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# Riesz potential and its commutators in generalized weighted Morrey spaces defined on Carleson curves

**Fatma Geleri**[a,*], **Canay Aykol**[b], **Javanshir J. Hasanov**[c,d]

[a]*Ankara University Graduate School of Natural and Applied Sciences, Ankara, Turkey*
[b]*Department of Mathematics, Faculty of Science, Ankara University, 06560 Ankara, Turkey*
[c]*Azerbaijan State Oil and Industry University, Baku, Azerbaijan*
[d]*Department of Science, Western Caspian University, Baku, Azerbaijan*

**Abstract.** In this paper we take up a characterization of weighted inequalities for Riesz potential and its commutators in generalized weighted Morrey spaces $\mathcal{M}^{p,\varphi}_{\omega}(\Gamma)$ defined on Carleson curves $\Gamma$ with the class of weights $F_{p,q}$. We prove the boundedness of Riesz potential $I^{\alpha}_{\Gamma}$ and its commutator $|b, I^{\alpha}_{\Gamma}|$ from the spaces $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\Gamma)$ to the spaces $\mathcal{M}^{p,\varphi_2}_{\omega_2}(\Gamma)$, where $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \alpha$, $(\varphi_1, \varphi_2)$ are positive measurable functions, $(\omega_1, \omega_2) \in F_{p,q}(\Gamma)$ and $b \in BMO(\Gamma)$.

## 1. Introduction

Morrey spaces were introduced by C.B. Morrey [31] in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations. Subsequently, Morrey spaces found important applications in the Navier- Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients and potential theory. Riesz potential $I^{\alpha}_{\Gamma}$ is one of the basic tools of harmonic analysis used in the solution of partial differential equations, and the boundedness of these operators in various function spaces has been studied by many mathematicians. Some of these spaces are generalized Morrey space $M^{p,\varphi}$, generalized weighted Morrey space $\mathcal{M}^{p,\varphi}_{\omega}$ and global generalized weighted Morrey space $GM_{p,\theta,\varphi,\omega}$. Coifman and Fefferman [9] obtained sufficient conditions for maximal and singular operator in weighted Lebesgue spaces and also Muckenhoupt and Wheeden [32] obtained weighted norm inequalities for fractional integrals. The boundedness of the Riesz potential in Morrey spaces was obtained by Adams [1]. In [16], Guliyev obtained the boundedness of Riesz potential, maximal and singular integral operators in generalized Morrey spaces. In 2012, Guliyev [17] generalized both the generalized Morrey space and the weighted Morrey space and defined the generalized weighted Morrey space. The boundedness of the Riesz potential in the generalized weighted Morrey space was proved in [4]. The boundedness of the Riesz potential and its commutator in global generalized weighted Morrey spaces with the Fefferman-Pong

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* Corresponding author: Fatma Geleri
*Email addresses:* fgeleri@ankara.edu.tr (Fatma Geleri), aykol@science.ankara.edu.tr (Canay Aykol), hasanovjavanshir@gmail.com (Javanshir J. Hasanov)
ORCID iDs: https://orcid.org/0009-0007-7016-0367 (Fatma Geleri), https://orcid.org/0000-0002-2854-6369 (Canay Aykol), https://orcid.org/0000-0002-3518-348X (Javanshir J. Hasanov)

weight class has been proven by Aykol, Geleri, Hasanov and Safarov in [5]. In [14], Riesz potential and its commutators in global generalized weighted Morrey spaces of homogeneous type with weights belonging to Fefferman-Pong class were proved.

Let $\Gamma = \{t \in \mathbb{C} : t = t(s), 0 \le s \le l \le \infty\}$ be a rectifiable Jordan curve in the complex plane with arc-length measure $\nu(t) = s$, where $l = \nu\Gamma$ lenghts of $\Gamma$.

We denote

$$\Gamma(t, r) = \Gamma \cap B(t, r), \quad t \in \Gamma \text{ and } r > 0,$$

where $B(t, r) = \{z \in \mathbb{C} : |z - t| < r\}$.

A rectifiable Jordan curve $\Gamma$ is called a Carleson curve (regular curve) if the condition

$$\nu\Gamma(t, r) \le c_0 r$$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ is independent of $t$ and $r$.

**Definition 1.1.** *Let $1 \le p < \infty$ and $0 \le \lambda \le 1$. We denote by $L_{p,\lambda}(\Gamma)$ the Morrey space, the set of all locally integrable functions $f$ on $\Gamma$ such that*

$$\left\|f\right\|_{L_{p,\lambda}(\Gamma)} = \sup_{t\in\Gamma,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))} < \infty.$$

*If $\lambda < 0$ or $\lambda > 1$, then $L_{p,\lambda}(\Gamma) = \Theta$, where $\Theta$ is the set of all functions equivalent to $0$ on $\Gamma$.*

Maximal operators and potential operators in various spaces defined on Carleson curves have been widely studied by many mathematicians (see, for example [6], [7], [23], [25], [26], [27], [29], [37]). Albrecht Böttcher and Yuri I. Karlovich showed that for general Carleson curves and general Muckenhoupt weights the sets in question were logarithmic leaves with a halo, and they presented final results concerning the shape of the halo in [7]. Dadashova, Aykol, Cakir and Serbetci studied the potential operator $I_\Gamma^\alpha$ in the modified Morrey space $\widetilde{L}_{p,\lambda}(\Gamma)$ and the spaces $BMO(\Gamma)$ defined on Carleson curves $\Gamma$ in [11]. In [19] Guliyev, Armutcu and Azeroglu established the boundedness of potential operator in the local generalized Morrey space $LM^{p,\varphi}_{\{t_0\}}(\Gamma)$ and the generalized Morrey space $M^{p,\varphi}(\Gamma)$ defined on Carleson curves $\Gamma$, respectively.

