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Three dimensional homogeneous hyperbolic Yamabe solitons

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Abstract. The present article deals with homogeneous hyperbolic Yamabe solitons of dimension three. We delve into unimodular as well as non-unimodular hyperbolic Yamabe solitons on Lorentzian manifolds. Hyperbolic Yamabe solitons on special three dimensional Lie groups have also been considered.

1. Introduction

A pseudo parabolic heat type initial value problem

$$\frac{\partial g}{\partial t}(t) = -2S, \qquad g(0) = g_0,$$

where g is the metric and S is the Ricci curvature, is well known as Ricci flow introduced by Hamilton [14, 15]. After the work of Hamilton, in 1960, H. Yamabe posed a problem which is known as Yamabe problem [28]. On the foundation work of Hamilton and Yamabe, R. Schoen [24] developed the notion of Yamabe flow. Further works on Yamabe flow can be found in [8, 12].

A Yamabe flow is represented by

$$\frac{\partial g}{\partial t}(t) = -2s(t)g(t),$$

where s is the scalar curvature. The Ricci flow and the Yamabe flow both are first order initial value problems. Self similar solutions of such a flow are known as solitons. In the geometric perspective, such solitons have been studied by a large number of researchers [12, 14, 17, 21, 26].

A hyperbolic geometric flow is a prototype of wave equations in $\mathbb{R}^n$. The notion of hyperbolic geometric flow is available in [11]. A hyperbolic geometric flow associated with the Ricci curvature has the nomenclature hyperbolic Ricci flow. A hyperbolic Ricci flow is more informative than the first order Ricci flow as it contains both the properties of a Ricci flow and the Einstein equation. Actually it is analogous to the non linear wave equation. The wave nature of the equation with two variables has been elaborated by Kong and collaborators [18, 20]. Azami [1, 2, 13] also analyzed different aspects of hyperbolic Ricci flow. It is to

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be further mentioned that in the paper [13] hyperbolic Ricci solitons have been studied on homogeneous Lorentzian manifolds. For details about Lorentzian manifolds we refer [9, 22, 27].

From the literature of soliton theory it is well known that in two dimensional situation Ricci solitons and Yamabe solitons are equivalent, but they are not so in higher dimensions. This phenomenon strikes us to extend the notion of hyperbolic Ricci solitons to the idea of hyperbolic Yamabe solitons. To this end let us first formulate the definition of a hyperbolic Yamabe soliton as follows.

The notions of hyperbolic Yamabe flow and hyperbolic Yamabe soliton have been introduced in [3] by Blaga and Özgür. The hyperbolic Yamabe flow is an evolution equation given by

$$\frac{\partial^2 g}{\partial t^2}(t) = -2s(t)g(t), \tag{1}$$

for a time depending (semi) Riemannian metric g on a smooth manifold M^n, where s denotes the scalar curvature of (M^n, g). In this connection, it is to be mentioned that some further developments of this topic can be found in [4–7, 19].

A solution $g(t)$ of the hyperbolic Yamabe flow on M^n is a Yamabe soliton if there exist a scalar positive function $\sigma(t)$ and a diffeomorphism ϕ_t on M^n such that

$$g(t) = \sigma(t)\phi_t^*(g_0), \qquad g(0) = g_0. \tag{2}$$

Without loss of generality, we may assume

$$\sigma(0) = 2, \quad \sigma'(0) = \lambda, \quad \sigma''(0) = -2\mu. \tag{3}$$

Using (1), (2) and (3), we get

$$£_V £_V g_0 + \lambda £_V g_0 = (\mu - s)g_0.$$

Ignoring the subscript of the metric, a Riemannian manifold (M^n, g) is called hyperbolic Yamabe soliton if there exist a vector field V on M^n and real scalars μ and λ such that

$$£_V £_V g + \lambda £_V g = (\mu - s)g. \tag{4}$$

The present article is constituted as follows:

After the introduction in Section 1, we study hyperbolic Yamabe solitons on three dimensional Riemannian manifolds in Section 2. We consider unimodular hyperbolic Yamabe solitons on Lorentzian manifolds in Section 3 and non-unimodular hyperbolic Yamabe solitons on Lorentzian manifolds in Section 4. Section 5 deals with hyperbolic Yamabe solitons on special three dimensional Lie groups.

2. Hyperbolic Yamabe Solitons On Three Dimensional Left-Invariant Riemannian metrics

Three dimensional Riemannian Lie groups were categorized by Milnor [21]. Sekigawa [25] established that a three dimensional simply connected and complete homogeneous manifold is either symmetric or a Lie group with left invariant Riemannian metric. The classification of three dimensional Riemannian Lie groups given by Milnor permits to determine all three dimensional homogeneous Riemannian manifolds. Rahmani [22] classified unimodular Lie groups in three dimensions that have a left invariant Lorentzian metric. Cordero and parker [10] investigated three-dimensional Lorentzian Lie groups, focusing on the symmetry groups of the sectional curvature in the various cases. Particularly, they obtained all the possible forms of a non unimodular Lie algebra. Calvaruso [9] has also worked on Three dimensional homogeneous Lorentzian manifolds.

Let G be a three dimensional connected Lie group with left invariant Riemannian metric. We discuss unimodular and non-unimodular cases separately.

2.1. Case of Uni modular:

Let us consider an orientation for the Lie algebra $\mathfrak{g}$ of G such that the bracket product is expressed by the cross product by the formula

$$[X, Y] = L[X \times Y],$$

where L is unique linear map from $\mathfrak{g}$ to itself. The Lie algebra $\mathfrak{g}$ is called unimodular if and only if this linear map is self-adjoint. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ consisting eigen vectors of L such that

$$[e_1, e_2] = m_3 e_3, \quad [e_2, e_3] = m_1 e_1, \quad [e_3, e_1] = m_2 e_2 \tag{5}$$

for three real constants m_1,m_2,m_3.

With respect to the basis $\{e_1, e_2, e_3\}$, Milnor [21] computed the components of the Ricci curvature as follows:

$$S_{ij} = \begin{bmatrix} \frac{1}{2}(m_1^2 - m_2^2 - m_3^3) + m_2 m_3 & 0 & 0 \\ 0 & \frac{1}{2}(m_2^2 - m_1^2 - m_3^2) + m_1 m_3 & 0 \\ 0 & 0 & \frac{1}{2}(m_3^2 - m_1^2 - m_2^2) + m_1 m_2 \end{bmatrix}. \tag{6}$$

The scalar curvature is

$$s = -\frac{1}{2}(m_1^2 + m_2^2 + m_3^2) + m_1 m_2 + m_2 m_3 + m_1 m_3. \tag{7}$$

Now, we choose an arbitrary vector field V such that $V = V_i e_i \in \mathfrak{g}$, where V_1, V_2 and V_3 are three real constant. Then using (5), we get

$$\pounds_V g = \begin{bmatrix} 0 & (m_1 - m_2)V_3 & (m_3 - m_1)V_2 \\ (m_1 - m_2)V_3 & 0 & (m_2 - m_3)V_1 \\ (m_3 - m_1)V_2 & (m_2 - m_3)V_1 & 0 \end{bmatrix} \tag{8}$$

Again with respect to the basis $\{e_1, e_2, e_3\}$ and using (5) and (8) we have

$$\begin{aligned}
(\pounds_V \circ \pounds_V)(g)_{11} &= 2V_2^2 m_3(m_3 - m_1) - 2V_3^2 m_2(m_1 - m_2), \\
(\pounds_V \circ \pounds_V)(g)_{22} &= -2V_1^2 m_3(m_2 - m_3) + 2V_3^2 m_1(m_1 - m_2), \\
(\pounds_V \circ \pounds_V)(g)_{33} &= 2V_1^2 m_2(m_2 - m_3) - 2V_2^2 m_1(m_3 - m_1,) \\
(\pounds_V \circ \pounds_V)(g)_{12} &= V_1 V_2 m_3(m_2 + m_1 - 2m_3), \\
(\pounds_V \circ \pounds_V)(g)_{13} &= V_1 V_3 m_2(m_1 - 2m_2 + m_3), \\
(\pounds_V \circ \pounds_V)(g)_{23} &= V_2 V_3 m_1(m_3 - 2m_1 + m_2).
\end{aligned} \tag{9}$$

Therefore, using (7), (8), (9) and from the equation (4) we get the following system of algebric equations

$$\begin{aligned}
&\bullet\ 2V_2^2 m_3(m_3 - m_1) - 2V_3^2 m_2(m_1 - m_2) = \mu + \frac{1}{2}(m_1^2 + m_2^2 + m_3^2) - m_1 m_2 - m_2 m_3 - m_3 m_1, \\
&\bullet\ -2V_1^2 m_3(m_2 - m_3) + 2V_3^2 m_1(m_1 - m_2) = \mu + \frac{1}{2}(m_1^2 + m_2^2 + m_3^2) - m_1 m_2 - m_2 m_3 - m_3 m_1, \\
&\bullet\ 2V_1^2 m_2(m_2 - m_3) - 2V_2^2 m_1(m_3 - m_1) = \mu + \frac{1}{2}(m_1^2 + m_2^2 + m_3^2) - m_1 m_2 - m_2 m_3 - m_3 m_1, \\
&\bullet\ V_1 V_2 m_3(m_2 + m_1 - 2m_3) + \lambda(m_1 - m_3)V_3 = 0, \\
&\bullet\ V_1 V_3 m_2(m_1 - 2m_2 + m_3) + \lambda(m_3 - m_1)V_2 = 0, \\
&\bullet\ V_2 V_3 m_1(m_3 - 2m_1 + m_2) + \lambda(m_2 - m_3)V_1 = 0.
\end{aligned} \tag{10}$$

Solving the above equation we get some solutions of (10), satisfying the following result.

