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A study of generalized Gamma-type operators

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Abstract. In this study, we discuss the approximation properties of Gamma operators $\mathbf{G}_\alpha$ for absolutely continuous and locally bounded functions by using results of probability theory and some inequalities with the method of Bojanic-Cheng. And then, metric form $\Omega_u(\xi, \lambda)$ is used with asymptotic formula combining to calculate an convergence rate asymptotically of Gamma operators $\mathbf{G}_\alpha$ for the bounded functions locally and also analysis techniques are used with Bojanic-Khan-Cheng's method to calculate an optimal convergence rate of Gamma operators $\mathbf{G}_\alpha$ for the functions which are absolutely continuous. Lastly, the convergence of the operators to a specific function is illustrated using Maple software.

1. Introduction

Lupaş and Müller [17] introduced Gamma operators which are most commonly used operators in approximation theory and have been used for calculating a better approximation to the target function. Zeng [27] studied some convergence properties of Gamma operators, e.g., the optimal convergence rate and asymptotic convergence rate for absolutely continuous and locally bounded functions, respectively. Karsli [13] discussed the convergence rate by a new defined Gamma operators for functions which have derivatives of bounded variation. Gupta [12] considered modified Gamma operators $\mathbf{G}_\alpha$ which are defined as:

$$\mathbf{G}_\alpha(\xi, u) = \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-\alpha s/u} \xi(s) ds. \tag{1}$$

The operators $\mathbf{G}_\alpha$ preserve the linear functions and they reduce to the following well-known operators in special cases:

1. If $\alpha = n$, we obtain the Post-Wider operators [9].
2. If $\alpha = nu$, we get the Rathore operators [10].

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Chen and Guo [6] discussed approximation properties of the Gamma operators for function with bounded variation on $[0, \infty)$. Özçelik et al.[20] introduced a modified Gamma operators and discussed a weighted approximation, Voronovskaya theorem, pointwise estimates and convergence rate. Rempulska and Skorupka [21] present new modification of Gamma operators which is defined for differentiable functions of polynomial weighted spaces and studied the approximation behaviour of these operators which are more optimal than the classical Gamma operators. Many researchers in the literature have examined the approximation properties of various Gamma operators (cf. [2–5, 7, 11, 14–16, 18, 22–26, 28]etc.) and reference therein.

Let ξ be a continuous function which is defined on $[0, \infty)$ and follows these condition:

$$|\xi(s)| \leq Me^{\beta s} \quad (M > 0,\ \beta \geq 0,\ s \to \infty).$$

In this study, the point-wise approximation behaviour of modified Gamma operators $\mathbf{G}_\alpha$ will be calculated for the spaces of functions which are locally bounded Ψ_B and absolutely continuous Ψ_{DB}. Then, Ψ_B and Ψ_{DB} are defined as:

$$\begin{aligned} \Psi_B &= \{\xi | \xi \text{ on every finite subinterval of } [0,\infty) \text{ is bounded.}\}, \\ \Psi_{DB} &= \left\{ \xi | \xi(u) - \xi(0) = \int_0^u q(s)ds; u \geq 0,\ q \text{ on every finite subinterval of } [0,\infty) \text{ is bounded.} \right\}. \end{aligned}$$

Furthermore, we defined the following metric form for a function $\xi \in \Psi_B$:

$$\Omega_u(\xi, \lambda) = \sup_{s \in [u-\lambda, u+\lambda]} |\xi(s) - \xi(u)|,$$

where $u \in [0, \infty)$ is fixed, $\lambda \geq 0$.
Then, we have

1. $\Omega_u(\xi, \lambda)$ is monotonic increasing with respect to λ.
2. If ξ is continuous at the point u, $\lim_{\lambda \to 0} \Omega_u(\xi, \lambda) = 0$.
3. If $\bigvee_a^b(\xi)$ denotes the total variation of ξ on $[a, b]$ and ξ is bounded variation on $[a, b]$, then $\Omega_u(\xi, \lambda) \leq \bigvee_{u-\lambda}^{u+\lambda}(\xi)$.

In Section 2, we used quantitative form of the central limit theorem for sign function to calculate an approximation formula asymptotically of Gamma operators $\mathbf{G}_\alpha$. After that, the metric form $\Omega_u(\xi, \lambda)$ combining with this asymptotic formula is used to calculate the convergence rate for the locally bounded function $\xi \in \Psi_B$ by Gamma operators $\mathbf{G}_\alpha$ at the point u where $\xi(u+)$ and $\xi(u-)$ exist. The absolute moment of first order for Gamma operators $\mathbf{G}_\alpha(|s-u|, u)$ in Section 3 is evaluated to get

$$\left| \mathbf{G}_\alpha(|s-u|, u) - \sqrt{\frac{2}{\alpha\pi}} u \right| \leq \frac{u}{15\alpha^{3/2}}. \tag{2}$$

Estimate (2) is optimal asymptotically and it gives better result than Bojanic and Khan result which is in ([4], Section 3.7) that

$$\mathbf{G}_\alpha(|s-u|, u) = \sqrt{\frac{2}{\alpha\pi}} u + O(\alpha^{-1}). \tag{3}$$

And then, analysis techniques combining with the Bojanic-Khan-Cheng's method and estimate (2) are used to calculate the convergence rate for absolutely continuous function $\xi \in \Psi_{DB}$ by Gamma operators $\mathbf{G}_\alpha$. The optimal asymptotic estimation is obtained.
Throughout this paper, $\alpha = \alpha(n)$ be a sequence such that $\alpha = \alpha(n) \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} \frac{n}{\alpha(n)} = l \in \mathbb{R}$.

