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A note on property (UW_{Π})

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Abstract. Let $\mathcal{H}$ be a complex infinite dimensional Hilbert space. For $T \in \mathcal{B}(\mathcal{H})$, T is said to satisfy property (UW_{Π}) if the complement in the approximate point spectrum of the Weyl essential approximate point spectrum coincides with the poles of T. In this paper, we deeply talk about the property (UW_{Π}) under some perturbations and property (UW_{Π}) for functions of operators. In addition, if T is Drazin invertible, then property (UW_{Π}) for functions of T can be transmitted to the functions of its Drazin inverse.

1. Introduction

In 1909, Weyl [10] examined the compact perturbations of some self-adjoint operators and found that the intersection of their spectrums consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. Later, the above observation was abstracted into the assertion "Weyl's theorem holds". In recent decades, many scholars pay attention to this theorem and a lot of excellent achievements followed. Also, combining with the relationships between different spectrum subsets, mathematicians put forward a series of variants. Then many researches emerged, mainly focusing on the perturbations, the functional calculus, and the operator matrices of Weyl type theorem (cf. [3, 4, 8, 9, 11, 12]). In [4], Berkani and Kachad introduced the definition of property (UW_{Π}) by means of the approximate point spectrum and the poles of some operator. Later, Zariouh gave another variant–property (Z_{Π_a}), using the Weyl spectrum and the left poles. In [3], the authors have talked about the relationships between those two properties and mainly talked about the stability of property (UW_{Π}) under some commuting perturbations and the preservation of property (UW_{Π}) under functional calculus. However, we find some errors therein, Theorem 2.5, Theorem 3.3 and Theorem 4.4, for example. In this paper, we will focus on this topic and put the matters in [3] right. To begin with, we introduce some terminology and notations.

$\mathbb{C}$, $\mathbb{N}$, $\mathbb{D}$ and $\mathbb{T}$ denote the set of all complex numbers, the set of all nonnegative integers, the unit disk and the unit circle, respectively. Let $\mathcal{H}$ be a complex infinite dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Without causing confusion, I denotes the identity mapping

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on some Hilbert space. Let $T \in \mathcal{B}(\mathcal{H})$. We use $N(T)$, $R(T)$ and $\sigma(T)$ denote the kernel, the range and the spectrum of T, respectively. If $R(T)$ is closed and $n(T) < \infty$, then T is said to be an upper semi-Fredholm operator; while T is said to be a lower semi-Fredholm operator if $d(T) < \infty$, where $n(T) = \dim N(T)$ and $d(T) = \operatorname{codim} R(T)$. Especially, if T is an upper semi-Fredholm operator with $n(T) = 0$, then T is called a bounded below operator. If T is upper semi-Fredholm or lower-Fredholm, then T is called a semi-Fredholm operator. Now, the index of T is defined by $\operatorname{ind}(T) = n(T) - d(T)$. If $\operatorname{ind}(T)$ is finite, then T is called a Fredholm operator. The approximate point spectrum $\sigma_a(T)$, the Weyl essential approximate point spectrum $\sigma_{ea}(T)$ and the Weyl spectrum $\sigma_w(T)$ are defined by

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a bounded below operator}\},$$

$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-Fredholm operator with } \operatorname{ind}(T - \lambda I) \leq 0\},$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Fredholm operator with } \operatorname{ind}(T - \lambda I) = 0\}.$$

The ascent and descent of T are closely related with the kernel and range of the power of T, which are defined by

$$\operatorname{asc}(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\},$$

$$\operatorname{des}(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}.$$

If the infimum does not exist, then we write $\operatorname{asc}(T) = \infty$ (resp. $\operatorname{des}(T) = \infty$). It is known from [6, Proposition 38.3] that $\operatorname{asc}(T) = \operatorname{des}(T)$ if they are finite meanwhile. In this case, T is said to be Drazin invertible. If $\operatorname{asc}(T) < \infty$ and $R(T^{\operatorname{asc}(T)+1})$ is closed, then T is said to be left Drazin invertible. If T is both Fredholm and Drazin invertible, then T is called a Browder operator. The Drazin spectrum $\sigma_D(T)$, the left Drazin spectrum $\sigma_{LD}(T)$ and the Browder spectrum $\sigma_b(T)$ are defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Drazin invertible operator}\},$$

$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a left Drazin invertible operator}\},$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Browder operator}\}.$$

$\Pi(T) = \sigma(T) \setminus \sigma_D(T)$ and $\Pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T)$ are called the set of all poles and the set of all left poles of T, respectively.

The following property has a fundamental role in local spectral theory. T is said to have single-valued extension property at $\lambda_0 \in \mathbb{C}$ (abbr. SVEP at λ_0), if for every open neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f : \mathcal{U} \to \mathcal{H}$ which satisfies the equation $(T - \lambda I)f(\lambda) \equiv 0$ for all $\lambda \in \mathcal{U}$ is the function $f \equiv 0$. T is said to have SVEP if it has SVEP at every $\lambda \in \mathbb{C}$. The interested reader could refer [1] for more details.

