*, x_{n+1} - x^*\rangle \\
&\le [1 - \sigma_n(\zeta-\delta)]\|x_n - x^*\|^2 + M_2\alpha_n\|x_n - x_{n-1}\| + M_3\beta_n\|w_{n-1} - x_{n-1}\| \\
&\quad +2\sigma_n\langle(f-\alpha F)x^*, x_{n+1} - x^*\rangle \\
&= [1 - \sigma_n(\zeta-\delta)]\|x_n - x^*\|^2 + \sigma_n(\zeta-\delta)\{\tfrac{M_2}{\zeta-\delta}\cdot\tfrac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| \\
&\quad +\tfrac{M_3}{\zeta-\delta}\cdot\tfrac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\| + \tfrac{2}{\zeta-\delta}\langle(f-\alpha F)x^*, x_{n+1} - x^*\rangle\} \quad \forall n \ge 1.
\end{aligned} \tag{3.32}$$

Step 3. We claim that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ to the VIP (3.4). Indeed, putting $\mathbf{\Lambda}_n = \|x_n - x^*\|^2$, we show the convergence of $\{\mathbf{\Lambda}_n\}$ to zero by the following two cases.

Case 1. Suppose that there exists an integer $m_0 \ge 1$ such that $\{\mathbf{\Lambda}_n\}$ is nonincreasing. Then the limit $\lim_{n\to\infty}\mathbf{\Lambda}_n = d < +\infty$ and $\lim_{n\to\infty}(\mathbf{\Lambda}_n - \mathbf{\Lambda}_{n+1}) = 0$. From (3.31) we obtain

$$\begin{aligned}
&(1-\sigma_n\zeta)\{(1-\lambda_n)\|w_n - p_n\|^2 + (1-\nu)[\|y_n - q_n\|^2 + \|y_n - p_n\|^2] \\
&\quad +(1-\gamma_n)\epsilon^2\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 + \gamma_n\epsilon^2\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\} \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \sigma_n M_5 \\
&= \mathbf{\Lambda}_n - \mathbf{\Lambda}_{n+1} + \sigma_n M_5.
\end{aligned}$$

Since $\nu \in (0,1), 0 < \liminf_{n\to\infty}(1-\lambda_n),\ 0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$, $\sigma_n \to 0$ and $\mathbf{\Lambda}_n - \mathbf{\Lambda}_{n+1} \to 0$, one has that

$$\lim_{n\to\infty}\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\| = \lim_{n\to\infty}\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n\| = 0, \tag{3.33}$$

and $\lim_{n\to\infty}\|w_n - p_n\| = \lim_{n\to\infty}\|y_n - q_n\| = \lim_{n\to\infty}\|y_n - p_n\| = 0$, which implies $\|x_n - p_n\| \le \|x_n - w_n\| + \|w_n - p_n\| \to 0 \quad (n\to\infty)$ and

$$\|w_n - q_n\| \le \|w_n - p_n\| + \|p_n - y_n\| + \|y_n - q_n\| \to 0 \quad (n\to\infty).$$

Noticing $u_n = q_n - \tau_{n,i_n}\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n$, $v_n = q_n - \tau_{n,\imath_n}\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n$ and the boundedness of $\{\tau_{n,i}\}$, from (3.33) we obtain that

$$\|q_n - u_n\| = \tau_{n,i_n}\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\| \to 0 \quad (n\to\infty),$$

and

$$\|q_n - v_n\| = \tau_{n,\imath_n}\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n}q_n\| \to 0 \quad (n\to\infty). \tag{3.34}$$

So it follows that

$$\begin{aligned}\|q_n - z_n\| &\le (1-\gamma_n)\|q_n - u_n\| + \gamma_n\|q_n - v_n\| \\ &\le \|q_n - u_n\| + \|q_n - v_n\| \to 0 \quad (n\to\infty),\end{aligned}$$

and

$$\begin{aligned}\|x_n - z_n\| &\le \|x_n - y_n\| + \|y_n - q_n\| + \|q_n - z_n\| \\ &\le \|x_n - p_n\| + \|p_n - y_n\| + \|y_n - q_n\| + \|q_n - z_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{3.35}$$

Since $x_n - z_n \to 0,\ \sigma_n \to 0$ and $\{f(x_n)\}, \{Fz_n\}$ are bounded, from Algorithm 3.1 we obtain that

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \|x_{n+1} - z_n\| + \|z_n - x_n\| \\ &\le \sigma_n(\|f(x_n)\| + \|\alpha F z_n\|) + \|z_n - x_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{3.36}$$

In addition, from the boundedness of $\{x_n\}$ it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (f-\alpha F)x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle (f-\alpha F)x^*, x_{n_k} - x^*\rangle. \tag{3.37}$$