2. Approximation for locally bounded functions

The convergence rate for function $\xi \in \Psi_B$ by Gamma operators $\mathbf{G}_\alpha$ is calculated in this part. We need some preliminary results to prove Theorem 2.4.

Lemma 2.1. *For $u \in (0,\infty)$, $i = 0, 1, 2, \cdots$, there holds*

$$\mathbf{G}_\alpha(s^i, u) = \frac{(\alpha)_i . u^i}{\alpha^i}. \tag{4}$$

Proof. By direct computation, we have

$$\begin{aligned}
\mathbf{G}_\alpha(s^{i+1}, u) &= \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-\alpha s/u} s^{i+1} ds \\
&= \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{i+\alpha} e^{-\alpha s/u} ds \\
&= \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{ut}{\alpha}\right)^{i+\alpha} e^{-t} \frac{u}{\alpha} dt \\
&= \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \left(\frac{u}{\alpha}\right)^{\alpha+i+1} \int_0^\infty t^{\alpha+i+1-1} e^{-t} dt \\
&= \left(\frac{u}{\alpha}\right)^{i+1} \times \frac{1}{\Gamma(\alpha)} \times \Gamma(\alpha + i + 1) \\
&= \frac{(\alpha)_{i+1} . u^{i+1}}{\alpha^{i+1}}.
\end{aligned}$$

□

Lemma 2.2. *For $u \in (0,\infty)$, we have*

$$\mathbf{G}_\alpha((s-u)^2, u) = \frac{u^2}{\alpha}; \tag{5}$$

$$\sqrt{\mathbf{G}_\alpha((s-u)^4, u)} \le \frac{4}{\alpha} u^2; \tag{6}$$

$$\sqrt{\mathbf{G}_\alpha((s-u)^6, u)} \le \frac{17}{\alpha^{3/2}} u^3; \tag{7}$$

$$\mathbf{G}_\alpha(e^{2\beta s}, u) \le (2e)^{2\beta u} \quad for\ \alpha > 2\beta u. \tag{8}$$

Proof. By direct computation and Lemma 2.1, we get

$$\begin{aligned}
\mathbf{G}_\alpha((s-u)^2, u) &= \frac{u^2}{\alpha}; \\
\mathbf{G}_\alpha((s-u)^4, u) &= \frac{3\alpha + 6}{\alpha^3} u^4; \\
\mathbf{G}_\alpha((s-u)^6, u) &= \frac{15\alpha^2 + 130\alpha + 120}{\alpha^5} u^6;
\end{aligned}$$

which satisfy eqs.(5)-(8). On the other side, if $\alpha > 2\beta u$ putting $s = \dfrac{\alpha u}{\alpha - 2\beta u}$, we have

$$\begin{aligned}
\mathbf{G}_\alpha(e^{2\beta s}, u) &= \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-\alpha s/u} e^{2\beta s} ds \\
&= \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-(\alpha/u - 2\beta)s} ds \\
&= \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{t}{(\frac{\alpha}{u} - 2\beta)}\right)^{\alpha-1} e^{-t} \frac{dt}{(\frac{\alpha}{u} - 2\beta)} \\
&= \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \times \left(\frac{u}{\alpha - 2\beta u}\right)^\alpha \int_0^\infty t^{\alpha-1} e^{-t} dt \\
&= \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \times \left(\frac{u}{\alpha - 2\beta u}\right)^\alpha \times \Gamma(\alpha) \\
&= \left(\frac{\alpha}{\alpha - 2\beta u}\right)^\alpha = \left(1 + \frac{2\beta u}{\alpha - 2\beta u}\right)^\alpha \leq (2e)^{2\beta u}.
\end{aligned}$$

□

In Lemma 2.3, we obtained the central limit theorem in the asymptotical form which is defined in probability theory. It is shown in Feller [[8], pp. 540-542].

Lemma 2.3. *Suppose that* $\{\zeta_i\}_{i=1}^\infty$ *be a sequence of random variables which are identically distributed and independent with the expectation* $E\zeta_1 = a_1$*, the variance* $E(\zeta_1 - a_1)^2 = \sigma^2 > 0, E(\zeta_1 - a_1)^4 < \infty$*, and let* F_α *stand for the distribution function of* $\sum_{i=1}^{[\alpha]} (\zeta_i - a_1)/\sigma\sqrt{\alpha}$*. If* F_α *is not a lattice distribution, then the following equation holds for all* $s \in (-\infty, +\infty)$ *:*

$$F_\alpha(s) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-u^2/2} du = \frac{E(\zeta_1 - a_1)^3}{6\sigma^3\sqrt{\alpha}} (1 - s^2) \frac{1}{\sqrt{2\pi}} e^{-s^2/2} + o(\alpha^{-1/2}). \tag{9}$$