For a Cauchy domain ([7]) Ω, if all the curves of $\partial\Omega$ are regular analytic Jordan curves, we say that Ω is an analytic Cauchy domain. For $T \in \mathcal{B}(\mathcal{H})$, if σ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subseteq \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$, where $\overline{\Omega}$ is the closure of Ω. We denote by $E(\sigma; T)$ the Riesz idempotent of T corresponding to σ, i.e.,

$$E(\sigma; T) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - T)^{-1} d\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. In this case, we have $\mathcal{H}(\sigma; T) = R(E(\sigma; T))$. Clearly, if $\lambda \in \operatorname{iso}\sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$. We write $\mathcal{H}(\lambda; T)$ instead of $\mathcal{H}(\{\lambda\}; T)$; if in addition, $\dim \mathcal{H}(\lambda; T) < \infty$, then $\lambda \in \sigma_0(T)$, where $\sigma_0(T)$ denotes the set of all normal eigenvalues of T. It is known to us all that $\sigma_0(T) = \sigma(T) \setminus \sigma_b(T)$.

Let $T \in \mathcal{B}(\mathcal{H})$. T is said to satisfy property (UW_Π) and denoted by $T \in (UW_\Pi)$, if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \Pi(T).$$

T is said to satisfy property (Z_{Π_a}) and denoted by $T \in (Z_{\Pi_a})$, if

$$\sigma(T) \setminus \sigma_w(T) = \Pi_a(T).$$

2. Property (UW_Π) and property (Z_{Π_a})

In [3, Theorem 2.5], it showed that property (UW_Π) implies property (Z_{Π_a}) and gave a characterization for the reverse implication in [3, Theorem 2.6]. However, this is not true. We explain it by the following example.

Let U be the right unilateral shift and $P \in \mathcal{B}(\ell^2)$ be a projection with $n(P) = d(P) = \infty$. Put $T = \begin{pmatrix} U & 0 \\ 0 & P \end{pmatrix}$. Then we have $\sigma(T) = \sigma_w(T) = \mathbb{D}$ and $\sigma_a(T) = \sigma_{ea}(T) = \{0\} \cup \mathbb{T}$. Also, $\Pi(T) = \emptyset$ and $\Pi_a(T) = \{0\}$. It follows that $T \in (UW_\Pi)$ but $T \notin (Z_{\Pi_a})$.

Meanwhile, if $T \in (Z_{\Pi_a})$, we could not get $T \in (UW_\Pi)$.

For example, let V be the left unilateral shift and $A \in \mathcal{B}(\ell^2)$ be defined by $A(x_1, x_2, x_3, \cdots) = (0, 0, x_1, x_2, x_3, x_4, \cdots)$. Put $T = \begin{pmatrix} V & 0 \\ 0 & A \end{pmatrix}$. Then we have $\sigma(T) = \sigma_w(T) = \sigma_a(T) = \mathbb{D}$ and $\sigma_{ea}(T) = \mathbb{T}$. So, $\Pi(T) = \Pi_a(T) = \emptyset$.

From those examples above, we see that there is no connection between property (UW_Π) and property (Z_{Π_a}).

For $T \in \mathcal{B}(\mathcal{H})$, we use T^* to denote the adjoint of T. Suppose that T^* has SVEP, then $T \in (Z_{\Pi_a})$ implies $T \in (UW_\Pi)$. Besides, if $[\sigma_a(T) \setminus \sigma_{ea}(T)] \cap \sigma_w(T) = \emptyset$ or $\sigma_w(T) \setminus \sigma_{ea}(T) = \sigma(T) \setminus \sigma_a(T)$, then $T \in (Z_{\Pi_a})$ implies $T \in (UW_\Pi)$.

Put $\sigma_c(T) = \{\lambda \in \mathbb{C} : R(T - \lambda I) \text{ is not closed}\}$. In the following, we give an equivalence for T to satisfy two properties at the same time.

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent.*

(1) $T \in (UW_\Pi) \cap (Z_{\Pi_a})$;

(2) $\sigma_b(T) = [\{\lambda \in \mathbb{C} : n(T - \lambda I) < \infty\} \cap \sigma_c(T)] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\} \cup [\{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I) = \infty\} \cap [\sigma(T) \setminus \Pi_a(T)]]$.

Proof. Necessity. Assume that λ does not belong to the right side in (2). Without loss of generality, suppose that $\lambda \in \sigma(T)$. Then we have $0 < n(T - \lambda I) \leq d(T - \lambda I)$. Now, we claim that $n(T - \lambda I) < \infty$. Otherwise, we could get $\lambda \in \Pi_a(T)$. Combining with $T \in (Z_{\Pi_a})$ we have $T - \lambda I$ is Weyl, a contradiction. Then $\lambda \notin \sigma_c(T)$. It follows that $T - \lambda I$ is upper semi-Fredholm and $\mathrm{ind}(T - \lambda I) \leq 0$. So, $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Using the fact that $T \in (UW_\Pi)$ we obtain $\lambda \notin \sigma_b(T)$.

