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Approximation properties of Meyer-König-Zeller type operators

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Abstract. In this paper, a new kind of Meyer-König-Zeller type operators will be constructed by their relationship with Baskakov type operators. Firstly, the Baskakov type operators preserving the function $e^{-\mu x}$ ($\mu > 0$) will be introduced by the dominated convergence theorem. Secondly, some approximation properties of this kind of new Baskakov operators will be discussed. Finally, with the help of a transformation from $[0, 1)$ to $[0, \infty)$, we will give the definition and the approximation theorem of the new Meyer-König-Zeller type operators.

1. Introduction

In 1960, Meyer-König-Zeller operators were proposed by Meyer-König W and Zeller K [28], for $f \in C_B[0, 1]$,

$$M_n(f; x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x), \quad 0 \le x < 1,$$

$$M_n(f; 1) := f(1), \quad m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}.$$

This class of operators is regarded as the most challenging ones. The difficulties of the Meyer-König-Zeller operators consist in their complexity of calculating central moments. Most researchers investigated only the estimation of central moments and the approximation properties of the classical Meyer-König-Zeller operators [24, 27, 30–32]. In this paper, using the relationship between Baskakov type operators and Meyer-König-Zeller type operators, we will constructed a new kind of Meyer-König-Zeller type operators and study their approximation properties.

In 1957, the Baskakov operators were proposed to approximate the continuous functions defined on the unbound interval $[0, \infty)$ [9]. The approximation properties of the Baskakov operators have been well studied in the last decades [9, 10, 15, 16, 19, 25]. The importance of the classic operators leads to the discovery of their numerous generalizations that improve the approximation accuracy, such as q-type

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operators, linear combinations of the operators, King type operators, etc. In 2003, King [26] first introduced a sequence of positive linear operators which preserve the function x^2. In recent years, many researchers [1–7, 13, 14, 18, 23, 29] focused their attention on the study of constructing King type operators and proving their properties. In 2022, Huang, Qi [23] introduced a new kind of Szász type operators $S_n^{\mu}(f;x)$ preserving the function $e^{-\mu x\ (\mu>0)}$: For $x \geq 0, \mu > 0, f \in C_B[0,\infty)$,

$$S_n^{\mu}(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-n\alpha_n(x)} \frac{(n\alpha_n(x))^k}{k!}, \tag{1}$$

where

$$\alpha_n(x) = \frac{\mu x}{n(1 - e^{-\frac{\mu}{n}})}, \ \lim_{n\to\infty} \alpha_n(x) = x.$$

It is known that both Szász and Gamma operators are limits, in an appropriate sense, of Baskakov operators. And the Baskakov operators can be established by the combination of Szász operators and Gamma operators [11, 12, 20]. Inspired by this, in this paper, another kind of Baskakov type operators preserving the function $e^{-\mu x}$ $(\mu > 0)$ are constructed by composition and analytical method.

By calculating, likewise the Post-Widder type operators $P_n^{\mu}(f;x)$ preserving the function $e^{-\mu x}$ $(\mu > 0)$ can be defined by: For $x \geq 0, \mu > 0, f \in C_B[0,\infty)$,

$$P_n^{\mu}(f;x) = \frac{1}{(n-1)!} \int_0^{\infty} f\left(\frac{\beta_n(x)\theta}{n}\right) \theta^{n-1} e^{-\theta} d\theta, \tag{2}$$

where

$$\beta_n(x) = \frac{e^{\frac{\mu x}{n}} - 1}{\frac{\mu}{n}}, \ \lim_{n\to\infty} \beta_n(x) = x.$$

By changing the variable, the operators $P_n^{\mu}(f,x)$ can also be represented by

$$P_n^{\mu}(f;x) = \frac{1}{(n-1)!} \int_0^{\infty} f(\theta) \left(\frac{n}{\beta_n(x)}\right)^n \theta^{n-1} e^{-\frac{n\theta}{\beta_n(x)}} d\theta. \tag{3}$$

According to integration by parts, for $n, \mu, x > 0, k = 0, 1 \cdots$, we can write

$$\int_0^{\infty} e^{-n\alpha_n(\theta)} \frac{(n\alpha_n(\theta))^k}{k!} \frac{1}{(n-1)!} \left(\frac{n}{\beta_n(x)}\right)^n \theta^{n-1} e^{-\frac{n\theta}{\beta_n(x)}} d\theta = \binom{n+k-1}{k} \frac{\gamma_n^k(x)}{(1+\gamma_n(x))^{n+k}},$$

here

$$\gamma_n(x) = \frac{e^{\frac{\mu x}{n}} - 1}{1 - e^{-\frac{\mu}{n}}}, \ \lim_{n\to\infty} \gamma_n(x) = x.$$

If we assume $S_n^{\mu}(|f|;\theta) < \infty, \theta \geq 0$, applying dominated convergence theorem together with simple arguments concerning power series, then

$$\int_0^{\infty} S_n^{\mu}(f;\theta) \frac{1}{(n-1)!} \left(\frac{n}{\beta_n(x)}\right)^n \theta^{n-1} e^{-\frac{n\theta}{\beta_n(x)}} d\theta = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \binom{n+k-1}{k} \frac{\gamma_n^k(x)}{(1+\gamma_n(x))^{n+k}}.$$

Hence, the Baskakov type operators that preserve $e^{-\mu x}$ $(\mu > 0)$ can be expressed as the composition of Post-Widder type operators and Szász type operators.

Definition 1.1. *For $x \geq 0, \mu > 0, f(x) \in C_B[0,\infty)$, $V_n^{\mu}(|f|;\theta) < \infty, \theta \geq 0$, the new kind of Baskakov operators are given by*

$$V_n^{\mu}(f;x) = P_n^{\mu}(S_n^{\mu}(f;\theta);x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \binom{n+k-1}{k} \frac{\gamma_n^k(x)}{(1+\gamma_n(x))^{n+k}}. \tag{4}$$

Further, we will study some approximation properties of the Baskakov type operators $V_n^{\mu}(f;x)$, such as direct theorem and Voronovskaja type weak inverse theorem. The complete structure of the manuscript constitutes four sections. The remaining parts of this paper are organized as follows. In section 2, the estimation of central moments of different operators will be given. In section 3, we shall discuss the approximation properties of Baskakov type operators. In section 4, through a transformation from $[0,1)$ to $[0,\infty)$, a new Meyer-König-Zeller type operators $M_n^{\mu}(f;x)$ will be given and investigated.