Since $\mathcal{H}$ is reflexive and $\{x_n\}$ is bounded, we may assume, without loss of generality, that $x_{n_k} \rightharpoonup \widetilde{x}$. Thus, from (3.37) one gets

$$\begin{aligned}\limsup_{n\to\infty}\langle (f-\alpha F)x^*, x_n - x^*\rangle &= \lim_{k\to\infty}\langle (f-\alpha F)x^*, x_{n_k} - x^*\rangle \\ &= \langle (f-\alpha F)x^*, \widetilde{x} - x^*\rangle.\end{aligned} \tag{3.38}$$

Since $p_n - y_n \to 0,\ q_n - y_n \to 0,\ w_n - p_n \to 0$ and $q_n - u_n \to 0$, by Lemma 3.5 we deduce that $\widetilde{x} \in \omega_w(\{x_n\}) \subset \Omega$. Hence from (3.4) and (3.38) one gets

$$\limsup_{n\to\infty}\langle (f-\alpha F)x^*, x_n - x^*\rangle = \langle (f-\alpha F)x^*, \widetilde{x} - x^*\rangle \le 0, \tag{3.39}$$

which together with (3.36), leads to

$$\begin{aligned}&\limsup_{n\to\infty}\langle (f-\alpha F)x^*, x_{n+1} - x^*\rangle \\ &= \limsup_{n\to\infty}[\langle (f-\alpha F)x^*, x_{n+1} - x_n\rangle + \langle (f-\alpha F)x^*, x_n - x^*\rangle] \\ &\le \limsup_{n\to\infty}[\|(f-\alpha F)x^*\|\|x_{n+1} - x_n\| + \langle (f-\alpha F)x^*, x_n - x^*\rangle] \le 0.\end{aligned} \tag{3.40}$$

Note that $\{\sigma_n(\zeta - \delta)\} \subset [0,1],\ \sum_{n=1}^{\infty}\sigma_n(\zeta-\delta) = \infty$, and

$$\limsup_{n\to\infty}\{\frac{M_2}{\zeta-\delta}\cdot\frac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| + \frac{M_3}{\zeta-\delta}\cdot\frac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\| + \frac{2}{\zeta-\delta}\langle (f-\alpha F)x^*, x_{n+1} - x^*\rangle\} \le 0.$$

Consequently, applying Lemma 2.4 to (3.32), one has $\lim_{n\to\infty}\|x_n - x^*\|^2 = 0$.

Case 2. Suppose that $\exists\{\mathbf{\Lambda}_{n_k}\} \subset \{\mathbf{\Lambda}_n\}$ s.t. $\mathbf{\Lambda}_{n_k} < \mathbf{\Lambda}_{n_k+1}\ \forall k \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\phi : \mathcal{N} \to \mathcal{N}$ by

$$\phi(n) := \max\{k \le n : \mathbf{\Lambda}_k < \mathbf{\Lambda}_{k+1}\}.$$

By Lemma 2.5, we get

$$\mathbf{\Lambda}_{\phi(n)} \le \mathbf{\Lambda}_{\phi(n)+1} \quad \text{and} \quad \mathbf{\Lambda}_n \le \mathbf{\Lambda}_{\phi(n)+1}.$$

From (3.31) we have

$$\begin{aligned}&(1-\sigma_{\phi(n)}\zeta)\{(1-\lambda_{\phi(n)})\|w_{\phi(n)} - p_{\phi(n)}\|^2 + (1-\nu)[\|y_{\phi(n)} - q_{\phi(n)}\|^2 + \|y_{\phi(n)} - p_{\phi(n)}\|^2] \\ &\quad + (1-\gamma_{\phi(n)})\epsilon^2\|\mathcal{T}^*_{i_{\phi(n)}}(I - S_{i_{\phi(n)}})\mathcal{T}_{i_{\phi(n)}}q_{\phi(n)}\|^2 + \gamma_{\phi(n)}\epsilon^2\|\mathcal{T}^*_{\imath_{\phi(n)}}(I - S_{\imath_{\phi(n)}})\mathcal{T}_{\imath_{\phi(n)}}q_{\phi(n)}\|^2\} \\ &\le \|x_{\phi(n)} - x^*\|^2 - \|x_{\phi(n)+1} - x^*\|^2 + \sigma_{\phi(n)}M_5 \\ &= \mathbf{\Lambda}_{\phi(n)} - \mathbf{\Lambda}_{\phi(n)+1} + \sigma_{\phi(n)}M_5,\end{aligned} \tag{3.41}$$