Let $f \in L_1^{loc}(\Gamma)$. The maximal operator $M_\Gamma$ is defined by

$$M_\Gamma f(t) = \sup_{t>0} (\nu\Gamma(t, r))^{-1} \int_{\Gamma(t,r)} |f(\tau)| d\nu(\tau).$$

**Theorem 1.2.** *[36] Let $1 < p < \infty$ and $(\omega_1, \omega_2) \in F_p(\Gamma)$. Then the operator $M_\Gamma$ is bounded from $L_{p,\omega_1}(\Gamma)$ to $L_{p,\omega_2}(\Gamma)$.*

**Corollary 1.3.** *Let $1 < p < \infty$ and $\omega \in F_p(\Gamma)$. Then the operator $M_\Gamma$ is bounded in $L_{p,\omega}(\Gamma)$.*

Let $0 \le \alpha < 1$ and $f$ be a locally integrable function on $\Gamma$. Then the fractional maximal function $M_\Gamma^\alpha$ is defined by

$$M_\Gamma^\alpha f(x) = \sup_{t>0} (\nu\Gamma(t, r))^{-1+\alpha} \int_{\Gamma(t,r)} |f(\tau)| d\nu(\tau).$$

Let $0 < \alpha < 1$ and $f \in L_1^{loc}(\Gamma)$. Then the potential operator $I_\Gamma^\alpha$ is defined by

$$I_\Gamma^\alpha f(t) = \int_\Gamma |t - \tau|^{\alpha-1} f(\tau) d\nu(\tau).$$

Adams type Sobolev-Morrey inequalities for the potential operators in Morrey space defined on Carleson curves were proved in [12].

**Theorem 1.4.** *[12] Let $\Gamma$ be a Carleson curve, $1 < p < \frac{1-\lambda}{\alpha}$, $0 < \alpha < 1$ and $0 \le \lambda \le 1-\alpha$. Then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of $I_\Gamma^\alpha$ from $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$.*

A measurable function $\omega : \Gamma \to [0,\infty]$ is referred to as a weight if $\omega^{-1}(\{0,\infty\})$ has measure zero. Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\Gamma$ be a rectifiable Jordan curve, and $\omega$ be a weight on $\Gamma$. Let the condition

$$\sup_{t\in\Gamma}\sup_{r>0} \frac{1}{r}\left(\int_{\Gamma(t,r)} \omega^p(y)d\nu(y)\right)^{1/p}\left(\int_{\Gamma(t,r)} \omega^{-p'}(y)d\nu(y)\right)^{1/p'} \tag{1}$$

is finite. The set of all weights $\omega$ on $\Gamma$ satisfying (1) is usually denoted by $A_p$ and referred to as the set of Muckenhoupt weights.

Let $1 \le p < \infty$, $\varphi$ be a positive measurable function on $(0,\infty)$ and $\omega$ be a weight. We define the generalized weighted Morrey space $\mathcal{M}_\omega^{p,\varphi}(\Gamma)$, the space of all functions $f \in L_{p,\omega}^{loc}(\Gamma)$ with finite norm

$$\|f\|_{\mathcal{M}_\omega^{p,\varphi}(\Gamma)} = \sup_{t\in\Gamma, r>0} \frac{r^{-\frac{1}{pk'}}}{\varphi(r)\|\omega\|_{L_{pk}(\Gamma(t,r))}} \|f\|_{L_{p,\omega}(\Gamma(t,r))},$$

where $L_{p,\omega}(\Gamma(t,r))$ denotes the weighted $L_p$-space of measurable functions $f$ for which

$$\|f\|_{L_{p,\omega}(\Gamma(t,r))} \equiv \|f\chi_{\Gamma(t,r)}\|_{L_{p,\omega}(\Gamma)} = \left(\int_{\Gamma(t,r)} |f(y)|^p \omega^p(y)d\nu(y)\right)^{\frac{1}{p}}.$$

Notice that if $\omega(x) = \chi_{\Gamma(t,r)}(x)$, then $\mathcal{M}_\omega^{p,\varphi}(\Gamma) = \mathcal{M}^{p,\varphi}(\Gamma)$ is the generalized Morrey space and if $\varphi(r) = r^{\frac{1-\lambda}{p}}$, then $\mathcal{M}_\omega^{p,\varphi}(\Gamma) = L_{p,\lambda}(\Gamma)$ is the classical Morrey space.

Two-weight norm inequalities for the operators of harmonic analysis on various function spaces were widely studied (see, for example [10, 15, 24, 28, 30]). The weighted norm inequalities with different types of weights on Morrey spaces were also studied (see, for example [20, 33, 37] ). The two-weight norm inequalities for the Hardy-Littlewood maximal function on Morrey spaces were obtained in [39]. The two-weight norm inequalities on weighted Morrey spaces for fractional maximal operators and fractional integral operators were obtained in [34]. Two-weight norm inequalities on generalized weighted Morrey spaces for maximal, Calderón-Zygmund operators and their commutators were obtained in [3].

In this paper, we give a new characterization of two-weighted inequalities for Riesz potential $I_\Gamma^\alpha$ and its commutators in generalized weighted Morrey spaces $\mathcal{M}_\omega^{p,\varphi}(\Gamma)$ defined on Carleson curves $\Gamma$ with the class of weights $F_{p,q}(\Gamma)$. We find the conditions for the boundedness of Riesz potential $I_\Gamma^\alpha$ and its commutator $|b, I_\Gamma^\alpha|$ from the generalized weighted Morrey spaces $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\Gamma)$ to the spaces $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\Gamma)$, where $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \alpha$, $(\varphi_1, \varphi_2)$ are positive measurable functions, $(\omega_1, \omega_2) \in F_{p,q}(\Gamma)$ and $b \in BMO(\Gamma)$.

In the sequel we use the letter $C$ for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence.

## 2. Background material

**Definition 2.1.** *The weight functions $(\omega_1, \omega_2)$ belong to the class $\widetilde{A}_p(\Gamma)$ for $1 \le p < \infty$, if the following statement*

$$\sup_{t\in\Gamma}\sup_{r>0} \frac{1}{r}\left(\int_{\Gamma(t,r)} \omega_2^p(y)d\nu(y)\right)^{\frac{1}{p}}\left(\int_{\Gamma(t,r)} \omega_1^{-p'}(y)d\nu(y)\right)^{\frac{1}{p'}}$$

*is finite.*

**Definition 2.2.** *The weight functions $(\omega_1, \omega_2)$ belong to the class $A_{p,q}(\Gamma)$ for $1 \le p, q < \infty$, if the following statement*

$$\sup_{t\in\Gamma}\sup_{r>0} r^{\frac{1}{p}-\frac{1}{q}-1}\left(\int_{\Gamma(t,r)} \omega_2^q(y)d\nu(y)\right)^{\frac{1}{q}}\left(\int_{\Gamma(t,r)} \omega_1^{-p'}(y)d\nu(y)\right)^{\frac{1}{p'}}$$

*is finite.*

**Definition 2.3.** *The weight functions $(\omega_1, \omega_2)$ belongs to the class $F_p(\Gamma)$ for $1 < p < \infty$, $1 < k < \infty$, if the following statement*

$$\sup_{t\in\Gamma}\sup_{r>0}\left(\frac{1}{r}\int_{\Gamma(t,r)} \omega_2^p(y)d\nu(y)\right)^{\frac{1}{p}}\left(\frac{1}{r}\int_{\Gamma(t,r)} \omega_1^{-p'k}(y)d\nu(y)\right)^{\frac{1}{p'k}}$$