Remark 1.2. *In this paper,* $C[0,\infty)$ *represents the space of continuous functions on* $[0,\infty)$*;* $C_B[0,\infty)$ *is the space of continuous bounded functions on* $[0,\infty)$*;* $C^*[0,\infty) := \{f \in C[0,\infty) : \lim_{n\to\infty} f(x)$ *exists and is limited* $\}$*, equipped with the uniform norm* $\|f\| := \sup_{x\in[0,\infty)} |f(x)|$.

Definition 1.3. *[16] Let r be a positive integer and* $\varphi(x)$ *be various step-weight functions.*
The moduli of smoothness are defined as follows: $f \in C[0,\infty)$,

$$\omega_{\varphi}^{r}(f;t) = \sup_{0<h\le t} \|\Delta_{h\varphi(x)}^{r} f(x)\|,$$

here $\Delta_{h\varphi(x)}^{r} f(x)$ *is the rth symmetric difference of a function f, given by*

$$\Delta_{h\varphi(x)}^{r} f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + \frac{rh\varphi(x)}{2} - kh\varphi(x)).$$

The K-Functional is given by:

$$K_{\varphi}^{r}(f;t^r) = \inf_{g}\{\|f-g\| + t^r\|\varphi^r g^{(r)}\| : g^{(r-1)} \in A.C._{loc.}, \|\varphi^r g^{(r)}\| < \infty\},$$

where $g^{(r-1)} \in A.C._{loc.}$ *means that g is* $r-1$ *times differentiable and* $g^{(r-1)}$ *is absolutely continuous in every closed finite interval.*

Remark 1.4. *[15, 16] The equivalence relationship between moduli of smoothness and K-Functional is: There exist some constants* $C > 0$ *and* t_0 *such that*

$$C^{-1}\omega_{\varphi}^{r}(f;t) \le K_{\varphi}^{r}(f;t^r) \le C\omega_{\varphi}^{r}(f;t), \quad 0 < t \le t_0.$$

2. The Estimation of Central Moments of Different Operators

Lemma 2.1. *[23] Let* $x \in [0,\infty), \mu > 0$*, one has*
$(1) S_n^{\mu}(1;x) = 1;$
$(2) S_n^{\mu}(t;x) = \alpha_n(x);$
$(3) S_n^{\mu}(t^2;x) = \alpha_n^2(x) + \frac{\alpha_n(x)}{n};$
$(4) S_n^{\mu}(t^3;x) = \alpha_n^3(x) + \frac{3\alpha_n^2(x)}{n} + \frac{\alpha_n(x)}{n^2};$
$(5) S_n^{\mu}(t^4;x) = \alpha_n^4(x) + \frac{6\alpha_n^3(x)}{n} + \frac{7\alpha_n^2(x)}{n^2} + \frac{\alpha_n(x)}{n^3}.$

Lemma 2.2. *[23] Let* $x \in [0,\infty), \mu > 0$*, one has*
(1) $\lim_{n\to\infty} nS_n^{\mu}((t-x);x) = \lim_{n\to\infty} n[\alpha_n(x) - x] = \frac{\mu x}{2};$
(2) $\lim_{n\to\infty} nS_n^{\mu}((t-x)^2;x) = \lim_{n\to\infty} n\left[\alpha_n^2(x) + \frac{\alpha_n(x)}{n} - 2x\alpha_n(x) + x^2\right] = x;$
(3) $\lim_{n\to\infty} n^2S_n^{\mu}((t-x)^4;x) = \lim_{n\to\infty} \Big\{n^2[\alpha_n^4(x) - 4x\alpha_n^3(x) + 6x^2\alpha_n^2(x) - 4x^3\alpha_n(x) + x^4] + n[6\alpha_n^3(x) - 12x\alpha_n^2(x) + 6x^2\alpha_n(x)] + [7\alpha_n^2(x) - 4x\alpha_n(x)] + \frac{\alpha_n(x)}{n}\Big\} = 3x^2.$

From the definition of $P_n^{\mu}(f;x)$ Eq. (2), by simple calculations, we can obtain the following lemma.

Lemma 2.3. *Let $x \in [0,\infty)$, $\mu > 0$, we have*
(1)$P_n^{\mu}(1;x) = 1$;
(2)$P_n^{\mu}(t;x) = \beta_n(x)$;
(3)$P_n^{\mu}(t^2;x) = \frac{n+1}{n}\beta_n^2(x)$;
(4)$P_n^{\mu}(t^3;x) = \frac{(n+1)(n+2)}{n^2}\beta_n^3(x)$;
(5)$P_n^{\mu}(t^4;x) = \frac{(n+1)(n+2)(n+3)}{n^3}\beta_n^4(x)$.

Lemma 2.4. *Let $x \in [0,\infty)$, $\mu > 0$, we obtain*
(1) $\lim\limits_{n\to\infty} nP_n^{\mu}((t-x);x) = \frac{\mu x^2}{2}$;
(2) $\lim\limits_{n\to\infty} nP_n^{\mu}((t-x)^2;x) = x^2$;
(3) $\lim\limits_{n\to\infty} n^2P_n^{\mu}((t-x)^4;x) = 3x^4$.

Proof. It is not difficult to show the first and second part of the Lemma 2.4, so we skip the details. It remains to show the third part. Recalling the definition of the operators Eq. (2) and using Lemma 2.3, we have

$$\begin{aligned}
&\lim_{n\to\infty} n^2P_n^{\mu}((t-x)^4;x)\\
=\ &\lim_{n\to\infty} n^2\left[P_n^{\mu}(t^4;x) - 4xP_n^{\mu}(t^3;x) + 6x^2P_n^{\mu}(t^2;x) - 4x^3P_n^{\mu}(t;x) + x^4\right]\\
=\ &\lim_{n\to\infty} n^2\left[\beta_n^4(x) - 4x\beta_n^3(x) + 6x^2\beta_n^2(x) - 4x^3\beta_n(x) + x^4\right]\\
&+\lim_{n\to\infty} n^2\left[\frac{6}{n}\beta_n^4(x) - 4x\frac{3}{n}\beta_n^3(x) + 6x^2\frac{1}{n}\beta_n^2(x)\right]\\
&+\lim_{n\to\infty} n^2\left[\frac{11}{n^2}\beta_n^4(x) - 4x\frac{2}{n^2}\beta_n^3(x)\right] + \lim_{n\to\infty} n^2\left[\frac{6}{n^3}\right]\beta_n^4(x)\\
:=\ &I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Using the Taylor formula for $e^{\frac{\mu x}{n}} - 1$, we will estimate $I_1 - -I_4$ respectively.