Theorem 2.1. *Assume a three dimensional Riemannian Lie group* $(G, \mathfrak{g})$ *where* $\mathfrak{g}$ *is its unimodular Lie algebra expressed by* (5) *with respect to a orthonormal basis* $\{e_1, e_2, e_3\}$. *Then a few nontrivial left invariant hyperbolic Yamabe solitons on* $\mathfrak{g}$ *are as follows*
(1) *if* $m_1 = m_2 = m_3$ *then* $\frac{3}{2}m_1^2 = \mu$ *and for all* $V \in \mathfrak{g}$ *and* $\lambda \in \mathbb{R}$,
(2) *if* $m_1 = m_3 \neq 0$; $\mu = 0$ *then* $V_1 = V_3 = 0$ *and for all* $V_2 \in \mathbb{R}$;
(3) *if* $m_2 = m_3 \neq 0$; $m_1 = 0$; $\mu = 0$ *then* $V_2 = V_3 = 0$ *and for all* $V_1 \in \mathbb{R}$ *and for all* $\lambda \in \mathbb{R}$,
(4) *if* $m_1 \neq 0$; $m_2 = m_3 = 0$; $\mu = -\frac{1}{2}m_1^2$ *then* $V_2 = V_3 = 0$ *and for all* $V_1 \in \mathbb{R}$ *and for all* $\lambda \in \mathbb{R}$.

2.2. Case of Non-unimodular :

Here we let a three dimensional non-unimodular Riemannian Lie algebra $\mathfrak{g}$ and $\mathfrak{u}$ is its two dimensional uni modular kernel. Then there exist an orthonormal basis $\{e_1, e_2, e_3\}$ such that e_1 is orthogonal to $\mathfrak{u}$ and $[e_1, e_2]$, $[e_1, e_3]$ are orthogonal to each other. Then bracket product is expressed as

$$\begin{aligned} &[e_1, e_2] = m_1 e_2 + m_2 e_3, \qquad [e_1, e_3] = m_3 e_2 + m_4 e_3, \\ &[e_2, e_3] = 0, \qquad m_1 + m_4 \neq 0, \qquad m_1 m_3 + m_2 m_4 = 0, \end{aligned} \tag{11}$$

for four real constants m_1, m_2, m_3, m_4.

With respect to the basis $\{e_1, e_2, e_3\}$, the Ricci components are given by [21]

$$S_{ij} = \begin{bmatrix} -m_1^2 - \frac{1}{2}m_2^2 - \frac{1}{2}m_3^2 - m_4^2 - m_2 m_3 & 0 & 0 \\ 0 & -m_1^2 - \frac{1}{2}m_2^2 + \frac{1}{2}m_3^2 - m_1 m_4 & 0 \\ 0 & 0 & -m_4^2 + \frac{1}{2}m_2^2 - \frac{1}{2}m_3^2 - m_1 m_4. \end{bmatrix} \tag{12}$$

and the scalar curvature is

$$s = -2m_1^2 - \frac{1}{2}m_2^2 - \frac{1}{2}m_3^2 - 2m_4^2 - 2m_1 m_4 - m_2 m_3. \tag{13}$$

Choosing an arbitrary vector field $V = V_i e_i \in \mathfrak{g}$, we have

$$£_V g = \begin{bmatrix} 0 & m_1 V_2 + m_3 V_3 & m_2 V_2 + m_4 V_3 \\ m_1 V_2 + m_3 V_3 & -2m_1 V_1 & -(m_2 + m_3) V_1 \\ m_2 V_2 + m_4 V_3 & -(m_2 + m_3) V_1 & -2m_4 V_1 \end{bmatrix}. \tag{14}$$

Using (11) and (14), we get

$$\begin{aligned} &(£_V \circ £_V)(g)_{11} = (4m_1 m_3 + 4m_2 m_4) V_2 V_3 + 2(m_1^2 + m_2^2) V_2^2 + 2(m_3^2 + m_4^2) V_3^2, \\ &(£_V \circ £_V)(g)_{22} = 4m_1^2 V_1^2 + 2m_2(m_2 + m_3) V_1^2, \\ &(£_V \circ £_V)(g)_{33} = 2m_3(m_2 + m_3) V_1^2 + 4m_4^2 V_1^2, \\ &(£_V \circ £_V)(g)_{12} = (-3m_1^2 - 2m_2^2 - m_2 m_3) V_1 V_2 + (-3m_1 m_3 - 2m_2 m_4 - m_3 m_4) V_1 V_3, \\ &(£_V \circ £_V)(g)_{13} = (-m_1 m_2 - 2m_1 m_3 - 3m_2 m_4) V_1 V_2 + (-m_2 m_3 - 2m_3^2 - 3m_4^2) V_1 V_3, \\ &(£_V \circ £_V)(g)_{23} = (m_1 m_2 + 3m_1 m_3 + 3m_2 m_4 + m_3 m_4) V_1^2. \end{aligned} \tag{15}$$

Equation (4) gives us the following system of equations

$$\begin{aligned}
&\bullet (4m_1m_3 + 4m_2m_4)V_2V_3 + 2(m_1^2 + m_2^2)V_2^2 + 2(m_3^2 + m_4^2)V_3^2 \\
&= \mu + 2m_1^2 + 2m_4^2 + \frac{1}{2}m_2^2 + \frac{1}{2}m_3^2 + 2m_1m_4 + m_2m_3, \\
&\bullet 4m_1^2V_1^2 + 2m_2(m_2 + m_3)V_1^2 - 2\lambda m_1V_1 \\
&= \mu + 2m_1^2 + 2m_4^2 + \frac{1}{2}m_2^2 + \frac{1}{2}m_3^2 + 2m_1m_4 + m_2m_3, \\
&\bullet 2m_3(m_2 + m_3)V_1^2 + 4m_4^2V_1^2 - 2\lambda m_4V_1 \\
&= \mu + 2m_1^2 + 2m_4^2 + \frac{1}{2}m_2^2 + \frac{1}{2}m_3^2 + 2m_1m_4 + m_2m_3, \\
&\bullet (-3m_1^2 - 2m_2^2 - m_2m_3)V_1V_2 + (-3m_1m_3 - 2m_2m_4 - m_3m_4)V_1V_3 + \lambda(m_1V_2 + m_3V_3) = 0, \\
&\bullet (-m_1m_2 - 2m_1m_3 - 3m_2m_4)V_1V_2 + (-m_2m_3 - 2m_3^2 - 3m_4^2)V_1V_3 + \lambda(m_2V_2 + m_4V_3) = 0, \\
&\bullet (m_1m_2 + 3m_1m_3 + 3m_2m_4 + m_3m_4)V_1^2 - \lambda(m_2 + m_3)V_1 = 0.
\end{aligned} \tag{16}$$

Solving (16), we find some solutions proving the follwing result.

Theorem 2.2. *Consider a three dimensional Riemannian Lie group* $(G, \mathfrak{g})$ *and* $\mathfrak{g}$ *is non uni modular Lie algebra followed by* (11) *with respect to the orthonormal basis* $\{e_1, e_2.e_3\}$. *Then a few nontrivial left invariant hyperbolic Yamabe soliton on* $\mathfrak{g}$ *are*

(1) *if* $m_1 = m_4 \neq 0$; $m_1 \neq 0$; $m_2 = m_3 = 0$; $\mu = -6m_1^2$; *then* $V_1 = \frac{\lambda}{2m_1}$ *and* $V_2 = 0$; $V_3 = 0$,
(2) *if* $m_2 = m_3$; $m_1 = m_4 = 0$; $\mu = -2m_2^2$ *then* $V_1 = V_2 = V_3 = 0$ *for all* $\lambda \in \mathbb{R}$,
(3) *if* $m_1 \neq 0$; $m_2 = m_3 = m_4 = 0$; $\mu = -2m_1^2$ *then* $V_1 = \frac{\lambda}{2m_1}$, $V_2 = 0$ *for all* $V_3 \in \mathbb{R}$,
(4) *if* $m_1 \neq 0$; $m_2 \neq 0$; $m_3 = m_4 = 0$; $\mu = -2m_1^2 - \frac{1}{2}m_2^2$ *then the system of equations has no solution.*

3. Uni modular Hyperbolic Yamabe Solitons On Three Dimensional Left-Invariant Lorentzian metric

In the Riemannian case, there exists an orthonormal basis $\{e_1, e_2.e_3\}$ but in the case of Lorentzian metric there exists a pseudo orthonormal basis $\{e_1, e_2.e_3\}$, with e_3 time like. The Lie algebra $\mathfrak{g}$ is unimodular if and only if the endomorphism L, defined by $[X, Y] = L(X \times Y)$, is auto adjoint [26]. In this cases four Different form (serge types) can occur. They are following:

$$\mathfrak{g}_g : \text{Serge type } \{3\} : \begin{bmatrix} q & p & -p \\ p & q & 0 \\ p & 0 & q \end{bmatrix}, \qquad \mathfrak{g} : \text{Serge type } \{1z\bar{z}\} : \begin{bmatrix} p & 0 & 0 \\ 0 & q & r \\ 0 & -r & q \end{bmatrix},$$

$$\mathfrak{g} : \text{Serge type } \{11,1\} : \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}, \qquad \mathfrak{g} : \text{Serge type } \{21\} : \begin{bmatrix} p & 0 & 0 \\ 0 & q & \delta \\ 0 & -\delta & q-2\delta \end{bmatrix}.$$

Seperately, we discuss all the cases.