Theorem 2.4. *Suppose that a function* $\xi \in \Psi_B$ *and let* $\xi(s) = O(e^{\beta s})$ *for some* $\beta \geq 0$ *as* $s \to \infty$*. If* $\xi(u+)$ *and* $\xi(u-)$ *define at a point* $u \in (0, \infty)$*, which is a fixed point. So* $\alpha > 2\beta u$*, we get*

$$\left| \mathbf{G}_\alpha(\xi, u) - \frac{\xi(u+) + \xi(u-)}{2} + \frac{\xi(u+) - \xi(u-)}{3\alpha\sqrt{2\pi}} \right| \leq \frac{5}{\alpha} \sum_{k=1}^{[\alpha]} \Omega_u(f_u, u/\sqrt{k}) + O(\alpha^{-1}), \tag{10}$$

where

$$f_u(t) = \begin{cases} \xi(s) - \xi(u+), & u < s < \infty; \\ 0, & s = u; \\ \xi(s) - \xi(u-), & 0 \leq s < u. \end{cases} \tag{11}$$

Proof. Let ξ satisfy the condition of Theorem 2.4, then ξ can be expressed as

$$\begin{aligned}
\xi(s) &= \frac{\xi(u+) + \xi(u-)}{2} + f_u(s) + \frac{\xi(u+) - \xi(u-)}{2} sign(s - u) \\
&\quad + \delta_u(s)\left[\xi(u) - \frac{\xi(u+) + \xi(u-)}{2}\right],
\end{aligned} \tag{12}$$

where $sign(s)$ is sign function, $\delta_u(s)$ is defined in (11) and
$\delta_u(s) = \begin{cases} 1, & s = u, \\ 0, & s \neq u. \end{cases}$ Obviously,

$$\mathbf{G}_\alpha(\delta_u, u) = 0. \tag{13}$$

Let us consider a sequence $\{\zeta_i\}_{i=1}^{\infty}$ of random variables which are independent for this Gamma distribution and their probability density functions are
$P_{\zeta_i}(s) = \begin{cases} \dfrac{\alpha}{u} e^{-\alpha s/u}, & s > 0, \\ 0, & s \le 0, \end{cases}$ where $u \in (0, \infty)$ is a variable. Hence by simple calculation we have

$$E(\zeta_1) = u, \;\; E(\zeta_1 - E\zeta_1)^2 = \sigma^2 = \frac{u^2}{\alpha}, \tag{14}$$

$$E(\zeta_1 - E\zeta_1)^3 = \frac{2u^3}{\alpha^2}, \quad E(\zeta_1 - E\zeta_1)^4 = \frac{9u^4}{\alpha^3} < \infty. \tag{15}$$

Let $\eta_\alpha = \displaystyle\sum_{i=1}^{[\alpha]} \zeta_i$ and F_α stand for the distribution function of $\sum_{i=1}^{[\alpha]} (\zeta_i - E\zeta_i)/\sigma\sqrt{\alpha}$. Then the probability distribution of the random variable η_α is
$P(\eta_\alpha \le y) = \left(\dfrac{\alpha}{u}\right)^{\alpha} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^y t^{\alpha-1} e^{-\alpha t/u} dt.$
Thus

$$\begin{aligned} \mathbf{G}_\alpha(sign(s-u), u) &= \left(\frac{\alpha}{u}\right)^{\alpha} \frac{1}{\Gamma(\alpha)} \int_u^{+\infty} s^{\alpha-1} e^{-\alpha s/u} ds - \left(\frac{\alpha}{u}\right)^{\alpha} \frac{1}{\Gamma(\alpha)} \int_0^u s^{\alpha-1} e^{-\alpha s/u} ds \\ &= 1 - 2P(\eta_\alpha \le u) = 1 - 2F_\alpha(0). \end{aligned} \tag{16}$$

By Lemma 2.3, combining eqs. (14), (15) with simple computations, we get

$$1 - 2F_\alpha(0) = \frac{-2E(\zeta_1 - a_1)^3}{6\sigma^3\sqrt{\alpha}} \frac{1}{\sqrt{2\pi}} + o(\alpha^{-1/2}) = \frac{-2}{3\alpha\sqrt{2\pi}} + O(\alpha^{-1}). \tag{17}$$

It follows from (12), (13), (16) and (17) that

$$\left| \mathbf{G}_\alpha(\xi, u) - \frac{\xi(u+) + \xi(u-)}{2} + \frac{\xi(u+) - \xi(u-)}{3\alpha\sqrt{2\pi}} \right| \;\; \le \;\; |\mathbf{G}_\alpha(f_u, u)| + O(\alpha^{-1}). \tag{18}$$

We need to evaluate $|\mathbf{G}_\alpha(f_u, u)|$. Let

$$K_\alpha(u, s) = P(\eta_\alpha \le s) = \left(\frac{\alpha}{u}\right)^{\alpha} \frac{1}{\Gamma(\alpha)} \int_0^s v^{\alpha-1} e^{-\alpha v/u} dv.$$

Then

$$\mathbf{G}_\alpha(f_u, u) = \int_0^{\infty} f_u(s) d_s K_\alpha(u, s). \tag{19}$$

Suppose $0 \le v \le s < u$, then, by Chebyshev inequality and finding that $E(\eta_\alpha - E\eta_\alpha)^2 = \dfrac{u^2}{\alpha}$, we get

$$K_\alpha(u, s) = P(\eta_\alpha \le s) = P(|\eta_\alpha - u| \ge u - s) \le \frac{u^2}{\alpha(u-s)^2}. \tag{20}$$