*is finite.*

**Definition 2.4.** *The weight functions $(\omega_1, \omega_2)$ belongs to the class $F_{p,q}(\Gamma)$ for $1 < p, q, k < \infty$, if the following statement*

$$\sup_{t\in\Gamma}\sup_{r>0} r^{\alpha-\frac{1}{p}+\frac{1}{q}-\frac{1}{qk}-\frac{1}{p'k}}\left(\int_{\Gamma(t,r)} \omega_2^{qk}(y)d\nu(y)\right)^{\frac{1}{qk}}\left(\int_{\Gamma(t,r)} \omega_1^{-p'k}(y)d\nu(y)\right)^{\frac{1}{p'k}}$$

*is finite.*

Let $f \in L_1^{loc}(\Gamma)$. Then the sharp maximal function $M_\Gamma^\sharp$ is defined by

$$M_\Gamma^\sharp f(t) = \sup_{t>0}(\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}|f(y) - f_{\Gamma(t,r)}|d\nu(y),$$

where $f_{\Gamma(t,r)}(t) = (\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)} f(y)d\nu(y)$.

**Definition 2.5.** *We define the $BMO(\Gamma)$ space as the set of all locally integrable functions $f$ such that*

$$\|f\|_{BMO} = \sup_{t\in\Gamma,\, r>0}(\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}|f(y) - f_{\Gamma(t,r)}|d\nu(y) < \infty$$

*or*

$$\|f\|_{BMO} = \inf_C \sup_{t\in\Gamma,\, r>0}(\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}|f(y) - C|d\nu(y) < \infty.$$

**Definition 2.6.** *Let $1 \le p < \infty$. We define the $BMO_{p,\omega}(\Gamma)$ space as the set of all locally integrable functions $f$ such that*

$$\|f\|_{BMO_{p,\omega}} = \sup_{t\in\Gamma,\, r>0}\frac{\|(f(\cdot) - f_{\Gamma(t,r)})\chi_{\Gamma(t,r)}\|_{L_{p,\omega}(\Gamma)}}{\|\omega\|_{L_p(\Gamma(t,r))}}$$

*or*

$$\|f\|_{BMO_{p,\omega}} = \sup_{t\in\Gamma,\, r>0}(\nu\Gamma(t,r))^{-1}\|(f(\cdot) - f_{\Gamma(t,r)})\chi_{\Gamma(t,r)}\|_{L_{p,\omega}(\Gamma)}\|\omega^{-1}\|_{L_{p'}(\Gamma(t,r))} < \infty.$$

**Theorem 2.7.** *[21] Let $1 \leq p < \infty$ and $\omega$ be a Lebesgue measurable function. If $\omega \in A_p(\Gamma)$, then the norms $\|\cdot\|_{BMO_{p,\omega}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

**Lemma 2.8.** *[22] Let $b \in BMO(\Gamma)$. Then there is a constant $C > 0$ such that*

$$\left|b_{B(t,r)} - b_{B(t,\tau)}\right| \leq C\|b\|_{BMO} \ln \frac{\tau}{r} \quad \textit{for} \quad 0 < 2r < \tau,$$

*where $C$ is independent of $b$, $r$ and $\tau$.*

Let $L_{\infty,v}(\mathbb{R}_+)$ be the weighted $L_\infty$-space with the norm

$$\|g\|_{L_{\infty,v}(\mathbb{R}_+)} = \operatorname*{ess\,sup}_{r>0} v(r)g(r).$$

We represent

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{r\to 0+} \varphi(r) = 0\right\}.$$

Let $u$ be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(r) := \|u\, g\|_{L_\infty(0,r)}, \quad r \in (0,\infty).$$

The following theorem was proved in [8].

**Theorem 2.9.** *[8] Suppose that $v_1$ and $v_2$ are non-negative measurable functions such that $0 < \|v_1\|_{L_\infty(0,r)} < \infty$ for every $r > 0$. Let $u$ be a continuous non-negative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L^\infty_{v_1}(\mathbb{R}_+)$ to $L^\infty_{v_2}(\mathbb{R}_+)$ on the cone $\mathbb{A}$ if and only if*

$$\left\|v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(0,\cdot)}\right)\right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H^*_w g(r) := \int_r^\infty g(\tau)w(\tau)d\tau, \quad 0 < r < \infty,$$

where $w$ is a weight.

The following theorem was proved in [18].

**Theorem 2.10.** *[18] Let $v_1$, $v_2$ and $w$ be weights on $(0,\infty)$ and $v_1(r)$ be bounded outside a neighborhood of the origin. The inequality*

$$ess \sup_{r>0} v_2(r) H^*_w g(r) \leq C ess \sup_{r>0} v_1(r)g(r)$$

*holds for some $C > 0$ for all non-negative and non-decreasing $g$ on $(0,\infty)$ if and only if*

$$B := \sup_{r>0} v_2(r) \int_r^\infty \frac{w(\tau)d\tau}{\operatorname*{ess\,sup}_{\tau<s<\infty} v_1(s)} < \infty.$$

## 3. Two-weighted inequalities for Riesz potential and its commutators in generalized weighted Morrey spaces defined on Carleson curves

In this section we prove the two-weighted inequalities for Riesz potential and its commutators in generalized weighted Morrey spaces defined on Carleson curves with the class of weights $F_{p,q}$.

**Theorem 3.1.** *[35] Let $0<\alpha<1$, $1<p<\frac{1}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\alpha$ and $(\omega_1,\omega_2)\in F_{p,q}(\Gamma)$. Then the operator $I_\Gamma^\alpha$ is bounded from $L_{p,\omega_1}(\Gamma)$ to $L_{q,\omega_2}(\Gamma)$.*

From the inequality $M_\Gamma^\alpha f(t)\le \omega_1^{\alpha-1}(I_\Gamma^\alpha)|f|(t)$, we get the following corollary.