which immediately yields

$$\lim_{n\to\infty}\|\mathcal{T}^*_{i_{\phi(n)}}(I-S_{i_{\phi(n)}})\mathcal{T}_{i_{\phi(n)}}q_{\phi(n)}\| = \lim_{n\to\infty}\|\mathcal{T}^*_{\imath_{\phi(n)}}(I-S_{\imath_{\phi(n)}})\mathcal{T}_{\imath_{\phi(n)}}q_{\phi(n)}\| = 0,$$

and $\lim_{n\to\infty}\|w_{\phi(n)}-p_{\phi(n)}\| = \lim_{n\to\infty}\|y_{\phi(n)}-q_{\phi(n)}\| = \lim_{n\to\infty}\|y_{\phi(n)}-p_{\phi(n)}\| = 0$, which hence leads to

$$\lim_{n\to\infty}\|x_{\phi(n)}-p_{\phi(n)}\| = \lim_{n\to\infty}\|w_{\phi(n)}-q_{\phi(n)}\| = 0.$$

Using the same inferences as in the proof of Case 1, we deduce that

$$\lim_{n\to\infty}\|q_{\phi(n)}-u_{\phi(n)}\| = \lim_{n\to\infty}\|q_{\phi(n)}-v_{\phi(n)}\| = 0,$$

$$\lim_{n\to\infty}\|x_{\phi(n)}-z_{\phi(n)}\| = \lim_{n\to\infty}\|x_{\phi(n)+1}-x_{\phi(n)}\| = 0,$$

and

$$\limsup_{n\to\infty}\langle(f-\alpha F)x^*, x_{\phi(n)+1}-x^*\rangle \le 0. \tag{3.42}$$

On the other hand, from (3.32) we obtain

$$\begin{aligned}\sigma_{\phi(n)}(\zeta-\delta)\mathbf{\Lambda}_{\phi(n)} \quad &\le \mathbf{\Lambda}_{\phi(n)} - \mathbf{\Lambda}_{\phi(n)+1} + \sigma_{\phi(n)}(\zeta-\delta)\{\tfrac{M_2}{\zeta-\delta}\cdot\tfrac{\alpha_{\phi(n)}}{\sigma_{\phi(n)}}\|x_{\phi(n)}-x_{\phi(n)-1}\| \\ &\quad + \tfrac{M_3}{\zeta-\delta}\cdot\tfrac{\beta_{\phi(n)}}{\sigma_{\phi(n)}}\|w_{\phi(n)-1}-x_{\phi(n)-1}\| + \tfrac{2}{\zeta-\delta}\langle(f-\alpha F)x^*, x_{\phi(n)+1}-x^*\rangle\},\end{aligned}$$

which implies

$$\begin{aligned}&\limsup_{n\to\infty}\mathbf{\Lambda}_{\phi(n)} \le \limsup_{n\to\infty}[\tfrac{M_2}{\zeta-\delta}\cdot\tfrac{\alpha_{\phi(n)}}{\sigma_{\phi(n)}}\cdot\|x_{\phi(n)}-x_{\phi(n)-1}\| \\ &\quad + \tfrac{M_3}{\zeta-\delta}\cdot\tfrac{\beta_{\phi(n)}}{\sigma_{\phi(n)}}\cdot\|w_{\phi(n)-1}-x_{\phi(n)-1}\| + \tfrac{2}{\zeta-\delta}\langle(f-\alpha F)x^*, x_{\phi(n)+1}-x^*\rangle] \le 0.\end{aligned}$$

Thus, $\lim_{n\to\infty}\|x_{\phi(n)}-x^*\|^2 = 0$. Also, note that

$$\begin{aligned}&\|x_{\phi(n)+1}-x^*\|^2 - \|x_{\phi(n)}-x^*\|^2 \\ &= 2\langle x_{\phi(n)+1}-x_{\phi(n)}, x_{\phi(n)}-x^*\rangle + \|x_{\phi(n)+1}-x_{\phi(n)}\|^2 \\ &\le 2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-x^*\| + \|x_{\phi(n)+1}-x_{\phi(n)}\|^2.\end{aligned} \tag{3.43}$$

Thanks to $\mathbf{\Lambda}_n \le \mathbf{\Lambda}_{\phi(n)+1}$, we get

$$\begin{aligned}&\|x_n-x^*\|^2 \le \|x_{\phi(n)+1}-x^*\|^2 \\ &\le \|x_{\phi(n)}-x^*\|^2 + 2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-x^*\| + \|x_{\phi(n)+1}-x_{\phi(n)}\|^2 \to 0 \quad (n\to\infty).\end{aligned}$$

That is, $x_n \to x^*$ as $n\to\infty$. This completes the proof.