Divide the integration of (19) into four parts as follows:

$$\int_0^{\infty} f_u(s) d_s K_\alpha(u, s) = \Delta_{1,\alpha}(f_u) + \Delta_{2,\alpha}(f_u) + \Delta_{3,\alpha}(f_u) + \Delta_{4,\alpha}(f_u),$$

$$\Delta_{1,\alpha}(f_u)=\int_0^{u-\frac{u}{\sqrt{\alpha}}} f_u(s)d_sK_\alpha(u,s),\qquad \Delta_{2,\alpha}(f_u)=\int_{u-\frac{u}{\sqrt{\alpha}}}^{u+\frac{u}{\sqrt{\alpha}}} f_u(s)d_sK_\alpha(u,s),$$

$$\Delta_{3,\alpha}(f_u)=\int_{u+\frac{u}{\sqrt{\alpha}}}^{2u} f_u(s)d_sK_\alpha(u,s),\qquad \Delta_{4,\alpha}(f_u)=\int_{2u}^{\infty} f_u(s)d_sK_\alpha(u,s).$$

We will evaluate $\Delta_{1,\alpha}(f_u), \Delta_{2,\alpha}(f_u), \Delta_{3,\alpha}(f_u)$ and $\Delta_{4,\alpha}(f_u)$. First, for $\Delta_{2,\alpha}(f_u)$, noting that $f_u(u)=0$, we have

$$|\Delta_{2,\alpha}(f_u)|\leq \int_{u-\frac{u}{\sqrt{\alpha}}}^{u+\frac{u}{\sqrt{\alpha}}} |f_u(s)-f_u(u)|d_sK_\alpha(u,s)\leq \Omega_u(f_u,u/\sqrt{\alpha}). \tag{21}$$

To calculate $|\Delta_{1,\alpha}(f_u)|=\left|\int_0^{u-\frac{u}{\sqrt{\alpha}}} f_u(s)d_sK_\alpha(u,s)\right|\leq \int_0^{u-\frac{u}{\sqrt{\alpha}}} \Omega_u(f_u,u-s)d_sK_\alpha(u,s)$.
Using integration by parts with $y=u-\frac{u}{\sqrt{\alpha}}$, we get

$$\int_0^{u-\frac{u}{\sqrt{\alpha}}} \Omega_u(f_u,u-s)d_sK_\alpha(u,s)\leq \Omega_u(f_u,u-y)K_\alpha(u,y)+\int_0^y K_\alpha(u,s)d_s\Omega_u(f_u,u-s). \tag{22}$$

Using inequality (20) and from (22), we have

$$|\Delta_{1,\alpha}(f_u)|\leq \Omega_u(f_u,u-y)\frac{u^2}{\alpha(u-y)^2}+\int_0^y \frac{u^2}{\alpha(u-s)^2}d_s(-\Omega_u(f_u,u-s)). \tag{23}$$

Since

$$\int_0^y \frac{d_s(-\Omega_u(f_u,u-s))}{(u-s)^2}=\frac{-\Omega_u(f_u,u-y)}{(u-y)^2}+\frac{\Omega_u(f_u,u)}{u^2}+\int_0^y 2\frac{\Omega_u(f_u,u-s)}{(u-s)^3}ds,$$

from (21), (22) it follows that

$$|\Delta_{1,\alpha}(f_u)|\leq \frac{1}{\alpha}\Omega_u(f_u,u)+\frac{2u^2}{\alpha}\int_0^{u-\frac{u}{\sqrt{\alpha}}} \frac{\Omega_u(f_u,u-s)}{(u-s)^3}ds.$$

Putting $s=u-\frac{u}{\sqrt{t}}$ for the last integral, we get

$$\int_0^{u-\frac{u}{\sqrt{\alpha}}} \frac{\Omega_u(f_u,u-s)}{(u-s)^3}ds=\frac{1}{2u^2}\int_1^\alpha \Omega_u\Big(f_u,\frac{u}{\sqrt{t}}\Big)dt.$$

Consequently

$$|\Delta_{1,\alpha}(f_u)|\leq \frac{1}{\alpha}\Big(\Omega_u(f_u,u)+\int_1^\alpha \Omega_u\Big(f_u,\frac{u}{\sqrt{t}}\Big)dt\Big). \tag{24}$$

Using the similar method to evaluate $|\Delta_{3,\alpha}(f_u)|$, we get

$$|\Delta_{3,\alpha}(f_u)|\leq \frac{1}{\alpha}\Big(\Omega_u(f_u,u)+\int_1^\alpha \Omega_u\Big(f_u,\frac{u}{\sqrt{t}}\Big)dt\Big). \tag{25}$$