$$\begin{aligned}
I_1 =\ &\lim_{n\to\infty}\left[n^2\beta_n^4(x) - 4n^2x\beta_n^3(x) + 6n^2x^2\beta_n^2(x) - 4n^2x^3\beta_n(x) + n^2x^4\right]\\
=\ &\lim_{n\to\infty}\left\{n^2\left[x^4 + 6x^2\left(\frac{\mu}{2n}x^2\right)^2 + 4x^3\left(\frac{\mu}{2n}x^2\right) + 4x^3\left(\frac{\mu^2}{6n^2}x^3\right) + o(n^{-2})\right]\right.\\
&-4n^2x\left[x^3 + 3x^2\left(\frac{\mu}{2n}x^2\right) + 3x\left(\frac{\mu}{2n}x^2\right)^2 + 3x^2\frac{\mu^2}{6n^2}x^3 + o(n^{-2})\right]\\
&+6n^2x^2\left[x^2 + 2x\left(\frac{\mu}{2n}x^2\right) + \left(\frac{\mu}{2n}x^2\right)^2 + 2x\frac{\mu^2}{6n^2}x^3 + o(n^{-2})\right]\\
&\left.+\left[-4n^2x^4 - 2n\mu x^5 - \frac{2}{3}\mu^2x^6 + o(1)\right] + n^2x^4\right\}\\
=\ &\lim_{n\to\infty}\left\{\left[n^2x^4 + \frac{3}{2}\mu^2x^6 + 2n\mu x^5 + \frac{2}{3}\mu^2x^6\right] + \left[-4n^2x^4 - 6n\mu x^5 - 3\mu^2x^6 - 2\mu^2x^6\right]\right.\\
&\left.+\left[6n^2x^4 + 6n\mu x^5 + \frac{3}{2}\mu^2x^6 + 2\mu^2x^6\right] + \left[-4n^2x^4 - 2n\mu x^5 - \frac{2}{3}\mu^2x^6\right] + n^2x^4 + o(1)\right\} = 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 =\ &\lim_{n\to\infty}\left\{6n\left[x + \frac{\mu}{2n}x^2 + o(n^{-1})\right]^4 - 12nx\left[x + \frac{\mu}{2n}x^2 + o(n^{-1})\right]^3 + 6nx^2\left[x + \frac{\mu}{2n}x^2 + o(n^{-1})\right]^2\right\}\\
=\ &\lim_{n\to\infty}\left\{6nx^4 + 12\mu x^5 - 12nx^4 - 18\mu x^5 + 6nx^4 + 6\mu x^5 + o(1)\right\} = 0,
\end{aligned}$$

and

$$I_3 = \lim_{n\to\infty}[11\beta_n^4(x) - 8x\beta_n^3(x)] = 3x^4,$$

$$I_4 = \lim_{n\to\infty} \frac{6}{n}\beta_n^4(x) = 0.$$

Therefore,

$$\lim_{n\to\infty} n^2 P_n^{\mu}((t-x)^4; x) = 3x^4.$$

□

Lemma 2.5. *Let $x \in [0,\infty)$, $\mu > 0$, examining the definition of the operators $P_n^{\mu}(f;x)$ Eq.(2), we have*
(1)$P_n^{\mu}(\alpha_n(\theta);x) = \gamma_n(x)$;
(2)$P_n^{\mu}(\alpha_n^2(\theta);x) = \frac{n+1}{n}\gamma_n^2(x)$;
(3)$P_n^{\mu}(\alpha_n^3(\theta);x) = \frac{(n+1)(n+2)}{n^2}\gamma_n^3(x)$;
(4)$P_n^{\mu}(\alpha_n^4(\theta);x) = \frac{(n+1)(n+2)(n+3)}{n^3}\gamma_n^4(x)$.

Lemma 2.6. *Let $x \in [0,\infty)$, $\mu > 0$, from the definition of the operators $V_n^{\mu}(f;x)$ Eq.(4), we have*
(1)$V_n^{\mu}(1;x) = 1$;
(2)$V_n^{\mu}(e^{-\mu t};x) = e^{-\mu x}$;
(3)$V_n^{\mu}(e^{-2\mu t};x) = \left[1 + (e^{\frac{\mu x}{n}} - 1)(1 + e^{-\frac{\mu}{n}})\right]^{-n}$.

Remark 2.7. $\lim_{n\to\infty} V_n^{\mu}(e^{-2\mu t};x) = e^{-2\mu x}$.

Lemma 2.8. *Let $x \in [0,\infty)$, $\mu > 0$, by the definition of the operators $V_n^{\mu}(f;x)$ Eq.(4), Lemma 2.1 and Lemma 2.5, we have*
(1)$V_n^{\mu}(t;x) = \gamma_n(x)$;
(2)$V_n^{\mu}(t^2;x) = \frac{n+1}{n}\gamma_n^2(x) + \frac{1}{n}\gamma_n(x)$;
(3)$V_n^{\mu}(t^3;x) = \frac{(n+1)(n+2)}{n^2}\gamma_n^3(x) + \frac{3(n+1)}{n^2}\gamma_n^2(x) + \frac{1}{n^2}\gamma_n(x)$;
(4)$V_n^{\mu}(t^4;x) = \frac{(n+1)(n+2)(n+3)}{n^3}\gamma_n^4(x) + \frac{6(n+1)(n+2)}{n^3}\gamma_n^3(x) + \frac{7(n+1)}{n^3}\gamma_n^2(x) + \frac{1}{n^3}\gamma_n(x)$.

Lemma 2.9. *Let $x \in [0,\infty)$, $\mu > 0$, we have*
(1) $\lim_{n\to\infty} nV_n^{\mu}((t-x);x) = \frac{\mu x^2 + \mu x}{2}$;
(2) $\lim_{n\to\infty} nV_n^{\mu}((t-x)^2;x) = x^2 + x$;
(3) $\lim_{n\to\infty} n^2 V_n^{\mu}((t-x)^4;x) = 3x^4 + 6x^3 + 3x^2$.