Remark 3.2. Compared with the corresponding results in Eslamian and Kamandi [25] and Ceng et al. [8], our results enhance and develop them in the following aspects.

(i) The problem of finding an element of $\bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)$ in [25] is extended to develop the BPVIP (3.4) with the GEPS and SCFPP constraints, i.e., the problem of finding $x^* \in \Omega = \mathrm{VI}(C,A)\cap\mathrm{Fix}(G)\cap(\bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i))$ such that $\langle(\alpha F - f)x^*, p-x^*\rangle \ge 0\ \forall p\in\Omega$. The inertial method with a correction term and a self-adaptive step size strategy in [25] is extended to develop our composite subgradient extragradient rule with inertial correction term for settling the BPVIP (3.4) with the GEPS and SCFPP constraints, which is based on the subgradient extragradient method with adaptive stepsizes, adaptive inertial correction term, and hybrid deepest-descent method.

(ii) The problem of finding a solution to the GEPS with the VIP and CFPP constraints in [8] is extended to develop the BPVIP (3.4) with the GEPS and SCFPP constraints, i.e., the problem of finding $x^* \in \Omega = \mathrm{VI}(C,A)\cap\mathrm{Fix}(G)\cap(\bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i))$ such that $\langle(\alpha F - f)x^*, p-x^*\rangle \ge 0\ \forall p\in\Omega$. The modified inertial subgradient extragradient algorithm for solving the GEPS with the VIP and CFPP constraints in [8] is extended to develop our composite subgradient extragradient rule with inertial correction term for settling the BPVIP (3.4) with the GEPS and SCFPP constraints, which is based on the subgradient extragradient method with adaptive stepsizes, adaptive inertial correction term, and hybrid deepest-descent method.

Remark 3.3. In particular, when $N = 1$, the above BPVIP (3.4) with the GEPS and SCFPP constraints, is reduced to the bilevel split pseudomonotone variational inequality problem (BSPVIP) with the GEPS constraint:

$$\text{Seek } x^* \in \Omega \text{ such that } \langle(\alpha F - f)x^*, y - x^*\rangle \geq 0 \ \ \forall y \in \Omega. \tag{3.44}$$

where $\Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(G) \cap \mathcal{T}_1^{-1}\mathrm{Fix}(S_1) = \mathrm{Fix}(G) \cap \{z \in \mathrm{VI}(C, A) : \mathcal{T}_1 z \in \mathrm{Fix}(S_1)\}$. In this case, Algorithm 3.1 is rewritten as

Algorithm 3.2.

Initialization: Let $\gamma > 0,\ \nu \in (0, 1),\ \ell \in (0, 1),\ \tau_1 \in [0, \infty),\ \bar{\alpha}, \bar{\beta} \in [0, 1]$ and $x_1, x_0, w_0 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Given the iterates $x_{n-1},\ x_n$ $(n \geq 1)$, compute x_{n+1} as follows:

Step 1. Put $w_n = x_n + \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1})$ and calculate

$$\begin{cases} p_n = \lambda_n w_n + (1 - \lambda_n)g_n, \\ h_n = T_{\eta_2}^{\Theta_2}(I - \eta_2 B_2)p_n, \\ g_n = T_{\eta_1}^{\Theta_1}(I - \eta_1 B_1)h_n, \end{cases}$$

where $\alpha_n \in [0, \overline{\alpha}_n]$ and $\beta_n \in [0, \overline{\beta}_n]$ such that $\overline{\alpha}_n = \begin{cases} \min\{\bar{\alpha}, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \bar{\alpha} & \text{otherwise,} \end{cases}$ and

$$\overline{\beta}_n = \begin{cases} \min\{\bar{\beta}, \frac{\varepsilon_n}{\|w_{n-1} - x_{n-1}\|}\} & \text{if } w_{n-1} \neq x_{n-1}, \\ \bar{\beta} & \text{otherwise.} \end{cases}$$

Step 2. Calculate $y_n = P_C(p_n - \zeta_n A p_n)$ and $q_n = P_{C_n}(p_n - \zeta_n A y_n)$ where ζ_n is the largest $\zeta \in \{\gamma, \gamma\ell, \gamma\ell^2, ...\}$ satisfying

$$\zeta\|Ap_n - Ay_n\| \leq \nu\|p_n - y_n\|,$$

and $C_n = \{y \in \mathcal{H} : \langle p_n - \zeta_n A p_n - y_n, y - y_n\rangle \leq 0\}$.