Finally, using Hölder inequality and the inequality (6), by hypothesis that $f_u(s)\leq M(e^{\beta s})$ as $s\to\infty$, (8), for $\alpha\geq 2\beta u$ we obtain

$$\begin{aligned}|\Delta_{4,\alpha}(f_u)| &\leq M\int_{2u}^{+\infty} e^{\beta s/\alpha} d_sK_\alpha(u,s)\\ &\leq \frac{M}{u^2}\int_0^{+\infty}(s/\alpha-u)^2 e^{\beta s/\alpha}d_sK_\alpha(u,s)\\ &\leq \frac{M}{u^2}\Big(\int_0^{+\infty}(s/\alpha-u)^4 d_sK_\alpha(u,s)\Big)^{1/2}\Big(\frac{M}{u^2}\int_0^{+\infty} e^{2\beta s/\alpha}d_sK_\alpha(u,s)\Big)^{1/2}\\ &\leq \frac{4M(2e)^{\beta u}}{\alpha}. \end{aligned} \tag{26}$$

Equations (21), (24)-(26) derive

$$\begin{aligned}|\mathbf{G}_\alpha(f_u,u)| &\leq |\Delta_{1,\alpha}(f_u)|+|\Delta_{2,\alpha}(f_u)|+|\Delta_{3,\alpha}(f_u)|+|\Delta_{4,\alpha}(f_u)|\\ &\leq \Omega_u(f_u,u/\sqrt{\alpha})+\frac{2}{\alpha}\Big(\Omega_u(f_u,u)+\int_1^{\alpha}\Omega_u\Big(f_u,\frac{u}{\sqrt{t}}\Big)dt\Big)+\frac{2M(2e)^{\beta u}}{\alpha}\\ &\leq \frac{5}{\alpha}\sum_{k=1}^{[\alpha]}\Omega_u(f_u,u/\sqrt{k})+\frac{4M(2e)^{\beta u}}{\alpha} \end{aligned} \tag{27}$$

From (18) and (27), we get the result of Theorem 2.4. □

Corollary 2.5. *Suppose that ξ be a function which have bounded variation at every subinterval of $[0,\infty)$. Suppose $\xi(s)=O(e^{\beta s})$ as $s\to\infty$ and for some $\beta\geq 0$. Then for $u\in(0,\infty)$ and $\alpha>2\beta u$, we get*

$$\begin{aligned}\left|\mathbf{G}_\alpha(\xi,u)-\frac{\xi(u+)+\xi(u-)}{2}+\frac{\xi(u+)-\xi(u-)}{3\alpha\sqrt{2\pi}}\right| &\leq \frac{5}{\alpha}\sum_{k=1}^{[\alpha]}\Omega_u(f_u,u/\sqrt{k})+O(\alpha^{-1})\\ &\leq \frac{5}{\alpha}\sum_{k=1}^{[\alpha]}\bigvee_{u-u/\sqrt{k}}^{u+u/\sqrt{k}}(f_u)+O(\alpha^{-1}). \end{aligned} \tag{28}$$

Corollary 2.6. *From Theorem 2.4, if $\Omega_u(f_u,\lambda)=o(\lambda)$, then*

$$\mathbf{G}_\alpha(\xi,u)=\frac{\xi(u+)+\xi(u-)}{2}-\frac{\xi(u+)-\xi(u-)}{3\alpha\sqrt{2\pi}}+o(\alpha^{-1/2}). \tag{29}$$

3. Approximation for absolutely continuous functions

In this part we calculate the convergence rate of Gamma operators $\mathbf{G}_\alpha$ for function $\xi\in\Psi_{DB}$. We need to evaluate the absolute moment of first order for the Gamma operators to prove Theorem 3.3: $\mathbf{G}_\alpha(|s-u|,u)$. In regards to this research, Bojanic and Khan [4] showed that

$$\mathbf{G}_\alpha(|s-u|,u)=\sqrt{\frac{2}{\alpha\pi}}u+O(\alpha^{-1}). \tag{30}$$

Here in the section below, we will provide an optimal estimate to $\mathbf{G}_\alpha(|s-u|,u)$.

Lemma 3.1. *For $u\in(0,\infty)$, there holds*

$$\mathbf{G}_\alpha(|s-u|,u)=\frac{2u\alpha^\alpha e^{-\alpha}}{\Gamma(\alpha+1)}. \tag{31}$$

Proof. From Lemma 2.1, using identity that $\mathbf{G}_\alpha(1,u) = 1$, $\mathbf{G}_\alpha(s,u) = u$, we get

$$\begin{aligned}\mathbf{G}_\alpha(|s-u|,u) &= \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty |s-u| s^{\alpha-1} e^{-\alpha s/u} ds \\ &= \left(\frac{\alpha}{u}\right)^\alpha \frac{1}{\Gamma(\alpha)} \Big(\int_0^u (u-s) s^{\alpha-1} e^{-\alpha s/u} ds - \int_u^\infty (u-s) s^{\alpha-1} e^{-\alpha s/u} ds \Big) \\ &= \left(\frac{\alpha}{u}\right)^\alpha \frac{2}{\Gamma(\alpha)} \int_0^u (u-s) s^{\alpha-1} e^{-\alpha s/u} ds \\ &= \frac{2u}{\Gamma(\alpha)} \int_0^\alpha v^{\alpha-1} e^{-v} dv - \frac{2u}{\Gamma(\alpha+1)} \int_0^\alpha v^\alpha e^{-v} dv.\end{aligned}$$

But

$$\int_0^\alpha v^{\alpha-1} e^{-v} dv = \alpha^{\alpha-1} e^{-\alpha} + \frac{1}{\alpha} \int_0^\alpha v^\alpha e^{-v} dv.$$

Thus

$$\mathbf{G}_\alpha(|s-u|,u) = \frac{2u\alpha^{\alpha-1}e^{-\alpha}}{\Gamma(\alpha)} = \frac{2u\alpha^\alpha e^{-\alpha}}{\Gamma(\alpha+1)}.$$

□

Corollary 3.2. *For $u \in (0,\infty)$, there holds*

$$\left| \mathbf{G}_\alpha(|s-u|,u) - \sqrt{\frac{2}{\alpha\pi}} u \right| \le \frac{u}{15\alpha^{3/2}}. \tag{32}$$

The best estimate is in (32), which cannot be asymptotically enhanced.