Sufficiency. It is obvious that $[\sigma_a(T) \setminus \sigma_{ea}(T)] \cup [\sigma(T) \setminus \sigma_w(T)] \subseteq \sigma_0(T)$. Suppose that $\lambda \in \Pi(T)$. Then we have $\lambda \in \mathrm{iso}\,\sigma(T)$ and $n(T - \lambda I) > 0$. It is clear that $\lambda \in \Pi_a(T)$. If $n(T - \lambda I) > d(T - \lambda I)$, then $T - \lambda I$ is lower semi-Fredholm and $\mathrm{ind}(T - \lambda I) > 0$. Using [5, VII, Proposition 6.9] we obtain $T - \lambda I$ is Weyl, a contradiction. If $n(T - \lambda I) < \infty$, then from $T - \lambda I$ is Drazin invertible we get $T - \lambda I$ is Browder. So, $\lambda \notin \sigma_c(T)$. From what we have talked above, we have $\lambda \notin \sigma_b(T)$. Suppose that $\lambda \in \Pi_a(T)$. Using a similar way, we also get $\lambda \notin \sigma_b(T)$. Therefore, $T \in (UW_\Pi) \cap (Z_{\Pi_a})$. □

Remark 2.2. *In Theorem 2.1, the four parts in (2) are essential. We will explain it from the following examples.*

(1) *Let T be a quasi-nilpotent operator with $0 < n(T) < \infty$. Then $\sigma(T) = \sigma_w(T) = \sigma_a(T) = \sigma_{ea}(T) = \sigma_D(T) = \sigma_{LD}(T) = \{0\}$. So, $T \in (UW_\Pi) \cap (Z_{\Pi_a})$. However, $\sigma_b(T) \neq \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\} \cup [\{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I) = \infty\} \cap [\sigma(T) \setminus \Pi_a(T)]]$.*

(2) *Let U be the right unilateral shift. Then $\sigma(U) = \sigma_w(U) = \mathbb{D}$ and $\sigma_a(U) = \sigma_{ea}(U) = \mathbb{T}$. So, $\Pi(U) = \Pi_a(U) = \emptyset$. Thus, $U \in (UW_\Pi) \cap (Z_{\Pi_a})$. But, $\sigma_b(U) \neq [\{\lambda \in \mathbb{C} : n(U - \lambda I) < \infty\} \cap \sigma_c(U)] \cup \{\lambda \in \mathbb{C} : n(U - \lambda I) > d(U - \lambda I)\} \cup [\{\lambda \in \mathbb{C} : n(U - \lambda I) = d(U - \lambda I) = \infty\} \cap [\sigma(U) \setminus \Pi_a(U)]]$.*

(3) *Let V be the left unilateral shift. It is obvious to check that $V \in (UW_\Pi) \cap (Z_{\Pi_a})$. But, $\sigma_b(V) \neq [\{\lambda \in \mathbb{C} : n(V - \lambda I) < \infty\} \cap \sigma_c(V)] \cup \{\lambda \in \sigma(V) : n(V - \lambda I) = 0\} \cup [\{\lambda \in \mathbb{C} : n(V - \lambda I) = d(V - \lambda I) = \infty\} \cap [\sigma(V) \setminus \Pi_a(V)]]$.*

(4) *Let S be a compact operator with infinite nullity. Then $\sigma(S) = \sigma_a(S)$, $\sigma_w(S) = \sigma_{ea}(S) = \{0\}$ and $\Pi(S) = \Pi_a(S) = \sigma(S) \setminus \{0\}$. So, $S \in (UW_\Pi) \cap (Z_{\Pi_a})$. However, $\sigma_b(S) \neq [\{\lambda \in \mathbb{C} : n(S - \lambda I) < \infty\} \cap \sigma_c(S)] \cup \{\lambda \in \sigma(S) : n(S - \lambda I) = 0\} \cup \{\lambda \in \mathbb{C} : n(S - \lambda I) > d(S - \lambda I)\}$.*

If T satisfies one property, then we give an equivalent characterization for T to satisfy the other property.

Theorem 2.3. *Suppose that $T \in (UW_{\Pi})$. Then $T \in (Z_{\Pi_a})$ if and only if $\Pi_a(T) = \sigma_0(T)$.*

Theorem 2.4. *Suppose that $T \in (Z_{\Pi_a})$. Then $T \in (UW_{\Pi})$ if and only if $\sigma_b(T) = \sigma_{ea}(T) \cup [\rho_a(T) \cap \sigma(T)]$.*

3. Property (UW_{Π}) under perturbations

In this section, we will focus on the stability of property (UW_{Π}) under commuting perturbations.