Step 3. Calculate $z_n = q_n - \tau_{n,1}\mathcal{T}_1^*(I - S_1)\mathcal{T}_1 q_n$, where $\tau_{n,1}$ is chosen to be the bounded sequence satisfying

$$0 < \epsilon \leq \tau_{n,1} \leq \frac{(1 - \xi_1)\|(I - S_1)\mathcal{T}_1 q_n\|^2}{\|\mathcal{T}_1^*(I - S_1)\mathcal{T}_1 q_n\|^2} - \epsilon \quad \text{if } (I - S_1)\mathcal{T}_1 q_n \neq 0,$$

otherwise set $\tau_{n,1} = \tau_1$.

Step 4. Calculate $x_{n+1} = \sigma_n f(x_n) + (I - \sigma_n \alpha F)z_n$.

Again set $n := n + 1$ and return to Step 1.

Theorem 3.2. Let $\{x_n\}$ be the sequence generated by Algorithm 3.2. Then $\{x_n\}$ converges strongly to the unique solution $z^* \in \Omega$ of the BSPVIP (3.44) with the GEPS constraint.

4. Viability and Performability

In what follows, we present an instance to show the viability and performability of the proposed rule. Put $\Theta_1 = \Theta_2 = 0,\ \alpha = 2,\ \eta_1 = \eta_2 = \frac{1}{3},\ \tau_i = \frac{1}{5},\ \bar{\alpha} = \bar{\beta} = \frac{1}{3},\ \gamma = 1,\ \nu = \ell = \frac{1}{2},\ \gamma_n = \lambda_n = \frac{2}{3},\ \varepsilon_n = \frac{1}{3(n+1)^2}$, and $\sigma_n = \frac{1}{3(n+1)}$ for all $n \geq 1$. First, we construct an instance of $\Omega = \mathrm{VI}(C, A) \cap \mathrm{Fix}(G) \cap (\bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)) \neq \emptyset$ with $G = T_{\eta_1}^{\Theta_1}(I - \eta_1 B_1)T_{\eta_2}^{\Theta_2}(I - \eta_2 B_2) = P_C(I - \eta_1 B_1)P_C(I - \eta_2 B_2)$, where $A : \mathcal{H} \to \mathcal{H}$ is pseudomonotone and Lipschitzian mapping, $B_1, B_2 : \mathcal{H} \to \mathcal{H}$ are two ism mappings, $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i$ is bounded linear operator, and $S_i : \mathcal{H}_i \to \mathcal{H}_i$ is ξ_i-demimetric mapping for $i = 1, 2, ..., N$.

Let $\mathcal{H}_1 = \mathcal{H} = \mathbf{R}$ and use $\langle a, b\rangle = ab$ and $\|\cdot\| = |\cdot|$ to denote its inner product and induced norm, respectively. Put $C = [-2, 2]$ and the starting points x_1, x_0, w_0 are arbitrarily chosen in C. Let $f(x) = F(x) = \frac{1}{2}x\ \forall x \in \mathcal{H}$, with

$$\delta = \frac{1}{2} < \zeta = 1 - \sqrt{1 - \alpha(2\eta - \alpha\kappa^2)} = 1 - \sqrt{1 - 2(2\cdot\frac{1}{2} - 2(\frac{1}{2})^2)} = 1.$$

Assume that $\mathcal{T}_1 x = x\ \forall x \in \mathcal{H}$. Let $B_1 x = B_2 x := Bx = x - \frac{1}{2}\sin x\ \forall x \in C$. Let the operators $A, S_r : \mathcal{H} \to \mathcal{H}$ be defined by

$$Ax := \frac{1}{1 + |\sin x|} - \frac{1}{1 + |x|} \quad \text{and} \quad S_r x := S_1 x = S_1 x = \frac{3}{5}x + \frac{1}{5}\sin x\ (r = 1, ..., N)\ \forall x \in \mathcal{H}.$$

We have the following conclusions:

(i) A is pseudomonotone and 2-Lipschitzian. Indeed, for each $v, w \in \mathcal{H}$ one has

$$\begin{array}{rl} \|Ax - Ay\| & \leq |\frac{\|y\|-\|x\|}{(1+\|y\|)(1+\|x\|)}| + |\frac{\|\sin y\|-\|\sin x\|}{(1+\|\sin t\|)(1+\|\sin x\|)}| \\ & \leq \frac{\|x-y\|}{(1+\|x\|)(1+\|y\|)} + \frac{\|\sin x-\sin y\|}{(1+\|\sin x\|)(1+\|\sin y\|)} \\ & \leq \|x-y\| + \|\sin x - \sin y\| \leq 2\|x-y\|. \end{array}$$

This ensures that A is 2-Lipschitzian. Let us show that A is pseudomonotone. For each $x, y \in \mathcal{H}$, it is readily known that

$$\begin{array}{c} \langle Ax, y - x\rangle = (\frac{1}{1+|\sin x|} - \frac{1}{1+|x|})(y - x) \geq 0 \\ \Rightarrow\ \langle Ay, y - x\rangle = (\frac{1}{1+|\sin y|} - \frac{1}{1+|y|})(y - x) \geq 0. \end{array}$$

(ii) B is $\frac{2}{9}$-ism. Indeed, since B is $\frac{1}{2}$-strongly monotone and $\frac{3}{2}$-Lipschitzian, we know that B is $\frac{2}{9}$-inverse-strongly monotone with $\rho = \sigma = \frac{2}{9}$.