3.1. *Lie Algebra* $\mathfrak{g}_g$*:*

For this algebra there exists $\{e_1, e_2, e_3\}$, a pseudo orthonormal basis with e_3 time like, so that

$$\begin{aligned}
\mathfrak{g}_g : \quad &[e_1, e_2] = m_1e_1 - m_2e_3, \qquad [e_1, e_3] = -m_1e_1 - m_2e_2, \\
&[e_2, e_3] = m_2e_1 + m_1e_2 + m_1e_3, \quad m_1 \neq 0.
\end{aligned} \tag{17}$$

With respect to the basis $\{e_1, e_2, e_3\}$, the Ricci curvature tensor of Lie algebra $\mathfrak{g}_g$ is characterized by [9]

$$S_{ij} = \begin{bmatrix} -\frac{1}{2}m_2^2 & -m_1m_2 & m_1m_2 \\ -m_1m_2 & -2m_1^2 - \frac{1}{2}m_2^2 & 2m_1^2 \\ m_1m_2 & 2m_1^2 & -2m_1^2 + \frac{1}{2}m_2^2 \end{bmatrix} \tag{18}$$

and the scalar curvature is

$$s = -4m_1^2 - \frac{1}{2}m_2^2. \tag{19}$$

Now choose an arbitrary vector field $V = V_ie_i \in \mathfrak{g}_g$, then we get

$$\pounds_V g = \begin{bmatrix} 2m_1(V_2 - V_3) & -m_1V_1 & m_1V_1 \\ -m_1V_1 & 2m_1V_3 & -m_1(V_2 + V_3) \\ m_1V_1 & -m_1(V_2 + V_3) & 2m_1V_2 \end{bmatrix} \tag{20}$$

Using (17) and (20), we obtain

$$\begin{aligned}
(\pounds_V \circ \pounds_V)(g)_{11} &= 4m_1^2V_2^2 - 8m_1^2V_2V_3 - 2m_1m_2V_1V_2 + 4m_1^2V_3^2 + 2m_1m_2V_1V_3, \\
(\pounds_V \circ \pounds_V)(g)_{22} &= 2m_1^2V_1^2 - 2m_1m_2V_1V_2 - 4m_1m_2V_1V_3 - 2m_1^2V_2V_3 + 2m_1^2V_3^2, \\
(\pounds_V \circ \pounds_V)(g)_{33} &= 2m_1^2V_1^2 - 4m_1m_2V_1V_2 - 2m_1m_2V_1V_3 + 2m_1^2V_2V_3 - 2m_1^2V_2^2, \\
(\pounds_V \circ \pounds_V)(g)_{12} &= -3m_1^2V_1V_2 + 3m_1m_2V_2V_3 + 3m_1^2V_1V_3 + m_1m_2V_2^2 + m_1m_2V_1^2 - 4m_1m_2V_3^2, \\
(\pounds_V \circ \pounds_V)(g)_{13} &= -3m_1^2V_1V_3 + 3m_1m_2V_2V_3 + 3m_1^2V_1V_2 - 4m_1m_2V_2^2 + m_1m_2V_3^2 - m_1m_2V_1^2, \\
(\pounds_V \circ \pounds_V)(g)_{23} &= -2m_1^2V_1^2 + m_1^2V_2^2 - m_1^2V_3^2 + 3m_1m_2V_1V_2 + 3m_1m_2V_1V_3.
\end{aligned} \tag{21}$$

Therefore using (19), (20) and (21), (4) gives following system of equations

$$\begin{aligned}
&\bullet\ 4m_1^2V_2^2 - 8m_1^2V_2V_3 - 2m_1m_2V_1V_2 + 4m_1^2V_3^2 + 2m_1m_2V_1V_3 + 2\lambda m_1(V_2 - V_3) = \mu + 4m_1^2 + \frac{1}{2}m_2^2, \\
&\bullet\ 2m_1^2V_1^2 - 2m_1m_2V_1V_2 - 4m_1m_2V_1V_3 - 2m_1^2V_2V_3 + 2m_1^2V_3^2 + 2\lambda m_1V_3 = \mu + 4m_1^2 + \frac{1}{2}m_2^2, \\
&\bullet\ 2m_1^2V_1^2 - 4m_1m_2V_1V_2 - 2m_1m_2V_1V_3 + 2m_1^2V_2V_3 - 2m_1^2V_2^2 + 2\lambda m_1V_2 = -\mu - 4m_1^2 - \frac{1}{2}m_2^2, \\
&\bullet\ -3m_1^2V_1V_2 + 3m_1m_2V_2V_3 + 3m_1^2V_1V_3 + m_1m_2V_2^2 + m_1m_2V_1^2 - 4m_1m_2V_3^2 - \lambda m_1V_1 = 0, \\
&\bullet\ -3m_1^2V_1V_3 + 3m_1m_2V_2V_3 + 3m_1^2V_1V_2 - 4m_1m_2V_2^2 + m_1m_2V_3^2 - m_1m_2V_1^2 + \lambda m_1V_1 = 0, \\
&\bullet\ -2m_1^2V_1^2 + m_1^2V_2^2 - m_1^2V_3^2 + 3m_1m_2V_1V_2 + 3m_1m_2V_1V_3 - \lambda m_1(V_2 + V_3) = 0.
\end{aligned} \tag{22}$$

Inview of (22), we have the following

Theorem 3.1. *Let $(G, \mathfrak{g}_g)$ be a three dimensional Lorentzian Lie group and $\mathfrak{g}_g$ is its unimodular Lie algebra defined by* (17) *with respect to a pseudo orthonormal basis $\{e_1, e_2, e_3\}$, with time like e_3. Then a few non trivial left invariant hyperbolic Yamabe solitons on $\mathfrak{g}_g$ are following*

(1) *if $m_1 = 0$; $m_2 \neq 0$ then $\mu = -\frac{1}{2}m_2^2$ and for all $V_1, V_2, V_3 \in \mathbb{R}$,*

(2) *if $m_1 \neq 0$; $m_2 = 0$; $\lambda = 0$; $\mu = -4m_1^2$ then $V_1 = 0$ and $V_2 = V_3$,*

(3) *if $m_1 \neq 0$; $m_2 \neq 0$ then $V_1 = V_2 = V_3 = 0$ is not a solution of* (22).

3.2. *Lie algebra $\mathfrak{g}$:*

There exists $\{e_1, e_2, e_3\}$, a pseudo orthonormal basis with e_3 time like, so that

$$\begin{aligned}
\mathfrak{g}: \quad &[e_1, e_2] = m_3e_2 - m_2e_3, \qquad [e_1, e_3] = -m_2e_2 + m_3e_3, \\
&[e_2, e_3] = m_1e_1, \qquad m_3 \neq 0.
\end{aligned} \tag{23}$$

The Ricci curvature tensor of Lorentzian Lie algebra $\mathfrak{g}$ is given by [9]

$$S_{ij} = \begin{bmatrix} -\frac{1}{2}m_1^2 - 2m_3^2 & 0 & 0 \\ 0 & \frac{1}{2}m_1^2 - m_1m_2 & m_3(m_1 - 2m_2) \\ 0 & m_3(m_1 - 2m_2) & -\frac{1}{2}m_1^2 + m_1m_2 \end{bmatrix} \tag{24}$$

and the scalar curvature is

$$s = -\frac{1}{2}m_1^2 - 2m_3^2. \tag{25}$$

Now for an arbitrary vector field $V = V_ie_i \in \mathfrak{g}$, the Lie derivative $\pounds_V g$ is

$$\pounds_V g = \begin{bmatrix} 0 & m_3V_2 + (m_1 - m_2)V_3 & (m_2 - m_1)V_2 - m_3V_3 \\ m_3V_2 + (m_1 - m_2)V_3 & -2m_3V_1 & 0 \\ (m_2 - m_1)V_2 - m_3V_3 & 0 & 2m_3V_1 \end{bmatrix} \tag{26}$$