We call $T \in \mathcal{B}(\mathcal{H})$ an a-isoloid operator if $\mathrm{iso}\sigma_a(T) \subseteq \sigma_p(T)$, where $\sigma_p(T)$ denotes the point spectrum of T. For a-isoloid operators, we firstly note the following facts.

(1) Property (UW_{Π}) is not stable under commuting compact perturbations.

For example, let T be an injective compact operator. Then T is an a-isoloid operator. Also, $\sigma(T) = \sigma_a(T)$, $\sigma_{ea}(T) = \{0\}$ and $\Pi(T) = \sigma(T) \setminus \{0\}$. So, $T \in (UW_{\Pi})$. Put $K = -T$. It is clear that $T + K \notin (UW_{\Pi})$.

(2) Property (UW_{Π}) is not stable under commuting finite rank perturbations.

For example, let U be the right unilateral shift and put $T = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$. Then $\sigma(T) = \mathbb{D}$ and $\sigma_a(T) = \sigma_{ea}(T) = \mathbb{T}$. So, T is an a-isoloid operator and $\Pi(T) = \emptyset$. Thus, $T \in (UW_{\Pi})$. Let $B \in \mathcal{B}(\ell^2)$ be defined by $B(x_1, x_2, x_3, \cdots) = (-\frac{x_1}{2}, 0, 0, \cdots)$ and put $F = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$. Then F is a finite rank operator and $TF = FT$. By direct calculation, we have $\sigma(T + F) = \mathbb{D}$, $\sigma_a(T + F) = \{\frac{1}{2}\} \cup \mathbb{T}$ and $\sigma_{ea}(T + F) = \mathbb{T}$. So, $T + F \notin (UW_{\Pi})$.

Moreover, this example shows that Theorem 3.3 and Lemma 4.7 in [3] are not true.

(3) Property (UW_{Π}) is not stable under commuting quasi-nilpotent perturbations.

For example, let T be a quasi-nilpotent operator with $0 < n(T) < \infty$. It is trivial to see that $T \in (UW_{\Pi})$. Put $Q = -T$. Then $T + Q \notin (UW_{\Pi})$.

(4) Property (UW_{Π}) is not stable under commuting nilpotent perturbations.

For example, let U be the right unilateral shift. It is elementary to check that $U \in (UW_{\Pi})$. Let $N \in \mathcal{B}(\ell^2)$ be defined by $N(x_1, x_2, x_3, \cdots) = (0, -x_1, 0, 0, \cdots)$. Then $N^2 = 0$ and $UN = NU$. It can be seen that $0 \in \sigma_a(U + N) \setminus \sigma_{ea}(U + N)$ but $U + N$ is not Weyl. So, $U + N \notin (UW_{\Pi})$.

For quasi-nilpotent operators, we have

Theorem 3.1. *Let Q be a quasi-nilpotent operator such that $Q \in (UW_{\Pi})$. If $KQ = QK$ and K^n is finite rank for some $n \in \mathbb{N}$, then $Q + K \in (UW_{\Pi})$.*

Proof. It is trivial that $\sigma_*(Q + K) = \sigma_*(Q)$, where $* \in \{b, ea, D\}$. Let $\lambda \in \sigma_a(Q + K) \setminus \sigma_{ea}(Q + K)$. Then $\lambda \notin \sigma_{ea}(Q)$. It follows that $Q - \lambda I$ is Browder and so is $Q + K - \lambda I$. Thus, $\lambda \in \Pi(Q + K)$. For the converse, if $\lambda \in \Pi(Q + K)$, then $Q - \lambda I$ is Drazin invertible. Since $Q \in (UW_{\Pi})$, it follows that $\lambda \neq 0$. So, $Q - \lambda I$ is Browder and so is $Q + K - \lambda I$. Thus, $\lambda \in \sigma_a(Q + K) \setminus \sigma_{ea}(Q + K)$. □

From Theorem 3.1 we obtain the following corollary.

Corollary 3.2. *([3, Corollary 3.2]) Let Q be a quasi-nilpotent operator such that $n(Q) < \infty$. If $KQ = QK$ and K^n is finite rank for some $n \in \mathbb{N}$, then $Q + K \in (UW_{\Pi})$.*

If we omit the quasi-nilpotent assumption in Theorem 3.1, we get

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ such that $\sigma(T) = \sigma_a(T)$ and $K \in \mathcal{B}(\mathcal{H})$ commuting with T such that K^n is finite rank for some $n \in \mathbb{N}$. If $T \in (UW_{\Pi})$, then $T + K \in (UW_{\Pi})$.*

Proof. Let $\lambda \in \sigma_a(T + K) \setminus \sigma_{ea}(T + K)$. Then $\lambda \notin \sigma_{ea}(T)$. So, $\lambda \in \rho_a(T) \cup [\sigma_a(T) \setminus \sigma_{ea}(T)]$. In both cases, we could get $T + K - \lambda I$ is Browder. So $\sigma_a(T + K) \setminus \sigma_{ea}(T + K) \subseteq \Pi(T + K)$. Similarly, we can get $\Pi(T + K) \subseteq \sigma_a(T + K) \setminus \sigma_{ea}(T + K)$. □

In the sequel, we talk about the quasi-nilpotent perturbations. For $T \in \mathcal{B}(\mathcal{H})$, the quasi-nilpotent part of T is defined by

$$H_0(T) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is known that T is quasi-nilpotent if and only if $H_0(T) = \mathcal{H}$. Firstly, we have the following lemma.