**Corollary 3.2.** *Let $0<\alpha<1$, $1<p<\frac{1}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\alpha$ and $(\omega_1,\omega_2)\in F_{p,q}(\Gamma)$. Then the operator $M_\Gamma^\alpha$ is bounded from $L_{p,\omega_1}(\Gamma)$ to $L_{q,\omega_2}(\Gamma)$.*

**Theorem 3.3.** *Let $0<\alpha<1$, $1<p<\frac{1}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\alpha$ and $(\omega_1,\omega_2)\in F_{p,q}(\Gamma)$. Then there exists a constant $C>0$ such that for an arbitrary $f\in L_{p,\omega_1}(\Gamma)$ the inequality*

$$\|I_\Gamma^\alpha f\|_{L_{q,\omega_2}(\Gamma(t,r))}\le Cr^{\frac{1}{qk'}}\|\omega_2\|_{L_{qk}(\Gamma(t,r))}\int_r^\infty \tau^{-\frac{1}{qk'}}\frac{\|f\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau} \tag{1}$$

*is hold.*

*Proof.* We represent $f$ as

$$f_1(y)=f(y)\chi_{\Gamma(t,2r)}(y) \text{ and } f_2(y)=f(y)\chi_{^c\Gamma(t,2r)}(y) \tag{2}$$

and have

$$I_\Gamma^\alpha f(y)=I_\Gamma^\alpha f_1(y)+I_\Gamma^\alpha f_2(y)$$

By Theorem 3.1, we obtain

$$\left\|I_\Gamma^\alpha f_1\right\|_{L_{q,\omega_2}(\Gamma(t,r))}\le C\left\|f_1\right\|_{L_{p,\omega_1}(\Gamma(t,r))}=C\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,2r))}.$$

Then

$$\left\|I_\Gamma^\alpha f_1\right\|_{L_{q,\omega_2}(\Gamma(t,r))}\le C\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,2r))}, \tag{3}$$

where the constant $C$ independent of $f$. With the help of (3) inequality we get

$$\left\|I_\Gamma^\alpha f_1\right\|_{L_{q,\omega_2}(\Gamma(t,r))}\le Cr^{\frac{1}{qk'}}\|\omega_2\|_{L_{qk}(\Gamma(t,r))}\int_r^\infty \tau^{-\frac{1}{qk'}}\frac{\|f\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}. \tag{4}$$

When $|t-z|\le r$ and $\left|z-y\right|\ge 2r$ we have $\frac{1}{2}\left|z-y\right|\le\left|t-y\right|\le\frac{3}{2}\left|z-y\right|$, and therefore

$$\left|I_\Gamma^\alpha f_2(z)\right|\le\int\limits_{^c\Gamma(t,2r)}\left|z-y\right|^{\alpha-1}\left|f(y)\right|d\nu(y)\le C\int\limits_{^c\Gamma(t,2r)}\left|t-y\right|^{\alpha-1}\left|f(y)\right|d\nu(y).$$

Then we obtain

$$\int\limits_{^c\Gamma(t,2r)}\left|t-y\right|^{\alpha-1}\left|f(y)\right|d\nu(y) \quad = \quad C\int\limits_{^c\Gamma(t,2r)}\left|f(y)\right|\left(\int\limits_{\left|t-y\right|}^{\infty}\tau^{\alpha-2}d\tau\right)d\nu(y)$$

$$
\begin{aligned}
&= C\int\limits_{2r}^{\infty}\tau^{\alpha-2}\left(\int\limits_{\{y\in\Gamma:2r\le|t-y|\le\tau\}}\left|f(y)\right|d\nu(y)\right)d\tau\\
&\le C\int\limits_{r}^{\infty}\tau^{\alpha-2}\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}\left\|\omega_1^{-1}\right\|_{L_{p'}(\Gamma(t,\tau))}d\tau\\
&\le C\int\limits_{r}^{\infty}\tau^{\alpha-2+\frac{1}{p'k'}}\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}\left\|\omega_1^{-1}\right\|_{L_{p'k}(\Gamma(t,\tau))}d\tau\\
&\le C\int\limits_{r}^{\infty}\tau^{\alpha-2+\frac{1}{p'k'}-\alpha+\frac{1}{p}-\frac{1}{q}+\frac{1}{p'k}+\frac{1}{qk}}\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}d\tau\\
&= C\int_r^{\infty}\tau^{-\frac{1}{qk'}}\frac{\|f\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}.
\end{aligned}
$$

Therefore we obtain

$$
\left\|I_\Gamma^\alpha f_2\right\|_{L_{q,\omega_2}(\Gamma(t,r))}\le Cr^{\frac{1}{qk'}}\|\omega_2\|_{L_{qk}(\Gamma(t,r))}\int_r^{\infty}\tau^{-\frac{1}{qk'}}\frac{\|f\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}. \tag{5}
$$

Hence we get which together with (4) and (5) yields (1). □

**Theorem 3.4.** *Let* $0<\alpha<1$, $1<p<\frac{1}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\alpha$ *and* $(\omega_1,\omega_2)\in F_{p,q}(\Gamma)$ *and the functions* $\varphi_1(r)$ *and* $\varphi_2(r)$ *fulfill the condition*

$$
\int_r^{\infty}\tau^{-\frac{1}{qk'}}\frac{\operatorname*{ess\,inf}_{\tau<r<\infty} r^{\frac{1}{pk'}}\varphi_1(r)\|\omega_1\|_{L_{pk}(\Gamma(t,r))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}\le C\varphi_2(r). \tag{6}
$$

*Then the operator* $I_\Gamma^\alpha$ *is bounded from* $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\Gamma)$ *to* $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\Gamma)$.

*Proof.* Let $f\in\mathcal{M}_{\omega_1}^{p,\varphi_1}$. From the definition of the norm of generalized weighted Morrey space we write

$$
\left\|I_\Gamma^\alpha f\right\|_{\mathcal{M}_{\omega_2}^{q,\varphi_2}(\Gamma(t,r))}=\sup_{t\in\Gamma,r>0}\frac{1}{r^{\frac{1}{qk'}}\varphi_2(r)\|\omega_2\|_{L_{qk}(\Gamma(t,r))}}\left\|I_\Gamma^\alpha f\right\|_{L_{q,\omega_2}(\Gamma(t,r))}. \tag{7}
$$