From (23) and (26), we get

$$\begin{aligned}
(\pounds_V \circ \pounds_V)(g)_{11} &= (2m_3^2 - 2m_2^2 + 2m_1m_2)V_2^2 + (-2m_2(m_1 - m_2) - 2m_3^2)V_3^2, \\
(\pounds_V \circ \pounds_V)(g)_{22} &= 4m_3^2V_1^2 + 2m_1m_3V_2V_3 + 2m_1(m_1 - m_2)V_3^2, \\
(\pounds_V \circ \pounds_V)(g)_{33} &= -4m_3^2V_1^2 - 2m_1(m_2 - m_1)V_2^2 + 2m_1m_3V_2V_3, \\
(\pounds_V \circ \pounds_V)(g)_{12} &= (-3m_3^2 + m_2(m_2 - m_1))V_1V_2 + (2m_2m_3 - m_1m_3)V_1V_3, \\
(\pounds_V \circ \pounds_V)(g)_{13} &= (-2m_2m_3 + m_1m_3)V_1V_2 + (3m_3^2 + m_2(m_1 - m_2))V_1V_3, \\
(\pounds_V \circ \pounds_V)(g)_{23} &= 2m_1(m_2 - m_1)V_2V_3 - m_1m_3V_3^2 - m_1m_3V_2^2.
\end{aligned} \tag{27}$$

Therefore using (4), we have the folllowing system of equations

$$\begin{aligned}
&\bullet\ (2m_3^2 - 2m_2^2 + 2m_1m_2)V_2^2 + (-2m_2(m_1 - m_2) - 2m_3^2)V_3^2 = \mu + \frac{1}{2}m_1^2 + 2m_3^2, \\
&\bullet\ 4m_3^2V_1^2 + 2m_1m_3V_2V_3 + 2m_1(m_1 - m_2)V_3^2 - 2\lambda m_3V_1 = \mu + \frac{1}{2}m_1^2 + 2m_3^2, \\
&\bullet\ -4m_3^2V_1^2 - 2m_1(m_2 - m_1)V_2^2 + 2m_1m_3V_2V_3 + 2\lambda m_3V_1 = -\mu - \frac{1}{2}m_1^2 - 2m_3^2, \\
&\bullet\ (-3m_3^2 + m_2(m_2 - m_1))V_1V_2 + (2m_2m_3 - m_1m_3)V_1V_3 + \lambda m_3V_2 + \lambda(m_1 - m_2)V_3 = 0, \\
&\bullet\ (-2m_2m_3 + m_1m_3)V_1V_2 + (3m_3^2 + m_2(m_1 - m_2))V_1V_3 + \lambda(m_2 - m_1)V_2 - \lambda m_3V_3 = 0, \\
&\bullet\ 2m_1(m_2 - m_1)V_2V_3 - m_1m_3V_3^2 - m_1m_3V_2^2 = 0.
\end{aligned} \tag{28}$$

By (28), we get the following

Theorem 3.2. *Let $(G, \mathfrak{g})$ be a three dimensional Lorentzian Lie group and $\mathfrak{g}$ is its unimodular Lorentzian Lie algebra expressed by* (23) *with respect to a pseudo orthonormal basis $\{e_1, e_2, e_3\}$, with time like vector field e_3. Then a few non trivial left invariant hyperbolic Yamabe soliton on $\mathfrak{g}$ are listed bellow*

(1) *if $m_1 = m_2 \neq 0$; $m_3 = 0$; $\mu = -\frac{1}{2}m_1^2$ then $V_1, V_2, V_3 \in \mathbb{R}$,*

(2) *if $m_3 \neq 0$; $m_1 = m_2 = 0$; $\mu = -2m_3^2$ then $V_1 = 0$, $V_2 = V_3$,*

(3) *if $m_1 \neq m_2$; $m_3 = 0$; $\mu = -\frac{1}{2}m_1^2$ then $V_2 = V_3 = 0$ and $V_1 \in \mathbb{R}$,*

(4) *if $\mu \neq 0$ then $V_1 = V_2 = V_3 = 0$ is not a solution of* (28).

3.3. *Lie algebra* $\mathfrak{g}$ *:*

For a pseudo orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 time like, the bracket product is described as

$$\mathfrak{g}: \quad [e_1, e_2] = -m_3e_3, \quad [e_1, e_3] = -m_2e_2, \quad [e_2, e_3] = m_1e_1. \tag{29}$$

With respect to the basis $\{e_1, e_2, e_3\}$, the component of the Ricci curvature are described by [9]

$$S_{ij} = \begin{bmatrix} -\frac{1}{2}(m_2^2 - m_1^2 + m_3^2) - m_2m_3 & 0 & 0 \\ 0 & \frac{1}{2}(m_1^2 - m_2^2 + m_3^2) - m_1m_3 & 0 \\ 0 & 0 & -\frac{1}{2}(m_3^2 - m_1^2 - m_2^2) + m_1m_2 \end{bmatrix} \tag{30}$$

and the scalar curvature is

$$s = \frac{3}{2}m_1^2 - \frac{1}{2}m_2^2 - \frac{1}{2}m_3^2 + m_1m_2 - m_2m_3 - m_1m_3. \tag{31}$$

Choosing an arbitrary vector field $V = V_ie_i \in \mathfrak{g}$, we have

$$£_Vg = \begin{bmatrix} 0 & (m_1 - m_2)V_3 & (m_3 - m_1)V_2 \\ (m_1 - m_2)V_3 & 0 & (m_2 - m_3)V_1 \\ (m_3 - m_1)V_2 & (m_2 - m_3)V_1 & 0 \end{bmatrix}. \tag{32}$$

Using (29) and (32), we obtain

$$\begin{aligned} (£_V \circ £_V)(g)_{11} &= -2m_3(m_3 - m_1)V_2^2 - 2m_2(m_1 - m_2)V_3^2, \\ (£_V \circ £_V)(g)_{22} &= 2m_3(m_2 - m_3)V_1^2 + 2m_1(m_1 - m_2)V_3^2, \\ (£_V \circ £_V)(g)_{33} &= 2m_2(m_2 - m_3)V_1^2 - 2m_1(m_3 - m_1)V_2^2, \\ (£_V \circ £_V)(g)_{12} &= -m_3(m_2 + m_1 - 2m_3)V_1V_2, \\ (£_V \circ £_V)(g)_{13} &= m_2(-2m_2 + m_3 + m_1)V_1V_3, \\ (£_V \circ £_V)(g)_{23} &= m_1(m_2 + m_3 - 2m_1)V_2V_3. \end{aligned} \tag{33}$$

Therefore from (4), we have the following system of equations

$$\begin{aligned} &\bullet -2m_3(m_3 - m_1)V_2^2 - 2m_2(m_1 - m_2)V_3^2 = \mu - \frac{3}{2}m_1^2 + \frac{1}{2}m_2^2 + \frac{1}{2}m_3^2 - m_1m_2 + m_2m_3 + m_1m_3, \\ &\bullet\ 2m_3(m_2 - m_3)V_1^2 + 2m_1(m_1 - m_2)V_3^2 = \mu - \frac{3}{2}m_1^2 + \frac{1}{2}m_2^2 + \frac{1}{2}m_3^2 - m_1m_2 + m_2m_3 + m_1m_3, \\ &\bullet\ 2m_2(m_2 - m_3)V_1^2 - 2m_1(m_3 - m_1)V_2^2 = -\mu + \frac{3}{2}m_1^2 - \frac{1}{2}m_2^2 - \frac{1}{2}m_3^2 + m_1m_2 - m_2m_3 - m_1m_3, \\ &\bullet -m_3(m_2 + m_1 - 2m_3)V_1V_2 + \lambda(m_1 - m_2)V_3 = 0, \\ &\bullet\ m_2(-2m_2 + m_3 + m_1)V_1V_3 + \lambda(m_3 - m_1)V_2 = 0, \\ &\bullet\ m_1(m_2 + m_3 - 2m_1)V_2V_3 + \lambda(m_2 - m_3)V_1 = 0. \end{aligned} \tag{34}$$

By virtue of (34), we obtain the following

Theorem 3.3. *Consider a three dimensional Lorentzian Lie group $(G, \mathfrak{g})$ and its unimodular Lorentzian Lie algebra $(G, \mathfrak{g})$ defined by (29) with respect to a pseudo orthonormal basis $\{e_1, e_2, e_3\}$, with time like vector field e_3. Then we have some non trivial left invariant hyperbolic Yamabe solitons on $\mathfrak{g}$, which are following*

(1) *if $m_1 = m_2 = m_3$ then $\mu = -\frac{1}{2}m_1^2$ for all $V \in \mathbb{R}$,*

(2) *if $m_1 = m_2$; $m_3 = 0$; $\mu = 2m_1^2$ then $V_1 = V_2 = 0$ and for all $V_3 \in \mathbb{R}$,*

(3) *if $m_1 = m_3$; $m_2 = 0$; $\mu = 0$; $\lambda = 0$ then $V_1^2 = V_3^2$ and $V_2 = 0$,*