Lemma 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$. If Q is quasi-nilpotent and $TQ = QT$, then $\sigma(T+Q) = \sigma(T)$ and $\sigma_a(T+Q) = \sigma_a(T)$.*

Proof. Without loss of generality, we show that $T+Q$ is bounded below if T is bounded below. Assume that T is bounded below. Then $H_0(T) = \{0\}$. In fact, since T is bounded below, there is some $c > 0$ such that $\|Tx\| \geq c\|x\|$, $\forall x \in \mathcal{H}$. Let $x \in H_0(T)$. Then we have $\lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0$. However, $\|T^n x\| \geq c^n \|x\|$. It follows that $c\|x\|^{\frac{1}{n}} \leq \|T^n x\|^{\frac{1}{n}}$. So, $x = 0$. Since $T+Q$ is upper semi-Fredholm, we only need to show $N(T+Q) = \{0\}$. Let $x \in N(T+Q)$. Then $Qx = -Tx$. So, $Q^n x = (-1)^n T^n x$. Since Q is quasi-nilpotent, we have $H_0(Q) = \mathcal{H}$. So, $\lim_{n\to\infty} \|Q^n x\|^{\frac{1}{n}} = 0$ and then $\lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0$. This means $x \in H_0(T)$. From what we have shown above, we get $x = 0$. So, $T+Q$ is bounded below.

If T is invertible, then $T+Q$ is Weyl. Since $T+Q$ is also bounded below, it follows that $T+Q$ is invertible. □

Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be polaroid (resp. a-polaroid) if $\mathrm{iso}\sigma(T) \subseteq \Pi(T)$ (resp. $\mathrm{iso}\sigma_a(T) \subseteq \Pi(T)$).

Theorem 3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a polaroid operator and satisfy property (UW_Π), Q be a quasi-nilpotent operator such that $TQ = QT$. Then $T+Q \in (UW_\Pi)$.*

Proof. It is known that $\sigma_*(T+Q) = \sigma_*(T)$, where $* \in \{ea, b\}$. If $\lambda \in \sigma_a(T+Q) \setminus \sigma_{ea}(T+Q)$, then from Lemma 3.1 we know that $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. It follows that $T - \lambda I$ is Browder since $T \in (UW_\Pi)$. So, $T + Q - \lambda I$ is Browder. Conversely, let $\lambda \in \Pi(T+Q)$, then $\lambda \in \mathrm{iso}\sigma(T+Q)$. By using Lemma 3.1 and the polaroidity of T we get $T - \lambda I$ is Drazin invertible. Then $T + Q - \lambda I$ is Browder again. □

From Theorem 3.3 we get the following result.

Corollary 3.6. *([1, Theorem 3.6]) Let $T \in (UW_\Pi)$. If $\mathrm{iso}\sigma_b(T) = \emptyset$ or $\mathrm{iso}\sigma_{ab}(T) = \emptyset$, then $T+Q \in (UW_\Pi)$ for every quasi-nilpotent Q with $TQ = QT$.*

Proof. From the conditions we can get T is polaroid. Then using Theorem 3.3 we get the conclusion. □

If $\mathrm{iso}\sigma(T) = \emptyset$, then we have $\Pi(T) = \emptyset$. Besides, suppose that there is an injective quasi-nilpotent Q such that $TQ = QT$, then from [3, Lemma 3.7] we obtain $\sigma_a(T) = \sigma_{ea}(T)$. Thus, $T \in (UW_\Pi)$. Also, from Theorem 3.3 we get $T + Q \in (UW_\Pi)$.

Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be finite-isoloid(abbr. f-isoloid) if every isolated point of $\sigma(T)$ is an isolated eigenvalue of finite multiplicity of T. If T is f-isoloid and there exists an injective quasi-nilpotent Q such that $TQ = QT$, then both T and $T+Q$ satisfy property (UW_Π). In fact, we could obtain $\mathrm{iso}\sigma(T) = \emptyset$. From what we have talked above, we get this result.

If $T - \lambda I$ is Fredholm for every nonzero λ, then T is called a Riesz operator. It is obvious that compact operators and quasi-nilpotent operators are Riesz operators.