Proof. We will deal only with (3). Recalling the definition of the operators $P_n^{\mu}(f;x)$ and Lemma 2.1-2.5, we have

$$\begin{aligned}
(3)\quad & \lim_{n\to\infty} n^2 V_n^{\mu}((t-x)^4;x) \\
= \quad & \lim_{n\to\infty} n^2\Bigg[P_n^{\mu}\left(\alpha_n^4(\theta) + \frac{6\alpha_n^3(\theta)}{n} + \frac{7\alpha_n^2(\theta)}{n^2} + \frac{\alpha_n(\theta)}{n^3};x\right) - 4xP_n^{\mu}\left(\alpha_n^3(\theta) + \frac{3\alpha_n^2(\theta)}{n} + \frac{\alpha_n(\theta)}{n^2};x\right) \\
& +6x^2P_n^{\mu}\left(\alpha_n^2(\theta) + \frac{\alpha_n(\theta)}{n};x\right) - 4x^3P_n^{\mu}(\alpha_n(\theta);x) + x^4\Bigg] \\
= \quad & \lim_{n\to\infty} n^2[\gamma_n^4(x) - 4x\gamma_n^3(x) + 6x^2\gamma_n^2(x) - 4x^3\gamma_n(x) + x^4] \\
& + \lim_{n\to\infty} n^2\left[\frac{6}{n}(\gamma_n^2(x) + \gamma_n(x))(\gamma_n^2(x) - 2x\gamma_n(x) + x^2)\right] \\
& + \lim_{n\to\infty} n^2\left[\frac{11}{n^2}\gamma_n^4(x) + \frac{18}{n^2}\gamma_n^3(x) + \frac{7}{n^2}\gamma_n^2(x) - \frac{8}{n^2}x\gamma_n^3(x) - \frac{12}{n^2}x\gamma_n^2(x) - \frac{4}{n^2}x\gamma_n(x)\right] \\
& + \lim_{n\to\infty} n^2\left[\frac{6}{n^3}\gamma_n^4(x) + \frac{12}{n^3}\gamma_n^3(x) + \frac{7}{n^3}\gamma_n^2(x) + \frac{\gamma_n(x)}{n^3}\right] \\
= \quad & 3x^4 + 6x^3 + 3x^2.
\end{aligned}$$

□

Lemma 2.10. *[8, 21] Let $\{A_n\}$ be a sequence of linear positive operators from $C^*[0,\infty)$ to $C^*[0,\infty)$, satisfying $\lim_{n\to\infty} A_n(e^{-kt};x) = e^{-kx}, k = 0,1,2$. Then the above convergence is uniform if and only if $\lim_{n\to\infty} A_n(f;x) = f(x)$ uniformly in $[0,\infty)$ for all $f \in C^*[0,\infty)$.*

3. Approximation Theorem of Baskakov Type Operators

Theorem 3.1. *Let $\mu > 0$, for any function $f(x) \in C^*[0,\infty)$, when $n \to \infty$, the sequence of linear positive operators $\{V_n^\mu(f;x)\}$ is convergent uniformly at $f(x)$ on the interval $[0,\infty)$.*

Proof. Recall Lemma 2.6, Lemma 2.10, therefore

$$(1)\|V_n^\mu(1;x)-1\| = \|P_n^\mu(S_n^\mu(1;\theta);x)-1\| = \sup_{x\in[0,\infty)} \left|P_n^\mu(S_n^\mu(1;\theta);x)-1\right| = 0.$$

(2) We will need the following expressions by Mathematica software or simple calculations.

$$\left[1+\frac{(e^{\frac{\mu x}{n}}-1)(1-e^{-\frac{1}{n}})}{(1-e^{-\frac{\mu}{n}})}\right]^{-n} = e^{-x} - \frac{(-1+\mu)(x^2e^{-x}+xe^{-x})}{2n}$$
$$- \quad \frac{(-1+\mu)(3x^4e^{-x}-2x^3e^{-x}-3\mu x^4e^{-x}-9x^2e^{-x}-2\mu x^3e^{-x}-4xe^{-x}+3\mu x^2e^{-x}+2\mu xe^{-x})}{24n^2}$$
$$+ \quad O(n^{-3}).$$

Therefore,

$$|V_n^\mu(e^{-t};x)-e^{-x}| = \left|\frac{1}{(n-1)!}\int_0^\infty e^{n\alpha_n(\frac{\beta_n(x)\theta}{n})(-1+e^{-\frac{1}{n}})}\theta^{n-1}e^{-\theta}d\theta - e^{-x}\right|$$
$$= \left|\frac{1}{(n-1)!}\int_0^\infty e^{-\theta\left[1+\frac{(e^{\frac{\mu x}{n}}-1)(1-e^{-\frac{1}{n}})}{(1-e^{-\frac{\mu}{n}})}\right]}\theta^{n-1}d\theta - e^{-x}\right| = \left|\left[1+\frac{(e^{\frac{\mu x}{n}}-1)(1-e^{-\frac{1}{n}})}{(1-e^{-\frac{\mu}{n}})}\right]^{-n} - e^{-x}\right|$$
$$\leq |-1+\mu|\Big(\frac{x^2e^{-x}}{2n}+\frac{xe^{-x}}{2n}+\frac{x^4e^{-x}}{8n^2}+\frac{x^3e^{-x}}{12n^2}+\frac{\mu x^4e^{-x}}{8n^2}+\frac{3x^2e^{-x}}{8n^2}+\frac{\mu x^3e^{-x}}{12n^2}+\frac{xe^{-x}}{6n^2}$$
$$+\frac{\mu x^2e^{-x}}{8n^2}+\frac{\mu xe^{-x}}{12n^2}\Big)+O(n^{-3})$$
$$\leq |-1+\mu|\Big(\frac{4}{2ne^2}+\frac{1}{2ne}+\frac{4^4}{8n^2e^4}+\frac{3^3}{12n^2e^3}+\frac{\mu 4^4}{8n^2e^4}+\frac{12}{8n^2e^2}+\frac{\mu 3^3}{12n^2e^3}+\frac{1}{6n^2e}$$
$$+\frac{4\mu}{8n^2e^2}+\frac{\mu}{12n^2e}\Big)+O(n^{-3}),$$