(4) *if $m_1 = m_3$; $m_2 = 0$; $\mu = 0$; $\lambda \neq 0$ then $V_1 = V_2 = V_3 = 0$.*

3.4. Lie algebra $\mathfrak{g}$:

For this Lie algebra there exists a pseudo orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 time like, such that

$$\begin{aligned} \mathfrak{g} : \quad &[e_1e_2] = -e_2 + (2\eta - m_2)e_3, \qquad [e_1, e_3] = -m_2e_2 + e_3, \\ &[e_2, e_3] = m_1e_1, \qquad \eta = \pm 1. \end{aligned} \tag{35}$$

With respect to the basis $\{e_1, e_2, e_3\}$, the Ricci curvature tensor of Lorentzian Lie algebra $\mathfrak{g}$ is characterized by [9]

$$S_{ij} = \begin{bmatrix} -\frac{1}{2}m_1^2 & 0 & 0 \\ 0 & \frac{1}{2}m_1^2 + 2\eta(m_1 - m_2) - m_1m_2 + 2 & m_1 + 2(\eta - m_2) \\ 0 & m_1 + 2(\eta - m_2) & -\frac{1}{2}m_1^2 + m_1m_2 + 2 - 2\eta m_2 \end{bmatrix} \tag{36}$$

and the scalar curvature is

$$s = 2\eta(m_1 - 2m_2) - \frac{1}{2}m_1^2 + 4. \tag{37}$$

For a left invariant vector field $V = V_ie_i \in \mathfrak{g}$, we find

$$£_Vg = \begin{bmatrix} 0 & -V_2 + (m_1 - m_2)V_3 & (m_2 - m_1 - 2\eta)V_2 - V_3 \\ -V_2 + (m_1 - m_2)V_3 & 2V_1 & 2\eta V_1 \\ (m_2 - m_1 - 2\eta)V_2 - V_3 & 2\eta V_1 & 2V_1 \end{bmatrix} \tag{38}$$

Using (35) and (38) and with respect to the basis $\{e_1, e_2, e_3\}$, we have

$$\begin{aligned}
&(£_V \circ £_V)(g)_{11} = (2 + 2(2\eta - m_2)(m_2 - m_1 - 2\eta))V_2^2 + (-4m_1 + 8m_2 - 8\eta)V_2V_3 \\
&+ (-2m_2(m_1 - m_2) - 2)V_3^2, \\
&(£_V \circ £_V)(g)_{22} = (4 - 4\eta(2\eta - m_2))V_1^2 + 2m_1(m_1 - m_2)V_3^2 - 2m_1V_2V_3, \\
&(£_V \circ £_V)(g)_{33} = (4m_2\eta - 4)V_1^2 - 2m_1(m_2 - m_1 - 2\eta)V_2^2 + 2m_1V_2V_3, \\
&(£_V \circ £_V)(g)_{12} = (-3 + (2\eta - m_2)(-m_2 + m_1 + 4\eta))V_1V_2 + (4\eta - 4m_2 + m_1)V_1V_3, \\
&(£_V \circ £_V)(g)_{13} = (-4m_2 + m_1 + 4\eta)V_1V_2 + (-2\eta m_2 + m_1m_2 - m_2^2 + 3)V_1V_3, \\
&(£_V \circ £_V)(g)_{23} = (-4\eta + 4m_2)V_1^2 + (2m_1(m_2 - m_1) - 2\eta m_1)V_2V_3 - m_1V_3^2 + m_1V_2^2.
\end{aligned} \tag{39}$$

Therefore (4) gives following

$$\begin{aligned}
&\bullet\ (2 + 2(2\eta - m_2)(m_2 - m_1 - 2\eta))V_2^2 + (-4m_1 + 8m_2 - 8\eta)V_2V_3 + (-2m_2(m_1 - m_2) - 2)V_3^2 \\
&= \mu - 2\eta(m_1 - 2m_2) + \frac{1}{2}m_1^2 - 4, \\
&\bullet\ (4 - 4\eta(2\eta - m_2))V_1^2 + 2m_1(m_1 - m_2)V_3^2 - 2m_1V_2V_3 + 2\lambda V_1 = \mu - 2\eta(m_1 - 2m_2) + \frac{1}{2}m_1^2 - 4, \\
&\bullet\ (4m_2\eta - 4)V_1^2 - 2m_1(m_2 - m_1 - 2\eta)V_2^2 + 2m_1V_2V_3 + 2\lambda V_1 = -\mu + 2\eta(m_1 - 2m_2) - \frac{1}{2}m_1^2 + 4, \\
&\bullet\ (-3 + (2\eta - m_2)(-m_2 + m_1 + 4\eta))V_1V_2 + (4\eta - 4m_2 + m_1)V_1V_3 - \lambda V_2 + \lambda(m_1 - m_2)V_3 = 0, \\
&\bullet\ (-4m_2 + m_1 + 4\eta)V_1V_2 + (-2\eta m_2 + m_1m_2 - m_2^2 + 3)V_1V_3 + \lambda(m_2 - m_1 - 2\eta)V_2 - \lambda V_3 = 0, \\
&\bullet\ (-4\eta + 4m_2)V_1^2 + (2m_1(m_2 - m_1) - 2\eta m_1)V_2V_3 - m_1V_3^2 + m_1V_2^2 + 2\lambda\eta V_1 = 0.
\end{aligned} \tag{40}$$

Using (40), we get the following

Theorem 3.4. *Let $(G, \mathfrak{g})$ be a three dimensional Lorentzian Lie group and its unimodular Lorentzian Lie algebra $\mathfrak{g}$ expressed by* (35) *with respect to a pseudo orthonormal basis $\{e_1, e_2, e_3\}$, with time like vector field e_3. Then some non trivial left invariant hyperbolic Yamabe solitons on $\mathfrak{g}$ are*
(1) *if $m_1 = 1$; $m_2 = 0$; $\eta = 1$; $\mu = \frac{11}{2}$ then $V_1 = V_2 = V_3 = 0$ for all $\lambda \in \mathbb{R}$,*
(2) *if $m_1 = 0$; $m_2 = 1$; $\eta = 1$; $\mu = 0$; $\lambda = 0$ then $V_1, V_2, V_3 \in \mathbb{R}$,*
(3) *if $m_1 = 0$; $m_2 = 1$; $\eta = 1$; $\mu = 0$; $\lambda \neq 0$ then $V_1 = V_2 = V_3 = 0$.*

4. Non uni-modular Hyperbolic Yamabe Solitons On Three Dimensional Left-Invariant Lorentzian metric

In this section we consider a three dimensional non unimodular Lorentzian lie algebra with respect to the pseudo orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time like. Here we discuss following three distinct cases.

4.1. Lie algebra $\mathfrak{g}$ *:*

For the Lie algebra $\mathfrak{g}$, there is a pseudo orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 time like, such that that

$$\mathfrak{g}: \quad \begin{aligned} &[e_1, e_2] = 0, \\ &[e_1, e_3] = m_1e_1 + m_2e_2, \\ &[e_2, e_3] = m_3e_1 + m_4e_2, \quad m_1 + m_4 \neq 0, \quad m_1m_3 + m_2m_4 = 0. \end{aligned} \tag{41}$$

The Ricci curvature tensor of Lie algebra $\mathfrak{g}$ is

$$S_{ij} = \begin{bmatrix} m_1^2 + \frac{1}{2}m_2^2 - \frac{1}{2}m_3^2 + m_1m_4 & 0 & 0 \\ 0 & m_1m_4 - \frac{1}{2}m_2^2 + \frac{1}{2}m_3^2 + m_4^2 & 0 \\ 0 & 0 & -m_1^2 - \frac{1}{2}m_2^2 - \frac{1}{2}m_3^2 - m_4^2 - m_2m_3 \end{bmatrix} \tag{42}$$

and the scalar curvature is

$$s = 2m_1m_4 - \frac{1}{2}m_2^2 - \frac{1}{2}m_3^2 - m_4^2 - m_2m_3. \tag{43}$$

Now choose a left invariant arbitrary vector field $V = V_ie_i \in \mathfrak{g}$, then we get

$$\pounds_V g = \begin{bmatrix} 2m_1V_3 & (m_2 + m_3)V_3 & -m_1V_1 - m_3V_2 \\ (m_2 + m_3)V_3 & 2m_4V_3 & -m_2V_1 - m_4V_2 \\ -m_1V_1 - m_3V_2 & -m_2V_1 - m_4V_2 & 0 \end{bmatrix} \tag{44}$$