We estimate $\left\|I_\Gamma^\alpha f\right\|_{L_{q,\omega_2}(\Gamma(t,r))}$ in (7) by means of Theorem 3.3 and Theorem 2.10. Taking $\nu_1(r)=\frac{1}{r^{\frac{1}{pk'}}\varphi_1(r)\|\omega_1\|_{L_{pk}}}$, $\nu_2(r)=\frac{1}{\varphi_2(r)}$, $g(r)=\|f\|_{L_{p,\omega_1}(\Gamma(t,r))}$, $w(\tau)=\tau^{-\frac{1}{qk'}-1}\|\omega_2\|^{-1}_{L_{qk}(\Gamma(t,\tau))}$, with inequality (6) we obtain

$$
\begin{aligned}
\left\|I_\Gamma^\alpha f\right\|_{\mathcal{M}_{\omega_2}^{q,\varphi_2}(\Gamma)} &\le C\sup_{t\in\Gamma,r>0}\frac{r^{\frac{1}{qk'}}\|\omega_2\|_{L_{qk}(\Gamma(t,r))}}{\varphi_2(r)r^{\frac{1}{qk'}}\|\omega_2\|_{L_{qk}(\Gamma(t,r))}}\int_r^{\infty}\tau^{-\frac{1}{qk'}}\frac{\|f\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}\\
&= C\sup_{t\in\Gamma,r>0}\frac{1}{\varphi_2(r)}\int_r^{\infty}\tau^{-\frac{1}{qk'}}\frac{\|f\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}\\
&\le C\sup_{t\in\Gamma,r>0}\frac{1}{r^{\frac{1}{pk'}}\varphi_1(r)\|\omega_1\|_{L_{pk}(\Gamma(t,r))}}\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,r))}\\
&= C\left\|f\right\|_{\mathcal{M}_{\omega_1}^{p,\varphi_1}(\Gamma)}.
\end{aligned}
$$

□

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In this section we consider commutators of the Riesz potential. Let $f \in L_1^{loc}(\Gamma)$ and $b \in BMO(\Gamma)$. Then commutators of the Riesz potential $[b, I_\Gamma^\alpha]$ are defined by the following equality

$$[b, I_\Gamma^\alpha]f(t) = \int\limits_\Gamma (b(t) - b(\tau))|t-\tau|^{\alpha-1} f(\tau)d\nu(\tau), \quad 0 < \alpha < 1.$$

Similarly, given a measurable function $b$ the operator $|b, I_\Gamma^\alpha|$ is defined by

$$|b, I_\Gamma^\alpha|f(t) = \int\limits_\Gamma |b(t) - b(\tau)||t-\tau|^{\alpha-1}|f(\tau)|d\nu(\tau), \quad 0 < \alpha < 1.$$

The maximal commutator $M_{b,\Gamma}$ is defined by

$$M_{b,\Gamma}(f)(t) := \sup_{r>0} (\nu\Gamma(t,r))^{-1} \int_\Gamma |b(t) - b(\tau)||f(\tau)|d\nu(\tau)$$

for all $t \in \Gamma$.

**Theorem 3.5.** *[2] Let $b \in BMO(\Gamma)$. Suppose that $Y$ is a Banach space of measurable functions defined by on $\Gamma$. Assume that $M$ is bounded on $Y$. Then the operator $M_{b,\Gamma}$ is bounded on $Y$, and the inequality*

$$\|M_{b,\Gamma} f\|_Y \leq c\,\|b\|_{BMO}\,\|f\|_Y$$

*holds with constant $c$ independent of $f$.*

**Corollary 3.6.** *Let $1 \leq p < \infty$, $b \in BMO(\Gamma)$ and $\omega \in F_p(\Gamma)$. Then the operator $M_{b,\Gamma}$ is bounded in $L_{p,\omega}(\Gamma)$.*

**Lemma 3.7.** *[13] Let $1 < s < \infty$ and $b \in BMO(\Gamma)$. Then*

$$M_\Gamma^\sharp(|b, I_\Gamma^\alpha|f(t)) \leq C\|b\|_{BMO}\left[\left(M_\Gamma|I_\Gamma^\alpha f(t)|^s\right)^{\frac{1}{s}} + \left(M_\Gamma^{s\alpha}|f(t)|^s\right)^{\frac{1}{s}}\right],$$

*where $C > 0$ is independent of $f$ and $t$.*

The following statement holds (see [38]):

**Proposition 3.8.** *[38] Let $1 < p < \infty$. Then for all $f \in L_{p(\cdot)}(\Gamma)$ and $g \in L_{p'}(\Gamma)$ there holds*

$$\left|\int_\Gamma f(y)g(y)d\nu(y)\right| \leq C\left|\int_\Gamma M_\Gamma^\sharp f(y) M_\Gamma g(y)d\nu(y)\right|$$

*with the constant $C > 0$ independent on $f$.*

**Lemma 3.9.** *Let $1 < p < \infty$ and $\omega \in F_p(\Gamma)$. Then*

$$\|f\|_{L_{p,\omega}} \leq C\|M_\Gamma^\sharp f\|_{L_{p,\omega}}$$

*with the constant $C > 0$ independent on $f$.*

*Proof.* From the following equivalence, we get

$$\|f\omega\|_{L_p} \approx \sup_{\|g\|_{L_{p'}} \leq 1}\left|\int_\Gamma f(y)g(y)\omega(y)d\nu(y)\right|.$$

According to Proposition 3.8,

$$\|f\omega\|_{L_{p(\cdot)}} \leq C \sup_{\|g\|_{L_{p'}} \leq 1} \left| \int_\Gamma M_\Gamma^\sharp f(y) M_\Gamma(g\omega)(y) d\nu(y) \right|.$$

By the Hölder inequality and Theorem 1.2, we derive

$$\begin{aligned} \|f\omega\|_{L_p} &\leq C \sup_{\|g\|_{L_{p'}} \leq 1} \|\omega M_\Gamma^\sharp f\|_{L_p} \left\|\omega^{-1} M_\Gamma(g\omega)\right\|_{L_{p'}} \\ &\leq C \sup_{\|g\|_{L_{p'}} \leq 1} \|\omega M_\Gamma^\sharp f\|_{L_p} \left\|g\right\|_{L_{p'}} \\ &\leq C\|\omega M_\Gamma^\sharp f\|_{L_p}. \end{aligned}$$

□

**Theorem 3.10.** *Let* $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \alpha$, $b \in BMO(\Gamma)$ *and* $(\omega_1, \omega_2) \in F_{p,q}(\Gamma)$, $\omega_1 \in F_p(\Gamma)$, $\omega_2 \in F_q(\Gamma)$. *Then the operator* $|b, I_\Gamma^\alpha|$ *is bounded from* $L_{p,\omega_1}(\Gamma)$ *to* $L_{q,\omega_2}(\Gamma)$.