which implies

$$\lim_{n\to\infty}\sup_{x\geq 0}|V_n^\mu(e^{-t};x)-e^{-x}| = 0.$$

Similarly,

$$\lim_{n\to\infty}\sup_{x\geq 0}|V_n^\mu(e^{-2t};x)-e^{-2x}| = 0.$$

The Theorem 3.1 is proved. □

Theorem 3.2. *Suppose $x > 0$, $f \in C_B[0,\infty)$, $f'(x)$ and $f''(x)$ exist. Then*

$$|V_n^\mu(f;x)-f(x)| \leq \frac{\alpha_n(x)}{x}\|f'\|(\beta_n(x)-x)+\frac{\alpha_n^2(x)}{x^2}\|f''\|(\beta_n(x)-x)^2+\frac{\gamma_n^2(x)}{n}\|f''\|$$
$$+ \quad C\omega^2_{\sqrt{x}}\left(f;\frac{\sqrt{\phi_n(x)+\nu_n^2(x)}}{\sqrt{x}}\right)+\omega_{\sqrt{x}}\left(f;\frac{\nu_n(x)}{\sqrt{x}}\right),$$

here the definitions of $\alpha_n(x), \beta_n x)$ *and* $\gamma_n(x)$ *are the same as the previous ones and* $\nu_n(x) = S_n^{\mu}(t-x;x)$, $\phi_n(x) = S_n^{\mu}((t-x)^2;x)$.

Proof. For the operators $V_n^{\mu}(f;x)$, we write

$$|V_n^{\mu}(f;x) - f(x)| \leq |P_n^{\mu}(S_n^{\mu}(f;\theta);x) - S_n^{\mu}(f;x)| + |S_n^{\mu}(f;x) - f(x)| := I_5 + I_6.$$

First, we estimate the part I_5. For $x > 0, \theta \geq 0$,

$$P_n^{\mu}(S_n^{\mu}(f;\theta);x) = \int_0^{\infty} S_n^{\mu}(f;\theta)\frac{1}{(n-1)!}\left(\frac{n}{\beta_n(x)}\right)^n \theta^{n-1}e^{-\frac{n\theta}{\beta_n(x)}}d\theta := \int_0^{\infty} S_n^{\mu}(f;\theta)g_n^{\beta_n(x)}(\theta)d\theta.$$

Expanding $f_{n,\mu}(\theta)$ by

$$f_{n,\mu}(\theta) = f_{n,\mu}(x) + f_{n,\mu}^{'}(x)(\theta - x) + \frac{1}{2}f_{n,\mu}^{''}(\xi)(\theta - x)^2,$$

here ξ is between θ and x, taking $f_{n,\mu} = S_n^{\mu}f$, we conclude that

$$S_n^{\mu}(f;\theta) = S_n^{\mu}(f;x) + S_n^{\mu'}(f;x)(\theta - x) + \frac{1}{2}S_n^{\mu''}(f;\xi)(\theta - x)^2.$$

Next, applying the operators P_n^{μ} to the above equation, we can deduce that

$$\begin{aligned} &\left|\int_0^{\infty} S_n^{\mu}(f;\theta)g_n^{\beta_n(x)}(\theta)d\theta - S_n^{\mu}(f;x)\right| \\ \leq\; & \|S_n^{\mu'}(f;x)\| \cdot \left|\int_0^{\infty}(\theta - x)g_n^{\beta_n(x)}(\theta)d\theta\right| + \frac{1}{2}\|S_n^{\mu''}(f;\xi)\| \cdot \left|\int_0^{\infty}(\theta - x)^2 g_n^{\beta_n(x)}(\theta)d\theta\right|. \end{aligned}$$

From Ref [17] Eq.(1) and the mean value theorem, there exists a point $\eta \in \left(\frac{k}{n}, \frac{k+1}{n}\right)$ such that

$$\begin{aligned} \left\|\frac{d}{dx}S_n^{\mu}(f;x)\right\| &= \left\|\frac{n\alpha_n(x)}{x}e^{-n\alpha_n(x)}\sum_{k=0}^{\infty}\left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right]\frac{(n\alpha_n(x))^k}{k!}\right\| \\ &= \left\|\frac{\alpha_n(x)}{x}e^{-n\alpha_n(x)}\sum_{k=0}^{\infty}\frac{(n\alpha_n(x))^k}{k!}n\frac{1}{n}f'(\eta)\right\| \leq \frac{\alpha_n(x)}{x}\|f'\|. \end{aligned}$$

Similarly, from Ref [17] Eq.(2) and the mean value theorem, there exist $\eta_1 \in \left(\frac{k+1}{n}, \frac{k+2}{n}\right)$, $\eta_2 \in \left(\frac{k}{n}, \frac{k+1}{n}\right)$, $\eta_3 \in (\eta_2, \eta_1)$, $0 \leq \eta_1 - \eta_2 \leq \frac{2}{n}$ such that

$$\begin{aligned} \left\|\frac{d^2}{dx^2}S_n^{\mu}(f;x))\right\| &= \left\|\left(\frac{n\alpha_n(x)}{x}\right)^2 e^{-n\alpha_n(x)}\sum_{k=0}^{\infty}\left[f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right]\frac{(n\alpha_n(x))^k}{k!}\right\| \\ &= \left\|\frac{n\alpha_n^2(x)}{x^2}e^{-n\alpha_n(x)}\sum_{k=0}^{\infty}[f'(\eta_1) - f'(\eta_2)]\frac{(n\alpha_n(x))^k}{k!}\right\| \\ &\leq \left\|\frac{n\alpha_n^2(x)}{x^2}e^{-n\alpha_n(x)}\sum_{k=0}^{\infty}\frac{2}{n}f''(\eta_3)\frac{(n\alpha_n(x))^k}{k!}\right\| = 2\frac{\alpha_n^2(x)}{x^2}\|f''\|. \end{aligned}$$

For all $x > 0$,

$$\int_0^{\infty}(\theta - x)g_n^{\beta_n(x)}(\theta)d\theta = \int_0^{\infty}\frac{1}{(n-1)!}\left(\frac{n}{\beta_n(x)}\right)^n \theta^n e^{-\frac{n\theta}{\beta_n(x)}}d\theta - x = \beta_n(x) - x,$$

and

$$\int_0^\infty (\theta - x)^2 g_n^{\beta_n(x)}(\theta)d\theta = \int_0^\infty \frac{1}{(n-1)!}\left(\frac{n}{\beta_n(x)}\right)^n \theta^{n+1} e^{-\frac{n\theta}{\beta_n(x)}} d\theta - 2x\beta_n(x) + x^2$$
$$= (\beta_n(x) - x)^2 + \frac{\beta_n^2(x)}{n}.$$