(iii) S_1 is a ξ_1-demicontractive mapping with $\xi_1 = \frac{1}{5}$ and $\mathrm{Fix}(S_1) = \{0\}$. Indeed, S_1 is a ξ_1-strictly pseudocontractive mapping with $\xi_1 = \frac{1}{5}$ because

$$\|S_1 x - S_1 y\|^2 = \|\frac{3}{5}(x - y) + \frac{1}{5}(\sin x - \sin y)\|^2 \leq \|x - y\|^2 + \frac{1}{5}\|(I - S)x - (I - S)y\|^2.$$

Consequently, $\Omega := \mathrm{VI}(C, A) \cap \mathrm{Fix}(G) \cap \mathcal{T}_1^{-1}\mathrm{Fix}(S_1) = \{0\} \neq \emptyset$, where $G = P_C(I - \eta_1 B_1)P_C(I - \eta_2 B_2) = [P_C(I - \frac{1}{3}B)]^2$. In addition, we observe that $0 < \frac{1}{5} = \epsilon \leq \tau_{n,1} \leq \frac{(1-\xi_1)\|(I-S_1)\mathcal{T}_1 q_n\|^2}{\|\mathcal{T}_1^*(I-S_1)\mathcal{T}_1 q_n\|^2} - \epsilon = \frac{3}{5}$ if $(I - S_1)\mathcal{T}_1 q_n \neq 0$, and $\tau_{n,1} = \tau_1 = \frac{1}{5}$ otherwise. So, we set $\tau_{n,1} = \frac{1}{5}\ \forall n \geq 1$. Also, it is clear that $\mathcal{T}_1^*(I - S_1)\mathcal{T}_1 q_n = (I - S_1)q_n$ and $x_{n+1} = \sigma_n f(x_n) + (I - \sigma_n\alpha F)z_n = \frac{1}{3(n+1)}\cdot\frac{1}{2}x_n + (1 - \frac{1}{3(n+1)})z_n$. In this case, putting

$$\alpha_n = \begin{cases} \min\{\frac{\frac{1}{3(n+1)^2}}{\|x_n - x_{n-1}\|}, \frac{1}{3}\} & \text{if } x_n \neq x_{n-1}, \\ \frac{1}{3} & \text{otherwise,} \end{cases}$$

and

$$\beta_n = \begin{cases} \min\{\frac{\frac{1}{3(n+1)^2}}{\|w_{n-1} - x_{n-1}\|}, \frac{1}{3}\} & \text{if } w_{n-1} \neq x_{n-1}, \\ \frac{1}{3} & \text{otherwise,} \end{cases}$$

we rewrite Algorithm 3.2 as follows

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1}), \\ p_n = \frac{2}{3}w_n + \frac{1}{3}Gp_n, \\ y_n = P_C(p_n - \zeta_n Ap_n), \\ q_n = P_{C_n}(p_n - \zeta_n Ay_n), \\ z_n = q_n - \frac{1}{5}(q_n - S_1 q_n), \\ x_{n+1} = \frac{1}{3(n+1)}\cdot\frac{1}{2}x_n + (1 - \frac{1}{3(n+1)})z_n \quad \forall n \geq 1, \end{cases}$$

where for each $n \geq 1$, η_n and C_n are chosen as in Algorithm 3.2. Therefore, using Theorem 3.2, we know that $\{x_n\}$ converges to $0 \in \Omega$.