*Proof.* Let $f \in L_{p,\omega_1}(\Gamma)$ and $b \in BMO(\Gamma)$. From Lemma 3.9 we obtain

$$\left\||b, I_\Gamma^\alpha|f\right\|_{L_{q,\omega_2}(\Gamma)} \leq C \left\|M_\Gamma^\# \left(|b, I_\Gamma^\alpha|f\right)\right\|_{L_{q,\omega_2}(\Gamma)}.$$

From Lemma 3.7 we have

$$\begin{aligned} \left\|M_\Gamma^\# \left(|b, I_\Gamma^\alpha|f\right)\right\|_{L_{q,\omega_2}(\Gamma)} &\leq C\|b\|_{BMO} \left\|\left(M_\Gamma \left|I_\Gamma^\alpha f\right|^s\right)^{\frac{1}{s}} + \left(M_\Gamma^{\alpha s} \left|f\right|^s\right)^{\frac{1}{s}}\right\|_{L_{q,\omega_2}(\Gamma)} \\ &\leq C\|b\|_{BMO} \left(\left\|\left(M_\Gamma \left|I_\Gamma^\alpha f\right|^s\right)^{\frac{1}{s}}\right\|_{L_{q,\omega_2}(\Gamma)} + \left\|\left(M_\Gamma^{\alpha s} \left|f\right|^s\right)^{\frac{1}{s}}\right\|_{L_{q,\omega_2}(\Gamma)}\right). \end{aligned}$$

From Corollary 1.3 and Theorem 3.1, we get

$$\left\|\left(M_\Gamma \left|I_\Gamma^\alpha f\right|^s\right)^{\frac{1}{s}}\right\|_{L_{q,\omega_2}(\Gamma)} \leq C \left\|\left(\left|I_\Gamma^\alpha f\right|^s\right)^{\frac{1}{s}}\right\|_{L_{q,\omega_2}(\Gamma)} \leq C \left\|I_\Gamma^\alpha f\right\|_{L_{q,\omega_2}(\Gamma)} \leq C \left\|f\right\|_{L_{p,\omega_1}(\Gamma)}.$$

From Corollary 3.2, we obtain

$$\left\|\left(M_\Gamma^{\alpha s} \left|f\right|^s\right)^{\frac{1}{s}}\right\|_{L_{q,\omega_2}(\Gamma)} \leq C \left\|\left(\left|f\right|^s\right)^{\frac{1}{s}}\right\|_{L_{p,\omega_1}(\Gamma)} \leq C \left\|f\right\|_{L_{p,\omega_1}(\Gamma)}.$$

Therefore we get

$$\left\||b, I_\Gamma^\alpha|f\right\|_{L_{q,\omega_2}(\Gamma)} \leq C\|b\|_{BMO} \left\|f\right\|_{L_{p,\omega_1}(\Gamma)}.$$

Thus the theorem has been proved. □

**Theorem 3.11.** *Let* $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \alpha$, $b \in BMO(\Gamma)$ *and* $(\omega_1, \omega_2) \in F_{p,q}(\Gamma)$, $\omega_1 \in F_p(\Gamma)$, $\omega_2 \in F_q(\Gamma)$. *Then*

$$\||b, I_\Gamma^\alpha|f\|_{L_{q,\omega_2}(\Gamma(t,r))} \leq Cr^{\frac{1}{qk'}} \|b\|_{BMO} \|\omega_2\|_{L_{qk}(\Gamma(t,r))} \int_r^\infty \tau^{-\frac{1}{qk'}} \left(1 + \ln\frac{\tau}{r}\right) \frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}} \frac{d\tau}{\tau}, \tag{8}$$

*where C does not depend on f and t.*

*Proof.* We represent $f$ as (2) and have

$$|b, I_\Gamma^\alpha| f(x) \leq |b, I_\Gamma^\alpha| f_1(x) + |b, I_\Gamma^\alpha| f_2(x).$$

By Theorem 3.10 we obtain

$$\||b, I_\Gamma^\alpha| f_1\|_{L_{q,\omega_2}(\Gamma(t,r))} \leq C\|b\|_{BMO}\|f_1\|_{L_{p,\omega_1}(\Gamma)} = C\|b\|_{BMO}\|f\|_{L_{p,\omega_1}(\Gamma(t,2r))},$$

where $C$ is independent of $f$. Then we get

$$\||b, I_\Gamma^\alpha| f_1\|_{L_{q,\omega_2}(\Gamma(t,r))} \leq Cr^{\frac{1}{qk'}}\|\omega_2\|_{L_{qk}(\Gamma(t,r))}\int_r^\infty \tau^{-\frac{1}{qk'}}\left(1+\ln\frac{\tau}{r}\right)\frac{\|f\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}. \tag{9}$$

When $|t-z| \leq r$ and $\left|z-y\right| \geq 2r$ we have $\frac{1}{2}\left|z-y\right| \leq \left|t-y\right| \leq \frac{3}{2}\left|z-y\right|$ and therefore we get

$$\begin{aligned}
|b, I_\Gamma^\alpha| f_2(z) &\leq C\int\limits_{{}^c\Gamma(t,2r)} \left|b(y)-b(z)\right|\left|z-y\right|^{\alpha-1}\left|f(y)\right|d\nu(y)\\
&\leq C\int\limits_{{}^c\Gamma(t,2r)} \left|b(y)-b(z)\right|\left|t-y\right|^{\alpha-1}\left|f(y)\right|d\nu(y).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\int\limits_{{}^c\Gamma(t,2r)} \left|b(y)-b(z)\right|\left|t-y\right|^{\alpha-1}\left|f(y)\right|d\nu(y)\\
&= C\int\limits_{2r}^{\infty}\tau^{\alpha-2}\left(\int\limits_{\{y\in\Gamma:2r\leq|t-y|\leq\tau\}}\left|b(y)-b(z)\right|\left|f(y)\right|d\nu(y)\right)d\tau\\
&\leq C\int\limits_{2r}^{\infty}\tau^{\alpha-2}\left(\int\limits_{\{y\in\Gamma:2r\leq|t-y|\leq\tau\}}\left|b(y)-b_{\Gamma(t,r)}\right|\left|f(y)\right|d\nu(y)\right)d\tau\\
&\quad + C\left|b(z)-b_{\Gamma(t,r)}\right|\int\limits_{2r}^{\infty}\tau^{\alpha-2}\left(\int\limits_{\{y\in\Gamma:2r\leq|t-y|\leq\tau\}}\left|f(y)\right|d\nu(y)\right)d\tau\\
&= J_1 + J_2.
\end{aligned}$$