From (41) and (44), we have

$$\begin{aligned} (\pounds_V \circ \pounds_V)(g)_{11} &= (4m_1^2 + 2m_2^2 + 2m_2m_3)V_3^2, \\ (\pounds_V \circ \pounds_V)(g)_{22} &= (2m_3(m_2 + m_3) + 4m_4^2)V_3^2, \\ (\pounds_V \circ \pounds_V)(g)_{33} &= 2(m_1^2 + m_2^2)V_1^2 + 2(m_3^2 + m_4^2)V_2^2 + 4(m_1m_3 + m_2m_4)V_1V_2, \\ (\pounds_V \circ \pounds_V)(g)_{12} &= (m_1(m_2 + m_3) + 2m_2m_4 + 2m_1m_3 + m_4(m_2 + m_3))V_3^2, \\ (\pounds_V \circ \pounds_V)(g)_{13} &= (-3m_1^2 - 2m_2^2 - m_2m_3)V_1V_3 + (-3m_1m_3 - 2m_2m_4 - m_3m_4)V_2V_3, \\ (\pounds_V \circ \pounds_V)(g)_{23} &= (-2m_1m_3 - m_1m_2 - 3m_2m_4)V_1V_3 + (-2m_3^2 - 3m_4^2 - m_2m_3)V_2V_3. \end{aligned} \tag{45}$$

Therefore using (43), (44) and (45) we get the following system of equations

$$\begin{aligned} &\bullet\ (4m_1^2 + 2m_2^2 + 2m_2m_3)V_3^2 + 2\lambda m_1V_3 = \mu - 2m_1m_4 + \frac{1}{2}m_2^2 + \frac{1}{2}m_3^2 + m_2m_3, \\ &\bullet\ (2m_3(m_2 + m_3) + 4m_4^2)V_3^2 + 2\lambda m_4V_3 = \mu - 2m_1m_4 + \frac{1}{2}m_2^2 + \frac{1}{2}m_3^2 + m_2m_3, \\ &\bullet\ 2(m_1^2 + m_2^2)V_1^2 + 2(m_3^2 + m_4^2)V_2^2 + 4(m_1m_3 + m_2m_4)V_1V_2 \\ &= -\mu + 2m_1m_4 - \frac{1}{2}m_2^2 - \frac{1}{2}m_3^2 - m_2m_3, \\ &\bullet\ (m_1(m_2 + m_3) + 2m_2m_4 + 2m_1m_3 + m_4(m_2 + m_3))V_3^2 + \lambda(m_2 + m_3)V_3 = 0, \\ &\bullet\ (-3m_1^2 - 2m_2^2 - m_2m_3)V_1V_3 + (-3m_1m_3 - 2m_2m_4 - m_3m_4)V_2V_3 - \lambda m_1V_1 - \lambda m_3V_2 = 0, \\ &\bullet\ (-2m_1m_3 - m_1m_2 - 3m_2m_4)V_1V_3 + (-2m_3^2 - 3m_4^2 - m_2m_3)V_2V_3 - \lambda m_2V_1 - \lambda m_4V_2 = 0. \end{aligned} \tag{46}$$

Solving equation (46), we have the following

Theorem 4.1. *Let* $(G, \mathfrak{g})$ *be a three dimensional Lorentzian Lie group and* $\mathfrak{g}$ *is its non uni-modular Lie algebra followed by* (41) *with respect to a pseudo orthonormal basis* $\{e_1, e_2, e_3\}$, *with time like* e_5. *Then a few non trivial left invariant hyperbolic Yamabe solitons on* $\mathfrak{g}$ *are*

(1) *if* $m_1 = m_4$; $m_2 = m_3 = 0$; $\lambda = 0$; $\mu = 2m_1^2$ *then* $V_1^2 = -V_2^2$ *and* $V_3 = 0$,

(2) *if* $m_2 = m_3$; $m_1 = m_4 = 0$; $\mu = -2m_2^2$; $\lambda = 0$ *then* $V_1 = V_2 = V_3 = 0$,

(3) *if* $m_2 = m_3 = m_4 = 0$; $m_1 \neq 0$; $\mu = 0$; *then* $V_1 = V_3 = 0$ *and* $V_2\mathbb{R}$,

(4) *if* $m_1 = m_4 \neq 0$; $m_2 = -m_3$; $\mu = 2m_1^2$ *then* $V_1 = V_2 = 0$ *and* $V_3 = -\frac{\lambda}{2m_1}$.

4.2. Lie algebra $\mathfrak{g}$ *:*

In $\mathfrak{g}$ there exists a pseudo orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 time like, so that

$$\begin{aligned} \mathfrak{g}: \quad & [e_1, e_2] = m_1 e_2 + m_2 e_3, \\ & [e_1, e_3] = m_3 e_2 + m_4 e_3, \\ & [e_2, e_3] = 0, \quad m_1 + m_4 \neq 0, \quad m_1 m_3 - m_2 m_4 = 0. \end{aligned} \tag{47}$$

With respect to the basis $\{e_1, e_2, e_3\}$, the Ricci tensor of Lie algebra $\mathfrak{g}$ is given by [9]

$$S_{ij} = \begin{bmatrix} \frac{1}{2}m_2^2 - m_1^2 + \frac{1}{2}m_3^2 - m_4^2 - m_2 m_3 & 0 & 0 \\ 0 & \frac{1}{2}m_2^2 - m_1^2 - \frac{1}{2}m_3^2 - m_1 m_4 & 0 \\ 0 & 0 & m_1 m_4 + \frac{1}{2}m_2^2 - \frac{1}{2}m_3^2 + m_4^2 \end{bmatrix} \tag{48}$$

and the scalar curvature is

$$s = -2m_1^2 + \frac{3}{2}m_2^2 - \frac{1}{2}m_3^2 - m_2 m_3. \tag{49}$$

For an arbitrary vector field $V = V_i e_i \in \mathfrak{g}$, we get

$$\pounds_V g = \begin{bmatrix} 0 & m_1 V_2 + m_3 V_3 & -m_2 V_2 - m_4 V_3 \\ m_1 V_2 + m_3 V_3 & -2m_1 V_1 & (m_2 - m_3) V_1 \\ -m_2 V_2 - m_4 V_3 & (m_2 - m_3) V_1 & 2m_4 V_1 \end{bmatrix} \tag{50}$$

Using (47) and (50), we have

$$\begin{aligned} (\pounds_V \circ \pounds_V)(g)_{11} &= 2m_1^2 V_2^2 + 2m_3^2 V_3^2 - 2m_4^2 V_3^2 - 2m_2^2 V_2^2 + (4m_1 m_3 - 4m_2 m_4) V_2 V_3, \\ (\pounds_V \circ \pounds_V)(g)_{22} &= (4m_1^2 - 2m_2(m_2 - m_3)) V_1^2, \\ (\pounds_V \circ \pounds_V)(g)_{33} &= (-2m_2 m_3 + 2m_3^2 - 4m_4^2) V_1^2, \\ (\pounds_V \circ \pounds_V)(g)_{12} &= (-3m_1^2 + 2m_2^2 - m_2 m_3) V_1 V_2 + (-3m_1 m_3 + 2m_2 m_4 - m_3 m_4) V_1 V_3, \\ (\pounds_V \circ \pounds_V)(g)_{13} &= (m_1 m_2 + 3m_2 m_4 - 2m_1 m_3) V_1 V_2 + (-2m_3^2 + 3m_4^2 + m_2 m_3) V_1 V_3, \\ (\pounds_V \circ \pounds_V)(g)_{23} &= (-m_1 m_2 + 3m_1 m_3 - 3m_2 m_4 + m_3 m_4) V_1^2. \end{aligned} \tag{51}$$

Therefore (4) gives following system of equations

$$\begin{aligned} &\bullet\ 2m_1^2 V_2^2 + 2m_3^2 V_3^2 - 2m_4^2 V_3^2 - 2m_2^2 V_2^2 + (4m_1 m_3 - 4m_2 m_4) V_2 V_3 \\ &= \mu + 2m_1^2 - \frac{3}{2}m_2^2 + \frac{1}{2}m_3^2 + m_2 m_3, \\ &\bullet\ (4m_1^2 - 2m_2(m_2 - m_3)) V_1^2 - 2\lambda m_1 V_1 = \mu + 2m_1^2 - \frac{3}{2}m_2^2 + \frac{1}{2}m_3^2 + m_2 m_3, \\ &\bullet\ (-2m_2 m_3 + 2m_3^2 - 4m_4^2) V_1^2 + 2\lambda m_4 V_1 = -\mu - 2m_1^2 + \frac{3}{2}m_2^2 - \frac{1}{2}m_3^2 - m_2 m_3, \\ &\bullet\ (-3m_1^2 + 2m_2^2 - m_2 m_3) V_1 V_2 + (-3m_1 m_3 + 2m_2 m_4 - m_3 m_4) V_1 V_3 + \lambda m_1 V_2 + \lambda m_3 V_3 = 0, \\ &\bullet\ (m_1 m_2 + 3m_2 m_4 - 2m_1 m_3) V_1 V_2 + (-2m_3^2 + 3m_4^2 + m_2 m_3) V_1 V_3 - \lambda m_2 V_2 - \lambda m_4 V_3 = 0, \\ &\bullet\ (-m_1 m_2 + 3m_1 m_3 - 3m_2 m_4 + m_3 m_4) V_1^2 + \lambda (m_2 - m_3) V_1 = 0. \end{aligned} \tag{52}$$