Generally, for the Riesz perturbations, we have

Theorem 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be an a-polaroid operator and satisfy property (UW_Π), R be a Riesz operator such that $TR = RT$ and $\sigma_a(T+R) = \sigma_a(T)$. Then $T+Q \in (UW_\Pi)$.*

Proof. From [2, Theorem 2.77, Corollary 2.81] we get $\sigma_*(T+R) = \sigma_*(T)$, where $* \in \{ea, b\}$. Let $\lambda \in \sigma_a(T+R) \setminus \sigma_{ea}(T+R)$, then we have $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Then from $T \in (UW_\Pi)$ we get $T - \lambda I$ is Browder and so is $T + R - \lambda I$. If $\lambda \in \Pi(T+R)$, then $\lambda \in \mathrm{iso}\sigma_a(T+R) = \mathrm{iso}\sigma_a(T)$. From T is an a-polaroid operator we obtain $T - \lambda I$ is Drazin invertible. Then we get $T + R - \lambda I$ is Browder again. □

4. Property (UW_Π) for functions of operators

For $T \in \mathcal{B}(\mathcal{H})$, we use $\mathrm{Hol}(\sigma(T))$ to denote the set of all analytic functions on some neighbourhood of $\sigma(T)$ and are not constant on every component of $\sigma(T)$.

Generally, there is no relationship between $T \in (UW_\Pi)$ and $f(T) \in (UW_\Pi)$, where $f \in \mathrm{Hol}(\sigma(T))$. We show this from the following examples. Let U be the right unilateral shift, V be the left unilateral shift and $P \in \mathcal{B}(\ell^2)$ be a projection with $n(P) = d(P) = \infty$. Put $T = \begin{pmatrix} U+I & 0 \\ 0 & V-I \end{pmatrix}$. Then $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda+1| \leq 1\} \cup \{\lambda \in \mathbb{C} : |\lambda-1| \leq 1\}$. So, $\Pi(T) = \emptyset$. Also, $\sigma_a(T) = \sigma_{ea}(T) = \{\lambda \in \mathbb{C} : |\lambda+1| \leq 1\} \cup \{\lambda \in \mathbb{C} : |\lambda-1| = 1\}$. It is obvious that $T \in (UW_\Pi)$. However, $T^2 \notin (UW_\Pi)$ since $T^2 - I$ is Weyl but not Browder.

Put $R = \begin{pmatrix} U+I & 0 \\ 0 & -P \end{pmatrix}$. Then we have $\sigma(R^2) = \{re^{i\theta} : r \leq 2(1+\cos\theta)\}$ is connected and $\sigma_a(R^2) = \sigma_{ea}(R^2)$. So, $R^2 \in (UW_\Pi)$. However, $R + I$ is Drazin invertible but not semi-Fredholm. Thus, $R \notin (UW_\Pi)$.

In this section, we aim to give the characterization for T such that $f(T) \in (UW_\Pi)$ for any $f \in \mathrm{Hol}(\sigma(T))$. First, for a given operator T and $f \in \mathrm{Hol}(\sigma(T))$, we have the following result.

Theorem 4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ and $f \in Hol(\sigma(T))$. If $f(T) \in (UW_\Pi)$, then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.*

Proof. It is elementary that $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$. Let $\mu \notin \sigma_{ea}(f(T))$. Then $\mu \in \rho_a(f(T)) \cup [\sigma_a(f(T)) \setminus \sigma_{ea}(f(T))]$. It follows from $f(T) \in (UW_\Pi)$ that $f(T) - \mu I$ is bounded below or Browder. Suppose that $f(T) - \mu I = (T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2}\cdots(T-\lambda_k I)^{n_k} g(T)$, where $\lambda_i \neq \lambda_i$ if $i \neq j$ and $g(T)$ is invertible. Then $T - \lambda_i I$ is bounded below or Browder. So, $\lambda_i \notin \sigma_{ea}(T)$, $1 \leq i \leq k$. Since $\mu = f(\lambda_i)$, we get $\mu \notin f(\sigma_{ea}(T))$. □

In [3, Theorem 4.4], it showed that for an a-isoloid operator T and $f \in \mathrm{Hol}(\sigma(T))$, if $T \in (UW_\Pi)$, then $f(T) \in (UW_\Pi)$ is equivalent with $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. But this is not true.

For example, let U be right unilateral shift and $P \in \mathcal{B}(\ell^2)$ be a projection with $0 < n(P) = d(P) < \infty$. Put $T = \begin{pmatrix} U & 0 \\ 0 & P-2I \end{pmatrix}$. Then we have $\sigma(T) = \{-2\} \cup \mathbb{D}$, $\sigma_a(T) = \{-2\} \cup \mathbb{T}$, $\sigma_{ea}(T) = \mathbb{T}$ and $\Pi(T) = \{-2\}$. So, T is a-isoloid and satisfies property (UW_Π). Also, $\mathrm{ind}(T - \lambda I) \leq 0$, $\forall \lambda \in \rho_{SF_+}(T)$. Here, $\rho_{SF_+}(T)$ denotes the upper semi-Fredholm domain of T. Set $p(z) = z(z+2)$, $z \in \mathbb{C}$. Then $\sigma_{ea}(p(T)) = p(\sigma_{ea}(T))$. However, $0 \in \sigma_a(p(T)) \setminus \sigma_{ea}(p(T))$ but $p(T)$ is not Weyl. Thus, $p(T) \notin (UW_\Pi)$.