To estimate $J_1$ :

$$\begin{aligned}
J_1 &= C\int\limits_{2r}^{\infty}\tau^{\alpha-2}\left(\int\limits_{\{y\in\Gamma:2r\leq|t-y|\leq\tau\}}\left|b(y)-b_{\Gamma(t,r)}\right|\left|f(y)\right|d\nu(y)\right)d\tau\\
&\leq C\int\limits_{2r}^{\infty}\tau^{\alpha-2}\left(\int\limits_{\{y\in\Gamma:2r\leq|t-y|\leq\tau\}}\left|b(y)-b_{\Gamma(t,\tau)}\right|\left|f(y)\right|d\nu(y)\right)d\tau\\
&\quad + C\int\limits_{2r}^{\infty}\tau^{\alpha-2}\left|b_{\Gamma(t,r)}-b_{\Gamma(t,\tau)}\right|\left(\int\limits_{\{y\in\Gamma:2r\leq|t-y|\leq\tau\}}\left|f(y)\right|d\nu(y)\right)d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq C\int_r^\infty \tau^{\alpha-2}\left\|b(.)-b_{\Gamma(t,\tau)}\right\|_{L_{p',\omega_1^{-1}}(\Gamma(t,\tau))}\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}d\tau\\
&\quad +C\int_r^\infty \tau^{\alpha-2}\left|b_{\Gamma(t,r)}-b_{\Gamma(t,\tau)}\right|\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}\left\|\omega_1^{-1}\right\|_{L_{p'}(\Gamma(t,\tau))}d\tau\\
&\leq C\|b\|_{BMO}\int_r^\infty \tau^{\alpha-2}\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}\left\|\omega_1^{-1}\right\|_{L_{p'}(\Gamma(t,\tau))}d\tau\\
&\quad +C\|b\|_{BMO}\int_r^\infty \tau^{\alpha-2}\ln\frac{\tau}{r}\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}\left\|\omega_1^{-1}\right\|_{L_{p'}(\Gamma(t,\tau))}d\tau\\
&= C\|b\|_{BMO}\int_r^\infty \tau^{\alpha-2}\left(1+\ln\frac{\tau}{r}\right)\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}\left\|\omega_1^{-1}\right\|_{L_{p'}(\Gamma(t,\tau))}d\tau\\
&\leq C\|b\|_{BMO}\int_r^\infty \tau^{\alpha-2+\frac{1}{p'k'}}\left(1+\ln\frac{\tau}{r}\right)\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}\left\|\omega_1^{-1}\right\|_{L_{p'k}(\Gamma(t,\tau))}d\tau\\
&\leq C\|b\|_{BMO}\int_r^\infty \tau^{\alpha-2+\frac{1}{p'k'}-\alpha+\frac{1}{p}-\frac{1}{q}+\frac{1}{p'k}+\frac{1}{qk}}\left(1+\ln\frac{\tau}{r}\right)\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}d\tau\\
&\leq C\|b\|_{BMO}\int_r^\infty \tau^{-\frac{1}{qk'}}\left(1+\ln\frac{\tau}{r}\right)\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}.
\end{aligned}$$

Then we obtain

$$J_1 \leq C\|b\|_{BMO}\int_r^\infty \tau^{-\frac{1}{qk'}}\left(1+\ln\frac{\tau}{r}\right)\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}. \tag{10}$$

Also

$$\begin{aligned}
\left|b(z)-b_{\Gamma(t,r)}\right| &= \left|b(z)-(\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}b(y)d\nu(y)\right|\\
&= \left|b(z)+(\nu\Gamma(t,r))^{-1}\left(\int_{\Gamma(t,r)}(b(z)-b(y))\,d\nu(y)-\int_{\Gamma(t,r)}b(z)d\nu(y)\right)\right|\\
&= (\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}(b(z)-b(y))\,d\nu(y)\\
&\leq \sup_{r>0}(\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}\left|b(z)-b(y)\right|d\nu(y)=M_b\chi_{\Gamma(t,r)}(z).
\end{aligned}$$

Then we obtain

$$\left|b(z)-b_{\Gamma(t,r)}\right| \leq \sup_{r>0}(\nu\Gamma(t,r))^{-1}\int_{\Gamma(t,r)}\left|b(z)-b(y)\right|d\nu(y)=M_b\chi_{\Gamma(t,r)}(z). \tag{11}$$

To estimate $J_2$ : (from (11) and Hölder inequality)

$$\begin{aligned}
J_2 &= C\left|b(z)-b_{\Gamma(t,r)}\right|\int_{2r}^{\infty}\tau^{\alpha-2}\left(\int_{\{y\in\Gamma:2r\le|t-y|\le\tau\}}\left|f(y)\right|d\nu(y)\right)d\tau\\
&\le C\left|b(z)-b_{\Gamma(t,r)}\right|\int_{r}^{\infty}\tau^{\alpha-2}\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}\left\|\omega_1^{-1}\right\|_{L_{p'}(\Gamma(t,\tau))}d\tau\\
&\le C\left|b(z)-b_{\Gamma(t,r)}\right|\int_{r}^{\infty}\tau^{\alpha-2+\frac{1}{p'k'}}\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}\left\|\omega_1^{-1}\right\|_{L_{p'k}(\Gamma(t,\tau))}d\tau\\
&\le C\left|b(z)-b_{\Gamma(t,r)}\right|\int_{r}^{\infty}\tau^{\alpha-2+\frac{1}{p'k'}-\alpha+\frac{1}{p}-\frac{1}{q}+\frac{1}{p'k}+\frac{1}{qk}}\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}d\tau\\
&\le C\left|b(z)-b_{\Gamma(t,r)}\right|\int_{r}^{\infty}\tau^{-\frac{1}{qk'}}\left(1+\ln\frac{\tau}{r}\right)\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}\\
&\le CM_b\chi_{\Gamma(t,r)}(z)\int_{r}^{\infty}\tau^{-\frac{1}{qk'}}\left(1+\ln\frac{\tau}{r}\right)\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}.
\end{aligned}$$

Then we obtain

$$J_2 = CM_b\chi_{\Gamma(t,r)}(z)\int_{r}^{\infty}\tau^{-\frac{1}{qk'}}\left(1+\ln\frac{\tau}{r}\right)\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}. \tag{12}$$

Therefore, we get

$$\begin{aligned}
|b, I_\Gamma^\alpha|f_2 &\le C\|b\|_{BMO}\int_{r}^{\infty}\tau^{-\frac{1}{qk'}}\left(1+\ln\frac{\tau}{r}\right)\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}\\
&\quad +CM_b\chi_{\Gamma(t,r)}(z)\int_{r}^{\infty}\tau^{-\frac{1}{qk'}}\left(1+\ln\frac{\tau}{r}\right)\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}.
\end{aligned}$$