Solving (52), we have the following

Theorem 4.2. *Let* $(G, \mathfrak{g})$ *be a three dimensional Lorentzian Lie group and* $\mathfrak{g}$ *is its non-unimodular Lie algebra defined by* (47) *with respect to a pseudo orthonormal basis* $\{e_1, e_2, e_3\}$, *with time like* e_3. *Then a few non trivial left invariant hyperbolic Yamabe solitons on* $\mathfrak{g}$ *are*

(1) *if* $m_1 = m_4 \neq 0$; $m_2 = m_3 = 0$; $\mu = -2m_1^2$; $\lambda = 0$ *then* $V_2^2 = V_3^2$ *and* $V_1 = 0$,
(2) *if* $m_2 = m_3 \neq 0$; $m_1 = m_4 = 0$; $\mu = 0$ *then* $V_2 = V_3 = 0$ *and* $V_1 \in \mathbb{R}$,
(3) *if* $m_1 = m_4 \neq 0$; $m_2 = m_3 = 0$; $V_2 = V_3 = 0$ *then* $\mu = -2m_1^2$ *and* $V_1 = \frac{\lambda}{2m_1}$,
(4) *if* $m_1 = m_2 = m_3 = m_4 \neq 0$; $\mu = -2m_1^2$ *then* $V_1 = \frac{\lambda}{2m_1}$ *and* $V_2 + V_3 = 1$.

4.3. Lie algebra $\mathfrak{g}$ *:*

There exists $\{e_1, e_2, e_3\}$, a pseudo orthonormal basis with e_3 time like, so that

$$\begin{aligned} \mathfrak{g}: \quad & [e_1, e_2] = -m_1e_1 - m_2e_2 - m_2e_3, \\ & [e_1, e_3] = m_1e_1 + m_2e_2, +m_2e_3 \\ & [e_2, e_3] = m_3e_1 + m_4e_2 + m_4e_3, \quad m_1 + m_4 \neq 0, \quad m_1m_3 = 0. \end{aligned} \tag{53}$$

With respect to the basis $\{e_1, e_2, e_3\}$, the Ricci curvature tensor of Lie algebra $\mathfrak{g}$ is

$$S_{ij} = \begin{bmatrix} -\frac{1}{2}m_3^2 & 0 & 0 \\ 0 & m_1m_4 - m_1^2 + \frac{1}{2}m_3^2 - m_2m_3 & m_1^2 - m_1m_4 + m_2m_3 \\ 0 & m_1^2 - m_1m_4 + m_2m_3 & m_1m_4 - m_1^2 - \frac{1}{2}m_3^2 - m_2m_3 \end{bmatrix} \tag{54}$$

and the scalar curvature is

$$s = -2m_1^2 - \frac{1}{2}m_3^2 - 2m_2m_3 + 2m_1m_4. \tag{55}$$

With respect to an arbitrary vector field $V = V_ie_i \in \mathfrak{g}$, we get

$$\begin{aligned} (£_Vg)(e_1, e_1) &= -2m_1(V_2 - V_3), \\ (£_Vg)(e_1, e_2) &= m_1V_1 - m_2V_2 + (m_2 + m_3)V_3, \\ (£_Vg)(e_1, e_3) &= -m_1V_1 + (m_2 - m_3)V_2 - m_2V_3, \\ (£_Vg)(e_2, e_2) &= 2m_2V_1 + 2m_4V_3, \\ (£_Vg)(e_2, e_3) &= -2m_2V_1 - m_4V_2 - m_4V_3, \\ (£_Vg)(e_3, e_3) &= 2m_2V_1 + 2m_4V_2. \end{aligned} \tag{56}$$

Using (53) and (56), we obtain

$$\begin{aligned} (£_V \circ £_V)(g)_{11} &= (4m_1^2 + 2m_2m_3)V_2^2 + (-8m_1^2 - 4m_2m_3)V_2V_3 + (4m_1^2 + 2m_2m_3)V_3^2, \\ (£_V \circ £_V)(g)_{22} &= (-2m_1m_2 - 2m_2m_4)V_1V_2 + (2m_1m_2 + 4m_1m_3 + 2m_2m_4)V_1V_3 \\ &\quad + (-2m_2m_4 - 2m_4^2)V_2V_3 + (2m_3(m_2 + m_3) + 2m_4^2)V_3^2 + 2m_1^2V_1^2, \\ (£_V \circ £_V)(g)_{33} &= 2m_1^2V_1^2 + (-2m_1m_2 - 2m_2m_4 + 4m_1m_3)V_1V_2 + (2m_1m_2 + 2m_2m_4)V_1V_3 \\ &\quad - 2m_3(m_2 - m_3)V_2^2 + (2m_2m_3 + 2m_4^2)V_2V_3, \\ (£_V \circ £_V)(g)_{12} &= (-3m_1^2 - m_2m_3)V_1V_2 + (m_2m_4 + m_1m_4)V_2^2 \\ &\quad + (-2m_1m_2 - 3m_1m_3 - 2m_2m_4 - m_3m_4)V_2V_3 + (3m_1^2 + m_2m_3)V_1V_3 \\ &\quad + (m_1(m_2 + m_3) + m_4(m_2 + m_3) + 2m_1m_3)V_3^2, \\ (£_V \circ £_V)(g)_{13} &= (3m_1^2 + m_2m_3)V_1V_2 + (-m_1m_2 + 3m_1m_3 - m_4(m_2 - m_3))V_2^2 \\ &\quad + (-m_1m_2 - m_2m_4)V_3^2 + (2m_1m_2 - 3m_1m_3 + 2m_2m_4 - m_3m_4)V_2V_3 \\ &\quad + (-3m_1^2 - m_2m_3)V_1V_3, \\ (£_V \circ £_V)(g)_{23} &= -2m_1^2V_1^2 + (m_2m_3 + m_4^2)V_2^2 + (-m_2m_3 - m_4^2)V_3^2 \\ &\quad + (2m_1m_2 + 2m_2m_4 - 2m_1m_3)V_1V_2 \\ &\quad + (-2m_1m_2 - 2m_2m_4 - 2m_1m_3)V_1V_3 - 2m_3^2V_2V_3. \end{aligned} \tag{57}$$

From the equations (53), (56) and (57), we have the following system of equations

$$\begin{aligned}
&\bullet\ (4m_1^2 + 2m_2m_3)V_2^2 + (-8m_1^2 - 4m_2m_3)V_2V_3 + (4m_1^2 + 2m_2m_3)V_3^2 - 2\lambda m_1(V_2 - V_3) \\
&= \mu + 2m_1^2 + \frac{1}{2}m_3^2 + 2m_2m_3 - 2m_1m_4, \\
&\bullet\ (-2m_1m_2 - 2m_2m_4)V_1V_2 + (2m_1m_2 + 4m_1m_3 + 2m_2m_4)V_1V_3 + (-2m_2m_4 - 2m_4^2)V_2V_3 \\
&+ (2m_3(m_2 + m_3) + 2m_4^2)V_3^2 + 2m_1^2V_1^2 + 2\lambda m_2V_1 + 2\lambda m_4V_3 \\
&= \mu + 2m_1^2 + \frac{1}{2}m_3^2 + 2m_2m_3 - 2m_1m_4, \\
&\bullet\ 2m_1^2V_1^2 + (-2m_1m_2 - 2m_2m_4 + 4m_1m_3)V_1V_2 + (2m_1m_2 + 2m_2m_4)V_1V_3 - 2m_3(m_2 - m_3)V_2^2 \\
&+ (2m_2m_3 + 2m_4^2)V_2V_3 + 2\lambda m_2V_1 + 2\lambda m_4V_2 = -\mu - 2m_1^2 - \frac{1}{2}m_3^2 - 2m_2m_3 + 2m_1m_4, \\
&\bullet\ (-3m_1^2 - m_2m_3)V_1V_2 + (m_2m_4 + m_1m_4)V_2^2 + (-2m_1m_2 - 3m_1m_3 - 2m_2m_4 - m_3m_4)V_2V_3 \\
&+ (3m_1^2 + m_2m_3)V_1V_3 + (m_1(m_2 + m_3) + m_4(m_2 + m_3) + 2m_1m_3)V_3^2 + \lambda m_1V_1 \\
&- \lambda m_2V_2 + \lambda(m_2 + m_3)V_3 = 0, \\
&\bullet\ (3m_1^2 + m_2m_3)V_1V_2 + (-m_1m_2 + 3m_1m_3 - m_4(m_2 - m_3))V_2^2 + (-m_1m_2 - m_2m_4)V_3^2 \\
&+ (2m_1m_2 - 3m_1m_3 + 2m_2m_4 - m_3m_4)V_2V_3 + (-3m_1^2 - m_2m_3)V_1V_3 - \lambda m_1V_1 \\
&+ \lambda(m_2 - m_3)V_2 - \lambda m_2V_3 = 0, \\
&\bullet\ -2m_1^2V_1^2 + (m_2m_3 + m_4^2)V_2^2 + (-m_2m_3 - m_4^2)V_3^2 + (2m_1m_2 + 2m_2m_4 - 2m_1m_3)V_1V_2 \\
&+ (-2m_1m_2 - 2m_2m_4 - 2m_1m_3)V_1V_3 - 2m_3^2V_2V_3 - 2\lambda m_2V_1 - \lambda m_4V_2 - \lambda m_4V_3 = 0.
\end{aligned} \tag{58}$$