Then,

$$I_5 \leq \frac{\alpha_n(x)}{x}\|f'\|(\beta_n(x) - x) + \frac{\alpha_n^2(x)}{x^2}\|f''\|(\beta_n(x) - x)^2 + \frac{\gamma_n^2(x)}{n}\|f''\|.$$

Second, we estimate the part I_6. It follows from Ref [22] that

$$I_6 \leq C\omega^2_{\sqrt{x}}\left(f; \frac{\sqrt{\phi_n(x) + \nu_n^2(x)}}{\sqrt{x}}\right) + \omega_{\sqrt{x}}\left(f; \frac{\nu_n(x)}{\sqrt{x}}\right),$$

here $\nu_n(x) = S_n^\mu(t - x; x)$, $\phi_n(x) = S_n^\mu((t - x)^2; x)$.
We complete the proof of Theorem 3.2. □

Theorem 3.3. *Let $f \in C_B[0, \infty)$, we have*

$$|V_n^\mu(f; x) - f(x)| \leq C\omega_\varphi^2\left(f; \frac{\sqrt{\pi_n^2(x) + \zeta_n(x) + \sqrt{\rho_n(x)}}}{\varphi(x)}\right) + \omega_\varphi\left(f; \frac{\pi_n(x)}{\varphi(x)}\right),$$

where $\pi_n(x) = V_n^\mu((t - x); x)$, $\zeta_n(x) = V_n^\mu((t - x)^2; x)$, $\rho_n(x) = V_n^\mu((t - x)^4; x)$, $\varphi(x) = \sqrt{x(1 + x)}$.

Proof. Noting that the first order moment of the operator $V_n^\mu(f; x)$ is not zero, we define the auxiliary operator $\overline{V_n^\mu}(f; x)$:

$$\overline{V_n^\mu}(f; x) = V_n^\mu(f; x) + f(x) - f(V_n^\mu(t; x)).$$

Obviously, $\overline{V_n^\mu}(1; x) = 1$, $\overline{V_n^\mu}((t - x); x) = 0$, this now implies

$$|V_n^\mu(f; x) - f(x)| = |V_n^\mu(f; x) - \overline{V_n^\mu}(f; x) + \overline{V_n^\mu}(f; x) - f(x)|$$
$$\leq |V_n^\mu(f; x) - \overline{V_n^\mu}(f; x)| + |\overline{V_n^\mu}(f; x) - f(x)| = |\overline{V_n^\mu}(f; x) - f(x)| + |f(V_n^\mu(t; x)) - f(x)|$$
$$= |\overline{V_n^\mu}(f; x) - f(x)| + |f(x + \pi_n(x)) - f(x)|.$$

On the other hand,

$$|\overline{V_n^\mu}(f; x) - f(x)| = |\overline{V_n^\mu}(f; x) - f(x) + \overline{V_n^\mu}(g; x) - \overline{V_n^\mu}(g; x) + g(x) - g(x)|$$
$$\leq |\overline{V_n^\mu}(f - g; x)| + |\overline{V_n^\mu}(g; x) - g(x)| + |f(x) - g(x)| \leq 4\|f - g\| + |\overline{V_n^\mu}(g; x) - g(x)|.$$

For $g' \in A.C._{loc.}$, $\|\varphi^2 g''\| < \infty$, it is clear that

$$g(t) - g(x) - (t - x)g'(x) = \int_x^t (t - u)g''(u)du.$$

Applying the operator $\overline{V_n^\mu}$ to the above equation, we write

$$|\overline{V_n^\mu}(g; x) - g(x)| \leq V_n^\mu\left(\left|\int_x^t |t - u| \cdot |g''(u)|du\right|; x\right) + \left|\int_x^{x+\pi_n(x)} |x + \pi_n(x) - u| \cdot |g''(u)|du\right|$$
$$\leq \|\varphi^2(x)g''\|\left[V_n^\mu\left(\left|\int_x^t \frac{|t - u|}{\varphi^2(u)}du\right|; x\right) + \left|\int_x^{x+\pi_n(x)} \frac{|x + \pi_n(x) - u|}{\varphi^2(u)}du\right|\right].$$

For $\varphi(x) = \sqrt{x(x+1)}$, u is between t and x,

$$\int_x^t \frac{|t-u|}{\varphi^2(u)}du \le \frac{(t-x)^2}{x(x+1)} + \frac{(t-x)^2}{t(x+1)},$$

we obtain

$$V_n^\mu\left(\left|\int_x^t \frac{|t-u|}{\varphi^2(u)}du\right|;x\right) \le \varphi^{-2}(x)V_n^\mu((t-x)^2;x) + \frac{1}{x+1}\sqrt{V_n^\mu((t-x)^4;x)}\sqrt{V_n^\mu\left(\frac{1}{t^2};x\right)},$$

and

$$S_n^\mu(\frac{1}{t^2};x) = \sum_{k=1}^{\infty}\frac{n^2}{k^2}e^{-n\alpha_n(x)}\frac{(n\alpha_n(x))^k}{k!} = \frac{1}{\alpha_n^2(x)}\sum_{k=1}^{\infty}e^{-n\alpha_n(x)}\frac{(n\alpha_n(x))^{k+2}}{(k+2)!}\cdot\frac{(k+1)(k+2)}{k^2} \le \frac{6}{\alpha_n^2(x)}.$$

Further,

$$V_n^\mu(\frac{1}{t^2};x) = P_n^\mu(S_n^\mu(\frac{1}{t^2};\theta);x) \le P_n^\mu\left(\frac{6}{\alpha_n^2(\theta)};x\right) = 6\frac{1}{(n-1)!}\int_0^\infty \frac{1}{\alpha_n^2(\frac{\beta_n(x)\theta}{n})}\theta^{n-1}e^{-\theta}d\theta$$

$$= \frac{6n^2}{(n-1)(n-2)}\left(\frac{1-e^{-\frac{\mu}{n}}}{e^{\frac{\mu x}{n}}-1}\right)^2 \le \frac{6n^2}{(n-1)(n-2)x^2} \le \frac{27}{x^2}(n\ge 3),$$

and

$$V_n^\mu\left(\left|\int_x^t \frac{|t-u|}{\varphi^2(u)}du\right|;x\right) \le \varphi^{-2}(x)V_n^\mu((t-x)^2;x) + \frac{3\sqrt{3}}{x(x+1)}\sqrt{V_n^\mu((t-x)^4;x)}$$
$$\le 7\varphi^{-2}(x)\left[\zeta_n(x) + \sqrt{\rho_n(x)}\right].$$