Example 1. In this example we will give a numerical illustration on R^k to show that our proposed algorithm is implementable on R^k, for $k = 100,\ 500, 1000$ and 5000 and compare its performance with Algorithms 3.1 and 3.2 of Xu et al. [39]. Let $f, F : R^k \to R^k$ be defined by $f(x) := \frac{1}{2}x$, $F(x) := 2x$. Let $C = \{x \in R^k : \|x\| \leq 2\}$, Θ_1 and Θ_2 be the zero bifunctions and $B_1 = B_2 = I$, where I is the identity map on R^k. Let $A : R^k \to R^k$ be defined by $A := (3 - \|x\|)x$. Fix $N = 5$. Let $S_i x = \frac{i}{5}x$ and $\mathcal{T}_i = ix$, $\alpha = 0.25$, $\eta_1 = 0.5$, $\eta_2 = 0.3$, $\tau_i = \frac{1}{5}$, $\bar{\alpha} = 0.1$, $\bar{\beta} = 0.3$, $\gamma = 1$, $\nu = \ell = \frac{1}{2}$, $\gamma_n = 0.5$, $\lambda_n = 0.2$, $\varepsilon_n = \frac{1}{(n+1)^{1.1}}$, and $\sigma_n = \frac{1}{(n+1)}$. In Algorithms 3.1 and 3.2 of Xu et al. [39], (XTL Alg. 3.1 and XTL Alg. 3.2 for short) we let $\gamma = 0.3$, $\lambda_1 = 0.4$, $\delta = 0.2$, $\psi = 1.2$, $\sigma_n = \frac{1}{(n+1)}$, $\eta_n = \frac{n}{2n+1}$, $\epsilon_n = \frac{100}{(n+1)^2}$, $\xi_n = \frac{1}{(n+1)^5}$, $Q = S_1$ and $B = F$. The initial points x_0, x_1 and g_0 are generated randomly in R^k. The numerical simulations is continued until $n = 1001$ or $E_n = \|x_{n+1} - x_n\| < 10^{-8}$ is not satisfied. The results of the numerical simulations are presented in Table 1 and Figures 1 and 2.

Table 1: Numerical performance of all algorithms in Example 1

Algorithms	$k = 100$		$k = 500$		$k = 1000$		$k = 5000$	
	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)	Iter.	Time (s)
Alg 3.1	22	0.0024	22	0.0034	23	0.0051	23	0.0065
Alg. 3.1 $\beta_n = 0$	24	0.0026	24	0.0039	26	0.0066	26	0.0081
Alg. 3.1 $\beta_n = \alpha_n = 0$	26	0.0036	26	0.0044	27	0.0076	27	0.0094
XTL Alg. 3.1	88	0.0156	89	0.0155	95	0.0162	104	0.0385
XTL Alg. 3.1	30	0.0052	30	0.0051	32	0.0045	34	0.0096

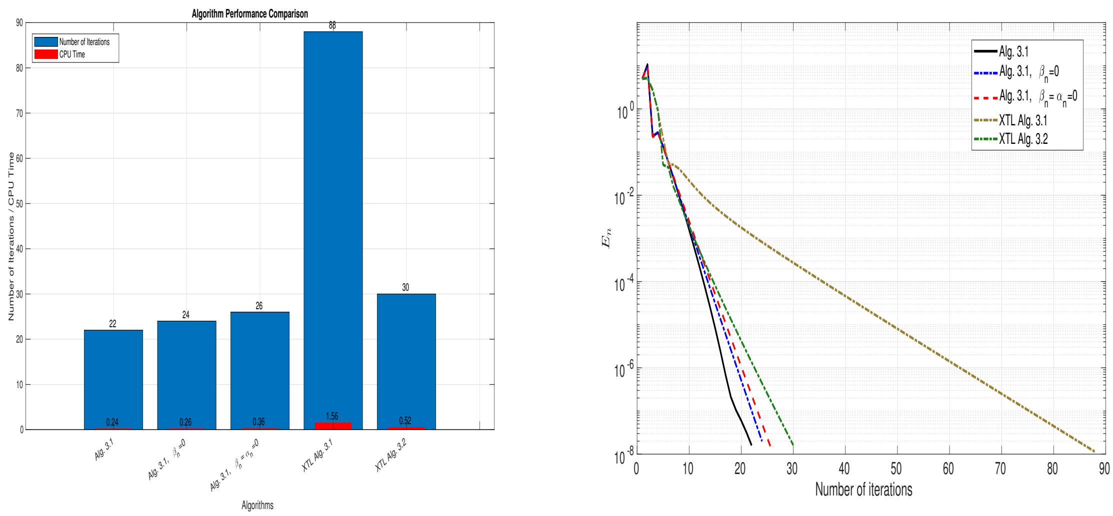

Figure 1: Graphical plot of the results in Table 1 Top: $k = 100$ Bottom: $k = 500$