Then by (10), (12) and Theorem 3.5 we have

$$\||b, I_\Gamma^\alpha|f_2\|_{L_{q,\omega_2}(\Gamma(t,r))} \le \|J_1\|_{L_{q,\omega_2}(\Gamma(t,r))} + \|J_2\|_{L_{q,\omega_2}(\Gamma(t,r))}$$

$$\begin{aligned}
\||b, I_\Gamma^\alpha|f_2\|_{L_{q,\omega_2}(\Gamma(t,r))} &\le Cr^{\frac{1}{qk'}}\|b\|_{BMO}\|\omega_2\|_{L_{qk}(\Gamma(t,r))}\int_{r}^{\infty}\tau^{-\frac{1}{qk'}}\left(1+\ln\frac{\tau}{r}\right)\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}\\
&\quad +C\|M_b\chi_{\Gamma(t,r)}(z)\|_{L_{q,\omega_2}(\Gamma(t,r))}\int_{r}^{\infty}\tau^{-\frac{1}{qk'}}\left(1+\ln\frac{\tau}{r}\right)\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}\\
&\le Cr^{\frac{1}{qk'}}\|b\|_{BMO}\|\omega_2\|_{L_{qk}(\Gamma(t,r))}\int_{r}^{\infty}\tau^{-\frac{1}{qk'}}\left(1+\ln\frac{\tau}{r}\right)\frac{\left\|f\right\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}}\frac{d\tau}{\tau}.
\end{aligned}$$

Hence

$$\||b, I_\Gamma^\alpha|f_2\|_{L_{q,\omega_2}(\Gamma(t,r))} \le Cr^{\frac{1}{qk'}} \|b\|_{BMO} \|\omega_2\|_{L_{qk}(\Gamma(t,r))} \int_r^\infty \tau^{-\frac{1}{qk'}} \left(1 + \ln \frac{\tau}{r}\right) \frac{\|f\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}} \frac{d\tau}{\tau}, \tag{13}$$

which togehter with (9) and (13) yields (8). □

Now, we prove the boundedness of commutators of the Riesz potential operator $|b, I_\Gamma^\alpha|$ from the generalized weighted Morrey spaces $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\Gamma)$ to the generalized weighted Morrey spaces $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\Gamma)$ . We find conditions on the functions $\varphi_1(r)$ and $\varphi_2(r)$ for the boundedness of $|b, I_\Gamma^\alpha|$ from $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\Gamma)$ to $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\Gamma)$.

**Theorem 3.12.** *Let $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \alpha$, $b \in BMO(\Gamma)$, $(\omega_1, \omega_2) \in F_{p,q}(\Gamma)$, $\omega_1 \in F_p(\Gamma)$, $\omega_2 \in F_q(\Gamma)$ and the functions $\varphi_1(r)$ and $\varphi_2(r)$ fulfill the condition*

$$\int_r^\infty \tau^{-\frac{1}{qk'}} \left(1 + \ln \frac{\tau}{r}\right) \frac{\operatorname*{ess\,inf}_{\tau<r<\infty} r^{\frac{1}{pk'}} \varphi_1(r) \|\omega_1\|_{L_{pk}(\Gamma(t,r))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}} \frac{d\tau}{\tau} \le C\varphi_2(r). \tag{14}$$

*Then the operator $|b, I_\Gamma^\alpha|$ is bounded from $\mathcal{M}_{\omega_1}^{p,\varphi_1}(\Gamma)$ to $\mathcal{M}_{\omega_2}^{q,\varphi_2}(\Gamma)$.*

*Proof.* Let $f \in \mathcal{M}_{\omega_1}^{p,\varphi_1}$. From the definition of the norm of generalized weighted Morrey space we write

$$\left\||b, I_\Gamma^\alpha|f\right\|_{\mathcal{M}_{\omega_2}^{q,\varphi_2}(\Gamma(t,r))} = \sup_{t\in\Gamma, r>0} \frac{1}{r^{\frac{1}{qk'}} \varphi_2(r) \|\omega_2\|_{L_{qk}(\Gamma(t,r))}} \left\||b, I_\Gamma^\alpha|f\right\|_{L_{q,\omega_2}(\Gamma(t,r))}. \tag{15}$$

We estimate $\left\||b, I_\Gamma^\alpha|f\right\|_{L_{q,\omega_2}(\Gamma(t,r))}$ in (15) by means of Theorem 3.11 and Theorem 2.10. Taking $\nu_1(r) = \frac{1}{r^{\frac{1}{pk'}} \varphi_1(r)\|\omega_1\|_{L_{pk}}}$, $\nu_2(r) = \frac{1}{\varphi_2(r)}$, $g(r) = \|f\|_{L_{p,\omega_1}(\Gamma(t,r))}$, $w(\tau) = \left(1 + \ln \frac{\tau}{r}\right) \tau^{-\frac{1}{qk'}-1} \|\omega_2\|^{-1}_{L_{qk}(\Gamma(t,\tau))}$, with inequality (14) and we obtain

$$\begin{aligned}
\left\||b, I_\Gamma^\alpha|f\right\|_{\mathcal{M}_{\omega_2}^{q,\varphi_2}(\Gamma)} &\le C\|b\|_{BMO} \sup_{t\in\Gamma, r>0} \frac{r^{\frac{1}{qk'}} \|\omega_2\|_{L_{qk}(\Gamma(t,r)}}{r^{\frac{1}{qk'}} \varphi_2(r) \|\omega_2\|_{L_{qk}(\Gamma(t,r))}} \int_r^\infty \tau^{-\frac{1}{qk'}} \left(1 + \ln \frac{\tau}{r}\right) \frac{\|f\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}} \frac{d\tau}{\tau} \\
&= C\|b\|_{BMO} \sup_{t\in\Gamma, r>0} \frac{1}{\varphi_2(r)} \int_r^\infty \left(1 + \ln \frac{\tau}{r}\right) \tau^{-\frac{1}{qk'}} \frac{\|f\|_{L_{p,\omega_1}(\Gamma(t,\tau))}}{\|\omega_2\|_{L_{qk}(\Gamma(t,\tau))}} \frac{d\tau}{\tau} \\
&\le C\|b\|_{BMO} \sup_{t\in\Gamma, r>0} \frac{1}{r^{\frac{1}{pk'}} \varphi_1(r) \|\omega_1\|_{L_{pk}(\Gamma(t,r))}} \|f\|_{L_{p,\omega_1}(\Gamma(t,r))} \\
&= C\|b\|_{BMO} \|f\|_{\mathcal{M}_{\omega_1}^{p,\varphi_1}(\Gamma)}.
\end{aligned}$$

□

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