Proof. By using Stirling's formula (cf[19]) and Lemma 3.1:

$$\Gamma(\alpha+1) = \sqrt{2\pi\alpha}(\alpha/e)^\alpha e^{c_\alpha}, \quad (12\alpha+1)^{-1} < c_\alpha < (12\alpha)^{-1},$$

we have

$$\sqrt{\frac{2}{\alpha\pi}} u - \mathbf{G}_\alpha(|s-u|,u) = \sqrt{\frac{2}{\alpha\pi}} u (1 - e^{-c_\alpha}),$$

and a simple calculation derives

$$\sqrt{2/\pi} \frac{u}{15\alpha^{3/2}} \le \sqrt{\frac{2}{\alpha\pi}} u (1 - e^{-c_\alpha}) \le \frac{u}{15\alpha^{3/2}}. \tag{33}$$

□

Theorem 3.3. *Suppose that ξ be a function which is belongs in Ψ_{DB} and suppose $\xi(s) \le Me^{\beta s}$ as $s \to \infty$ and for some $\beta \ge 0$, $M > 0$. If $h(u+)$ and $h(u-)$ exists at a point $u \in (0,\infty)$ which is fixed point then for $\alpha > 2\beta u$ and $h = \xi'$ as ξ is absolutely continuous on every closed subinterval of* $[0,\infty)$, *we get*

$$\left| \mathbf{G}_\alpha(\xi,u) - \xi(u) - \frac{(h(u+) - h(u-))u}{\sqrt{2\pi\alpha}} \right| \le \frac{5u}{\alpha} \sum_{k=1}^{[\sqrt{\alpha}]} \Omega_u(\psi_u, u/k) + \frac{|h(u+) - h(u-)|u + 17M(2e)^{\beta u}}{\alpha^{3/2}}, \tag{34}$$

$$\psi_u(s) = \begin{cases} h(s) - h(u+), & u < s < \infty; \\ 0, & s = u; \\ h(s) - h(u-), & 0 \le s < u. \end{cases} \tag{35}$$

Proof. By direct computation, we get that

$$\mathbf{G}_\alpha(\xi, u) - \xi(u) = \frac{h(u+) - h(u-)}{2}\mathbf{G}_\alpha(|s-u|, u) - L_{\alpha,u}(\psi_u) + R_{\alpha,u}(\psi_u) + T_{\alpha,u}(\psi_u), \tag{36}$$

where

$$\begin{aligned} L_{\alpha,u}(\psi_u) &= \int_0^u \Big(\int_s^u \psi_u(w)dw\Big) d_s K_\alpha(u,s), \\ R_{\alpha,u}(\psi_u) &= \int_u^{2u} \Big(\int_u^s \psi_u(w)dw\Big) d_s K_\alpha(u,s), \\ T_{\alpha,u}(\psi_u) &= \int_{2u}^{+\infty} \Big(\int_u^s \psi_u(w)dw\Big) d_s K_\alpha(u,s). \end{aligned}$$

Integration by parts gives

$$\begin{aligned} L_{\alpha,u}(\psi_u) &= \int_0^u \Big(\int_s^u \psi_u(w)dw\Big) d_s K_\alpha(u,s) \\ &= \int_s^u \psi_u(w)dw K_\alpha(u,s)\Big|_0^u + \int_0^u K_\alpha(u,s)\psi_u(s)ds \\ &= \int_0^u K_\alpha(u,v)\psi_u(v)dv \\ &= \Big(\int_0^{u-u/\sqrt{\alpha}} + \int_{u-u/\sqrt{\alpha}}^u\Big) K_\alpha(u,v)\psi_u(v)dv. \end{aligned}$$

Note that $K_\alpha(u,v) \le 1$ and $\psi_u(u) = 0$, it follows that

$$\Big|\int_{u-u/\sqrt{\alpha}}^u K_\alpha(u,v)\psi_u(v)dv\Big| \le \frac{u}{\sqrt{\alpha}}\Omega_u\Big(\psi_u, \frac{u}{\sqrt{\alpha}}\Big) \le \frac{2u}{\alpha}\sum_{k=1}^{[\sqrt{\alpha}]}\Omega_u\Big(\psi_u, \frac{u}{k}\Big).$$

On the other side, using change of variable $v = u - u/w$ and by inequality (20), we have

$$\begin{aligned} \Big|\int_0^{u-u/\sqrt{\alpha}} K_\alpha(u,v)\psi_u(v)dv\Big| &\le \frac{u^2}{\alpha}\int_0^{u-u/\sqrt{\alpha}} \frac{\Omega_u(\psi_u, u-v)}{(u-v)^2}dv \\ &= \frac{u}{\alpha}\int_1^{\sqrt{\alpha}} \Omega_u(\psi_u, u/w)dw \le \frac{u}{\alpha}\sum_{k=1}^{[\sqrt{\alpha}]}\Omega_u(\psi_u, u/k). \end{aligned}$$