If we replace "T is a-isoloid" by "$\sigma_a(T) = \sigma(T)$", then we get

Theorem 4.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and $f \in Hol(\sigma(T))$. If $T \in (UW_\Pi)$ and $\sigma_a(T) = \sigma(T)$, then the following statements are equivalent.*

(1) $f(T) \in (UW_\Pi)$;

(2) $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.

Proof. (1) ⇒ (2) holds from Theorem 4.1.

(2) ⇒ (1) Let $\mu \in \sigma_a(f(T)) \setminus \sigma_{ea}(f(T))$ and suppose that $f(T) - \mu I = (T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2}\cdots(T-\lambda_k I)^{n_k} g(T)$, where $\lambda_i \neq \lambda_i$ if $i \neq j$ and $g(T)$ is invertible. Then $\lambda_i \in \rho_a(T) \cup [\sigma_a(T) \setminus \sigma_{ea}(T)]$, $1 \leq i \leq k$. Then from $T \in (UW_\Pi)$ and $\sigma_a(T) = \sigma(T)$ we get $T - \lambda_i I$ is Browder. So, $f(T) - \mu I$ is Browder. If $\mu \in \Pi(f(T))$ and $f(T) - \mu I$ has the decomposition as above, then $T - \lambda_i I$ is Drazin invertible, $1 \leq i \leq k$. From $T \in (UW_\Pi)$ we get $\lambda_i \notin \sigma_{ea}(T)$. So, $\mu = f(\lambda_i) \notin f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$. Thus, $\mu \in \sigma_a(f(T)) \setminus \sigma_{ea}(f(T))$. □

Generally, we have the following result.

Theorem 4.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $f(T) \in (UW_\Pi)$ for any $f \in Hol(\sigma(T))$ if and only if the following statements hold:*

(1) $T \in (UW_\Pi)$;

(2) $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for any $f \in Hol(\sigma(T))$;

(3) if $\sigma_0(T) \neq \emptyset$, then $\sigma_a(T) = \sigma(T)$.

Proof. Sufficiency. We divide it into the following two cases.

Case 1. $\sigma_0(T) = \emptyset$.

Now, from (1) we have $\sigma_a(T) = \sigma_{ea}(T)$ and $\Pi(T) = \emptyset$. So, $\Pi(f(T)) = \emptyset$ for any $f \in \mathrm{Hol}(\sigma(T))$. Since $\sigma_a(T)$ satisfies spectral mapping theorem, combining with (2) we get $\sigma_a(f(T)) = \sigma_{ea}(f(T))$ for any $f \in \mathrm{Hol}(\sigma(T))$. So, $f(T) \in (UW_\Pi)$.

Case 2. $\sigma_0(T) \neq \emptyset$.

Now, from (3) we have $\sigma_a(T) = \sigma(T)$. Then from Theorem 4.2 we get $f(T) \in (UW_\Pi)$ for any $f \in \mathrm{Hol}(\sigma(T))$.

Necessity. (1) Take $f_1(z) = z, z \in \mathbb{C}$. Then $T = f_1(T) \in (UW_\Pi)$.

(2) It suffices to show $\mathrm{ind}(T - \lambda I)\mathrm{ind}(T - \mu I) \geq 0$ for any pair of $\lambda, \mu \in \rho_{SF_+}(T)$. If not, then there exist $\lambda, \mu \in \rho_{SF_+}(T)$ such that $\mathrm{ind}(T - \lambda I) = m$ and $\mathrm{ind}(T - \mu I) < 0$, where m is a positive integer. If $\mathrm{ind}(T - \mu I) = -\infty$, then put $f_2(z) = (z - \lambda)(z - \mu), z \in \mathbb{C}$; if $\mathrm{ind}(T - \mu I) = -n$, where n is a positive integer, then put $f_2(z) = (z - \lambda)^n (z - \mu)^m, z \in \mathbb{C}$. In both cases we could get $0 \in \sigma_a(f_2(T)) \setminus \sigma_{ea}(f_2(T))$. So, $f_2(T)$ is Browder and then $T - \lambda I$ is Browder, a contradiction.

(3) Assume $\alpha \in \sigma_0(T)$. If $\sigma_a(T) \neq \sigma(T)$, then there is $\beta \in \sigma(T) \setminus \sigma_a(T)$. Put $f_3(z) = (z - \alpha)(z - \beta), z \in \mathbb{C}$. Then we have $0 \in \sigma_a(f_3(T)) \setminus \sigma_{ea}(f_3(T))$. So, $f_3(T)$ is Browder. It follows that $T - \beta I$ is Browder and so is invertible, a contradiction. □

If $T \in \mathcal{B}(\mathcal{H})$ is Drazin invertible with Drazin inverse S, then $\mathrm{asc}(T) = \mathrm{des}(T) = p$ for some $p \in \mathbb{N}$. Now, $R(T^p)$ is closed and $\mathcal{H} = N(T^p) \oplus R(T^p)$. Under this space decomposition, $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where T_1 is nilpotent and T_2 is invertible. Now, $S = \begin{pmatrix} 0 & 0 \\ 0 & T_2^{-1} \end{pmatrix}$.