Similarly,

$$\left|\int_x^{x+\pi_n(x)} \frac{|x+\pi_n(x)-u|}{\varphi^2(u)}du\right| \le \varphi^{-2}(x)\pi_n^2(x) + \frac{1}{x+1}\cdot\frac{1}{x+\pi_n(x)}\pi_n^2(x) \le 2\varphi^{-2}(x)\pi_n^2(x).$$

Therefore,

$$|\overline{V_n^\mu}(g;x) - g(x)| \le 7\|\varphi^2(x)g''\|\varphi^{-2}(x)\left[\pi_n^2(x) + \zeta_n(x) + \sqrt{\rho_n(x)}\right],$$

and

$$|\overline{V_n^\mu}(f;x) - f(x)| \le 4\|f-g\| + 7\|\varphi^2(x)g''\|\varphi^{-2}(x)\left[\pi_n^2(x) + \zeta_n(x) + \sqrt{\rho_n(x)}\right],$$

$$|f(x+\pi_n(x)) - f(x)| = \left|f\left(x + \varphi(x)\frac{\pi_n(x)}{\varphi(x)}\right) - f(x)\right| \le \omega_\varphi\left(f;\frac{\pi_n(x)}{\varphi(x)}\right).$$

To conclude, we have obtained

$$|V_n^\mu(f;x) - f(x)| \le 4\|f-g\| + 7\|\varphi^2(x)g''\|\varphi^{-2}(x)\left[\pi_n^2(x) + \zeta_n(x) + \sqrt{\rho_n(x)}\right] + \omega_\varphi\left(f;\frac{\pi_n(x)}{\varphi(x)}\right)$$

$$\le C\omega_\varphi^2\left(f;\frac{\sqrt{\pi_n^2(x) + \zeta_n(x) + \sqrt{\rho_n(x)}}}{\varphi(x)}\right) + \omega_\varphi\left(f;\frac{\pi_n(x)}{\varphi(x)}\right).$$

□

Theorem 3.4. *Let $f' \in C_B[0,\infty)$, for a positive linear operator V_n^μ, we have*

$$|V_n^\mu(f;x) - f(x)| \le |\pi_n(x)|\cdot|f'(x)| + 2\sqrt{\zeta_n(x)}\omega\left(f';\sqrt{\zeta_n(x)}\right),$$

here $\pi_n(x)$ and $\zeta_n(x)$ are given in Theorem 3.3.

Proof. From the following relation

$$V_n^{\mu}(f(t)-f(x);x) = f'(x)V_n^{\mu}((t-x);x) + V_n^{\mu}\left(\int_x^t [f'(u)-f'(x)]\,du;x\right),$$

for any $\xi > 0$,

$$\left|\int_x^t [f'(u)-f'(x)]du\right| \le \omega(f';\xi)\left(\frac{(t-x)^2}{\xi} + |t-x|\right).$$

Thus

$$|V_n^{\mu}(f;x)-f(x)| \le |f'(x)|\cdot|V_n^{\mu}((t-x);x)| + \omega(f';\xi)\left(\frac{1}{\xi}V_n^{\mu}((t-x)^2;x) + V_n^{\mu}(|t-x|;x)\right).$$

Choosing $\xi = \sqrt{\zeta_n(x)}$ and using the Hölder inequality, we conclude that

$$|V_n^{\mu}(f;x)-f(x)| \le |\pi_n(x)|\cdot|f'(x)| + 2\sqrt{\zeta_n(x)}\,\omega\left(f';\sqrt{\zeta_n(x)}\right).$$

□

Theorem 3.5. *Let $f'' \in C_B[0,\infty)$, then*

$$\lim_{n\to\infty} n[V_n^{\mu}(f;x)-f(x)] = \frac{\mu x^2+\mu x}{2}f'(x) + (x^2+x)f''(x).$$

Proof. We recall the linear properties of the operator V_n^{μ} and the expansion of f,

$$V_n^{\mu}(f;x)-f(x) = f'(x)V_n^{\mu}((t-x);x) + \frac{f''(x)}{2}V_n^{\mu}((t-x)^2;x) + V_n^{\mu}(h(t,x)(t-x)^2;x),$$

where $h(t,x) = \frac{1}{2}[f''(\xi)-f''(x)]$, ξ is between x and t. For $\delta > 0$, we have

$$V_n^{\mu}(|h(t,x)|(t-x)^2;x) \le \omega(f'';\delta)V_n^{\mu}((t-x)^2;x) + \frac{1}{\delta}\omega(f'';\delta)V_n^{\mu}(|t-x|^3;x).$$

Choosing $\delta = \frac{1}{\sqrt{n}}$, it follows from Lemma 2.9 that

$$\begin{aligned}\lim_{n\to\infty} nV_n^{\mu}(h(t,x)(t-x)^2;x) &= \lim_{n\to\infty}\omega(f'';\delta)nV_n^{\mu}((t-x)^2;x) + \lim_{n\to\infty}\omega(f'';\delta)n^{\frac{3}{2}}V_n^{\mu}(|t-x|^3;x)\\ &\le \lim_{n\to\infty}\omega(f'';\delta)\lim_{n\to\infty}\sqrt{nV_n^{\mu}((t-x)^2;x)}\cdot\lim_{n\to\infty}\sqrt{n^2V_n^{\mu}((t-x)^4;x)}\\ &= 0.\end{aligned}$$

Combining Lemma 2.9, the proof of Theorem 3.5 can be completed. □

4. Approximation Properties of Meyer-König-Zeller Type operators

In this part, $[0,1)$ is the domain of Meyer-König-Zeller operators with independent variables x and function $f(x)$. Correspondingly, $[0,\infty)$ is the domain of Baskakov operators with independent variables y and function $g(y)$.