Thus, it satisfies that

$$|L_{\alpha,u}(\psi_u)| \le \frac{3u}{\alpha}\sum_{k=1}^{[\sqrt{\alpha}]}\Omega_u(\psi_u, u/k). \tag{37}$$

A similar estimation gives

$$|R_{\alpha,u}(\psi_u)| \leq \frac{3u}{\alpha}\sum_{k=1}^{[\sqrt{\alpha}]}\Omega_u(\psi_u, u/k). \tag{38}$$

At last, using inequality (7) and (8) and assuming that $\xi(s) \leq Me^{\beta s}(M > 0, \beta \geq 0)$ and $s > \alpha u$, we get

$$\begin{aligned}
|T_{\alpha,u}(\psi_u)| &\leq M\int_{2u}^{+\infty} e^{\beta s/\alpha} d_s K_\alpha(u,s) \\
&\leq \frac{M}{u^3}\int_{2u}^{+\infty}(s/\alpha - u)^3 e^{\beta s/\alpha} d_s K_\alpha(u,s) \\
&\leq \frac{M}{u^3}\Big(\int_0^{+\infty}(s/\alpha - u)^6 d_s K_\alpha(u,s)\Big)^{1/2}\Big(\frac{M}{u^2}\int_0^{+\infty} e^{2\beta s/\alpha} d_s K_\alpha(u,s)\Big)^{1/2} \\
&\leq \frac{17M(2e)^{\beta u}}{\alpha^{3/2}}.
\end{aligned} \tag{39}$$

Theorem 3.3 now follows from (32), (36)-(39) and by a simple computation. The proof of the theorem is complete.

In the last section, we demonstrate that the Theorem 3.3 estimate is optimal asymptotically. Directly by the calculation, we get that

$$|s - u| - |0 - u| = \int_0^s sign(w - u)dw, \quad s \in [0, \infty).$$

We take $\xi(s) = |s - u|$, then $h(s) = sign(s - u), h(u+) - h(u-) = 2, \psi_u \equiv 0$ in Theorem 3.3. Therefore, by simple computation and from(34),(33), we get

$$\sqrt{2/\pi}\frac{u}{15\alpha^{3/2}} \leq \left|\mathbf{G}_\alpha(|s - u|, u) - \sqrt{\frac{2}{\alpha\pi}}u\right| \leq \frac{2u + 17M(2e)^{\beta u}}{\alpha^{3/2}}. \tag{40}$$

Consequently, (34) cannot be asymptotically improved further. □

Remark 3.4. *Theorem 2.4 and Theorem 3.3 satisfies condition $\alpha > 2\beta s$ because the approximation function ξ that follows the growth condition: $\xi(s) \leq Me^{\beta s}$ for some $\beta \geq 0$ and $M > 0$ as $s \to \infty$ is considered in Theorem 2.4 and 3.3. In case, if $\beta = 0$, then ξ is bounded function on* $[0, \infty)$.

Remark 3.5. *Let us consider function ξ with derivatives of bounded variation i.e ξ is belongs to Ψ_{DB}. Hence, Theorem 3.3 is a special case of the approximation of functions whose derivatives is bounded variation. Theorem 3.3 is superior than a result of [4] in this case. Also, the asymptotically optimal estimation is obtain in Theorem 3.3.*

Example 3.6. *For $\alpha = 10, 15, 20$, the convergence of absolutely continuous function* $\xi(u) = \begin{cases} u\sin\left(\frac{1}{u}\right), & u > 0, \\ 0, & u \leq 0. \end{cases}$ *and* $f(u) = \begin{cases} u\cos\left(\frac{1}{u}\right), & u > 0, \\ 0, & u \leq 0. \end{cases}$ *by $\mathbf{G}_\alpha(\xi, u)$ is shown in Figure 1 and 2, respectively.*