In [3], it has shown that property (UW_Π) is transmitted from Drazin invertible operator to its Drazin inverse. Moreover, we have

Theorem 4.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible with Drazin inverse S. Then*

(1) $T \in (Z_{\Pi_a})$ if and only if $S \in (Z_{\Pi_a})$;

(2) $f(T) \in (UW_\Pi)$ if and only if $f(S) \in (UW_\Pi)$ for any $f \in Hol(\sigma(T)) \cap Hol(\sigma(S))$.

Proof. (1) If T is invertible, then $S = T^{-1}$. The conclusion holds clearly. In the following, we assume T is not invertible.

Let $\lambda \in \sigma(S) \setminus \sigma_w(S)$. We assume that $\lambda \neq 0$. Then as in the matrix representation of S above, we have $T_2^{-1} - \lambda I$ is not invertible but Weyl. So, $\frac{1}{\lambda} I - T_2$ is not invertible but Weyl. It follows that $\frac{1}{\lambda} \in \sigma(T) \setminus \sigma_w(T)$. From $T \in (Z_{\Pi_a})$ we get $\frac{1}{\lambda} I - T$ is Browder. So, $T_2^{-1} - \lambda I$ is Browder and then $S - \lambda I$ is Browder. Let $\lambda \in \Pi_a(S)$. We assume that $\lambda \neq 0$ again. Then $\mathrm{asc}(S - \lambda I) = p$ and $R((S - \lambda I)^{p+1})$ is closed. Now, $S - \lambda I = \begin{pmatrix} -\lambda I & 0 \\ 0 & T_2^{-1} - \lambda I \end{pmatrix}$. So, $\mathrm{asc}(T_2^{-1} - \lambda I) = p$ and $R((T_2^{-1} - \lambda I)^{p+1})$ is closed. It follows that $\mathrm{asc}(T_2 - \frac{1}{\lambda} I) = p$ and $R((T_2 - \frac{1}{\lambda} I)^{p+1})$ is closed. Since $T - \frac{1}{\lambda} I = \begin{pmatrix} T_1 - \frac{1}{\lambda} I & 0 \\ 0 & T_2 - \frac{1}{\lambda} I \end{pmatrix}$ and $T_1 - \frac{1}{\lambda} I$ is invertible, we get $\mathrm{asc}(T - \frac{1}{\lambda} I) = p$ and $R((T - \frac{1}{\lambda} I)^{p+1})$ is closed. So, $\frac{1}{\lambda} \in \Pi_a(T)$. Combining with $T \in (Z_{\Pi_a})$ we get $T - \frac{1}{\lambda} I$ is Browder. So, $S - \lambda I$ is also Browder. Therefore, $S \in (Z_{\Pi_a})$. The reverse implication can be obtained in a similar way.

(2) It is obvious that $T \in (UW_\Pi)$. Then from [3, Theorem 4.10] we get $S \in (UW_\Pi)$. Suppose that $\sigma_0(S) \neq \emptyset$. Then $\sigma_0(T) \neq \emptyset$. In fact, let $\lambda \in \sigma_0(S)$. If $\lambda = 0$, then $0 < \dim N(T^p) < \infty$. It follows that $\dim N(T) > 0$ and so T is not invertible. Now, T_1 is Browder. Combining with the fact that T_2 is invertible we get T is Browder. So, $0 \in \sigma_0(T)$. Assume that $\lambda \neq 0$. Now, $S - \lambda I = \begin{pmatrix} -\lambda I & 0 \\ 0 & T_2^{-1} - \lambda I \end{pmatrix}$. It follows that $T_2^{-1} - \lambda I$ is Browder but not invertible. Then $\frac{1}{\lambda} I - T_2$ is Browder but not invertible. Since $\frac{1}{\lambda} I - T = \begin{pmatrix} \frac{1}{\lambda} I - T_1 & 0 \\ 0 & \frac{1}{\lambda} I - T_2 \end{pmatrix}$ and $\frac{1}{\lambda} I - T_1$ is invertible, we obtain $\frac{1}{\lambda} I - T$ is Browder but not invertible, i.e., $\frac{1}{\lambda} \in \sigma_0(T)$. Now, from Theorem 4.3 we get $\sigma_a(T) = \sigma(T)$. Using a similar way we get $\sigma_a(S) = \sigma(S)$. Moreover, $\sigma_{ea}(f(S)) = f(\sigma_{ea}(S))$ for any $f \in \mathrm{Hol}(\sigma(T)) \cap \mathrm{Hol}(\sigma(S))$. Using Theorem 4.3 again we obtain $f(S) \in (UW_\Pi)$. □

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