Definition 4.1. *The transformation σ: $[0,1) \longrightarrow [0,\infty)$ is given by*

$$y = \sigma(x) = \frac{x}{1-x},$$

and its inverse transformation σ^{-1}: $[0,\infty) \longrightarrow [0,1)$ is given by

$$x = \sigma^{-1}(y) = \frac{y}{1+y}.$$

Definition 4.2. *Let $x \in [0,1)$, $y \in [0,\infty)$, $\lambda(x) = 1-x$. For functions $f(x)$ and $g(y)$, we define an operator τ: $[0,\infty) \longrightarrow [0,1)$:*

$$f(x) = \tau(g)(x) = \lambda(x)(g \circ \sigma)(x),$$

and its inverse operator τ^{-1}: $[0,1) \longrightarrow [0,\infty)$ is given by:

$$g(y) = \tau^{-1}(f)(y) = \frac{1}{\lambda \circ \sigma^{-1}(y)}(f \circ \sigma^{-1})(y).$$

Definition 4.3. *By the relationship between Baskakov type operators and Meyer-König-Zeller operators, the definition of the new type Meyer-König-Zeller operators is as following:*

$$M_n^{\mu}(f;x) = \frac{1}{\lambda(x)[1+\gamma_n(\frac{x}{1-x})]}\tau(V_n^{\mu}(\tau^{-1}(f)))(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}[\delta_n(x)]^k[1-\delta_n(x)]^{n+1},$$

where

$$\delta_n(x) = \frac{e^{\frac{\mu x}{n(1-x)}}-1}{e^{\frac{\mu x}{n(1-x)}}-e^{-\frac{\mu}{n}}}; \quad \lim_{n\to\infty}\delta_n(x) = x.$$

Remark 4.4. *From the above transformation σ and operator τ, we conclude that*

$$\begin{aligned}
&\frac{1}{\lambda(x)[1+\gamma_n(\frac{x}{1-x})]}\tau(V_n^{\mu}(\tau^{-1}(f)))(x) = \frac{1}{\lambda(x)[1+\gamma_n(\frac{x}{1-x})]}\lambda(x)(V_n^{\mu}(\tau^{-1}(f))\circ\sigma)(x)\\
=\ & \frac{1}{1+\gamma_n(y)}V_n^{\mu}(g;y) = \frac{1}{1+\gamma_n(y)}\sum_{k=0}^{\infty} g\left(\frac{k}{n}\right)\binom{n+k-1}{k}[\gamma_n(y)]^k[1+\gamma_n(y)]^{-n-k}\\
=\ & \sum_{k=0}^{\infty}\frac{1}{\lambda\circ\sigma^{-1}\left(\frac{k}{n}\right)}(f\circ\sigma^{-1})\left(\frac{k}{n}\right)\binom{n+k-1}{k}[\gamma_n(y)]^k[1+\gamma_n(y)]^{-n-k-1}\\
=\ & \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}\left[\frac{\gamma_n(y)}{1+\gamma_n(y)}\right]^k\left[1-\frac{\gamma_n(y)}{1+\gamma_n(y)}\right]^{n+1}\\
=\ & \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}[\delta_n(x)]^k[1-\delta_n(x)]^{n+1} = M_n^{\mu}(f;x).
\end{aligned}$$

Theorem 4.5. *Let $f(x) \in C_B[0,1)$, $g(y) \in C_B[0,\infty)$. According to Definition 4.1-4.2, one has*

$$|M_n^{\mu}(f;x) - f(x)| \le |V_n^{\mu}(g;y) - g(y)| + \|g\|\frac{\mu}{n} + O(n^{-2}).$$

Proof. Using Mathematica software or simple calculations, we have

$$\frac{1-e^{-\frac{\mu}{n}}}{e^{\frac{\mu y}{n}}-e^{-\frac{\mu}{n}}} = \frac{1}{1+y} - \frac{\mu y}{2(1+y)}\cdot\frac{1}{n} + O(n^{-2}).$$

Recalling the boundedness of $g(y)$ and according to Definition 4.2 and 4.3, we have

$$\begin{aligned}
& |M_n^{\mu}(f;x) - f(x)| \\
= & \left|\frac{1}{1+\gamma_n(y)} V_n^{\mu}(g;y) - \frac{1}{1+y} g(y)\right| \\
= & \left|\frac{1}{1+\gamma_n(y)} (V_n^{\mu}(g;y) - g(y)) + \frac{1}{1+\gamma_n(y)} g(y) - \frac{1}{1+y} g(y)\right| \\
\leq & \left|\frac{1}{1+\gamma_n(y)}\right| \cdot |V_n^{\mu}(g;y) - g(y)| + \left|\frac{1}{1+\gamma_n(y)} - \frac{1}{1+y}\right| \cdot |g(y)| \\
\leq & |V_n^{\mu}(g;y) - g(y)| + \|g\| \left|\frac{1-e^{-\frac{\mu}{n}}}{e^{\frac{\mu y}{n}} - e^{-\frac{\mu}{n}}} - \frac{1}{1+y}\right| \\
\leq & |V_n^{\mu}(g;y) - g(y)| + \|g\| \frac{\mu y}{2(1+y)} \cdot \frac{1}{n} + O(n^{-2}) \\
\leq & |V_n^{\mu}(g;y) - g(y)| + \|g\| \frac{\mu}{n} + O(n^{-2}).
\end{aligned}$$

□

Noting that the approximation theorems of Meyer-König-Zeller operators can be obtained easily by the corresponding approximation theorem of Baskakov operators, for example, we shall state the result in the following corollary from Theorem 3.3.

Corollary 4.6. *Suppose* $f(x) \in C_B[0,1)$, $g(y) \in C_B[0,\infty)$, *then*

$$|M_n^{\mu}(f;x) - f(x)| \leq C\omega_{\varphi}^2\left(g; \frac{\sqrt{\pi_n^2(y) + \zeta_n(y) + \sqrt{\rho_n(y)}}}{\varphi(y)}\right) + \omega_{\varphi}\left(g; \frac{\pi_n(y)}{\varphi(y)}\right) + \|g\|\frac{\mu}{n} + O(n^{-2}),$$

where $\pi_n(y) = V_n^{\mu}((t-y);y)$, $\zeta_n(y) = V_n^{\mu}((t-y)^2;y)$, $\rho_n(y) = V_n^{\mu}((t-y)^4;y)$, $\varphi(y) = \sqrt{y(1+y)}$, $y = \frac{x}{1-x}$.

Data Availability

No data were used to support this study.

Declarations

Conflict of interest

The authors declare no conflict of interest in this paper.

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