After solving (58), we get the following

Theorem 4.3. *Let $(G, \mathfrak{g})$ be a three dimensional Lorentzian Lie group and $\mathfrak{g}$ is its non-unimodular Lie algebra defined by* (53) *with respect to a pseudo orthonormal basis $\{e_1, e_2, e_3\}$, with time like e_3. Then a few non trivial left invariant hyperbolic Yamabe solitons on $\mathfrak{g}$ are*

(1) *if $m_2 = m_4$; $m_1 = m_3 = 0$; $\mu = 0$; $\lambda = 0$; then $V_2 = V_3 = 0$ and $V_1 \in \mathbb{R}$,*
(2) *if $m_2 \neq 0$; $m_4 \neq 0$; $m_1 = m_3 = 0$; $\mu = 0$ then $V_2 = 0$; $V_3 = -\frac{\lambda}{m_4}$ and $V_1 \in \mathbb{R}$,*
(3) *if $m_1 \neq 0$; $m_2 = m_3 = m_4 = 0$; $\mu = -2m_1^2$ then $V_1 = 0$ and $V_2 = V_3$.*

5. Hyperbolic Yamabe Solitons on Three dimensional Special Lie Groups

Assume a Lie algebra $\mathfrak{g}$ (dim ≥ 2) described by

$$[u, v] = l(u)v - l(v)u, \qquad u, v \in \mathfrak{g}, \tag{59}$$

where l is a linear map, $l : \mathfrak{g} \to \mathbb{R}$. The sectional curvature of that Lie algebra $\mathfrak{g}$ followed by (59) is constant [21], $K = -||l||^2$. Now we choose a basis $\{e_1, e_2, e_3\}$ for a three dimensional lie algebra $\mathfrak{g}$. Then from (59), we have

$$\begin{aligned}
\mathfrak{g}: \quad & [e_1, e_2] = m_2e_1 - m_1e_2, \\
& [e_1, e_3] = m_3e_1 - m_1e_3, \\
& [e_2, e_3] = m_3e_2 - m_2e_3,
\end{aligned} \tag{60}$$

for three real constants $m_1 = l(e_1), m_2 = l(e_2), m_3 = l(e_3)$

When we assume left invariant Riemannian metric g on $\mathfrak{g}$, we choose an orthonormal basis $\{e_1, e_2, e_3\}$. Then we have Ricci curvature tensor, $\pounds_V g$ and $(\pounds_V \circ \pounds_V)(g)$ with respect to the basis $\{e_1, e_2, e_3\}$ and any left invariant vector $V = V_ie_i$.

Now equation (4) gives the following system of equations

$$
\begin{aligned}
&\bullet (4m_2^2+2m_1^2)V_2^2+(4m_3^2+2m_1^2)V_3^2+8m_2m_3V_2V_3+2m_1m_2V_1V_2+2m_1m_3V_1V_3\\
&+2\lambda m_2V_2+2\lambda m_3V_3=\mu+6m_1^2+6m_2^2+6m_3^2,\\
&\bullet (4m_1^2+2m_2^2)V_1^2+(4m_3^2+2m_2^2)V_3^2+8m_1m_3V_1V_3+2m_1m_2V_1V_2+2m_2m_3V_2V_3\\
&+2\lambda m_1V_1+2\lambda m_3V_3=\mu+6m_1^2+6m_2^2+6m_3^2,\\
&\bullet (4m_1^2+2m_3^2)V_1^2+(4m_2^2+2m_3^2)V_2^2+8m_1m_2V_1V_2+2m_1m_3V_1V_3+2m_2m_3V_2V_3\\
&+2\lambda m_1V_1+2\lambda m_2V_2=\mu+6m_1^2+6m_2^2+6m_3^2,\\
&\bullet (-3m_1^2-3m_2^2)V_1V_2-3m_1m_3V_2V_3-3m_2m_3V_1V_3-m_1m_2V_1^2-m_1m_2V_2^2\\
&+2m_1m_2V_3^2-\lambda m_2V_1-\lambda m_1V_3=0,\\
&\bullet (-3m_1^2-3m_3^2)V_1V_3-3m_2m_3V_1V_2-3m_1m_2V_2V_3-m_1m_3V_1^2-m_1m_3V_3^2\\
&+2m_1m_3V_2^2-\lambda m_3V_1-\lambda m_1V_3=0,\\
&\bullet (-3m_2^2-3m_3^2)V_2V_3-3m_1m_3V_1V_2-3m_1m_2V_1V_3-m_2m_3V_2^2-m_2m_3V_3^2\\
&+2m_2m_3V_1^2-\lambda m_3V_2-\lambda m_2V_3=0.
\end{aligned}
\tag{61}
$$

and if we assume left invariant Lorentzian metric g on $\mathfrak{g}$ is expressed by (60), we choose a pseudo orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time like, then for any left invariant vector $V = V_ie_i \in \mathfrak{g}$, we get the Ricci tensor, $\pounds_V g$ and $(\pounds_V \circ \pounds_V)(g)$.

Now, equation (4) gives us the following

$$
\begin{aligned}
&\bullet (4m_2^2+2m_1^2)V_2^2+(4m_3^2-2m_1^2)V_3^2+8m_2m_3V_2V_3+2m_1m_2V_1V_2+2m_1m_3V_1V_3\\
&+2\lambda m_2V_2+2\lambda m_3V_3=\mu+2(m_1^2+m_2^2-m_3^2),\\
&\bullet (4m_1^2+2m_2^2)V_1^2+(4m_3^2-2m_2^2)V_3^2+8m_1m_3V_1V_3+2m_1m_2V_1V_2+2m_2m_3V_2V_3\\
&+2\lambda m_1V_1+2\lambda m_3V_3=\mu+2(m_1^2+m_2^2-m_3^2),\\
&\bullet (-4m_1^2+2m_3^2)V_1^2+(-4m_2^2+2m_3^2)V_2^2-8m_1m_2V_1V_2-2m_1m_3V_1V_3-2m_2m_3V_2V_3\\
&-2\lambda m_1V_1-2\lambda m_2V_2=-\mu-2(m_1^2+m_2^2-m_3^2),\\
&\bullet (-3m_1^2-3m_2^2)V_1V_2-3m_1m_3V_2V_3-3m_2m_3V_1V_3-m_1m_2V_1^2-m_1m_2V_2^2\\
&-2m_1m_2V_3^2-\lambda m_2V_1-\lambda m_1V_3=0,\\
&\bullet (3m_1^2-3m_3^2)V_1V_3-3m_2m_3V_1V_2+3m_1m_2V_2V_3-m_1m_3V_1^2+m_1m_3V_3^2\\
&+2m_1m_3V_2^2-\lambda m_3V_1+\lambda m_1V_3=0,\\
&\bullet -3m_1m_3V_1V_2+3m_1m_2V_1V_3+(3m_2^2-3m_3^2)V_2V_3-m_2m_3V_2^2+m_2m_3V_3^2\\
&+2m_2m_3V_1^2-\lambda m_3V_2+\lambda m_2V_3=0.
\end{aligned}
\tag{62}
$$

From the above, we see that there is a small change of sign between (61) and (62) due to the differences of signature of the metrics and this slight change is accountable for difference of the solution. Solving (61) and (62), we prove the following.

Theorem 5.1. *Consider a three dimensional Lie algebra expressed by* (60).

(1) *If g is left invariant Riemannian metric with respect to a orthonormal basis $\{e_1, e_2, e_3\}$. Then a few nontrivial left invariant hyperbolic Yamabe soliton on $\mathfrak{g}$ are as follows*
(*i*) *if $m_1 = m_2 = m_3$ and $\mu = -18m_1^2$ then $V_1 = V_2 = V_3 = 0$,*
(*ii*) *if $m_1 = m_2 = 0$; $m_3 \neq 0$; $\mu = -6m_3^2$; $\lambda = 0$ then $V_3 = 0$ and $V_1^2 = -V_2^2$,*
(*iii*) *if $m_1 = m_3 = 0$; $\mu = -6m_2^2$; $\lambda = 0$ then $V_1 = V_2 = 0$ and $V_3 \in \mathbb{R}$.*

(2) *If g is left invariant Lorentzian metric with respect to a pseudo orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time like. Then a few nontrivial left invariant hyperbolic Yamabe solitons on $\mathfrak{g}$ are as follows*
(i) *if $m_1 = m_2 = m_3$ and $\mu = -2m_1^2$ then $V_1 = V_2 = V_3 = 0$,*
(ii) *if $m_1 = m_2 = 0$; $m_3 \neq 0$; $\mu = 2m_3^2$; then $V_1 = 0$ and $V_2^2 = V_3^2$.*

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