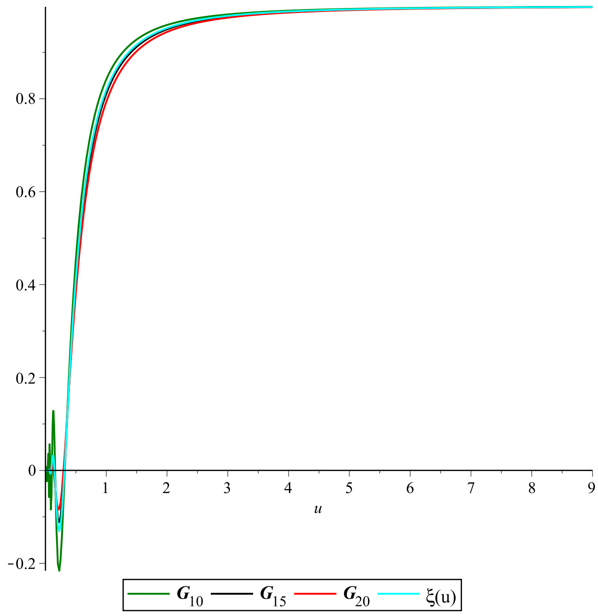

Figure 1: Approximation Process

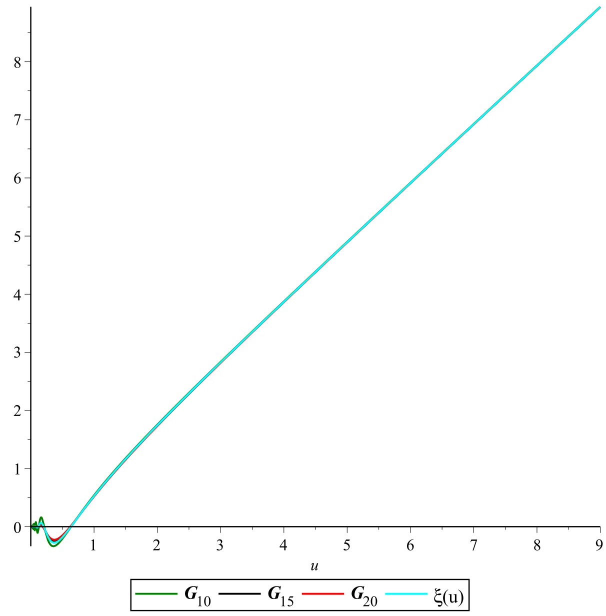

Figure 2: Approximation Process

Example 3.7. *For $\alpha = 10, 15, 20$, the convergence of absolutely continuous function* $\xi(u) = \begin{cases} u^4 \sin\left(\frac{1}{u}\right), & u > 0, \\ 0, & u \leq 0. \end{cases}$ *by* $\mathbf{G}_\alpha(\xi, u)$ *is shown in Figure 3*

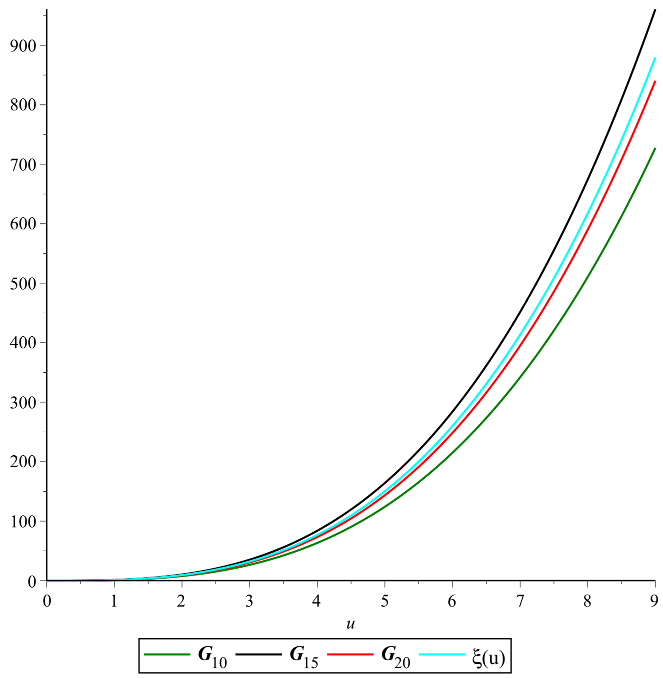

Figure 3: Approximation Process

Example 3.8. *For* $\alpha = 10, 15, 20$, *the convergence by* $\mathbf{G}_\alpha(\xi, u)$ *for locally bounded function to* $\xi(u) = \dfrac{1}{(1+u^2)}$ *is shown in Figure 4*

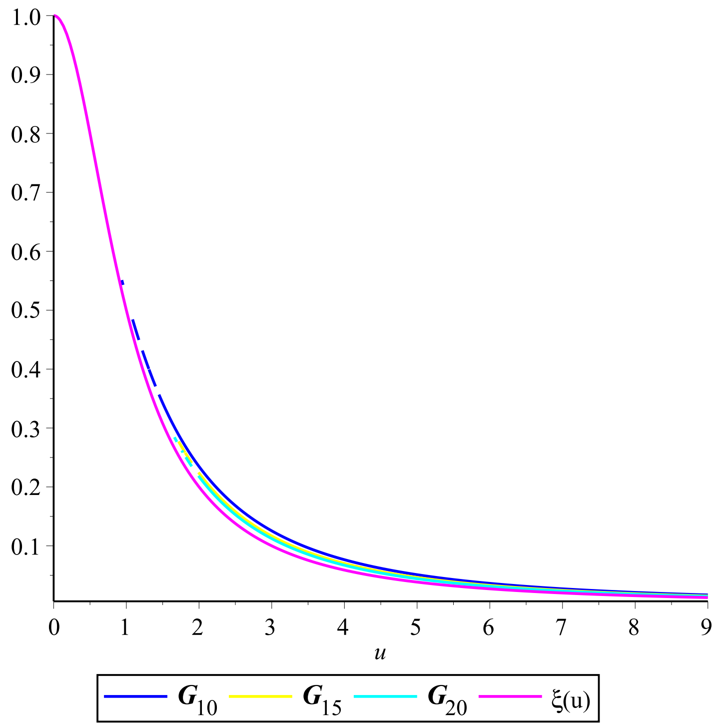

Figure 4: Approximation Process

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