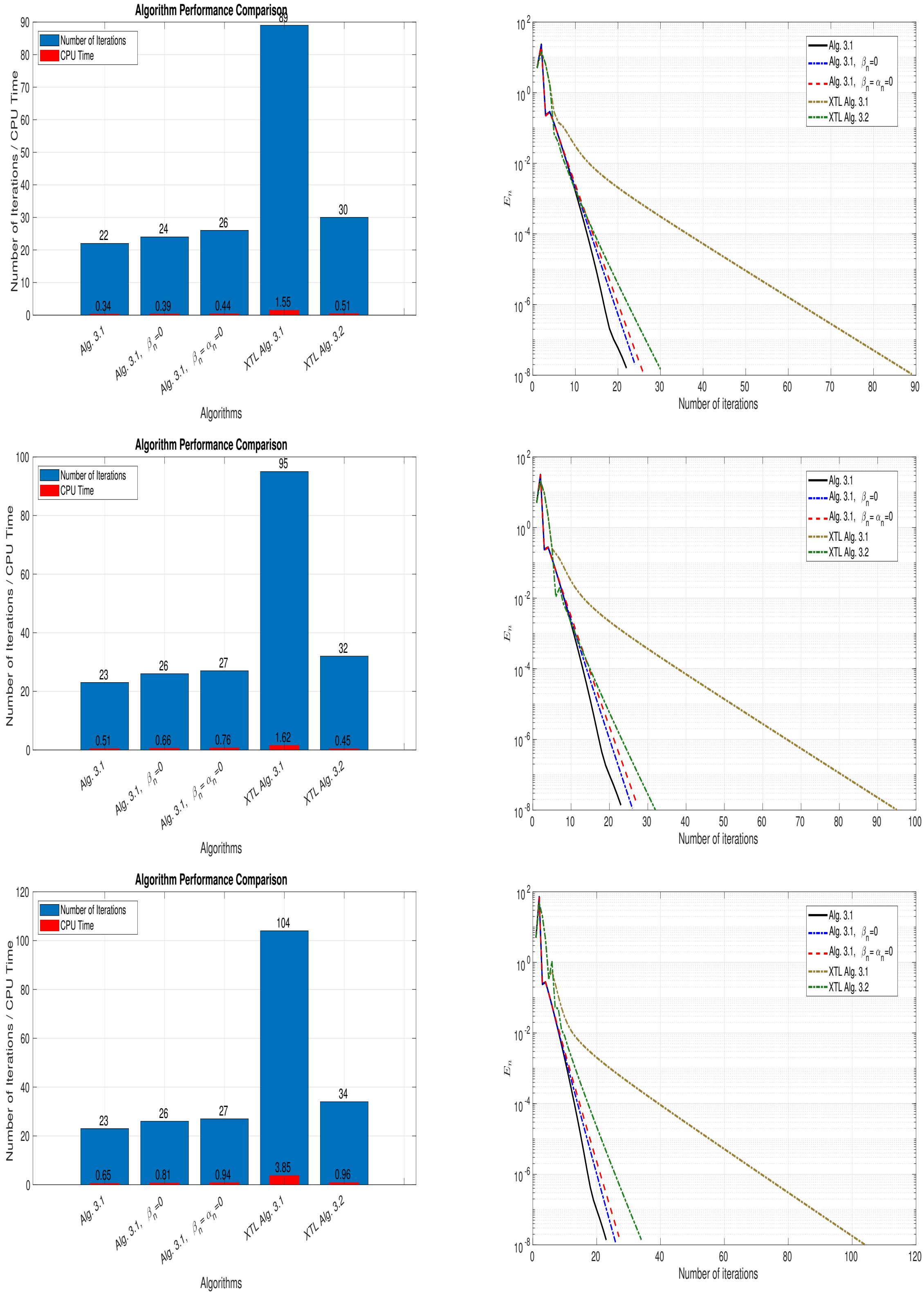

Figure 2: Graphical plot of the results in Table 1 Top: $k = 500$, Middle $k = 1000$ Bottom: $k = 5000$

5. Conclusions

With the help of the modified inertial subgradient extragradient algorithm in [8], we have devised a composite subgradient extragradient rule with inertial correction term to solve the BPVIP (3.4) with the constraints of the GEPS and SCFPP involving demimetric mappings in real Hilbert spaces. The BPVIP is composed of the upper-level VIP whose solution set is $\mathrm{VI}(\Omega, \alpha F - f)$ and the lower-level VIP whose solution set is $\mathrm{VI}(C, A)$. Furthermore, we have established a strong convergence theorem under certain appropriate conditions. Meanwhile, we have applied our main outcome for solving a bilevel split pseudomonotone variational inequality problem (BSPVIP) with the GEPS constraint. The problem considered in this paper has potential applications in real-world problems such as image recognition, signal processing, machine learning, and so on. Additionally, an illustrated instance is provided to support the implementation and performance of the proposed rule. Our rule incorporates an adaptive stepsize technique without the prior knowledge of the operator norm and exploits the inertial technique with a correction term to speed up the convergence of the proposed algorithm.

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